EDUCATIVE COMMENTARY ON JEE 2007 MATHEMATICS PAPERS

Contents

Unlike last year, this year the JEE candidates were allowed to take the question papers with them. Last year an application under the Right to Information Act had to be filed just to get the official text of the question paper. Welcome as the change is, one hopes that in future it will be followed by a display of the correct answers on the official websites of JEE, immediately after the examination. This will allow the public at large to point out errors, if any, which can arise because of misprints, ambiguous interpretations, and once in a rare while, because of some mistake in papersetting. Ideally, after displaying the correct answers, comments from the public may be invited within a reasonable time, studied and then the answers should be frozen. This would be the most equitable procedure. It will also give the candidates an early, realistic idea of their scores, so that they can act accordingly. (For example, if they realise that their chances of success are poor, they can concentrate on other examinations, or start preparing for the next JEE in case another attempt is allowed.)

In the commentary below, even though the official text of the questions is now available, it is reproduced for ready reference before giving the answer and comments. The questions in Paper 1 are numbered serially from 1 to 22 while those in Paper 2 have been numbered from 23 to 44. In the actual examination, they were numbered from 45 to 66 in each paper and the order was different in different versions. In each paper there are four sections with 9 questions of 3 marks each in Section I, 4 questions of 3 marks each in Section II, 6 questions of 4 marks each in Section III and 3 questions of 6 marks each in Section IV. Assuming that a candidate gives equal time to all three subjects in each paper, this means that for each mark he has just about 45 seconds.

As in the case of the educative commentaries on the JEE papers of the last few years, the references given here refer to the author's *Educative JEE Mathematics*, unless otherwise stated.

PAPER 1

Contents

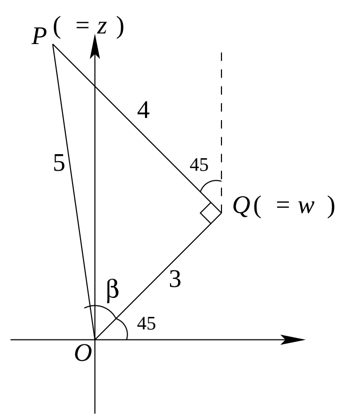
SECTION I

Each question has only one correct answer.

Q.1 A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. From there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. Then position of P in the Argand diagram is

(A) $3e^{i\pi/4} + 4i$ (B) $(3 - 4i)e^{i\pi/4}$
(C) $(4 + 3i)e^{i\pi/4}$ (D) $(3 + 4i)e^{i\pi/4}$

Answer and Comments: (D). A very straightforward problem once you know what an Argand diagram is. Denote the origin by O and the intermediate point (reached by the first walk) as Q and denote the complex number corresponding to Q by w. Then $e^{i\pi/4}$ is a complex number of unit length along the direction $\overrightarrow{OQ}$ Since the length of OQ is 3 units, we get



$$w = 3e^{i\pi/4} \qquad (1)$$

The line from Q to P makes an angle $3\pi/4$ with the x-axis. So, $e3\pi i/4$ is a complex number of unit length in its direction. As the length of QP is 4 units, we get

$$z - w = 4e^{3\pi i/4} \qquad (2)$$

From (1) and (2), we get

$$\begin{aligned} z = w + (z - w) &= 3e^{i\pi/4} + 4e^{3\pi i/4} \\ &= e^{i\pi/4}(3 + e^{i\pi/2}) \\ &= e^{i\pi/4}(3 + 4i) \end{aligned} \tag{3}$$

It is tempting to try to do the problem a little 'cleverly' by observing that since $\angle OQP$ is a right angle, OP is of 5 units length. So, if α is the angle made by OP with the positive x-axis then

$$z = 5e^{i\alpha} \tag{4}$$

To find α, we let $\beta = \angle POQ$. Then $\alpha = \beta + \pi/4$. So, (4) gives

$$z = 5e^{i\beta}e^{i\pi/4} = 5e^{i\pi/4}(\cos\beta + i\sin\beta) \tag{5}$$

From the right angled triangle OQP, we get $\cos\beta = \frac{3}{5}$ and $\sin\beta = \frac{4}{5}$. If we substitute these values into (5), we get the answer. But this method is not necessarily simpler. Moreover, if in the original problem, the journey had more than two parts, then the alternate solution will be very messy to say the least. The first solution, however, goes through no matter how many parts the journey has. Suppose the man walks along straight line segments from O to P_1, then from P_1 to P_2, and so on and the last segment is from P_{n-1} to P_n. Call O as P_0 and let $z_0(= 0), z_1, z_2, \ldots, z_n$ be the complex numbers corresponding to $P_0, P_1, \ldots, P_n$ respectively. If we are given α_k as the angle which the k-th segment makes with the x-axis, and r_k as the length of this segment, then we have

$$z_k - z_{k-1} = r_k e^{i\alpha_k} \tag{6}$$

for $k = 1, 2, \ldots, n$. Adding these equations and putting $z_0 = 0$ gives

$$z_n = \sum_{k=1}^{n} r_k e^{i\alpha_k} \tag{7}$$

as the position of the point reached at the end of the journey. So, this is a problem where a straightforward approach is also easier and more general.

Q.2 The number of solutions of the pair of equations

$$2\sin^2\theta - \cos 2\theta = 0 \tag{1}$$

$$\text{and}\quad 2\cos^2\theta - 3\sin\theta = 0 \tag{2}$$

in the interval $[0, 2\pi]$ is

(A) zero (B) one (C) two (D) four

Answer and Comments: (C). A straightforward problem about trigonometric equations. The first equation can be written as a quadratic in $\sin\theta$ viz.

$$2\sin^2\theta - 1 + 2\sin^2\theta = 0 \tag{3}$$

or $\sin^2\theta = \frac{1}{4}$. Hence $\sin\theta = \pm\frac{1}{2}$. Each possibility has two solutions in the interval $[0, 2\pi]$. But we have to see which of these also satisfies (2). This can be done even by a simple substitution (i.e. without solving (2)). Both the values of $\sin\theta$ give $\cos^2\theta = \frac{3}{4}$. But only $\sin\theta = \frac{1}{2}$ satisfies (2). So the common solutions are those of $\sin\theta = \frac{1}{2}$. The interval $[0, 2\pi]$ has two such solutions. (They can be identified explicitly as $\pi/6$ and $5\pi/6$. But that is not asked in the problem. In fact for every value λ with $|\lambda| < 1$, the equation $\sin\theta = \lambda$ has exactly two solutions in the interval $[0, 2\pi]$ as one sees easily from the graph of the sine function. The problem would have been a little more testing if the value of $\sin\theta$ had come out to be some number λ of this type for which the two values of θ cannot be identified easily, e.g. $\sin\theta = \frac{1}{3}$.

Q.3 A hyperbola, having the transverse axis of length $2\sin\theta$ is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is

(A) $x^2\text{cosec}^2\theta - y^2\sec^2\theta = 1$ (B) $x^2\sec^2\theta - y^2\text{cosec}^2\theta = 1$
(C) $x^2\sin^2\theta - y^2\cos^2\theta = 1$ (D) $x^2\cos^2\theta - y^2\sin^2\theta = 1$

Answer and Comments: (A). Like some other JEE problems in the past, this one appears to be designed more to test the vocabulary about conics! The equation of the ellipse, when cast in the standard form, is

$$\frac{x^2}{4} + \frac{y^2}{3} = 1 \tag{1}$$

An easy calculation gives the eccentricity of this ellipse as $\sqrt{1 - \frac{3}{4}} = \frac{1}{2}$. Hence its foci lie at the points $(\pm 1, 0)$ on the x-axis. Therefore these are also the foci of the hyperbola we are after. In particular we see that its axes are along the co-ordinate axes and hence its equation can be taken in the standard form, viz.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{2}$$

Here a, b are two unknowns. To determine them, we need two equations in a and b. One of these is provided by the condition that the foci of (2) are at $(\pm 1, 0)$ while the other is provided by the condition that the transverse axis has length $2\sin\theta$. The latter implies, instantaneously, that $a = \sin\theta$. Since the foci are at $(\pm 1, 0)$ we get $ae = 1$, i.e. $e\sin\theta = 1$ or, $e = \text{cosec}\ \theta$, where e is the eccentricity of the hyperbola (and not of the ellipse). But we also have another relation involving the eccentricity, say e, of (2), viz.

$$b^2 = a^2(e^2 - 1) \tag{3}$$

Putting $a = \sin\theta$ and $e = \text{cosec}\ \theta$, we get $b = \sin\theta\sqrt{\text{cosec}^2\theta - 1} = \sin\theta\cot\theta = \cos\theta$. So, the equation of the hyperbola is

$$\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1 \tag{4}$$

which is the same as (A).

In the statement of the problem, it is not given that the hyperbola is in the standard form (2). This needs some reasoning as given above. In a multiple choice test, a candidate can simply *assume* that the hyperbola is in the standard form and still get the correct answer. In fact, some candidates may not even do that consciously. For them, a hyperbola is *always* in the form (2). Since no reasoning needs to be given, there is no way to distinguish between these three types of students. In fact, the sincere ones will be fools because they will be spending more time than the unscrupulous ones. To avoid this, the question could have been designed so that neither the ellipse nor the hyperbola has its axes along the coordinate axes. But then the time given to answer the question would be inadequate. This is the price to pay when the paper-setters have to design the questions so as to conform to a certain given pattern which does not permit awarding credit proportional to the time needed.

Q.4 The number of distinct real values of λ , for which the vectors $-\lambda^2\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-\lambda^2\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-\lambda^2\hat{k}$ are coplanar, is

(A) zero (B) one (C) two (D) three

Answer and Comments: (C). Questions of this type are so common that hardly any commentary is needed as far as the method is concerned. The easiest way is to set the determinant of the coefficients equal to zero. That is,

$$\begin{vmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0 \tag{1}$$

When expanded, this becomes a cubic in λ^2 and hence a sixth degree equation in λ, which, after some simplification, becomes

$$(\lambda^2+1)(-\lambda^4+\lambda^2+2) = 0 \tag{2}$$

which further factorises as

$$(\lambda^2+1)^2(\lambda^2-2) = 0 \tag{3}$$

As λ is real, the only possible roots are $\lambda = \pm\sqrt{2}$. So (1) has two distinct real roots.

Note that the problem does not ask to solve (2). All that is needed is the number of roots it has. This leads to some short cuts to the solution, of varying degrees of legitimacy. On the face of it, the equation (2) could have as many as six real roots. But since the equation is a cubic in λ^2, its roots pair off as negatives of each other with the rider that if at all 0 is a root then it must be a multiple root. But one sees from the determinant in (1) that 0 is not a root. By inspection, we can identify $\lambda^2 = 2$ as a solution since in that case the three rows of the determinant add to the identically 0 row. A clever student can tick-mark the correct answer even at this stage, because since 0 is not a root, if at all there are any other real roots, then then there would be at least two more real roots and hence the answer to the problem would be at least four. But as none of the given alternatives exceeds three, the correct answer must be two.

A student who can reason like this deserves to be rewarded in terms of the time he saves. But once again, the sad reality is that because of the objective format of the testing, there is no way to distinguish such a student from one who, after observing that the determinant vanishes for $\lambda^2 = 2$, simply ticks the correct answer without any further thought.

Q.5 The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$

(A) on the left of $x = c$	(B) on the right of $x = c$
(C) at no point	(D) at all points

Answer and Comments: (A). The most straightforward way is to actually find the point of intersection of the two lines, viz. the tangent, say L_1, at (c, e^c) and the secant, say L_2, joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$. Since $\frac{d}{dx}e^x = e^x$ for all x, the equation of L_1 is

$$y - e^c = e^c(x - c) \tag{1}$$

The equation of L_2 is

$$y - e^{c-1} = \frac{e^{c+1} - e^{c-1}}{2}(x - c + 1) \tag{2}$$

Let $P = (x_0, y_0)$ be the point of intersection of L_1 and L_2. The question does not ask us to find P explicitly in terms of c. All that is needed is its x- coordinate. Eliminating y between (1) and (2) and solving for x, we get

$$x_0 = c + \frac{2 - \frac{1}{e} - e}{e - \frac{1}{e} - 2} \tag{3}$$

The problem is now reduced to determining the sign of the second term on the R. H. S. As a consequence of the A.M.-G.M. inequality, the numerator is negative since $e \neq 1$. For this we do not need even the approximate value of e. However, we do need it for the sign of the denominator. It is enough to know that $e > 2.7$. For, then we have $e - 2 > 0.7$ and also $\frac{1}{e} < \frac{1}{2} = 0.5$. Thus we see that the denominator is positive and therefore the second term on the R.H.S. of (3) is negative. Therefore, P, the point of intersection, lies on the left of the line $x = c$.

The method above is straightforward and would have applied to any (differentiable) function instead of the function $f(x) = e^x$. But there is a more instructive and elegant way to do the problem by observing some properties of this function. The question deals with the relationship between the tangent and a chord of the graph of this function. So the property most relevant in this connection is the concavity of the function $y = e^x$ as defined on p. 504. The second derivative of the function e^x is e^x itself and it is positive everywhere. So, the function is concave upward on every interval and hence, in particular on the interval $[c-1, c+1]$, which is presently relevant. Further, the first derivative, which is also e^x, is also positive everywhere. So the function $y = e^x$ is increasing on every interval. Therefore its graph is qualitatively similar to that in Figure (a) on p. 503.

Let us interpret the problem in terms of the graph of the function $y = e^x$. Call the points $(c-1, e^{c-1})$, $(c+1, e^{c+1})$ and (c, e^c) on this graph as P_1, P_2 and P_3 respectively. The question deals with the point of intersection, say P, of the tangent L_1 at P_3 with the secant L_2 passing through P_1 and P_2. As the function is concave upwards, the point P_3 lies below the secant L_2. So it is obvious that P will lie to the left or to the right of P_3 depending on whether the slope of L_1 is less than or greater than that of L_2. (The situation would have been exactly the opposite had the function been concave downwards. However, whether the function is increasing or decreasing has no role here.)

This key observation leads to the solution without any unnecessary computation, that is without finding P, and indeed, without even finding its x-coordinate. We already know the slopes of L_1 and L_2 as e^c and $\dfrac{e^{c+1} - e^{c-1}}{2}$ respectively. In fact, that's how we obtained equations (1) and (2) above. To decide which of these two slopes is bigger, we divide both by e^c which is positive. Then the comparison reduces to that between 1 and $(e - 1/e)/2$, or equivalently, between 2 and $e - \frac{1}{e}$. We already saw that $e - \frac{1}{e} - 2 > 0$. Hence we get the same answer as before, viz. (A).

Actually, the two solutions are not radically different. If we want to give a proof of the key observation above, we would need to compute the x-coordinate of P and that would indeed require us to write down (1), (2) and (3). But the point is that in the second solution this work is avoided by using an easy and obvious consequence of concavity upwards. The first solution is very mechanical in its approach. Once again, being a multiple choice test, there is no way to tell which method a candidate has followed.

Q.6 $\displaystyle\lim_{x \to \pi/4} \frac{\int_2^{\sec^2 x} f(t)\, dt}{x^2 - \frac{\pi^2}{16}}$ equals

(A) $\frac{8}{5} f(2)$ (B) $\frac{2}{\pi} f(2)$ (C) $\frac{2}{\pi} f(\frac{1}{2})$ (D) $4f(2)$

Answer and Comments: (A). Obviously the key idea is the second form of the fundamental theorem of calculus (p. 628). The denominator tends to 0. If we factor it as $(x - \pi/4)(x + \pi/4)$, the second factor poses no problem. The first factor (the villain) has to be tackled by combining it with the numerator. Specifically, call the desired limit as L. Then we have

$$L = \frac{2}{\pi} L_1 \tag{1}$$

where

$$L_1 = \lim_{x \to \pi/4} \frac{\int_2^{\sec^2 x} f(t)\, dt}{x - \frac{\pi}{4}} \tag{2}$$

Thus the problem now reduces to finding the limit L_1. It can be recognised immediately as the derivative of a certain function at the point $\frac{\pi}{4}$. To be precise,

$$L_1 = g'(\pi/4) \tag{3}$$

where

$$g(x) = \int_2^{\sec^2 x} f(t)\, dt \tag{4}$$

To differentiate g w.r.t. x we put $u = \sec^2 x$ and write $g(x)$ as $h(u)$ where $h(u) = \displaystyle\int_2^u f(t)\, dt$ and apply the Chain Rule. By the second form of the fundamental theorem of calculus, we have

$$h'(u) = f(u) \tag{5}$$

and so by the chain rule,

$$g'(x) = h'(u)\frac{du}{dx} = f(u) 2 \sec x \sec x \tan x = 2\sec^2 x \tan x f(\sec^2 x) \tag{6}$$

Putting $x = \pi/4$, we get

$$L_1 = g'(\pi/4) = 4f(2) \tag{7}$$

and hence finally from (1), $L = \dfrac{8}{\pi} f(2)$.

Note that we have not used the L'Hôpital's rule, a hot favorite with most students. Instead, we have factored the denominator and evaluated the troublesome part of the limit by recognising it as a derivative. L'ôpital's rule can, of course, be used if we can't think of this trick. The calculation is mechanical. The first derivative of the denominator is $2x$ which tends to a finite non-zero limit as x tends to $\pi/4$. So we need to apply the rule only once. The derivative of the numerator is found by combining the chain rule and the second form of the fundamental theorem of calculus.

So even with the L'Hôpital's rule, the computations are essentially the same as before and we get

$$L = \lim_{x \to \pi/4} \frac{(2\sec x)(\sec x \tan x) f(\sec^2 x)}{2x} = \frac{4f(2)}{\pi/2} = \frac{8}{\pi} f(2) \tag{8}$$

So, this is yet another instance where L'Hôpital's rule is not absolutely essential but convenient. The question does not specify what properties the function $f(t)$ has. The very fact that the statement of the question deals with its integral can be taken to mean that $f(t)$ is integrable on every interval, or at least on an interval containing the point 2. But, to apply the fundamental theorem of calculus you need a little more than mere integrability. For example, continuity of the integrand is sufficient. It would have been nice had the question stated this explicitly. But perhaps such a degree of theoretical perfection is hardly to be expected at the JEE level.

