EDUCATIVE COMMENTARY ON JEE 2008 MATHEMATICS PAPERS

Contents

The pattern of JEE 2008 closely resembles that of JEE 2007. As in 2007, this year too there were two papers, each covering all the three subjects, viz. Mathematics, Physics and Chemistry. All the questions are of the multiple choice type and nowhere the candidate has to show his work. The only departure is that while last year each subject in each paper had 22 questions, this year it has 23 questions in Paper 1 and 22 in Paper 2. Also, unlike last year, the first paper contains a few questions where more than one of the given answers is correct while the second paper contains a few questions of matching the pairs where, the same entry in each column may have several matches in the other column. These features seem to have been revived from JEE 2006. In effect they increase the actual number of questions the candidate has to think about because one single question amounts to several subquestions, some of which may be totally unrelated to each other. One saving feature is that there is no negative marking for these questions (even though there is negative marking for all other questions). Moreover, for matching questions in Section IV of Paper 2, some partial credit is allowed for every statement from Column I which is correctly matched. Apparently, this is in an answer to the criticism that in the past there was some unfairness in such questions because even one failure wiped out the good performance on the remaining subquestions, even though the two may be quite unrelated.

Together, each subject in Paper 1 has 82 marks while that in Paper 2 has 81 marks. So, as in 2007, a candidate who apportions his time equally gets just about 45 seconds for each mark.

By the time the present commentary was prepared, the IITs have officially displayed on their websites the questions and the model answers (but without any justifications). The numbering of the questions in each paper follows that on the websites except where the comments of some question refer to those of some other, in which case the latter is shifted to an earlier place if necessary. As in the case of the educative commentaries on the JEE papers of the last few years, the references given here refer to the author's *Educative JEE Mathematics*, unless otherwise stated.

QUESTIONWISE PAGINATION

Paper 1

Paper 2

PAPER 1

Contents

SECTION I

Straight Objective Type

This section contains 6 multiple choice questions. Each question has 4 choices out of which **ONLY ONE** is correct. There are 3 marks for a correct answer, 0 marks if the question is not answered and -1 mark in all other cases.

Q.1 Consider the two curves $C_1 : y^2 = 4x$ and $C_2 : x^2 + y^2 - 6x + 1 = 0$. Then,

(A) C_1 and C_2 touch each other only at one point

(B) C_1 and C_2 touch each other exactly at two points

(C) C_1 and C_2 intersect (but do not touch) exactly at two points

(D) C_1 and C_2 neither intersect nor touch each other.

Answer and Comments: (B). The curve C_1 is a standard parabola while C_2 is a circle centred at the point $(3, 0)$ on the x-axis. As a result, both the curves are symmetric about the x-axis. Also the first curve meets the x-axis only at the origin which does not lie on the second curve. So, the curves C_1 and C_2 do not meet anywhere on their axis of symmetry. It follows that they must meet at an even number of points, with half of them lying above the x-axis and the other half being their reflections into the x-axis. Moreover, if they touch each other at some point then they will also touch each other at its reflection.

Unfortunately, this reasoning, based solely on the symmetry of the two curves eliminates only one possibility, viz. (A). To decide which of the remaining alternatives is the right answer, it is better to actually find the points of intersection and then see if the curves have the same slope at these common points. For the given two curves, this is very easy to do. If we solve their equations simultaneously, we get $x = 1$ and $y = \pm 2$. So the points of intersection are $(1, 2)$ and $(1, -2)$ (which agrees with our observation that they are reflections of each other in the x-axis.) Because of symmetry, it suffices to check if the two curves touch each other either one of these two points. We choose it to be the point $(1, 2)$. At this point the tangent to the curve $y^2 = 4x$ has slope 1 as can be checked by implicit differentiation (which gives $2yy' = 4$ and hence $y' = 2/y$). The slope of the tangent to the curve at the point $(1, 2)$ can also be obtained by implicit differentiation. But since C_2 is known to be a circle centred at $(3, 0)$, we can also get the slope of the tangent at $(1, 2)$ from the slope of the radius through $(1, 2)$. Either way the slope is 1. So the two curves have a common tangent at the point $(1, 2)$ and hence touch each other at this point. As noted earlier, it follows they also touch each other at $(1, -2)$. So (B) holds.

In this problem it was very easy to explicitly find the points of intersection of the two curves. The problem would have been more interesting (and challenging) if this were not so. Suppose, for example, that we keep the first curve C_1 as it is but let C_2 be the curve $y^2 = f(x)$ where $f(x)$ is some (differentiable) function of x. (In the given problem $f(x) = 6x - x^2 - 1$.) Again we assume that $f(0) \neq 0$ so that the origin is not a common point of the two curves. Once again, by symmetry, if (x_0, y_0) is a point of intersection of C_1 and C_2, then so is $(x_0, -y_0)$. At any such point we must have $f(x_0) = 4x_0$, or equivalently, if we define $g(x) = f(x) - 4x$ then x_0 is a root of the equation

$$g(x) = 0 \tag{1}$$

So the two curves will intersect if and only if Equation (1) has a positive root. In the present problem, $f(x) = 6x - x^2 - 1$ and so (1) reduced to a quadratic $x^2 - 2x + 1 = 0$ which is very easy to solve. But if $f(x)$ is a more complicated function, then we may not be able to solve (1) explicitly. Even then the existence of a positive solution of (1