Q.7 Let $f(x)$ be differentiable on the interval $(0, \infty)$ and $f(1) = 1$. Suppose $\lim\limits_{t \to x} \dfrac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$. Then $f(x)$ is

(A) $\dfrac{1}{3x} + \dfrac{2x^2}{3}$ (B) $\dfrac{-1}{3x^3} + \dfrac{4x^2}{3}$ (C) $\dfrac{-1}{x} + \dfrac{2}{x^2}$ (D) $\dfrac{1}{x}$

Answer and Comments: (A). This is another problem where the key idea is to recognise a given limit as some derivative. Rewriting $t^2 f(x) - x^2 f(t)$ as $x^2 t^2 \left(\dfrac{f(x)}{x^2} - \dfrac{f(t)}{t^2} \right)$ we see that the given limit is nothing but $-x^4 \dfrac{d}{dx} \left(\dfrac{f(x)}{x^2} \right)$. On computing this derivative, the given condition about the function $f(x)$ translates as

$$x^2 f'(x) - 2x f(x) = -1 \tag{1}$$

As in the last question, we could have arrived at (1) more efficiently using L'Hôpital's rule to find the limit in the statement of the problem. Care has to be taken to differentiate w.r.t. t rather than w.r.t. x since the variable is t and x is a constant as far as the limit is concerned. In detail,

$$\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = \lim_{t \to x} \frac{2t f(x) - x^2 f'(t)}{1} = 2x f(x) - x^2 f'(x) \tag{2}$$

from which (1) follows immediately.

Whichever way we arrive at (1), it is a differential equation and now the problem is reduced to solving it subject to the initial condition $f(1) = 1$. Let us call $f(x)$ as y. Then $f'(x)$ becomes $\dfrac{dy}{dx}$ and the equation takes a more familiar form after dividing it throughout by x^2.

$$\frac{dy}{dx} - 2\frac{1}{x} y = -\frac{1}{x^2} \tag{3}$$

This is a linear differential equation. Instead of writing its solution using the readymade formula (38) on p. 707, let us first identify the integrating factor as $e^{F(x)}$ where $F(x) = \int -2\frac{dx}{x} = -\ln(x^2)$. Hence the integrating factor is $\dfrac{1}{x^2}$. Multiplying (2) throughout by $1/x^2$ we get

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = -\frac{1}{x^4} \tag{4}$$

The L. H. S. is simply $\dfrac{d}{dx}\left(\dfrac{y}{x^2}\right)$. So, the general solution of (1) is

$$f(x) = y = x^2 \int -\frac{dx}{x^4} = x^2\left(\frac{1}{3x^3} + c\right) \tag{5}$$

The initial condition $f(1) = 1$ determines c as $\dfrac{2}{3}$. Hence, finally,

$$f(x) = \frac{1}{3x} + \frac{2x^2}{3} \tag{6}$$

Note that (4) could have been obtained directly from (1) by dividing it throughout by x^4. But it would take an unusual perceptivity to conceive this trick. For most mortals, going through (3) is the right thing to do.

The question is good because it asks you to first construct a differential equation in a somewhat unusual way (far different than constructing it from some geometric property of the graph of $f(x)$, which is very common) and then to solve it. But on a proportionate basis the time allowed for it is only a couple of minutes, which is far too short. Also an unscrupulous student can bypass the second part (viz. solving (1)) by simply substituting the given answers in (1) one-by-one. The order of the answers would benefit such a student because the very first alternative is the right one. To avoid such sneaky attempts, the question could have asked for some numerical value, say $f(2)$. That would mean a little more work for the sincere student. But in a competitive examination, that is a lesser evil than giving an unfair advantage to dishonesty.

Q.8 One Indian and four American men and their wives are to be seated randomly around a circular table. Then the conditional probability that the Indian man is seated adjacent to his wife, given that each American man is seated adjacent to his wife is

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{2}{5}$ (D) $\dfrac{1}{5}$

Answer and Comments: (C). Although posed as a probability problem, this problem is essentially a counting problem, requiring us to count certain kinds of arrangements. As it often happens in counting problems, the answer is very simple if you can identify the essence. Note that it is not at all important in this problem to count the total number of circular arrangements of the 10 persons, (which, incidentally, is 9!). Our concern is solely with the set, say S, of those arrangements where every American couple sits together. So, we might as well regard each such couple as a single object. There are four such objects. In addition, we have the Indian man and his wife. So we have in all 6 objects and the number of their circular permutations is 5!. But, in each such arrangement, members of the same couple can exchange their seats. As there are 4 such couples these exchanges can occur in 2^4 different ways. All put together, we get

$$|S| = 2^4 \times 5! \tag{1}$$

Now let A be the subset of S consisting of those arrangements (already from S) in which the Indian man also sits next to his wife. Then $|A|$ can be calculated in the same manner as we obtained $|S|$. A is effectively the set of all circular arrangements of 5 objects, with the understanding that each arrangement can give rise to 2^5 similar arrangements. Hence

$$|A| = 2^5 \times 4! \tag{2}$$

The desired probability is merely the ratio $\dfrac{|A|}{|S|}$ which comes out to be $\dfrac{2}{5}$.

There is a clear duplication of reasoning in deriving (1) and (2). Still, it is a good problem, which can be done within a couple of minutes if the key idea (viz. treating a couple as a single object) strikes you quickly. The computations involved are simple and not prone to errors. (Note that it is in general not a good idea to expand the factorials in (1) and (2). The time in doing so is a waste because later many of their factors are going to get cancelled. The same holds for the powers 2^4 and 2^5. This is a general rule applicable to many counting problems and especially to probability problems.)

Q.9 Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\dfrac{\alpha}{2}$, 2β be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is

(A) $\dfrac{2}{9}(p-q)(2q-p)$ (B) $\dfrac{2}{9}(q-p)(2p-q)$

(C) $\dfrac{2}{9}(q-2p)(2q-p)$ (D) $\dfrac{2}{9}(2p-q)(2q-p)$

Answer and Comments: (D). There is a clever way to choose this answer among the given ones. Here we have two monic quadratic equations. The relationship between them is that one of the roots of the second equation is half of one of the roots of the first, while the other root of the second equation is twice the other root of the first. As a result, the product of the two roots is the same for both the equations. This obvious conclusion is reflected in the fact that the constant term of both the equations is the same, viz. r. But a more subtle observation is that this relationship between the two equations is symmetric. That is, if we interchange the two equations, then too one of the roots of the second equation will be half of one of the roots of the first and the second one will be twice. (Basically, this happens because the numbers $1/2$ and 2 are reciprocals of each other. Had the roots of the second equation been, say $\alpha/2$ and 3β, then there would have been no symmetry. Of course, in that case the constant terms of the two equations would have been different.) As a result, the roles of p and q are symmetric in this problem. Therefore, an expression for r in terms of p and q will necessarily be symmetric w.r.t. p and q, i.e. should remain unchanged if p and q are interchanged. Among the given alternatives, only (D) satisfies this condition. So, if at all one of them is correct, it has to be (D).

It is hard to say if the paper-setters simply missed this sneaky reasoning or whether they decided to keep the problem as it is despite noticing it. In that case, the intention of the problem is not so much to test quadratic equations but some other type of cleverness, which sometimes pays off.

Anyway, let us also arrive at the answer honestly. From the formulas for the sum and the product of the roots of a quadratic, we get a system of three equations in the two unknowns α and β, viz.

$$\alpha + \beta = p \tag{1}$$

$$\frac{\alpha}{2} + 2\beta = q \tag{2}$$

$$\alpha\beta = r \tag{3}$$

If we solve (1) and (2) for α and β (in terms of p and q), then by putting these values in (3) we shall get an expression for r in terms of p and q. Both (1) and (2) are linear equations in α and β. Sparing the details, the solution is

$$\alpha = \frac{2(2p-q)}{3}, \quad \beta = \frac{(2q-p)}{3} \tag{4}$$

So, finally, $r = \alpha\beta = \frac{2}{9}(2p-q)(2q-p)$.

SECTION II

Assertion - Reason Type

This section contains four questions numbered 10 to 13. Each question contains STATEMENT-1 (Assertion) and STATEMENT-2 (Reason). Each question has four choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

Q.10 Let $H_1, H_2, \ldots, H_n$ be mutually exclusive and exhaustive events with $P(H_i) > 0$, $i = 1, 2, \ldots, n$. Let E be any other event with $0 < P(E) < 1$.

STATEMENT-1 : $P(H_i/E) > P(E/H_i)P(H_i)$ for $i = 1, 2, \ldots, n$.

because

STATEMENT-2 : $\sum_{i=1}^{n} P(H_i) = 1$.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (B). The truth of Statement-2 follows from the law of disjunction and the fact that the events $H_1, H_2, \ldots, H_n$ are mutually exclusive and exhaustive. The truth of the first statement requires the law of conditional probability. Using it we have,

$$P(H_i/E) = \frac{P(H_i \cap E)}{P(E)} \tag{1}$$

$$P(E/H_i)P(H_i) = \frac{P(E \cap H_i)}{P(H_i)}P(H_i) = P(E \cap H_i) \tag{2}$$

Now, $E \cap H_i$ and $H_i \cap E$ are the same events. So, the numerator of the R.H.S. of (1) equals the R.H.S. of (2). But the denominator $P(E)$ is given to be less than 1. So the R.H.S. of (1) is bigger than that of (2). Thus we see that Statement-1 and Statement-2 are both true. Now the question remains whether the second statement is an explanation of the first one. The answer is in the negative because in proving Statement-1, we nowhere used Statement-2. In fact, in Statement-1, only one H_i is involved at a time and its truth for each such H_i is quite independent of the mutual relationship among the events $H_1, H_2, \ldots, H_n$.

The wording of the question is confusing because in the preamble, the word 'reason' is used, which has the connotation of an implication. Call Statement-1 and Statement-2 as p and q respectively. Then to say that q is a (valid) reason for p means that the implication statement $q \longrightarrow p$ is true. Logically, $q \longrightarrow p$ means $p \vee \neg q$, or verbally, 'either p holds or else q fails'. So, whenever p is true, the implication statement $q \longrightarrow p$ is automatically true regardless of whether q is true or not. A candidate who takes this interpretation is likely to mark (A) as the correct answer. But he will be marked wrong even though he has the correct thinking. This is unfair.

Q.11 Tangents are drawn from the point $(17, 7)$ to the circle $x^2 + y^2 = 169$.

STATEMENT-1 : The tangents are mutually perpendicular.

because

STATEMENT-2 : The locus of the point from which mutually perpendicular tangents can be drawn to the circle is $x^2 + y^2 = 338$.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (A). The locus referred to in Statement-2 is a circle concentric with the given circle and having a radius which is $\sqrt{2}$ times the radius of the given circle. This is a well-known fact which follows simply from the Pythagoras theorem and elementary properties of tangents. (The locus is sometimes called the **director circle** of the given circle. Of course you don't have to know all this to answer the question correctly.) So, the locus in Statement-2 is the circle $x^2+y^2 = 2\times 169 = 338$. Hence Statement-2 is true. Also since $17^2 + 7^2 = 289 + 49 = 338$, we see that Statement-1 is also true. And since we have derived it from Statement-2, it follows that the latter is an explanation of the former. (Unlike in the last question, there is no confusion here.)

Q.12 Let the vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{RS}$, $\overrightarrow{ST}$, $\overrightarrow{TU}$, and $\overrightarrow{UP}$ represent the sides of a regular hexagon.

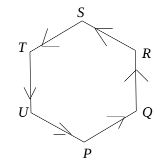
STATEMENT-1 : $\overrightarrow{PQ} \times (\overrightarrow{RS} + \overrightarrow{ST}) \neq \vec{\mathbf{0}}$

because

STATEMENT-2 : $\overrightarrow{PQ} \times \overrightarrow{RS} = \vec{\mathbf{0}}$ and $\overrightarrow{PQ} \times \overrightarrow{ST} \neq \vec{\mathbf{0}}$

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (C). The entire problem is focused on the vanishing of the cross product of two (non-zero) vectors. This happens if and only if the vectors have either the same or the opposite directions.



In the present case, the vectors $\overrightarrow{PQ}$ and $\overrightarrow{ST}$ are oppositely directed and so their cross product is $\mathbf{0}$. This means that the second part of Statement-2 is false. That makes Statement-2 false regardless of whether the first part is true. Still, for the sake of completeness, we note that it is also false since the sides PQ and RS are not parallel to each other. As for Statement-1, the vector $\overrightarrow{RS} + \overrightarrow{ST}$ is simply the vector $\overrightarrow{RT}$. Its cross product with $\overrightarrow{PQ}$ is non-zero because the two do not represent parallel sides of the polygon. Hence Statement-1 is true.

Q.13 Let $F(x)$ be an indefinite integral of $\sin^2 x$.

STATEMENT-1 : The function $F(x)$ satisfies $F(x+\pi) = F(x)$ for all real x.

because

STATEMENT-2 : $\sin^2(x + \pi) = \sin^2 x$ for all real x.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (D). Trivially, Statement-2 is true because $\sin(x + \pi) = -\sin x$ for all real x. To decide if Statement-1 is true, one can first find $F(x)$ explicitly. It comes out to be $\frac{x}{2} - \frac{\sin 2x}{4} + c$ for some constant c. Clearly because of the first term, $F(x)$ will not satisfy the given equality for any x. But if one understands what the question is really asking then it is easy to answer it even without finding $F(x)$ explicitly. The integrand is a periodic function of x with period π. The question then asks whether an antiderivative of a peiodic function is necessarily periodic. A similar question about derivatives has an affirmative answer and so it is tempting to think that the same holds for integrals too. But this is not so. In the present problem, the easiest way to see this is that since the integrand is positive (except at all multiples of π), its integral over every interval will be positive. Hence the function defined by the integral will be strictly increasing. Obviously such a function cannot be periodic. To put it in symbols, for every real x we have

$$F(x+\pi) - F(x) = \int_x^{x+\pi} \sin^2 t \, dt \qquad (1)$$

which is positive since the integrand is positive for all $t \in [x, x+\pi]$ except at one or two points. Hence $F(x+\pi) > F(x)$ for all real x. This latter method is applicable even when an indefinite integral cannot be found in a closed form, as would be the case if instead of $\sin^2 x$, we had the function $e^{\sin x}$. It is periodic (with period 2π). But it is positive everywhere and hence its integral cannot be periodic.

SECTION III

Linked Comprehension Type

This section contains two paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (A), (B), (C) and (D), out of which **ONLY ONE** is correct.

Paragraph for Question No.s 14 to 16.

Let V_r denote the sum of the first r terms of an arithmetic progression (A.P.) whose first term is r and whose common difference is $(2r-1)$. Let $T_r = V_{r+1} - V_r - 2$ and $Q_r = T_{r+1} - T_r$ for $r = 1, 2, \ldots$

Q.14 The sum $V_1 + V_2 + \ldots + V_n$ is

(A) $\frac{1}{12}n(n+1)(3n^2 - n + 1)$ (B) $\frac{1}{12}n(n+1)(3n^2 + n + 2)$

(C) $\frac{1}{2}n(2n^2 - n + 1)$ (D) $\frac{1}{3}(2n^2 - 2n + 3)$

Answer and Comments: (B). Using the standard formula for the sum of the terms of an A.P. we first find a closed form expression for V_r for $r = 1, 2, \ldots$, viz.

$$V_r = r^2 + (2r-1)\frac{r(r-1)}{2} = \frac{2r^3 - r^2 + r}{2} \qquad (1)$$

It is now very easy to find an expression for $V_1 + V_2 + \ldots + V_n$ using the formulas for the sums $\sum_{r=1}^{n} r^3$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$. All these three formulas are very standard and together give

$$\begin{aligned}
V_1 + V_2 + \ldots + V_n &= \sum_{r=1}^{n} r^3 - \frac{1}{2}\sum_{r=1}^{n} r^2 + \frac{1}{2}\sum_{r=1}^{n} r \\
&= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4} \\
&= \frac{n(n+1)}{12}[3n(n+1) - (2n+1) + 3] \\
&= \frac{1}{12}n(n+1)(3n^2+n+2) \qquad (2)
\end{aligned}$$

which tallies with (B).

Q.15 T_r is always

(A) an odd number (B) an even number
(C) a prime number (D) a composite number

Answer and Comments: (D). We have already obtained a closed form expression for V_r in (1). Using it we get

$$\begin{aligned}
T_r &= V_{r+1} - V_r - 2 \\
&= \frac{2(r+1)^3 - (r+1)^2 + (r+1)}{2} - \frac{2r^3 - r^2 + r}{2} - 2 \\
&= \frac{2(3r^2+3r+1) - 2r - 1 + 1 - 4}{2} \\
&= \frac{6r^2+4r-2}{2} \\
&= 3r^2 + 2r - 1 \qquad (3)
\end{aligned}$$

This expression factors as $(3r-1)(r+1)$. As r is a positive integer, both the factors are integers greater than 1. So, this is a proper factorisation of T_r. Thus it is a composite number.

Q.16 Which of the following is a correct statement?

(A) $Q_1, Q_2, Q_3, \ldots$ are in an A.P. with common difference 5
(B) $Q_1, Q_2, Q_3, \ldots$ are in an A.P. with common difference 6
(C) $Q_1, Q_2, Q_3, \ldots$ are in an A.P. with common difference 11
(D) $Q_1 = Q_2 = Q_3 = \ldots$

Answer and Comments: (B). In the last question we had to find a closed form expression for T_r from a closed form expression for V_r which was obtained in the earlier problem. In the present problem we have to begin by obtaining a closed form expression for Q_r from a closed form expression for T_r, viz. (3) above. Thus

$$\begin{aligned} Q_r &= T_{r+1} - T_r \\ &= [3(r+1)^2 + 2(r+1) - 1] - [3r^2 + 2r - 1] \\ &= 6r + 5 \end{aligned} \tag{4}$$

From this it is immediate that the terms $Q_1, Q_2, Q_3, \ldots$ form an A.P. with common difference 6. But since in this paragraph, so many times we have taken the difference of the consecutive terms of a sequence, we might as well do it once more to get a formal proof! Thus,

$$Q_{r+1} - Q_r = 6(r+1) + 5 - (6r+5) = 6 \tag{5}$$

which is independent of r. Hence the Q's form an A.P. with common difference 6.

In the good old days of JEE, when there were no multiple choice questions, all the three questions of this paragraph would have been combined together into a single question (requiring about 7 to 8 minutes work) asking the candidate to show that the Q's form an A.P. To answer this single question the candidate would have to go through (1) and (3) anyway. There could be some partial credit for reaching these steps. In the present set-up of the JEE, all questions have to be multiple choice. Moreover, they have to be short, usually permitting a candidate only about two to three minutes per question (sometimes even less). As a result, paper-setters are forced to carve out these steps as separate questions and add a little twist of factorisation of (3). It is solely to make this factorisation possible that in the statement of the problem (or rather, the 'paragraph') T_r has been defined as $V_{r+1} - V_r - 2$ and not as $V_{r+1} - V_r$, which would be more natural. The term -2 has no function in the main theme of the paragraph.

It is instructive to look at the main theme of the paragraph a little more in detail. The sequence $V_1, V_2, \ldots, V_r, \ldots$ has been given in an indirect manner. But Equation (1) above expresses V_r directly in terms of r. Note that this expression is a cubic polynomial in r. Now the sequence $T_1, T_2, \ldots, T_r, \ldots$ is defined by taking the difference of the consecutive terms of the sequence $V_1, V_2, \ldots, V_r, \ldots$ (and adding -2, which is extraneous, as just noted). From (3) we see that T_r is a quadratic polynomial in r. Next, the sequence $Q_1, Q_2, \ldots, Q_r, \ldots$ is obtained from the sequence $T_1, T_2, \ldots, T_r, \ldots$, in a similar manner, viz. by taking the difference between the consecutive terms. And as we proved in (4), Q_r is a polynomial of degree 1 in r.

This suggests that a more general result may be true. That indeed turns out to be true as shown in the following theorem.

Theorem: Suppose in the sequence $A_1, A_2, \ldots, A_r, \ldots$, A_r is a polynomial in r of degree $k \geq 1$ (say). Define

$$B_r = A_{r+1} - A_r \tag{6}$$

for $r = 1, 2, 3, \ldots$. Then B_r is a polynomial of degree $k-1$ in r.

Proof: We are given that there is some polynomial, say $f(x) = a_k x^k + a_{k-1}x^{k-1} + \ldots + a_1 x + a_0$ with $a_k \neq 0$ such that

$$A_r = f(r) = a_k r^k + a_{k-1} r^{k-1} + \ldots + a_1 r + a_0 \tag{7}$$

for all $r = 1, 2, \ldots$. Substituting this into (6) we get

$$\begin{aligned} B_r &= f(r+1) - f(r) \\ &= a_k[(r+1)^k - r^k] + a_{k-1}[(r+1)^{k-1} - r^{k-1}] + \ldots + a_2 r + a_1 \end{aligned} \tag{8}$$

This is clearly a polynomial in r of degree at most k. But if we apply the binomial theorem, we see that the expression in the first bracket is a polynomial of degree $k-1$ (with leading term kr^{k-1}). Since the expressions in all other brackets are polynomials of degrees $k-2$ or less, and $a_k \neq 0$, we see that B_r is a polynomial in r of degree $k-1$ (with leading coefficient ka_k). ∎

This theorem is hardly profound. But its significance is noteworthy. A perceptive reader will hardly fail to notice that ka_k is also the leading coefficient in the derivative $f'(x)$ which is a polynomial of degree $k-1$. It is not true, of course, that $B_r = f'(r)$ for all r, as can be shown by simple examples. For example if $f(x) = x^2$, then $f'(x) = 2x$ but $A_r = r^2$ and $B_r = A_{r+1} - A_r = 2r + 1 \neq f'(r)$. But even the fact that B_r, as a polynomial in r has the same leading coefficient as $f'(x)$ has a certain significance which deserves to be looked into.

By very definition, for every real c, $f'(c)$ is the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \tag{9}$$

Here x is a continuous variable and can assume values that are arbitrarily close but not equal to c. And when x is very close to c (but not equal to c), the ratio $\dfrac{f(x) - f(c)}{x - c}$ can be taken as a good approximation to the derivative $f'(c)$.

Let us now see what will be the discrete analogue of such a limit. We replace the continuous variable x (which can take any real value) by a discrete variable, say n, which takes only positive integers as values. A sequence is, by definition, a function of such a discrete variable. Of course, we generally prefer to denote the terms of a sequence by symbols like $A_1, A_2, \ldots, A_n, \ldots$ instead of $f(1), f(2), \ldots, f(n), \ldots$. Now, fix any particular value of n, say r. If we try to define the derivative of the sequence $\{A_n\}_{n \geq 1}$ at r, a straight analogy with (9) would give

$$A'_r = \lim_{n \to r} \frac{A_n - A_r}{n - r} \tag{10}$$

But the trouble is that this limit makes no sense. The variable n *does not* assume values arbitrarily close to r. Of course, it does assume the value r. But that is totally irrelevant as far as the limit is concerned. Excluding it, the closest n can come to r is when it equals either $r+1$ or $r-1$. Accordingly, the ratio in the R.H.S. of (10) will be either $A_{r+1} - A_r$ or $A_r - A_{r-1}$. These expressions are called respectively, the r-th **forward** and the r-th **backward differences** of the sequence $\{A_n\}_{n \geq 1}$. These differences can be taken as the discrete analogues of the concept of the right handed and the left handed derivatives. (They are not exact analogues, but only crude ones because of the inherent difficulty in defining the limit above. So, they resemble the derivatives in some but not in all respects. Note, for example, that in general the forward and the backward differences are different from each other, in sharp contrast with the two sided derivatives which are always equal for a differentiable function.)

In the theorem above, the terms of the sequence $\{B_n\}_{n \geq 1}$ are nothing but the forward differences of the sequence $\{A_n\}_{n \geq 1}$. This sequence is therefore called the **first forward derived sequence** of the sequence $\{A_n\}_{n \geq 1}$. Similarly, the sequence $\{A_n - A_{n-1}\}_{n \geq 2}$ (which is defined only for $n \geq 2$), is called the **first backward derived sequence** of $\{A_n\}_{n \geq 1}$. (The term 'derived sequence', used without qualification, usually means the forward derived sequence.) Note that unlike derivatives which are defined only for differentiable functions, these derived sequences make sense for *every* sequence because they are defined by a simple process of subtraction, no limiting process being involved. Repeating this construction, one can define the second derived sequences, the third derived sequences and so on.

The theorem above can be paraphrased to say that if the terms of a sequence are given by a polynomial of degree k then those of its first derived sequence are given by a polynomial of degree $k-1$. The derived sequences share a few other properties of derivatives too. For example, it is immediate that a sequence is constant if and only if its derived sequence is identically 0. There is also an analogue of the Leibnitz rule. But these things are beyond our scope. We mention them because the central theme of the present paragraph is the (forward) derived sequence. In fact, if the intention of these so called paragraphs is to test 'comprehension', it would have been better if the preamble of the present paragraph had defined the concept of the forward derived difference and framed some of their properties as questions.

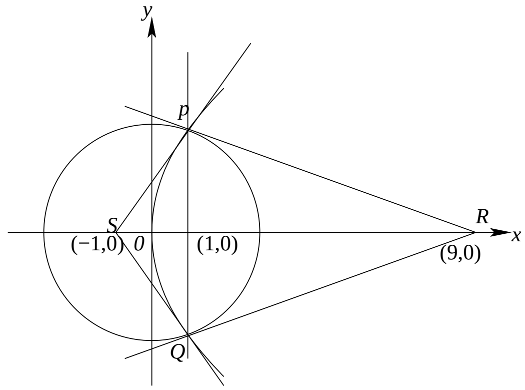
Paragraph for Question No.s 17 to 19

Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S.

Q.17 The ratio of the areas of the triangles PQS and PQR is

(A) $1 : \sqrt{2}$ (B) $1 : 2$ (C) $1 : 4$ (D) $1 : 8$

Answer and Comments: (C). Solving the two equations simultaneously, we get the points of intersection as $P = (1, 2\sqrt{2})$ and $Q = (-1, 2\sqrt{2})$. As the circle is symmetric about the x-axis and these two points are also symmetrically located w.r.t. the x-axis, it is obvious that the tangents to the circle from P and Q will meet on the x-axis. The same is also true of the parabola. This need not have been stated in the question. To locate the point R we take the tangent to the circle at P and take its point of intersection with the x-axis. (We could have as well taken Q instead of P.)



The slope of the tangent to the circle $x^2 + y^2 = 9$ at P is $-\dfrac{1}{2\sqrt{2}}$ and so its equation is

$$y - 2\sqrt{2} = -\frac{1}{2\sqrt{2}}(x - 1) \tag{1}$$

Hence the point R comes out to be $(9, 0)$. Similarly, to get the point S we first consider the tangent to the parabola $y^2 = 8x$ at the point $P = (1, 2\sqrt{2})$. The slope is $\dfrac{4}{2\sqrt{2}} = \sqrt{2}$ and so its equation is

$$y - 2\sqrt{2} = \sqrt{2}(x - 1) \tag{2}$$

which meets the x-axis when $x = -1$. So $S = (-1, 0)$. The question deals with the ratio of the areas of the triangles PQS and PQR. As we know the coordinates of all the four points P, Q, R and S, we can determine both the areas and then take their ratio. But that is a waste. A more efficient method is to note that both the triangles have a common side, viz. PQ. Hence the ratio of their areas is the same as the ratio of their altitudes on this common side. In the present case these altitudes can be determined easily by inspection because the equation of the line PQ is simply $x = 1$. So the desired altitudes are determined by the x-coordinates of the points S and R. They come out to be 2 and 8 respectively. So their ratio is $1 : 4$ and this is also the ratio of the areas of the triangles PQS and PQR.

Q.18 The radius of the circumcircle of the triangle PRS is

(A) 5 (B) $3\sqrt{3}$ (C) $3\sqrt{2}$ (D) $2\sqrt{3}$

Answer and Comments: (B). There are many formulas for the circumradius of a triangle. In the present case, we can easily determine the sides of the triangle PRS since we know the coordinates of the vertices. So, if we can find the sine of any one of its angles we shall get the circumradius. We choose $\angle PSR$ for this. Its since $PS = \sqrt{4+8} = 2\sqrt{3}$, we have

$$\sin(\angle PSR) = \frac{2\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} \tag{3}$$

The opposite side, viz. PR equals $\sqrt{64+8} = \sqrt{72} = 6\sqrt{2}$. Hence the circumradius is $\dfrac{6\sqrt{2}}{2\sqrt{2}/\sqrt{3}} = 3\sqrt{3}$.

Q.19 The radius of the incircle of the triangle PQR is

(A) 4 (B) 3 (C) $\frac{8}{3}$ (D) 2

Answer and Comments: (D). Once again, there are several formulas for the inradius of a triangle. In the present case we can easily find the area, say Δ, as well as the three sides and hence the semi-perimeter, say s, of the triangle. So the most convenient choice is to use the fact that the inradius equals $\dfrac{\Delta}{s}$. Since the side PQ equals $4\sqrt{2}$ and the altitude from R is 8 we have

$$\Delta = \frac{1}{2} \times 4\sqrt{2} \times 8 = 16\sqrt{2} \tag{4}$$

sq. units. In the answer to the last question we already found PR as $6\sqrt{2}$. By symmetry, RQ also equals this. So

$$s = \frac{1}{2}(4\sqrt{2} + 12\sqrt{2}) = 8\sqrt{2} \tag{5}$$

units. Hence the inradius equals $\dfrac{16\sqrt{2}}{8\sqrt{2}} = 2$ units.

Unlike the last paragraph, the present one has no coherent theme. Rather, it is a hotch-potch of some simple coordinate geometry and trigonometry. In the heydays of the JEE, there often used to be some interesting problems about solutions of triangles. (See Chapter 11 for many such problems.) Often these problems were about some general triangle and numerical data, if any, came in only at the end. Obviously, such problems required perceptivity in addition to some skill in formula manipulation. But such full length problems have no place in the present JEE set-up. So, what we see here are three straightforward, numerical problems. They do demand some thinking. For example, in Q.17, equating the ratio of the areas to that of the altitudes simplifies the work. But this kind of thinking is not mandatory, as it often used to be in the problems in the past. Nor is it adequately rewarding, because, since the coordinates of the vertices of the triangles are known, calculating their areas using the determinant formula is not so prohibitively long. And, in any case, now that the candidates do not show their work, it is impossible to tell if a candidate did a problem with imaginative thinking or just by brute force.

SECTION IV

Matching Answers Type

This section contains 3 questions. Each question contains entries given in two columns, those in **Column I** being labelled A,B,C,D while those in **Column II** are labelled p,q,r,s. An entry in either column may have more than correct matches in the other. Indicate all the correct matchings (and no others) by writing ordered pairs of the form (X,y) where X is an entry in **Column I** and y is an entry in **Column II**.

Q.20 The entries in **Column I** are certain definite integrals while those in **Column II** are some real numbers. Match each integral with its value.

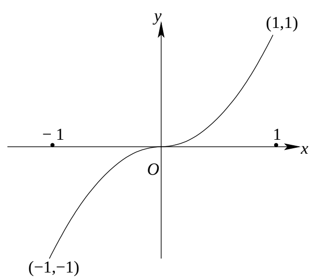
Column I	**Column II**
(A) $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2}$	(p) $\frac{1}{2}\log(\frac{2}{3})$
(B) $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$	(q) $2\log(\frac{2}{3})$
(C) $\displaystyle\int_{2}^{3} \frac{dx}{1-x^2}$	(r) $\frac{\pi}{3}$
(D) $\displaystyle\int_{1}^{2} \frac{dx}{x\sqrt{x^2-1}}$	(s) $\frac{\pi}{2}$

Answer and Comments: (A,s), (B,s), (C,p), (D,r). The integrands in the first two integrals have standard antiderivatives, viz. $\tan^{-1} x$ and $\sin^{-1} x$ respectively. That in the third can be resolved into partial fractions as $\dfrac{1/2}{1+x} + \dfrac{1/2}{1-x}$. For (D), the substitution $x = \sec\theta$ converts it to $\displaystyle\int_0^{\pi/3} 1 \ d\theta$. Therefore the values of the integrals come out to be $\frac{\pi}{2}, \frac{\pi}{2}, \frac{1}{2}\ln(\frac{2}{3})$ and $\frac{\pi}{3}$ respectively. It is truly shocking to see such simple and straightforward integrals in JEE, when not too long ago (e.g. in 2002) questions have been asked where the substitution that will do the trick is far from obvious. Again, this has to be blamed on the fact that in the new set-up of the JEE, there is no room for challenging, full length questions. So such a question is replaced by four dwarfs. But obviously it is wrong to equate a candidate who can do these four questions with one who could do a challenging integral in the past.

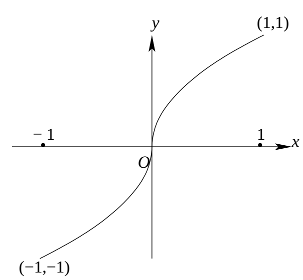
Q.21 The entries in **Column I** are some functions of a variable x while those in **Column II** are certain properties of functions. Match the functions with the properties they have. ($[x]$ denotes the greatest integer less than or equal to x.)

Column I	**Column II**
(A) $x\|x\|$	(p) continuous in $(-1, 1)$
(B) $\sqrt{\|x\|}$	(q) differentiable in $(-1, 1)$
(C) $x + [x]$	(r) strictly increasing in $(-1, 1)$
(D) $\|x-1\| + \|x+1\|$	(s) not differentiable at least at one point in $(-1, 1)$

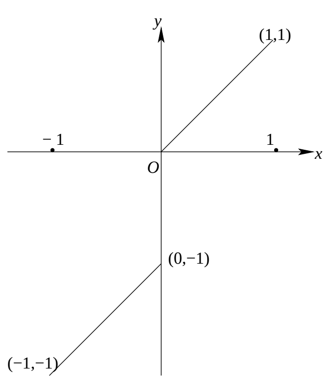
Answer and Comments: (A,p), (A,q), (A,r), (B, p), (B,s), (C,r), (C,s), (D,p), (D,q). In essence there are 16 true/false type questions here, of the form whether function X has property y. The functions are unrelated to each other. But the properties have some logical linkages. Clearly q and s are logical negations of each other and so every function has exactly one of these. Similarly, q implies p and so a function which has the property q will also have property p.



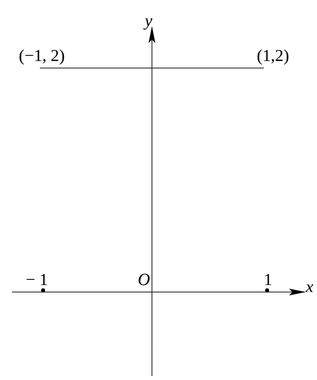
(A)



(B)



(C)



(D)

These observations, coupled with graphs of these functions (which are shown above) give the answers immediately. For questions like this, the multiple choice format is really suitable, because even though the proofs are not difficult, writing them out in full is time consuming and tedious. The function in (D) is constant on $(-1, 1)$ since $-1 < x < 1$ implies $|x - 1| = 1 - x$ and $|x + 1| = x + 1$. All the other three functions are differentiable with positive derivatives and hence strictly increasing on the intervals $(-1, 0)$ as well as $(0, 1)$. Also they are continuous on each of these intervals. So we only need to check their behaviour at 0. The function in (A) equals x^2 for $x > 0$ and $-x^2$ for $x < 0$. Both the right and left handed derivatives exist at 0 and equal 0 each. Hence the function is differentiable at 0. (Note that the function $|x|$ is not differentiable at 0. In fact, this is the most standard example to show that in general continuity does not imply differentiability. But multiplication by the factor x serves to smoothen the function a little by removing the kink in its graph. So the resulting function is differentiable at 0. However, it is not second differentiable.)

The function in (B) is continuous at 0 but not differentiable since neither the left nor the right handed derivative exists. (Both of them are infinite.) Finally, the function in (C) equals x for $x \in (0, 1)$ and equals $x - 1$ for $x \in (-1, 0)$. Hence it is discontinuous at 0 because the right and the left sided limits are 0 and -1 respectively. However, the value of the function at 0 lies between these two limits. So, the function is strictly increasing not only on each of the subintervals $(-1, 0)$ and $(0, 1)$ but on the entire interval $(-1, 1)$.

Q.22 Consider the following linear equations

$$
\begin{aligned}
ax + by + cz &= 0 \\
bx + cy + az &= 0 \\
cx + ay + bz &= 0
\end{aligned}
$$

Column I lists some conditions on a, b, c while **Column II** lists some statements about what the equations represent. Match each condition with its implication(s).

Column I		Column II	
(A)	$a+b+c \neq 0$ and $a^2+b^2+c^2 = ab+bc+ca$	(p)	the equations represent planes meeting only at a single point
(B)	$a+b+c = 0$ and $a^2+b^2+c^2 \neq ab+bc+ca$	(q)	the equations represent the line $x=y=z$
(C)	$a+b+c \neq 0$ and $a^2+b^2+c^2 \neq ab+bc+ca$	(r)	the equations represent identical planes
(D)	$a+b+c = 0$ and $a^2+b^2+c^2 = ab+bc+ca$	(s)	the equations represent the whole of the three dimensional space

Answer and Comments: (A,r), (B,q), (C,p), (D,s). Each of the three given equation represents a plane passing through the origin in the three dimensional space. But this plane degenerates into the whole space when the coefficients a, b, c are all 0. Except for this case, the three planes will meet either at a single point or on a single line or they may represent identical planes (e.g. when $a = b = c \neq 0$). These four possibilities correspond to the entries in **Column II** and the problem asks you to decide which possibility holds when. The crucial observation is that the determinant of the coefficients of the system has a well-known factorisation. Specifically,

$$\begin{aligned} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} &= 3abc - a^3 - b^3 - c^3 \\ &= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \qquad (1) \end{aligned}$$

As there are two factors on the R.H.S., logically there are four possibilities depending upon which of these factors vanishes. These four possibilities correspond to the entries in **Column I**. When both the factors are non-zero, so is the determinant. Hence the system has a unique solution, which geometrically means that the three planes meet only at the origin. Thus we see that (C) implies (p).

In all the other three cases the determinant vanishes, and so there will be some degeneracy. Note that the second factor in the R.H.S. of (1) can be written as

$$a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2] \qquad (2)$$

As a, b, c are all real, the expression in the brackets vanishes when and only when $a = b = c$. So in (A) and (D), we are given that $a = b = c$. This makes the three equations identical. In fact, each is the equation $a(x+y+z) = 0$. Moreover in this case $a+b+c = 3a$. In (A) we have $a \neq 0$

and so all three equations represent the same plane, viz. $x + y + z = 0$. However, in (D), we have $a = 0$ and so every equation degenerates to $0x + 0y + 0z = 0$ which is an identity satisfied by all points of the three dimensional space. Therefore, (A) implies (r) while (D) implies (s).

(B) is the most interesting case. Here we have $a+b+c = 0$. As a result, any point which satisfies any two of the three equations will automatically satisfy the third one. So, we focus our attention to the solution set of any two of the equations, say the first two equations. Since a, b, c are not all equal, at least two of them are non-zero and so certainly, each equation represents a (non-degenerate) plane. We claim that the first two equations represent distinct planes. For otherwise, their coefficients will be proportional and as a result we shall have $b^2 = ac, c^2 = ab$ and $a^2 = bc$ and hence $a^2 + b^2 + c^2 = ab + bc + ca$, a contradiction to the second part of (B). Therefore in (B), the three equations together represent some straight line, say L in space. That this line is, in fact, the line $x = y = z$ follows any point which satisfies $x = y = z$ satisfies all three equations since $a + b + c = 0$. Therefore, the line L surely contains all points on the line $x = y = z$. Therefore the two lines must be the same.

Those who are familiar with a little bit of linear algebra and in particular with the concept of the rank of a matrix (introduced on p. 109) can do the problem more efficiently. Let A be the 3×3 matrix of the coefficients of the given system of the three linear equations in three unknowns x, y, z. That is,

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \tag{3}$$

In (1) we computed the determinant of A. This matrix A defines what is called a linear transformation from the three dimensional vector space $\mathbb{R}^3$ to itself. The set of solutions of the system constitutes a certain subspace, say V, of the domain space $\mathbb{R}^3$. (This subspace is often called the **null space** of the linear transformation, because it consists of those points where the linear transformation assumes the value 0, i.e. the null value. The dimension of the null space is called the **nullity** of the linear transformation.) In the present case, since the null space V is a subspace of $\mathbb{R}^3$ its possible dimensions are $0, 1, 2$ and 3 and these correspond to the entries in **Column II** in that order. (In fact, this explains why the entries in **Column II** are ordered in a particular manner.)

There is a standard theorem of linear algebra, often called the **rank nullity theorem** which says that the nullity, that is, the dimension of the solution space V is $n - r$ where n is the dimension of the domain vector space (in our problem $n = 3$) and r is the rank of the matrix A. The rank can be defined intuitively as the maximum number of independent or ‘really different’ equations in the system. (For example, if one of the equations is a linear combination of some others, then it is superfluous and does not contribute to the rank.) The truth of this theorem is intuitively obvious because every linear equation puts one constraint on the coordinates x, y, z and thereby reduces the dimension of the solution space by 1, but a superfluous equation does not decrease the solution space and hence does not reduce the dimension.

This theorem is of little use unless we have some way to determine the rank of a matrix. There is a standard method for this using what is called **row reduction** of the matrix. But it will take us too far afield to discuss it. Instead we resort to the definition given in the footnote on p. 109, according to which the rank r of A is the size of the largest square submatrix of A with a non-zero determinant. This definition is generally not very convenient because there are so many square submatrices of a given matrix and it is hardly practical to evaluate their determinants. But in the present case since the only possible values of r are 0,1, 2 and 3, such a determination is not so difficult.

Suppose the 3×3 matrix A is itself non-singular. By (1), this corresponds to the entry (C) in **Column I**. In this case the rank r equals 3 and so by the rank nullity theorem, the solution space V has dimension $3-3=0$. In other words it consists of a single point, viz. the origin. This is the possibility (p) in the second column. At the other extreme, we have a situation where the rank is 0. This would mean that the coefficients a, b, c are all 0 (because they can be equated with the determinants of the 1×1 submatrices of A.) In this case both the factors of the R.H.S. of (1) vanish and so we are in (D) of the first column. Here the solution space has dimension $3 - 0 = 3$ which corresponds to (s) in **Column II**.

The two remaining cases are $r = 1$ and $r = 2$. To say that A has rank 1 means that at least one of the coefficients a, b, c is non-zero but that the determinant of any 2×2 submatrix vanishes. By inspection, this is equivalent to saying that the expressions $a^2 - bc, b^2 - ac$ and $c^2 - ab$ all vanish. Hence $a^2 + b^2 + c^2 = ab + bc + ca$. But, as noted before, that forces $a = b = c$ and that puts us in (A) in **Column I**. Hence (A) implies (r).

Finally, suppose A has rank 2. This means A is singular but at least one of the expressions $a^2 - bc, b^2 - ac$ and $c^2 - ab$ is non-zero. This also implies that $a^2 + b^2 + c^2 \neq ab + bc + ca$ because otherwise we shall have $a = b = c$ which would mean all these three expressions vanish. Since the R.H.S. of (1) vanishes while the last factor does not, we must have $a+b+c = 0$. So we are in (B) in **Column I**. By the theorem quoted above, the solution space V has dimension 1. So, it is a straight line L. That this line is $x = y = z$ is proved the same way as in the earlier solution.

This is really a very good problem requiring knowledge of solutions of systems of linear equations, a little bit of solid coordinate geometry and some algebraic manipulations. Note that the fact that the coefficients are real was vitally used in both the solutions. Without it, the condition $a^2 + b^2 + c^2 = ab + bc + ca$ will not imply $a = b = c$. (The rank nullity theorem cited above also holds for complex vector spaces. But in the solution based on it we also need something more.)

It may be argued that those who are exposed to the rank nullity theorem (which is not a part of the JEE syllabus) will have an unfair advantage. But nothing can be done about it in an examination where all that matters is whether a candidate has got the right answer rather than how he has obtained it. In the past some candidates appeared for JEE more than twice while simultaneously studying more advanced mathematics. So the possibility of some of them being exposed to the rank nullity theorem was a real one. Now that a candidate can appear for JEE only twice, it is unlikely that he has been exposed to this theorem in a regular course work. And, if he has acquired it all on his own, why not reward him?

PAPER 2

Contents

SECTION I

Each question has only one correct answer.

Q.23 Let $f(x) = \dfrac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$ and $g(x) = f \circ f \circ f \circ \ldots \circ f(x)$, where in the composite, f occurs n times. Then $\displaystyle\int x^{n-2} g(x)\, dx$ equals

(A) $\dfrac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}} + K$ (B) $\dfrac{1}{(n-1)}(1+nx^n)^{1-\frac{1}{n}} + K$

(C) $\dfrac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}} + K$ (D) $\dfrac{1}{(n+1)}(1+nx^n)^{1+\frac{1}{n}} + K$

Answer and Comments: (A). Multiple choice questions of this kind lend themselves rather readily to dishonest attempts. In the present problem, the simplest case is $n = 2$. But even in this simplest case all the four alternatives given are distinct. So, whichever of them is correct for $n = 2$ will have to be correct for all n, if at all one of them is the correct answer for all n. So, it suffices to do only the case $n = 2$. In this case we have

$$f(x) = \frac{x}{\sqrt{1+x^2}} \tag{1}$$

The function $g(x)$ is given to be $f(f(x))$. Calling $f(x)$ as u for simplicity, we have

$$\begin{aligned} g(x) &= f(f(x)) = f(u) = \frac{u}{\sqrt{1+u^2}} \\ &= \frac{x/\sqrt{1+x^2}}{\sqrt{1+\frac{x^2}{1+x^2}}} \\ &= \frac{x}{\sqrt{1+2x^2}} \end{aligned} \tag{2}$$

As $n = 2$, we have $x^{n-2} = 1$ and hence $\int x^{n-2} g(x)\, dx$ is the same as $\int g(x)\, dx$, i.e. $\int \frac{x}{\sqrt{1+2x^2}}\, dx$. With the substitution $v = x^2$ (or even directly after you gain some practice with such standard substitutions) this indefinite integral comes out to be $\frac{1}{2}(1+2x^2)^{1/2}$. This tallies only with (A).

Again, it is not clear whether the paper-setters simply missed this short cut or whether they really intended it. If they noticed it and did not want it, they could have simply changed the fake answer (B) to $\frac{1}{n}(1+nx^2)^{1-\frac{1}{n}}$, which coincides with the correct answer (A) for $n = 2$. When a multiple choice question involves some parameter (in this case n), care has to be taken to see that the correct answer is not revealed by working out the problem for some special value of the parameter. Special values of the parameter can, of course, weed out some alternatives. But a candidate who adopts this strategy (which is really unscrupulous) should be forced to work out at least two special cases. Giving him the answer for working out just one special case of the problem serves only to add to the injustice meted out to a scrupulous candidate.

Let us now do the problem honestly. It is tempting to try induction on n. But there is an inherent difficulty here. The function g is obtained by taking the composite of the function f with itself n times (to be fussy, only $n-1$ times). But this function f itself depends on the parameter n. So, the function $g(x)$ for a particular value of n, say $n = k$ is not easily related to $g(x)$ for $n = k-1$ or for any lower values of n. Hence induction is not the right tool here. Moreover, in induction you have to know the answer beforehand. You can only *verify* its truth.

Let us, therefore, try the problem without induction on n. Since the function $f(x)$ itself depends on the parameter n, it would be better to denote it by $f_n(x)$. Similarly, let us denote $g(x)$ by $g_n(x)$. As we commented above, there is no easy relationship between $g_n(x)$ and $g_{n-1}(x)$ because the functions $f_n(x)$ and $f_{n-1}(x)$ are not the same. We are given that g_n is obtained by composing f_n with itself n times. To handle it effectively, let us first introduce some notation. For any function h, the composites $h \circ h$, $h \circ h \circ h$ (whenever defined) are often denoted by h^2, h^3 etc. (It is unfortunate that this notation is also used for the powers of the value of the function. For example, by $\sin^2 x$ we normally understand $(\sin x)^2$. However, for the purpose of this problem, let $\sin^2$ stand for the composite of the sine function with itself, i.e. $\sin^2(x) = \sin(\sin x)$.)

Now, for a fixed n, let us see what the functions $f_n^2, f_n^3, \ldots$ come out to be. Obviously,

$$f_n^1(x) = f_n(x) = \frac{x}{(1+x^n)^{1/n}} \tag{3}$$

The composite $f_n(f_n(x))$ can be calculated in the same manner as (2) above and gives

$$\begin{aligned} f_n^2(x) = f_n^2(x) &= \frac{x/(1+f_n(x))^{1/n}}{\left(1+\frac{x^n}{1+x^n}\right)^{1/n}} \\ &= \frac{x}{(1+2x^n)^{1/n}} \qquad (4) \end{aligned}$$

The next in line is $f_3(x)$. A straightforward computation, similar to the one above gives

$$\begin{aligned} f_n^3(x) &= f_n(f_n^2(x)) \\ &= \frac{x/(1+2x^n)^{1/n}}{\left(1+\frac{x^n}{1+2x^n}\right)^{1/n}} \\ &= \frac{x}{(1+3x^n)^{1/n}} \qquad (5) \end{aligned}$$

The past three formulas have a clear pattern and suggest that we must have

$$f_n^k(x) = \frac{x}{(1+kx^n)^{1/n}} \qquad (6)$$

for every $k = 1, 2, 3, \ldots$. Note that here the parameter n is fixed and the variable is k which ranges over the set of positive integers. The formula above can be proved by induction on k. The case $k = 1$ is precisely (3). The proof of the inductive step is straightforward and resembles the calculation in (5) with 2 being replaced by k. Specifically,

$$\begin{aligned} f_n^{k+1}(x) &= f_n(f_n^k(x)) \\ &= \frac{x/(1+kx^n)^{1/n}}{\left(1+\frac{x^n}{1+kx^n}\right)^{1/n}} \qquad (7) \\ &= \frac{x}{(1+(k+1)x^n)^{1/n}} \qquad (8) \end{aligned}$$

where we have used (6) as the induction hypothesis in (7). We are interested only in one case of () viz. where $k = n$, because in this case $f_n^n(x)$ is $g_n(x)$ which is given as $g(x)$ in the statement of the problem. Thus,

$$g(x) = g_n(x) = f_n^n(x) = \frac{x}{(1+nx^n)^{1/n}} \qquad (9)$$

Therefore the problem now reduces to finding $\displaystyle\int \frac{x^{n-1}}{(1+nx^n)^{1/n}}\,dx$. This can be done by putting $1 + nx^n = t$. Then $dt = n^2x^{n-1}dx$ and we get

$$\int x^{n-2}g(x)\,dx = \int \frac{x^{n-1}}{(1+nx^n)^{1/n}}\,dx$$

$$
\begin{aligned}
&= \frac{1}{n^2}\int \frac{dt}{t^{1/n}} \\
&= \frac{1}{n^2}\frac{t^{-\frac{1}{n}+1}}{1-\frac{1}{n}} + K \\
&= \frac{1}{n(n-1)}(1+nx^n)^{1-1/n} + K \qquad (10)
\end{aligned}
$$

Although the problem is superficially a problem on integration, the integration part of the problem is relatively minor. The real crux of the problem is that the integrand is given to you in a clumsy form and you have to reduce it to a manageable form. There is no obvious way to do it. One has to first try a few special cases (such as (3), (4), (5)), recognise the pattern that emerges, guess (6) and prove it by induction. After doing this, the integration per se is simple. In a conventional examination, where a candidate has to show all his work, this would have been a very good full length question. But in its present form, its goodness is marred by several factors. First, a student who simply guesses (6) but does not bother to establish it by induction gains an unfair advantage over a scrupulous student in terms of the time saved. And, as noted before, a student can simply sneak in by doing only the special case $n = 2$, without appreciating the difficulty of the general case.

Q.24 The letters of the word **COCHIN** are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. Then the number of words that appear before the word **COCHIN** is

(A) 360 (B) 192 (C) 96 (D) 48

Answer and Comments: (C). Let S be the set of all permutations of the letters of the word **COCHIN**. The given word has 6 letters with one of them, viz. **C** occurring twice. Hence $|S| = 6!/2! = 360$. The question asks the cardinality of the subset, say T, of S, consisting of those members of S which will be listed before **COCHIN** in a dictionary. Among the letters of this word, **C** appears before any others in the alphabetical order while **O** appears in the end. So it is clear that every member in T has to begin with a **C**, but that its second letter can be **C, H, I, N** or **O**. Let us divide the set T into five mutually disjoint subsets T_1, T_2, T_3, T_4 and T_5 depending upon whether its second letter is **C, H, I, N** or **O** respectively. Then we have

$$|T| = |T_1| + |T_2| + |T_3| + |T_4| + |T_5| \qquad (1)$$

Computation of the first four terms on the R.H.S. is easy. In each case we know the first two letters and the remaining four letters could be permuted in any order. Also they are all distinct since the only repeater, viz. **C** has already appeared at the start. It follows that for $i = 1, 2, 3, 4$,

$$|T_i| = 4! \qquad (2)$$

In finding $|T_5|$ we have to be careful. Since every member of T_5 matches with the given word **COCHIN** in the first two letters, in order to decide whether it will appear before or after **COCHIN**, we shall have to take into account its third letter, which can be **C, H, I** or **N**. Luckily the third letter of **COCHIN** (viz. **C**) is also the 'smallest' among these four. Hence every word in T_5 must begin with **COC**. We shall now have to consider its fourth letter which could be either **H, I** or **N**. In order that the word comes before **COCHIN**, the fourth letter will have to be **H** because it is the 'smallest' among these three and is also the fourth letter of **COCHIN**. Hence every member of T_5 begins with **COCH**. Its fifth letter will have to be **I**, because it is the 'smaller' of **I** and **N**. For the last letter we have no choice but to let it be **N**. But then the word formed is **COCHIN** which is the original word. So it cannot precede itself.

Summing up, the set T_5 is empty and hence

$$|T_5| = 0 \tag{3}$$

Substituting (2) and (3) into (1) gives 96 as the answer.

The tricky part of the problem is to get (3). An alert student could have observed it immediately after recognising that the letters in **COCHIN** after the second letter are in their natural alphabetical order. If instead, the given word were, say **COLABA** then many different cases would have to be considered.

The problem is a good one for a multiple choice question because although the reasoning looks long when expressed in words, doing it mentally does not take long. The computations involved are reasonable. The key idea is to 'divide and rule', i.e. break up the set T into several parts suitably and find the cardinality of each part. The only objection, perhaps, is that a candidate has to know not only the way words are listed in a dictionary, but will also have to know the order of the letters in a particular language, viz. English. Considering that the JEE question paper is printed both in English and in Hindi, it may be argued that those who studied in English get a slight advantage over the others since they are so used to English dictionaries. This is a very minor and non-mathematical objection. Still, the papersetters could have avoided it by replacing the letters by single digit numbers whose relative order is the same as the alphabetical order of the letters of the given word. For example, let **C, H, I, N, O** correspond to the digits 2, 4, 5, 6, 8. Then **COCHIN** corresponds to the six digit number 282456 and the problem could have asked to find out how many of the six digit numbers obtained by permutations of the digits of this number are less than this number. Mathematically the problem would be the same. But it will be independent of any particular language.

Q.25 Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Which of the following is correct?

(A) $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a} = \vec{0}$ (B) $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a} \neq \vec{0}$
(C) $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{a} \times \vec{c} \neq \vec{0}$ (D) $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$ and $\vec{c} \times \vec{a}$ are mutually perpendicular.

Answer and Comments: (B). In problems like this the symmetry of the data and also of the answers provided play a crucial role. We are given that

$$\vec{a} + \vec{b} + \vec{c} = \vec{0} \tag{1}$$

If we take the cross product of the vector $\vec{a} + \vec{b} + \vec{c}$ with $\vec{a}$ we get

$$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{0} \tag{2}$$

Using standard properties of the cross product, this gives

$$\vec{a} \times \vec{b} - \vec{c} \times \vec{a} = \vec{0} \tag{3}$$

i.e. $\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$. Because of cyclic symmetry we must therefore have

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a} \tag{4}$$

So the correct answer has to be either (A) or (B). It only remains to check whether each of these three terms is the zero vector or not. As all the three vectors are unit vectors, the only way the cross product of two of them vanish is when they are parallel. So we shall have

$$\vec{a} = \alpha\vec{c} \tag{5}$$

$$\text{and} \quad \vec{b} = \beta\vec{c} \tag{6}$$

for some scalars α and β. Moreover since $1 = |\vec{a}| = |\alpha||\vec{c}| = |\alpha|$, we must have $\alpha = \pm 1$. Similarly $\beta = \pm 1$. But then (1), (5) and (6) together would give

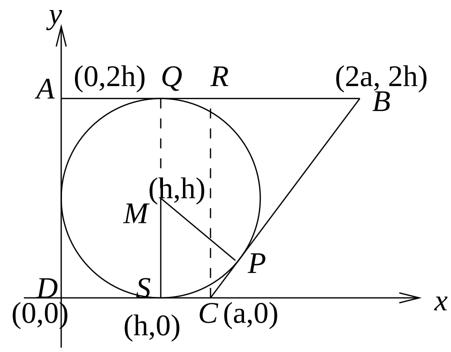
$$1 + \alpha + \beta = 0 \tag{7}$$

which is impossible for any of the four combination of the values of α and β. So, we eliminate (A) and then (B) is the right answer. We could have as well given this argument without (6). Since $\alpha = \pm 1$, $\vec{a} = \pm\vec{c}$ and so $\vec{a} + \vec{c}$ is either $2\vec{c}$ or $\vec{0}$. Therefore $|\vec{a} + \vec{c}| = 2$ or 0. But by (1), $|\vec{a} + \vec{c}| = |-\vec{b}| = |\vec{b}| = 1$, a contradiction.

A somewhat novel way of attacking the problem is to interpret the data geometrically. Since all the three vectors are unit vectors and (1) holds, they represent the three (directed) sides of an equilateral triangle. But then the angle between every two of them is $\pm 60°$ depending on the orientation of the triangle. But in neither case the cross product will vanish. In fact all the three cross products will equal $\pm\frac{\sqrt{3}}{2}\vec{k}$ where $\vec{k}$ is a vector perpendicular to the plane of the triangle. With this approach we not only get the correct answer but we get the value of each term. This alternate geometric solution makes the problem interesting.

Q.26 Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is

(A) 3 (B) 2 (C) $\frac{3}{2}$ (D) 1



Answer and Comments: (B). From the data, $ABCD$ is a trapezium with height AD. For notational simplicity, let us call this height $2h$ rather than h. Note that h is also the radius of the circle which touches the three sides AB, AD and CD of the quadrilateral $ABCD$. So our problem is to find h from the rest of the data.

Let us introduce an auxiliary variable. Let a be the length of the side CD. Then AB has length $2a$ and the formula for the area of the trapezium gives $h(a + 2a) = 18$ i.e.

$$ah = 6 \tag{1}$$

We need one more equation in a and h. This has to come from the fact that the fourth side BC also touches the circle above. The key to the solution lies in how we convert this information into an equation in a and h. For this we have at least two methods available.

The safest method is to use coordinates. Taking D as the origin and the axes as shown in the figure the equation of the circle is

$$(x - h)^2 + (y - h)^2 = h^2 \tag{2}$$

The points C and B are $(a, 0)$ and $(2a, 2h)$ respectively and so the equation of the line BC is

$$2h(x - a) - ay = 0 \tag{3}$$

We can now get an equation in a and h if we apply the condition for tangency for the line (3) to touch the circle (2). Those who remember such conditions can do it instantly. Even otherwise we can get it by equating the perpendicular distance of M, the centre of the circle, from the line (3) with the radius, h. Since $M = (h, h)$, we get

$$\frac{|2h^2 - 3ah|}{\sqrt{4h^2 + a^2}} = h \tag{4}$$

which on squaring and simplifying becomes

$$8a^2 = 12ah \tag{5}$$

It is trivial to solve (1) and (4) together to get $h = 2, a = 3$.

An alternate solution is to use pure geometry and some very simple properties of tangents to a circle. Specifically, all that we need is that the two tangents to a circle from a point outside it are equal. Let the circle touch the sides AB, BC and CD at Q, P and S respectively. Then we have

$$BP = BQ \quad = \quad 2a - h \tag{6}$$

$$\text{and} \quad CP = CS \quad = \quad a - h \tag{7}$$

Adding,

$$BC = 3a - 2h \tag{8}$$

If we can get some other expression (in terms of a and h) for the length of the side BC then, equating it with (8) will give us the desired equation in a and h. Here again, we have a choice. Since we already know the coordiantes of B and C, the distance formula gives

$$BC = \sqrt{a^2 + 4h^2} \tag{9}$$

But if we are committed to use only pure geometry and therefore to shun coordinates, then we can do some work which essentially amounts to deriving the distance formula. Specifically, drop a perpendicular CR from C to AB. Then the triangle CRB is right angled with $CR = 2h$ and $RB = AB - AR = AB - DC = 2a - a = a$. So we can also get (9) from the Pythagoras theorem.

The problem is a good test of a candidate's ability to visualise the data and pick the right tools from a huge collection of formulas and results in geometry. Although it is not a difficult problem, the time allowed (viz. a little over two minutes) is far too little for those who try it honestly. Once again, the multiple choice format permits a sneaky solution. But this time even the sneaky solution requires some work, viz. drawing a correct diagram and also coming up with (1). Thereafter a short cut can be taken as follows. It is obvious from the diagram that since the side BC touches the inscribed circle, the diameter must lie between the lengths of the parallel sides CD and AB. Therefore, $a < 2h < 2a$ or equivalently,

$$h < a < 2h \tag{10}$$

Putting the four given possible values of h viz. $h = 3, 2, \frac{3}{2}$ and 1 into (1), the corresponding values of a come out to be $2, 3, 6$ and 18. Obviously only $h = 2$ satisfies the double inequality (10). So, if at all one of the answers is correct, it has to be (B).

Q.27 $\frac{d^2x}{dy^2}$ equals

(A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-1}$ (D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

Answer and Comments: (D). If y is a function of x, then under certain conditions x can be expressed as a function of y. If y is a differentiable function of x then in general x may not be a differentiable function of y (a standard counterexample being $y = x^3$, where the inverse function is not differentiable at $y = 0$). But in case it is a differentiable function, then the derivatives $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are reciprocals of each other. This fact is often used. The present question deals with the relationship between the second derivatives $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$.

A straight analogy with the relationship between the first derivatives suggests that (A) is the correct answer. This is especially likely to appeal to those who think that the derivative $\frac{dy}{dx}$ is simply the ratio of the expressions dy and dx. The fact is that even though the derivative $\frac{dy}{dx}$ is, by definition, the limit of the ratio $\frac{\Delta y}{\Delta x}$ (often called the **incrementary ratio**), the usual law that the limit of a ratio is the ratio of the limits does not apply here because the limit of the denominator (and also that of the numerator) is 0. In fact, although the expressions dy and dx (often called the **differentiables**) can be given meanings of their own, they are far beyond the JEE level.

The point is that the answer cannot be obtained by a blind analogy with the relationship between the first derivatives $\frac{dy}{dx}$ and $\frac{dx}{dy}$. In other words, (A) may not be the correct answer. Its falsity can also be by a simple example such as $y = x^2$. We define it only for $x > 0$, so that there is no problem in defining the inverse function, viz. $x = \sqrt{y}$. Here the second derivative $\frac{d^2y}{dx^2}$ is a constant, viz. 2. But the second derivative $\frac{d^2x}{dy^2}$ is certainly not a constant. So they cannot be reciprocals of each other.

Once again, the tragedy is that the correct answer is revealed by trying all the four alternatives in this very special case. A direct computation from $x = \sqrt{y} = y^{1/2}$ gives

$$\begin{aligned} \frac{dx}{dy} &= \frac{1}{2}y^{-1/2} && (1) \\ \frac{d^2x}{dy^2} &= -\frac{1}{4}y^{-3/2} && (2) \\ \text{and} \quad \frac{dy}{dx} &= 2x = 2y^{1/2} && (3) \end{aligned}$$

We already know $\frac{d^2y}{dx^2} = 2$ and hence also that (A) is not the right answer. Straight substitution from (3) reduce the expressions in (B), (C) and (D) to $-4y^{3/2}$, $8y$ and $-\frac{1}{4}y^{-3/2}$ respectively. Only the last one tallies with (2). So, (D) is the only right answer in this case and therefore has to be *the* right answer to the question.

For an honest solution, note that, by definition, $\frac{d^2x}{dy^2}$ is $\frac{d}{dx}\left(\frac{dx}{dy}\right)$. We can replace $\frac{dx}{dy}$ by $\frac{1}{dy/dx}$. We have to differentiate this expression w.r.t. y. For this we apply the chain rule. That is, we differentiate it w.r.t. x and then multiply the result by $\frac{dx}{dy}$, or equivalently, by $\frac{1}{dy/dx}$. A straightforward computation using the quotient rule for derivatives now gives

$$\begin{aligned} \frac{d^2x}{dy^2} &= \frac{d}{dy}\left(\frac{1}{\frac{dy}{dx}}\right) \\ &= \frac{1}{(dy/dx)}\frac{d}{dx}\left(\frac{1}{\frac{dy}{dx}}\right) \\ &= \left(\frac{dy}{dx}\right)^{-1} \times (-1) \times \left(\frac{dy}{dx}\right)^{-2} \times \frac{d^2y}{dx^2} \\ &= -\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-3} && (4) \end{aligned}$$

which tallies with (D). Thus the honest solution is not so time consuming. In fact, it takes less time than the sneaky solution given above because in that solution, even the fake answers had to be verified. So, this is one of those problems where dishonesty is possible but does not pay. Still it is an option available for those who get confused when derivatives, and especially the second derivatives, w.r.t. unusual variables are involved. There is really no reason to get confused because the chain rule allows you to differentiate a given expression with a variable of your choice.

Q.28 The differential equation $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{y}$ determines a family of circles with

(A) variable radii and a fixed centre at $(0, 1)$
(B) variable radii and a fixed centre at $(0, -1)$
(C) fixed radius 1 and variable centres along the x-axis
(D) fixed radius 1 and variable centres along the y-axis

Answer and Comments: (C). A very straightforward problem about the family of curves represented by a first order differential equation. Also the d.e. is easy to solve since it can be cast in the separate variable form as

$$\frac{y}{\sqrt{1-y^2}} = dx \tag{1}$$

Integrating both the sides the general solution comes as

$$-\sqrt{1-y^2} = x + c \tag{2}$$

where c is an arbitrary constant. After squaring, this can be rewritten as

$$(x+c)^2 + y^2 = 1 \tag{3}$$

which represents a circle of a fixed radius 1 and a variable centre $(-c, 0)$ which lies on the x-axis for all values of c.

Q.29 If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\dfrac{z}{1-z^2}$ lie on

(A) a line not passing through the origin
(B) $|z| = \sqrt{2}$
(C) the x-axis
(D) the y-axis

Answer and Comments: (D). The condition $z \neq \pm 1$ ensures that the expression $\dfrac{z}{1-z^2}$ is defined. We are further given that $|z| = 1$. As a result $\dfrac{1}{z}$ equals $\overline{z}$. This observation with a clever algebraic manipulation gives

$$\begin{aligned} \frac{z}{1-z^2} &= \frac{1}{\frac{1}{z} - z} \\ &= \frac{1}{\overline{z} - z} \end{aligned} \tag{1}$$

Now, for any complex number z, $\overline{z} - z$ is always purely imaginary. Hence so is its reciprocal (which is defined except when $z = \pm 1$). Therefore the given expression is always purely imaginary and hence represents a point on the y-axis.

This solution is very short but a little tricky. But the trick of replacing $\overline{z}$ by $1/z$ for a point on the unit circle is a familiar one. (See for example the solution using complex numbers to the Main Problem of Chapter 10.) For those who cannot come up with this trick, the best method is to write z in the polar form, i.e. as $e^{i\theta}$ for some (real) θ. Then we get

$$\begin{aligned} \frac{z}{1-z^2} &= \frac{e^{i\theta}}{1-e^{2i\theta}} \\ &= \frac{\cos\theta + i\sin\theta}{(1-\cos 2\theta) - i\sin 2\theta} \\ &= \frac{\cos\theta + i\sin\theta}{2\sin\theta(\sin\theta - i\cos\theta)} \\ &= \frac{i}{2\sin\theta} \qquad (2) \end{aligned}$$

which is always a purely imaginary number.

Finally, for those who have the (unhealthy) habit of converting every problem about complex numbers to a problem about real numbers, let $z = x + iy$ as usual. Then, $|z| = 1$ implies $x^2 + y^2 = 1$ and we have

$$\begin{aligned} \frac{z}{1-z^2} &= \frac{x+iy}{(1-x^2+y^2) - 2ixy} \\ &= \frac{(x+iy)[(1-x^2+y^2)+2ixy]}{(1-x^2+y^2)^2 + 4x^2y^2} \\ &= \frac{x(1-x^2+y^2) - 2xy^2}{(1-x^2+y^2)^2+4x^2y^2} + \frac{y(1-x^2+y^2)+2x^2y}{(1-x^2+y^2)^2+4x^2y^2} \\ &= \frac{x(1-x^2-y^2)}{(1-x^2+y^2)^2+4x^2y^2} + \frac{y(1-x^2+y^2)+2x^2y}{(1-x^2+y^2)^2+4x^2y^2} \qquad (3) \end{aligned}$$

But we are given that $x^2 + y^2 = 1$. So the real part of the R.H.S. of (3) vanishes. Therefore the expression $\frac{z}{1-z^2}$ is purely imaginary for the given values of z.

Q.30 Let E^c denote the complement of an event E. Let E, F, G be pairwise independent events with $P(G) > 0$ and $P(E \cap F \cap G) = 0$. Then $P(E^c \cap F^c | G)$ equals

(A) $P(E^c) + P(F^c)$ (B) $P(E^c) - P(F^c)$
(C) $P(E^c) - P(F)$ (D) $P(E) - P(F^c)$

Answer and Comments: (C). Yet another problem about conditional probability. The papersetters have been careful to specify that $P(G) > 0$

as otherwise the conditional probability w.r.t. G makes no sense. But there is a slight ambiguity in the statement of the problem. On the face of it, the expression $P(E^c \cap F^c|G)$ could mean either $P((E^c \cap F^c)|G)$ or $P(E^c \cap (F^c|G))$ and it is not immediately clear which is the intended meaning. But the latter meaning can be ruled out because $F^c|G$ is really not an event. Even though we talk of its probability, it is a conditional probability. Specifically, it is the probability of the event $P(F^c \cap G)$ *given* G. To elucidate more, suppose the events E, F, G can be identified with certain subsets (also denoted by the same symbols) of some sample space, say X. Then $F^c = X - F$ is also an event in the sample space X. But $F^c|G$ cannot be identified with any subset of X and hence it is not an event in the sample space X. If we want to view this conditional probability as the probability of some event in some sample space, we shall have to take G as the sample space. Then $F^c|G$ may be identified with the event $F^c \cap G$ of this new sample space. (Note that while reducing the sample space from X to G, we shall have to multiply the probabilities or the 'weights' of the various elements of G by the factor $1/P(G)$ each, because even though $P(G)$ may be less than 1 when G is a subset of the old sample space X, $P(G)$ has to equal 1 when G itself is the sample space.)

Such difficulties are going to bother only those who think very finely. In a problem like this, such fine thinking is more a liability than an asset because although the ambiguity can be resolved as explained above, it takes some precious time to do so, which is really wasted because those who are blissfully ignorant of such possible ambiguities will run farther in the race. That is why, one wishes that the papersetters had made their intention unambiguous by appropriately inserting parentheses.

Anyway, we interpret the expression $P(E^c \cap F^c|G)$ to mean simply $P((E^c \cap F^c)|G$. Then by the law of conditional probability, we have

$$P(E^c \cap F^c|G) = \frac{P(E^c \cap F^c \cap G)}{P(G)} \tag{1}$$

We now apply DeMorgan's laws (or rather, their translations for probability) to replace the event $E^c \cap F^c$ by the event $(E \cup F)^c$. We also note that $P(G \cap (E \cup F)^c$ equals $P(G) - P(G \cap (E \cup F)$. Combined with the distributivity of the intersection over union, we get

$$\begin{aligned} P(E^c \cap F^c|G) &= \frac{P(E \cup F)^c \cap G}{P(G)} \\ &= \frac{P(G) - P(G \cap (E \cup F))}{P(G)} \\ &= \frac{P(G) - P((G \cap E) \cup (G \cap F))}{P(G)} \end{aligned} \tag{2}$$

To make further progress, we apply the law of probability for the disjunction of the events $G \cap E$ and $G \cap F$, according to which,

$$P((G \cap E) \cup (G \cap F))) = P(G \cap E) + P(G \cap F) - P((G \cap E) \cap (G \cap F)) \quad (3)$$

The last term is the same as $P(E \cap F \cap G)$ which is given to be 0. Hence from (2) and (3),

$$P(E^c \cap F^c|G) = \frac{P(G) - P(E \cap G) - P(E \cap F)}{P(G)} \quad (4)$$

Note that so far we have not used the hypothesis that the events E, F, G are pairwise disjoint (another term for mutually disjoint). We now put it to to use to replace $P(E \cap G)$ and $P(F \cap G)$ by $P(E)P(G)$ and $P(F)P(G)$ respectively. Then the calculation above reaches to

$$\begin{aligned} P(E^c \cap F^c|G) &= \frac{P(G) - P(G)P(E) - P(G)P(F)}{P(G)} \\ &= 1 - P(E) - P(F) \quad (5) \end{aligned}$$

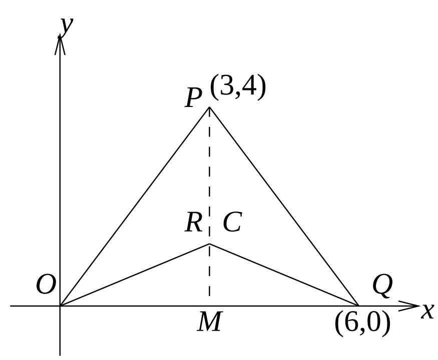
Note that so far all the expressions have been symmetric w.r.t. E and F. We can now replace either $1 - P(E)$ with $P(E^c)$ or $1 - P(F)$ by $E(F^c)$. (These are nothing but complementary probabilities.) Accordingly the answer can be either $P(E^c) - P(F)$ or $P(F^c) - P(E)$. Both are correct. But only the first one is given as one of the alternatives, viz. (C).

The problem is a good combination of the various laws of probability such as the conditional probability, the law of disjunction, the complementary probability. It also tests knowledge of independence of events. Overall it is an excellent problem for a full length question. Even as a multiple choice it is a good question as there is no sneaky path. But once again, the time given is far too inadequate.

Q.31 Let $O(0,0), P(3,4), Q(6,0)$ be the vertices of the triangle OPQ. The point R inside the triangle is such that the triangles OPR, PQR, OQR are of equal areas. The coordinates of R are

(A) $(4/3, 3)$ (B) $(3, 2/3)$ (C) $(3, 4/3)$ (D) $(4/3, 2/3)$

Answer and Comments: (C). This question is really a trick played by the papersetters. Success lies in quick recognising how the point R is related to the triangle OPQ. It is well known that the centroid of a triangle has the property that all the three triangles formed by the sides with opposite vertex at the centroid have equal areas. This follows, for example, from the fact that the centroid divides every median in the ratio 2 : 1.



For example, let C be the centroid of the triangle OPQ and let OM be the median through P. Then we have $PC : CM = 2 : 1$ which implies

$$\Delta OPC : \Delta OCM = 2 : 1 \tag{1}$$

and hence

$$\Delta OPC = \frac{2}{3}\Delta OPM \tag{2}$$

But we also have

$$\Delta OPM = \Delta QPM = \frac{1}{2}\Delta OPQ \tag{3}$$

From (2) and (3),

$$\Delta OCP = \frac{1}{3}\Delta OPQ \tag{4}$$

By symmetry the same assertion holds for ΔOCQ and ΔPCQ. So the three triangles OCP, PCQ and OCQ have equal areas (each equaling one third of the area of the triangle OPQ).

What the present question demands is the converse of the result above. We are given that R is a point inside the triangle OPQ with the property that the triangles ORP, PRQ and ORQ have equal areas. Obviously, each of these areas equals one third of the area of the triangle OPQ. We want to show that R must coincide with C, the centroid. The proof is easy. Since the triangles OCQ and ORQ have the same areas and a common base OQ, the distances of R and C from the line OQ must be equal. Further they are both on the same sides of this line. So R must lie on a line, say L_1 through C which is parallel to the side OQ. By a similar reasoning, R must also lie on a line, say L_2 through C which is parallel to the side PQ. But these two lines can meet only at one point. Hence R must be the same as the point C.

Once R is identified as the centroid of the triangle OPQ, the answer is instantaneous. We simply take the arithmetic means of the coordinates of the vertices to determine the coordinates of R.

Very few students will bother to prove the converse. Once they recall the stated property of the centroid, they will simply identify the point R with the centroid of the triangle OPQ without any further thinking and they will be rewarded since it is only the answer and not the justification that matters.

The question is tricky because if one is asked to prove that the centroid of a triangle has the stated property, that is fairly straightforward. In the present question, even if you have known this property, it might not strike you that it is an indirect way of specifying the centroid. This is so because this property is not one that is used many times, unlike for example, the property of an angle bisector that every point on it is equidistant from the two arms of the angle. he moment you are given that a point is equidistant from two given lines, you immediately realise that it lies on the bisector of one of the angles between the two lines. Recognising that R is the centroid of OPQ from the knowledge that the triangles ORP, PRQ and ORQ have equal areas is not so easy. It takes a different frame of mind to dig out this particular property of the centroid. It is somewhat like this. If somebody asks you the maiden name of your uncle's wife, you may be able to answer it correctly. But if this is not the name by which she is generally called in your family, then even if you come across that name somewhere, you may not immediately be reminded of your uncle's wife.

Fortunately even for those who cannot identify R with the centroid of the triangle OPQ, everything is not lost. One way out is to recognise that the side OQ has length 6 while the altitude through P has length 4. Hence the area of the triangle OPQ is 12 units. It follows that the area of ORQ is 4 units and hence that the perpendicular distance of R from OQ is $2 \times \frac{4}{6}$ i.e. $\frac{4}{3}$. But since the side OQ lies along the x-axis, $4/3$ is also the y-coordinate of R. Among the given answers this holds only in (C). So, no further work is necessary. Yet another lapse on the part of the papersetters.

But even if you do not avail of this short cut, the solution can also be completed honestly. We have already found the y-coordinate of R as $4/3$. Let h be its x-coordinate. Then $R = (h, 4/3)$. The area of the triangle OPR is already known to be 4 units. So, using the determinant formula for the area, we get

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ h & 4/3 & 1 \\ 3 & 4 & 1 \end{vmatrix} = \pm 4 \tag{5}$$

which gives $4h - 4 = \pm 8$. This means h equals either 3 or -1. The latter possibility is ruled out because since the x-coordinates of all the three vertices O, P, Q are non-negative, so will be that of any point inside the triangle. So, we have $R = (3, 4/3)$.

SECTION II

Assertion - Reason Type

This section contains four questions numbered 32 to 35. Each question contains STATEMENT-1 (Assertion) and STATEMENT-2 (Reason). Each question has four choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

Q.32 Consider the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

STATEMENT-1 : The parametric equations of the line of intersection of the given planes are $x = 3 + 4t, y = 1 + 2t, z = 15t$.

because

STATEMENT-2 : The vector $14\hat{i} + 2\hat{j} + 15\hat{k}$ is parallel to the line of intersection of the two planes.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (D). Call the given two planes as P_1 and P_2 and their line of intersection as L. The equations of P_1 and P_2 are

$$\begin{aligned} 3x - 6y - 2z &= 15 \qquad (1) \\ \text{and} \quad 2x + y - 2z &= 5 \qquad (2) \end{aligned}$$

Let L' be the line whose parametric equations are

$$x = 3 + 14t, \quad y = 1 + 2t, \quad z = 15t \qquad (3)$$

Statement-1 says that L' coincides with L. This can be done by actually finding the parametric equations of L and then comparing them with (3). But that is hardly necessary. A line is completely determined by any two of its points. If both these points lie in a plane, so does the entire line. So, in the present case, it suffices to take any two points on L' and check if they satisfy both (1) and (2). For this we feed any two values of the parameter t in (3). The simplest choice is $t = 0$ which gives $x = 3, y = 1, z = 0$. Straight substitution into (1) gives $9 - 6 = 15$, which is false. So the point $(3, 1, 0)$ on L' does not lie in the plane P_1 and hence certainly not on the line L. So without any further checking we see that Statement-1 is false. (If this point did lie on P_1, we would further have to see if it also lies on P_2. If not, the statement is false. If it did, it still does not follow that $L = L'$. We shall have to check with some other value of t, say $t = 1$. Alternately, we can substitute (3) into (1) and (2) and see if both are satisfied for every value of the parameter t.)

Now, as for Statement-2, let $\mathbf{u}$ be a vector parallel to the line L. Then $\mathbf{u}$ is parallel to $\mathbf{u}_1 \times \mathbf{u}_2$ where $\mathbf{u}_1, \mathbf{u}_2$ are vectors perpendicular to the planes P_1 and P_2 respectively. These vectors can be identified by collecting the coefficients of x, y, z from (1) and (2) respectively. Thus we have

$$\begin{aligned} \mathbf{u}_1 &= 3\hat{i} - 6\hat{j} - 2\hat{k} \\ \mathbf{u}_2 &= 2\hat{i} + \hat{j} - 2\hat{k} \end{aligned}$$

$$\text{and hence} \quad \mathbf{u} \quad = \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\hat{i} + 2\hat{+}15\hat{k} \qquad (4)$$

So Statement-2 is true. (Note that it is a pure coincidence that the vector in (4) is exactly the one given in the statement of the problem. Statement-2 would be true even if the two are merely scalar multiples of each other.)

The problem is too straightforward to merit much comment. It would have been a more interesting problem if Statement-2 were replaced by its negation. In that case, Statement-2 (in the new form) would have been false and would have been a valid explanation for Statement-1 (which will remain false). In order that the line L' given by (3) be the line of intersection of the planes P_1 and P_2, two conditions have to be satisfied: (i) the vector $\mathbf{u}_1 \times \mathbf{u}_2$ is parallel to L' and (ii) any one point of L' lies on both P_1 and P_2. In the modified form of the question, (i) would have failed and therefore automatically Statement-1 would have been false. This would have really tested a candidate's understanding of assertion and reason. As it stands, the problem consists of two separate true or false type questions.

Q.33 STATEMENT-1 : The curve $y = \dfrac{-x^2}{2} + x + 1$ is symmetric w.r.t. the line $x = 1$.

because

STATEMENT-2 : A parabola is symmetric about its axis.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (A). Evidently, Statement-2 is true, being a standard property of parabolas. As it stands, Statement-1 means that if a point P lies on the given curve then so does its reflection in the line $x = 1$. This can be verified by direct substitution. Let $P = (h, k)$. Then evidently, $Q = (2 - h, k)$. Now suppose P lies on the given curve. Then we have

$$k = \frac{-h^2}{2} + h + 1 \qquad (1)$$

To check if Q also lies on the same curve we shall have to see if the equation

$$k = \frac{(2-h)^2}{2} + (2-h) + 1 \qquad (2)$$

is true. If we expand $(2 - h)^2$ and use (1), we indeed see that this is the case. So Statement-1 is also true. But so far we have checked the truths of the two statements independently of each other. We now have to see if Statement-2 is a correct explanation of Statement-2. For this we have to see two things, viz. whether the curve is a parabola, and secondly whether $x = 1$ is its axis. This can be done by recasting the equation of the curve so that instead of x it appears in terms of $x - 1$. (In effect, we are shifting the origin to $(1, 0)$. But that need not be done elaborately.)

$$\begin{aligned} y &= -\frac{x^2}{2} + x + 1 \\ &= -\frac{[(x-1)+1]^2}{2} + (x-1) + 2 \\ &= -\frac{(x-1)^2}{2} + \frac{3}{2} \qquad (3) \\ & \qquad (4) \end{aligned}$$

Or in other words,

$$y - \frac{3}{2} = -\frac{1}{2}(x-1)^2 \qquad (5)$$

This is the equation of a vertically downward parabola with centre at $(1, 3/2)$ and axis along $x - 1 = 0$ i.e. along $x = 1$. So, we see that Statement-2 is indeed a correct explanation of Statement-1.

Of course, we could have as well bypassed (1) and (2) because built in (5) is also a proof that the curve is symmetric about the line $x = 1$. The problem could have been made more intersecting if the curve in Statement-1 were some other curve (e.g. an ellipse or a hyperbola with one of the axis along the line $x = 1$. In that case too both statements would be true. But Statement-2 would not be a correct explanation for Statement-1.

Q.34 Let $f(x) = 2 + \cos x$ for all real x. STATEMENT-1 : For each real t, there exists a point c in $[t, t + \pi]$ such that $f'(c) = 0$.

because

STATEMENT-2 : $f(t) = f(t + 2\pi)$ for each real t.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (B). Once again, the truth of Statement-2 is immediate by the fact that 2π is a period of the cosine function. It is tempting to try to prove Statement-1 using Lagrange Mean Value Theorem. This approach would be perfectly feasible if we had $f(t) = f(t + \pi)$, i.e. $\cos t = \cos(t + \pi)$. But this is not the case. That does not however, mean that Statement-1 is false. Just because a statement cannot be proved by one method does not mean it is false. It may sometimes be possible to prove it some other way. It is only when you find a counterexample that you can definitely say that the statement is false. In the present case, this means that if you can find even one value of t for which there exists no $c \in [t, t+\pi]$ for which $f'(c) = 0$, then Statement-1 is false.

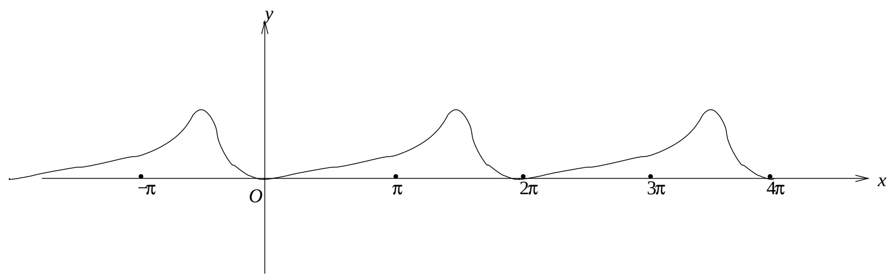
But before trying to find such a counterexample, let us give another try to prove it. The Mean Value Theorem is a general theorem which is applicable to any function which satisfies certain continuity and differentiability conditions. For a specific given function whose derivative is easy to find, questions about the derivative are better answered by directly studying the behaviour of this derivative, rather than by applying MVT. That is what happens in the present problem. We have $f'(x) = -\sin x$ for all x. So, $f'(c)$ is zero precisely when $c = n\pi$ for some integer n. Statement-1 now amounts to saying that every interval of the form $[t, t+\pi]$ contains some integral multiple of π. But this is true because the length of the interval is π. (Note that it is crucial that we are taking a closed interval. A semi-open interval will also do. But the result may be false for an open interval of length π, the simplest counterexample being the interval $(0, \pi)$.)

Thus we see that Statement-1 is also true. Now the question is whether Statement-2 is a correct explanation for Statement-2. If, for example, Statement-1 can be proved using the property of $f(x)$ given in Statement-2, then Statement-2 would explain Statement-1. In the present case, the property in Statement-2 is that the function $f(x)$ is periodic with period 2π. So if we can derive Statement-2 solely from the periodicity of f (with period 2π, then it will be a correct explanation. In other words, the problem now is that if we are given an abstract function $f(x)$ which has 2π as a period, then can we prove that for every t, the interval $[t, t+\pi]$ must contain at least one zero of $f'(x)$? Trivially, the answer is in the negative because mere periodicity does not even imply continuity, let alone differentiability. So, as it stands, Statement-2 is not a correct explanation for Statement-1.

The problem could be made a little more challenging by specifying that in Statement-2, the function $f(x)$ is not only periodic with period 2π but is also differentiable everywhere. We claim that even then it does not imply Statement-1. Note that now we do not know much about the particular function $f'(x)$. So we cannot say much about its vanishing except what is implied by some general results like the MVT. Now, periodicity of f does imply that for every t, $f(t) = f(t+2\pi)$ and hence there is some $c \in (t, t+\pi)$ such that $f'(c) = 0$. But this is no good. We would like to apply MVT to the interval $[t, t+\pi]$ and now we are helpless because periodicity of f implies very little about the relationship between $f(t)$ and $f(t+\pi)$.

Of course, this still does not mean that Statement-2 *cannot* imply

Statement-1. Just because we cannot do something does not mean that nobody can. The only way to assert the impossibility is, once again, by giving a counterexample. That is, we must give an example of a (differentiable) function $f(x)$ with period 2π for which Statement-1 fails. It is not easy to come up with such an example from the standard periodic functions such as the sines and cosines. But it can be constructed by first conceiving its graph. As f is periodic, its values on the interval $[0, 2\pi]$ determine its values everywhere. Care has to be taken to see that $f(2\pi) = f(0)$. Also, f has to be chosen to be differentiable on $(0, 2\pi)$ and further the right handed derivative of f are 0 must match its left handed derivative at 2π (so as to ensure that f remains differentiable at these two points when extended periodically over the entire real line). The best thing is to draw the graph so that it touches the x-axis at these two points. One such graph is shown in the figure below.



For the function shown here $f'(c) = 0$ only when c is an even multiple of π or when it is of the form $3\pi/2 + 2k\pi$ for some integer k. Note in particular that f' never vanishes on the open interval $(0, 3\pi/2)$. This interval has length greater than π. So by taking t to be any value between 0 to $\pi/2$, (e.g. $t = \pi/4$) we get a closed interval $[t, t + \pi]$ which contains no c for which $f'(c) = 0$.

The computational part of this problem is very minimal. It is a reasoning oriented problem. Specifically, it tests the ability to distinguish between the truth of a statement and the relationship of its truth with that of some other statement. In a conventional examination, a candidate will have to give all the reasoning above to claim full credit. So this is another instance where a multiple choice format unduly rewards a student who is good in computation but not sharp in reasoning. For example, a student who naively gives the correct answer by merely thinking that Statement-2 was never used in the proof of Statement-1 gets the correct answer without even really understanding the question. Even for those who understand the question, the papersetters have made the life unduly easy by not specifying differentiability of f in Statement-2.

Q.35 The lines $L_1 : y-x = 0$ and $L_2 : 2x+y = 0$ intersect the line $L_3 : y+2 = 0$

at P and Q respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R.

STATEMENT-1 : The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$.

because

STATEMENT-2 : In any triangle, bisector of an angle divides the triangle into two similar triangles.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (C). Once again, it is a good idea to first tackle Statement-2. It is a very standard property of the bisector of an angle of a triangle that it divides the opposite side in the ratio of the adjacent sides. But nowhere does it say that the two triangles formed by the bisector are similar. Surely, one pair of their angles are equal, each being half the angle that gets bisected. But they both have one angle equal to an angle of the original triangle and there is no reason why these have to be equal, unless the original triangle is isosceles. So Statement-2 is trivially false. And that renders the problem trivial, because, out of the four alternatives given, only in (C) Statement-2 is false. So regardless of whether Statement-1 is true or not, (C) is the right answer.

From the point of view of scoring, it is foolish to spend time checking the truth of (C). Although there have been several other questions so far where the answer could be obtained with an unwarranted short cut, nothing ca come close to this problem. In those problems, even thinking of the short cut required some perceptivity (e.g. Q.9) or some computation (Q.13). In the present question, the falsity of Statement-2 is instantaneous, and thereafter you need not even read Statement-1. That essentially makes the problem a giveaway.

Anyway, from an educative point of view, let us see whether Statement-1 is true. The line L_3 is simply $y = -2$. The equations of L_1 and L_2 can be recast as $y = x$ and $y = -2x$ respectively. So the pints P and Q come out to be $P = (-2, -2)$ and $Q = (1, -2)$. Evidently L_1 and L_2 meet at the origin O. he angle bisector of $\angle POQ$ is given to meet PQ are R. So, by the property of angle bisectors mentioned above, we have

$$PR : RQ = OP : OQ \tag{1}$$

By a direct calculation $OP = 2\sqrt{2}$ and $OQ = \sqrt{5}$. Putting these values into (1) we see that Statement-1 is true.

SECTION III

Linked Comprehension Type

This section contains two paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (A), (B), (C) and (D), out of which **ONLY ONE** is correct.

Paragraph for Question No.s 36 to 38

Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For $n \geq 2$, let A_{n-1} and H_{n-1} have arithmetic, geometric and harmonic means as A_n, G_n, H_n respectively.

Q.36 Which of the following statements is correct?

(A) $G_1 > G_2 > G_3 > \ldots$ (B) $G_1 < G_2 < G_3 < \ldots$
(C) $G_1 = G_2 = G_3 = \ldots$ (D) $G_1 < G_3 < G_5 < \ldots$ and $G_2 > G_4 > G_6 > \ldots$

Answer and Comments: (C). Call the two numbers as a and b with $0 < a < b$. Then, by definition,

$$H_1 = \frac{2ab}{a+b}, \quad G_1 = \sqrt{ab}, \quad A_1 = \frac{a+b}{2} \tag{1}$$

As the question deals with the sequence $G_1, G_2, G_3, \ldots$, we first compute G_2 from (1). As G_2 is given to be the G.M. of A_1 and H_1 we get

$$G_2 = \sqrt{\frac{a+b}{2} \, \frac{2ab}{a+b}} = \sqrt{ab} \tag{2}$$

So, we notice that G_2 is simply G_1. This will not be true of A_2 and H_2. They will be in general different from A_1 and H_1 respectively. But since G_3 is obtained from A_2 and H_2 in precisely the same manner as G_2 has been obtained from A_1 and H_1, we see that G_3 will also equal G_2. Continuing like this we shall get that all G_n's will be equal. So (C) is the correct answer.

This is a good problem, requiring very little computation once the key idea strikes you, viz. that the G.M. of the A.M. and H.M. of two numbers is the same as the G.M. of the original two numbers. This fact is fairly well known. But success lies in digging it out at the right moment.

Q.37 Which of the following statements is correct?

(A) $A_1 > A_2 > A_3 > \ldots$ (B) $A_1 < A_2 < A_3 < \ldots$
(C) $A_1 > A_3 > A_5 > \ldots$ and $A_2 < A_4 < A_6 > \ldots$ (D) $A_1 < A_3 < A_5 > \ldots$ and $A_2 > A_4 > A_6 > \ldots$

Answer and Comments: (A). This time we have to compare the successive arithmetic means. By definition, we have

$$2A_n = A_{n-1} + H_{n-1} \tag{3}$$

for every $n \geq 2$. We now apply the A.M.-G.M.-H.M. inequality by which,

$$H_{n-1} < G_{n-1} < H_{n-1} \tag{4}$$

We are not interested in the middle term. But from the other two terms and (3) we get

$$2A_n = A_{n-1} + H_{n-1} < 2A_{n-1} \tag{5}$$

which implies that $A_n < A_{n-1}$. Thus the sequence $A_1, A_2, A_3, \ldots$ is strictly decreasing.

Questions based on the A.M.-G.M. inequality are very common. The present question requires in addition, the G.M.-H.M. inequality, which, although a direct consequence of the A.M.-G.M. inequality (and, in fact equivalent to it), is used comparatively less frequently. As in the last question, once the key idea strikes, very little work is needed.

Q.38 Which of the following statements is correct?

(A) $H_1 > H_2 > H_3 > \ldots$	(B) $H_1 < H_2 < H_3 < \ldots$
(C) $H_1 > H_3 > H_5 > \ldots$ and $H_2 < H_4 < H_6 > \ldots$	(D) $H_1 < H_3 < H_5 > \ldots$ and $H_2 > H_4 > H_6 > \ldots$

Answer and Comments: (B). This time we have to compare the successive harmonic means. We can proceed as in the last question. But there is a better way which uses the results of the last two questions. We already know that $G_1 = G_2 = G_3 = \ldots$. Let us denote this common value by G. (Actually, G equals $\sqrt{ab}$ as in the solution go Q.36. But that is not very vital here. What matters is that it is a fixed number, independent of n.) We also know that

$$A_n H_n = G_n^2 \tag{6}$$

(which was crucially used in the solution to Q.36). So,

$$H_n = \frac{G^2}{A_n} \tag{7}$$

In the last question, we proved that the A_n's are in a strictly descending order. Hence their reciprocals are in a strictly ascending order. When they are multiplied by the constant G^2 (which is positive), the order remains unaffected. So we conclude that the harmonic means are in a strictly ascending order.

The three parts of the paragraph are closely related and together constitute a very good bunch of simple questions. There is an interesting theme in the subject matter of this paragraph. The very idea of a mean (of whatever kind) is to replace extremes by something in between. In other words, taking means is a sort of a moderation process. So if it is applied repeatedly, the results will be closer and closer to each other than the original numbers. To be precise, if a, b are the original numbers with $0 < a < b$ then for every n we have

$$a \; < H_1 \; < \ldots < H_{n-1} \; < H_n \; < \ldots < \; G \; < \ldots A_n \; < A_{n-1} \; < \ldots \; < A_1 \; < b \quad (8)$$

where $G = \sqrt{ab}$ is the geometric mean of a and b. Thus we see that the sequence $\{A_n\}$ is strictly monotonically decreasing and bounded below (by G). Similarly, the sequence $\{H_n\}$ is strictly monotonically increasing and bounded above, again by G. The difference $A_n - H_n$ is, therefore getting smaller and smaller as n increases. But we can actually show that it tends to 0 as n tends to ∞. This can be done by showing that both the sequences $\{A_n\}$ and $\{H_n\}$ converge to the same limit, viz. G. The proof requires a very basic property of the real number system, viz. its completeness (Chapter 16, p. 583). Because of this property, both the sequences converge to some limits, say A^* and H^* respectively. From (8) we also have

$$H^* \; \leq \; G \; \leq A^* \tag{9}$$

We in fact, claim that equality holds throughout. For this we use (3) and (6) together to get

$$2A_n = A_{n-1} + \frac{G^2}{A_{n-1}} \tag{10}$$

Taking limits as $n \to \infty$ we get

$$2A^* = A^* + \frac{G^2}{A^*} \tag{11}$$

which gives $(A^*)^2 = G^2$ and hence $A^* = \pm G$. But as the original numbers a, b are both positive, so are G and all A_n's. So $A^* \geq 0$ and therefore $A^* = G$. Taking the limits of both the sides in (6) we also get $H^* = \dfrac{G^2}{A^*} = G$. So both the sequences $\{A_n\}$ and $\{H_n\}$ converge to a common limit G as was to be proved.

Summing up, if we start with two fixed positive numbers and recursively define sequences by taking the arithmetic and the harmonic mean then, both these sequences have a common limit and the limit equals the geometric mean of the two numbers we started with. Note that the geometric means are not involved in the construction.

Let us now do an analogous search by recursively taking the arithmetic and the geometric means of two positive real numbers. So, once again, let $0 < a < b$. Let $a_1 = \dfrac{a+b}{2}$ and $g_1 = \sqrt{ab}$. For every $n > 1$ we now define

$$a_n \quad = \quad \frac{a_{n-1} + g_{n-1}}{2} \tag{12}$$

$$\text{and} \quad g_n \quad = \quad \sqrt{a_{n-1} g_{n-1}} \tag{13}$$

Note that a_1 and g_1 coincide, respectively, with A_1 and G_1 as defined in the present paragraph. But this is not true for the subsequent means because they are defined in a different manner. (For example, $G_2 = \sqrt{ab}$ as we saw earlier. But $g_2 = \sqrt{a_1 g_1} = \sqrt{\frac{(a+b)\sqrt{ab}}{2}} = \frac{\sqrt{a+b}}{\sqrt{2}} a^{1/4} b^{1/4}$ which is quite different from G_2.)

Because of the A.M.-G.M. inequality, it is not hard to show that for every n,

$$a < g_1 < g_2 < \ldots < g_{n-1} < g_n < a_n < a_{n-1} < \ldots < a_2 < a_1 < b \quad (14)$$

Thus, $\{g_n\}$ is a strictly monotonically increasing and $\{a_n\}$ is a strictly monotonically decreasing sequence. Trivially, both the sequences are bounded since all their terms lie in the interval $[a, b]$. So each is convergent. Further, using either (12) or (13), it is easy to show that their limits are equal. This common limit is called the **arithmetic-geometric mean** of the real numbers a and b, sometimes denoted by $M(a, b)$. But there is no easy formula to express it directly in terms of a and b. The great mathematician Gauss proved the surprising formula

$$M(a,b) = \frac{\pi}{4\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{(a+b)^2 - (a-b)^2 \sin^2\theta}}} \quad (15)$$

This is a very interesting formula. But the integral cannot be evaluated exactly since there is no antiderivative in a closed form. All one can do is to find its approximate values using techniques such as the trapezoidal rule or the Simpson's rule (p. 653).

We can similarly define the harmonic-geometric mean of a and b as the common limit of sequences recursively defined by taking the harmonic and the geometric means. But again, there is no easy formula for it. However, the harmonic-arithmetic mean of two numbers is simply their geometric mean. That is basically the theme of the present paragraph, if we ignore the question of convergence of the sequences $\{A_n\}$ and $\{H_n\}$.

Paragraph for Question No.s 39 to 41

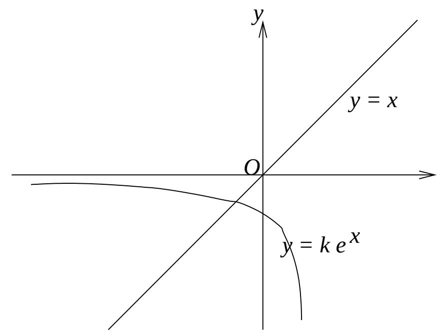
If a continuous function f defined on the real line $I\!R$, assumes positive and negative values in $I\!R$ then the equation $f(x) = 0$ has one root in $I\!R$. For example, if it is known that a continuous function f on $I\!R$ is positive at some point and its minimum value is negative then the equation $f(x) = 0$ has a root in $I\!R$.

Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

Q.39 The line $y = x$ meets the line $y = ke^x$ for $k \leq 0$ at

(A) no points (B) one point
(C) two points (D) more than two points

Answer and Comments: (B). The graph of the exponential function e^x is very standard. Modifying it suitably, we get the graph of $y = ke^x$ for a given real constant k. For $k < 0$, the graph has the shape shown. For $k = 0$, it degenerates into the straight $y = 0$. From the graph it is very obvious that it intersects the line $y = x$ at exactly one point.



While an argument like this is perfectly valid (especially when you don't have to give *any* reasoning!), it is not called an analytic proof, because it presumes that the graph is drawn accurately. But to draw the graph accurately, you first need to have the properties of the exponential function analytically established. So, even when something looks obvious from a graph, one should be prepared to back it by an analytical proof.

In the present case, finding the points of intersection of the two graphs $y = ke^x$ and $y = x$ is equivalent to finding the rots of the equation $f(x) = 0$ where we set

$$f(x) = ke^x - x \tag{1}$$

And this is where the result stated in the preamble becomes relevant. (By the way, this result is the well known Intermediate Value Property on p. 584.) We see that $f(0) = k < 0$. On the other hand, as $x \to -\infty$, $e^x \to 0$ and hence $f(x) \to \infty$. In particular, this means that we can find some R such that for all $x < R$, $f(x) > 0$. Hence by the property in the preamble, f vanishes at least once. Translated in terms of the graphs, the graph of $y = ke^x$ meets that of $y = x$ at at least one point.

To show that there cannot be more than one point of intersection, amounts to showing that the function $f(x) = ke^x - x$ cannot have more than one zero. This does not quite follow from the property in the preamble. Here we need some other theorems based on the derivatives. Note that $f(x)$ is differentiable everywhere with

$$f'(x) = ke^x - 1 \tag{2}$$

The exponential function assumes only positive values. As $k \leq 0$, we see from (2) that $f'(x) < 0$ for all real x. So, the function $f(x)$ is strictly decreasing on $I\!R$. Such a function has to be one-to-one. In particular, it cannot have more than one zero.

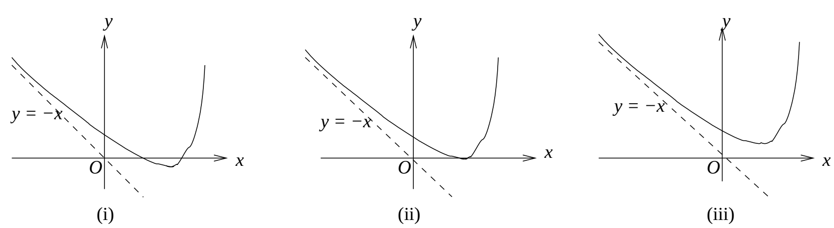
Q.40 The positive value of k for which $ke^x - x = 0$ has only one root is

(A) $\dfrac{1}{e}$ (B) 1 (C) e (D) $\log_e 2$

Answer and Comments: (A). This time we are dealing with the same function as in the last question, but the constant k is now positive and that changes its behaviour drastically. Obviously, $f(x) > 0$ for all $x \leq 0$. But the behaviour of $f'(x)$ is now different. When k was not positive, $f'(x)$ was negative for all x and so the function $f(x)$ was strictly decreasing on the entire real line. But when $k > 0$, from (2) we see that $f'(x) = 0$ when $x = -\ln k$. Further, since e^x is a strictly increasing function of x, we also see from (2) that $f'(x) < 0$ for $x < -\ln k$ and $f'(x) > 0$ for $x > -\ln k$. As a result, $f(x)$ is strictly decreasing on $(-\infty, -\ln k)$ and strictly increasing on $(-\ln k, \infty)$. So, $f(x)$ will have an absolute minimum at $x = -\ln k$. Also we have

$$f(-\ln k) = ke^{-\ln k} + \ln k = 1 + \ln k \tag{3}$$

Thus we conclude that the minimum value of $f(x)$ on $\mathbb{R}$ is $1 + \ln k$ and it is attained at $x = -\ln k$. To get a qualitative idea of the shape of the graph of $f(x)$ we observe that since $k > 0$, $f(x) \to \infty$ as $x \to -\infty$. But more is true. As $x \to -\infty$, ke^x gets very small. So the graph of $y = f(x)$ will lie only slightly above the line $y = -x$. In fact, the vertical distance between the two will tend to 0 as $x \to -\infty$. As a result, the line $y = -x$ is an asymptote of the graph (p. 235). On the other hand, for large positive values of x, e^x and hence ke^x is much bigger than x. (This can be seen by observing that $e^x > \dfrac{x^2}{2}$ for all $x > 0$. Or one can apply the L'ôpital's rule to see that $ke^x/x \to \infty$ as $x \to \infty$.) Therefore $f(x) \to \infty$ very rapidly as $x \to \infty$.



We now get a good idea of the graph of $y = f(x)$. Its shape is qualitatively the same for all positive values of k. But regarding where it cuts the x-axis, several cases arise depending on what $\ln k$ is. They are:

(i) If $\ln k < -1$, i.e. if $k < 1/e$, then the minimum value of $f(x)$ is negative and the second part of the preamble of the paragraph applies. In this case, the graph of $y = f(x)$ cuts the x-axis at two distinct points. Or in other words, the equation $f(x) = 0$ has exactly two roots.

(ii) $\ln k = -1$, i.e. $k = 1/e$. In this case the graph of $f(x)$ touches the x-axis at $(1, 0)$. And the function $f(x)$ has only one zero.

(iii) $\ln k > -1$, i.e. $k > 1/e$. In this case the minimum of the function is positive and the graph lies entirely above the x-axis.

Clearly, Case (ii) contains the answer to the present question.

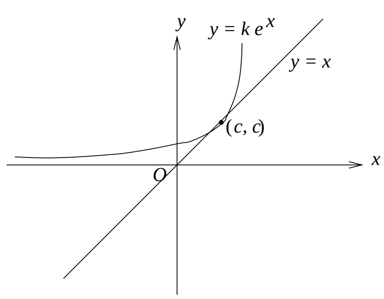
Q.41 For $k > 0$, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is :

(A) $(0, 1/e)$ (B) $(1/e, 1)$ (C) $(1/e, \infty)$ (D) $(0, 1)$

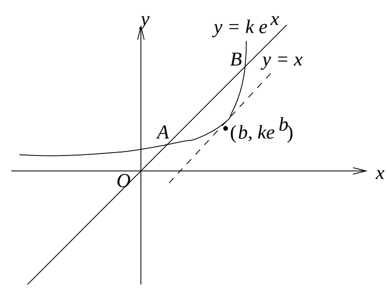
Answer and Comments: (A). The explanation was already given in the discussion of Case (i) in the solution to the last question.

The common theme of this paragraph is the behaviour of the graph of the function $ke^x - x$ for various values of the parameter k. For $k \leq 0$, the function is strictly decreasing on the entire real line. (In the degenerate case when $k = 0$, $f(x)$ is merely $-x$ and the graph is a straight line.) When $k > 0$, the function has a global minimum. But the number of zeros of $f(x)$ depends on the value of k. In a conventional examination all the three questions could have been clubbed together into a single question and the candidates could also have been asked to draw the graph of $y = f(x)$ for various values of k.

Questions 40 and 41 directly deal with the function $f(x)$. It is not clear why Question 39 has been cast in a different form, viz. to find the number of points of intersection of the curve $y = ke^x$ and the line $y = x$. Probably the idea is to give some variety to the question as otherwise there is considerable repetition of work in the three questions. We gave two solutions for Q.39. One was based on drawing the graphs $y = ke^x$ and $y = x$ and then appealing to visual intuition. In the analytic solution we converted the problem to finding the number of zeros of the function $f(x) = ke^x - x$. In essence the second solution was the analytic version of the first solution which was very intuitive in nature.



(a)



(b)

Let us now go the other way round and convert the analytic solutions of Questions 40 and 41 to their intuitive versions. That is, we view the problem of finding the roots of the equation $ke^x - x = 0$ as the problem of finding the points of intersection of the two curves $y = ke^x$ and $y = x$ and tackle it by drawing graphs. We already have the graph of $y = ke^x$ for $k < 0$ in the figure accompanying the solution to Q.39. The graph of $y = ke^x$ when $k > 0$ will be the mirror image of this in the x-axis. The line $y = x$ will be entirely below this graph for some values of k (e.g. if $k \geq 1$). For some other values of k, it will cut the graph in two distinct points as Figure (b). But these two points can coincide for some value(s) of k as shown in Figure (a) and Q.40 asks precisely when this happens. Geometrically, it is obvious that in this case the line $y = x$ must touch the graph $y = ke^x$. Let this point of contact be (c, c) for some $c > 0$. Then the point (c, c) lies on the graph $y = e^x$ and so

$$c = ke^c \tag{4}$$

But since the two curves touch each other, their slopes at this common point must also coincide. That gives

$$ke^c = 1 \tag{5}$$

Solving (4) and (5) together, we get $c = 1$ and $k = 1/e$. So, for this value of k, the function $f(x) = ke^x - x$ has exactly one root. This gives the same answer to Q.40 as before.

Now, for Q.41, we look at Figure (b). We are given that the line $y = x$ meets the curve $y = ke^x$ at two distinct points. By the geometric version of the Lagrange's MVT, between these two points, there exists some point, say (b, ke^b) on this curve the tangent at which is parallel to the chord AB. This means

$$ke^b = 1 \tag{6}$$

From the figure it is clear that the portion of the curve $y = ke^x$ between the points A and B lies below the chord AB. This is yet another fact which looks very obvious from the graph. For a rigorous proof, one notes that since $k > 0$, the function $y = ke^x$ is concave upwards. Therefore, in addition to the equality (6), we also have an inequality, viz.

$$ke^b < b \tag{7}$$

where the inequality is strict because the point (b, ke^b) is not on the line $y = x$. (6) and (7) together imply $b > 1$. But by (6) again, $b = -\ln k$. So we have $-\ln k > 1$, or $\ln k < -1$ which is equivalent to $k < 1/e$. Thus the values of k for which the line $y = x$ cuts the curve $y = ke^x$ in two distinct points for $k \in (0, 1/e)$. Translated back, these are also the values for which the function $f(x) = ke^x - x$ has two roots.

Note that the analytical version and the graphic version are not radically different since both go through the same steps. The graphic proof is more vivid and easier to conceive. The fact that such proofs are not accepted in 'polished' mathematics is really not a serious drawback. Actually, even a mature mathematician rarely conceives a proof i an analytical form. He often draws graphs and visually sees what is happening. This inspires the graphic proof. Then he translates it into an analytical proof.

SECTION IV

Matching Answers Type

This section contains 3 questions. Each question contains entries given in two columns, those in **Column I** being labelled A,B,C,D while those in **Column II** are labelled p,q,r,s. An entry in either column may have more than correct matches in the other. Indicate all the correct matchings (and no others) by writing ordered pairs of the form (X,y) where X is an entry in **Column I** and y is an entry in **Column II**

Q.42 **Column I** contains four incomplete statements. Match them with their correct completions in **Column II**.

Column I	**Column II**
(A) Two intersecting circles	(p) have a common tangent
(B) Two mutually external circles	(q) have a common normal
(C) Two circles, one strictly inside the other	(r) do not have a common tangent
(D) Two branches of a hyperbola	(s) do not have a common normal

Answer and Comments: (A,p), (A,q), (B,p), (B,q), (C,q), (C,r), (D,q), (D,r). The parts regarding circles are trivial and common observations. *Any two* circles in the same plane have a common normal, viz. the line joining their centres. (When the circles are concentric, any line through their common centre is a normal to both of them.) As for a common tangent, it is obvious that the only time it false to exist is when one circle lies completely inside the other. (It is interesting to recall here that in the 1993 JEE, a question asked for a common tangent of two circles whose equations were given. But because of a misprint, it turned out that one of the circles was strictly inside the other!)

That leaves us only with the last statement, involving a hyperbola. Since the question asks only for pure geometric properties of the hyperbola, we are free to take its equation in the standard form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1}$$

The two branches are characterised by $x \geq a$ (the right branch) and $x \leq -a$ (the left branch). The x-axis is a normal to both the branches. The only part left is to check if they have a common tangent. For this, it is convenient to take the parametric form of (1) given by

$$x = a\sec\theta, \quad y = b\tan\theta \tag{2}$$

Here the parameter θ varies in $(-\pi/2, \pi/2)$ for the right branch and in $(\pi/2, 3\pi/2)$ for the left branch. Now suppose the two branches have a common tangent, touching the right branch at $P = (a\sec\alpha, b\tan\alpha)$ and the left branch at $Q = (a\sec\beta, b\tan\beta)$ where $\alpha \in (-\pi/2, \pi/2)$ and $\beta \in (\pi/2, 3\pi/3)$. Equating the slopes of the tangents at these two points we get

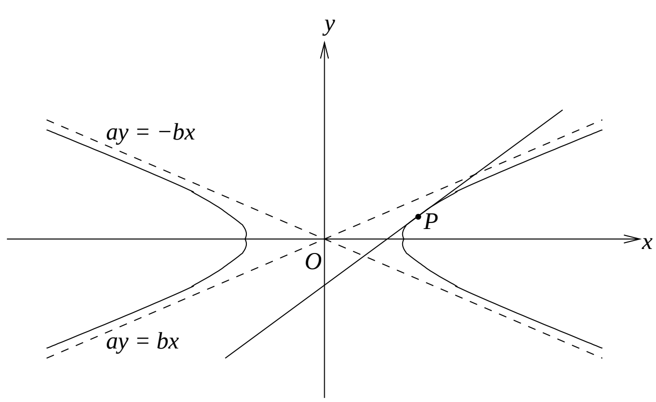
$$\frac{b\sec^2\alpha}{a\tan\alpha\sec\alpha} = \frac{b\sec^2\beta}{a\tan\beta\sec\beta} \tag{3}$$

which simplifies to $\sin\alpha = \sin\beta$. Hence we must have

$$\beta = n\pi + (-1)^n\alpha \tag{4}$$

for some integer n. Because of the restrictions on α and β, the only way this can happen is if $\alpha + \beta = \pi$. But in that case the points P and Q are $(a\sec\alpha, b\tan\alpha)$ and $(-a\sec\alpha, -b\tan\alpha)$. The slope of the line PQ is, therefore $\dfrac{2b\tan\alpha}{2a\sec\alpha} = \dfrac{b\sin\alpha}{a}$, which does not tally with the slope of the tangent at P, viz. $\dfrac{b}{a\sin\alpha}$.

Thus we have proved that the two branches of the hyperbola (1) can have no common tangent. This is also obvious intuitively if one looks at the diagram of the hyperbola in which the two asymptotes $y = \pm\dfrac{b}{a}x$ are also shown. The numerical value of the slope of the tangent at any point is greater than that of the asymptotes. So the tangent is steeper than the asymptotes and hence cannot intersect the other branch of the hyperbola.



There is also a super-intelligent way to show that the two branches of the hyperbola (1) cannot have a common tangent. The key idea is to recognise that a tangent is a line which meets a curve in two coincident points. (This idea was also used in the graphic solution to Q.40.) So, if a line touches a curve at two distinct points then it meets that curve in four points, paired off in two pairs of coincident points. But since (1) is a second degree equation in both x and y, this is impossible. To elaborate, suppose the equation of a straight line is

$$y = mx + c \tag{5}$$

To get the common points of this line with the hyperbola, we substitute (5) into (1) and get

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1 \tag{6}$$

This equation is a quadratic in x and hence has two roots which could be real and distinct, real and coincident or complex. Accordingly, the line cuts the curve in two distinct points or touches it at one point or does not meet it at all. In no case, can it be a tangent to the curve at two distinct points.

Whichever way one attempts it, this last part is the only non-trivial part of the entire question. It is unthinkable that a candidate who can do this part will be unable to do the other parts. Since there is no part marking to the question, one wonders what is the purpose of clubbing together these parts. It only forces a good student to waste his time in doing some trivial work. And the irony is that if he commits a silly clerical error in filling the boxes corresponding to the right answers, then he gets a duck as per the rules because the evaluation is done by machines. So, one single lapse can spoil some other good work even when the two are completely unrelated. This is almost like reducing the perfect score of a student in one subject to zero because he has failed in some other subject.

Q.43 Let (x, y) be such that

$$\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$$

Column I contains four incomplete statements. Match them with their correct completions in **Column II**.

Column I	**Column II**
(A) If $a = 1$ and $b = 0$, then (x, y)	(p) lies on the circle $x^2 + y^2 = 1$
(B) If $a = 1$ and $b = 1$, then (x, y)	(q) lies on $(x^2 - 1)(y^2 - 1) = 0$
(C) If $a = 1$ and $b = 2$, then (x, y)	(r) lies on $y = x$
(D) If $a = 2$ and $b = 2$, then (x, y)	(s) lies on $(4x^2 - 1)(y^2 - 1) = 0$

Answer and Comments: (A,p), (B,q), (C,p), (D,s). This is a question about inverse trigonometric functions. The equation in the preamble and hence its solution set depends on the values of the parameters a and b. However, the problem does not require us to solve the equation in its generality. Instead, we are given four special cases of the equation. The simpler approach is to tackle them one-by-one. This is also the approach a candidate is most likely to take because in many other matching pairs type questions, the various parts are often unrelated and each requires separate work (e.g. see Q. 20 and 21). So, even if there is some common method to attack all the parts, a candidate's mind is not channeled to try it.

So, we first tackle each statement in **Column I** separately. In (A), the equation reduces to

$$\sin^{-1} x + \cos^{-1} y = 0 \tag{1}$$

or equivalently, $\sin^{-1} x = -\cos^{-1} y$. Taking sines of both the sides we get

$$x = -\sin(\cos^{-1} y) = \pm\sqrt{1-y^2} \tag{2}$$

which implies that $x^2 + y^2 = 1$.

In (B), the given equation reduces to

$$\sin^{-1} x + \cos^{-1} y + \cos^{-1}(xy) = \frac{\pi}{2} \tag{3}$$

or equivalently,

$$\cos^{-1} y + \cos^{-1}(xy) = \frac{\pi}{2} - \sin^{-1} x \tag{4}$$

Taking cosines of both the sides we get

$$yxy \pm \sqrt{1-y^2}\sqrt{1-x^2y^2} = x \tag{5}$$

or equivalently,

$$x(y^2-1) = \pm\sqrt{1-y^2}\sqrt{1-x^2y^2} \tag{6}$$

Squaring, $x^2(y^2-1)^2 = (1-y^2)(1-x^2y^2)$. So either $1-y^2 = 0$ or $1-x^2y^2 = x^2(1-y^2)$. The latter means $1 = x^2$. Hence at least one of $(1-x^2)$ and $(1-y^2)$ has to vanish, which is equivalent to (q).

Next, in (C) the equation to be solved reduces to

$$\sin^{-1} x + \cos^{-1} y + \cos^{-1}(2xy) = \frac{\pi}{2} \tag{7}$$

Rewriting as before and taking cosines of both the sides gives (analogously to (5)),

$$2xy^2 \pm \sqrt{1-y^2}\sqrt{1-4x^2y^2} = x \tag{8}$$

Simplifying as before this becomes

$$x^2(1-2y^2) = (1-y^2)(1-4x^2y^2) \tag{9}$$

which, after expansion and cancellation, reduces to

$$x^2 = 1 - y^2 \tag{10}$$

So the correct alternative is (p) again.

Finally, in (D), similar steps give

$$2xy^2 - \pm\sqrt{1-y^2}\sqrt{1-4x^2y^2} = 2x \tag{11}$$

which becomes

$$4x^2(1-y^2)^2 = (1-y^2)(1-4x^2y^2) \tag{12}$$

which gives either $1-y^2 = 0$ or $4x^2(1-y^2) = 1-4x^2y^2$. The latter reduces to $1-4x^2 = 0$. The two possibilities together are equivalent to (s) in **Column II**.

Note that there is considerable duplication of work, especially in (B), (C) and (D). We could have avoided it by first solving the given equation for any general values of thew parameters a and b. We begin by rewriting it as

$$\cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2} - \sin^{-1}(ax) \tag{13}$$

Taking cosines of both the sides we get

$$bxy^2 \pm \sqrt{1-y^2}\sqrt{1-b^2x^2y^2} = ax \tag{14}$$

Simplifying and squaring gives

$$x^2(1-by^2)^2 = (1-y^2)(1-b^2x^2y^2) \tag{15}$$

This simplifies to

$$a^2x^2 + (b^2-2ab)x^2y^2 + y^2 = 1 \tag{16}$$

Now, one by one, we put the values of a and b given in the statements in **Column I**. In (A), we have $a = 1, b = 0$ and we simply get $x^2 + y^2 = 1$. In (B), we have $a = 1, b = 1$ and so $x^2 - x^2y^2 + y^2 = 1$ which can be recast as $(1-x^2)(1-y^2) = 0$. In (C), $a = 1, b = 2$ and so $x^2 + y^2 = 1$ once again. Finally, in (D), $a = 2, b = 2$ giving $4x^2 - 4x^2y^2 + y^2 = 1$ which can be recast as $(1-4x^2)(1-y^2) = 0$. This solution involves the same reasoning as before but is shorter because duplication of work has been avoided. The ability to detect an opportunity for avoiding duplication of work is a valuable asset both in mathematics and in real life. The present problem is ideal to reward those who have this ability. But once again, since there is no way to find out how the candidates arrived at the answer it is impossible to really find out who are these gifted candidates.

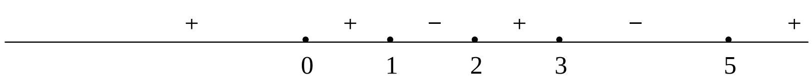
Q.44 Let $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 5x + 6}$. **Column I** contains some incomplete statements. Match them with their correct completions in **Column II**.

Column I	**Column II**
(A) If $-1 < x < 1$, then $f(x)$ satisfies	(p) $0 < f(x) < 1$
(B) If $1 < x < 2$, then $f(x)$ satisfies	(q) $f(x) < 0$
(C) If $3 < x < 5$, then $f(x)$ satisfies	(r) $f(x) > 0$
(D) If $x > 5$, then $f(x)$ satisfies	(s) $f(x) < 1$

Answer and Comments: (A,p), (A,r), (A,s), (B,q), (B,s), (C,q), (C,s), (D,p), (D,r), (D,s). Note that in the second column, (p) is equivalent to (r) and (s) put together. So, if (p) holds for some entry in the first column, then we automatically know that (r) and (s) also hold. Also if (q) holds then neither (p) nor (r) can hols. But (s) must hold. These logical interdependencies reduce the work a little. The given expression for $f(x)$ is a ratio of two quadratic expressions for in x. So, the sign of $f(x)$ depends on the signs of the numerator and the denominator. Both are easy to determine because the quadratics factor easily, giving

$$f(x) = \frac{(x-1)(x-5)}{(x-2)(x-3)} \tag{1}$$

As both the numerator and the denominator have only simple roots, we expect sign changes to occur at each one of them, i.e. at $1, 2, 3$ and 5. Clearly $f(x) > 0$ for $x < 1$ as all factors are negative and similarly $f(x) > 0$ for $x > 5$ as all the factors are positive. Hence $f(x)$ is negative on $(1, 2)$ and $(3, 5)$ and positive on $(2, 3)$. (See the figure below.)



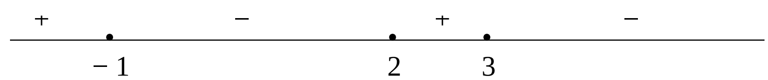
sign of f(x)

Therefore we have the pairs (A,r), (B,q), (C,q) and (D,r). Also the first inequality in (p) holds in (A) and (D).

We now have to compare the values of $f(x)$ with 1. For this we first consider

$$\begin{aligned} f(x) - 1 &= \frac{x^2 - 6x + 5}{x^2 - 5x + 6} - 1 \\ &= \frac{-x - 1}{x^2 - 5x + 6} \\ &= \frac{-(x+1)}{(x-2)(x-3)} \end{aligned} \tag{2}$$

The denominator is negative on $(2, 3)$ and positive outside the interval $[2, 3]$ while the numerator is positive for $x < -1$ and negative for $x > -1$. Put together $f(x) - 1$ is positive for $x < -1$, negative for $-1 < x < 2$, positive again for $2 < x < 3$ and negative for $x > 3$. (See the figure below.)



sign of f(x) − 1

So (s) and the second part of (p) hold true for all the four entries in **Column I**. This completes the solution.

This is a simple problem especially since questions based on the same key idea have been asked before in JEE (e.g. Problem 6.25(c) on p. 230). As in the last problem, the work involved is repetitious because you have to give very similar reasonings for all the four entries of the first column. But in the last problem, there was a way to avoid the duplication of work. In the present question, there is no such way.

CONCLUDING REMARKS

The general comments made in the *Educative Commentary on JEE 2006 Mathematics Paper* about the undesirability of making the entire question paper objective type apply to the present paper too and need not be repeated. Last year, fill in the blank questions were asked where the answers were numerical and could be indicated by filling in the appropriate bubbles in the Objective Response Sheet (ORS). In such questions, you cannot get the correct answer merely by eliminating the others. You often have to do the entire work, except that you do not show it. This year, no such questions are asked. The reason is not clear. It could be the criticism that as there is nothing to guide a sincere candidate, a silly slip of computation (or even of filling the bubbles) could cost him heavily unlike in the conventional type examination where the work shown by him could earn him substantial partial credit. Anyway, such questions constituted only a small part of the last year's question paper and so their deletion this year does not make much difference.

The most noticeable difference between the two papers is that both the 2007 papers are considerably simpler than those of the 2006 paper. In fact sometimes this simplicity borders on triviality. Take for example, the definite integrals in Q.20. Evaluation of an integral like $\int_{-1}^{1} \frac{dx}{1+x^2}$ might have come as the last step of some bigger problem such as evaluation of some area or the evaluation of some more complicated integral. But they have never been asked by themselves.

This is also probably the first time that the JEE paper contains not a single problem where areas have to be evaluated using definite integrals. Such questions, even when fairly straightforward, often demand fair length of work. Last year the candidates were asked to find the area bounded by the parabolas $-4y^2 = x$ and $x-1 = -5y^2$ but the time allowed (on a proportionate basis) was less than a minute! Naturally this invited a lot of criticism. But it is hard to hold the papersetters responsible for this mockery because they probably had to work under severe constraints and therefor had no freedom to award a question credit proportional to the time it would reasonably take. Apparently, this year the papersetters have registered their protest by simply not asking any such question!

Also gone are any questions in number theory, matrices or combinatorial identities. Q. 40 involves the minimum of a function in an indirect manner. Except for that there are no questions on maxima and minima. The probalility questions are domainated by conditional probability. Inequalities are touched upon (in Q. 37, 38 and 44) but only in a very predictable manner. These problems come nowhere close to those requiring ingenious regrouping of terms such as Problem 11 in JEE 2004 Main Paper.

But perhaps this is the price to be paid when the papersetters are forced to replace full length questions where the candidates have show their work and are given a reasonable time for it, by questions that have to be finished in a minute or two. No challenging problems can be asked without inviting severe criticism. So the only option left is to ask a large number of extremely straightforward questions which makes speed a dominant consideration rather than quality of thinking. One is reminded of a story where the father of a girl of a marriageable age asks his friend to find her a suitable man. The friend instead sends two boys of ages 12 each (or perhaps three boys of 8 each) saying that together they are as good as a young man of 24.

Not surprisingly, the candidates were very happy this year that they got an easy question paper. But can the ability to evaluate the four trivial integrals in Q.20 be equated with the ability to evaluate even one of the many challenging integrals that have been asked in the past?

Given the constraints, the papersetters have done a commendable job in some questions. For example, Q. 22 is a really good question. It is probably the best question among all. Some of the paragraphs have some interesting coherent themes. Q. 9 and Q. 23 could have become good questions if care had been taken to prevent sneaky answers.

But overall, the questions are too elementary and straightforward to warrant a detailed analysis of the qualities of each question as was done in the commentary last year. If this year's JEE is an indication of the way JEE will be in the years to come, one doubts for how long it will be able to retain its prestige in the academic circles.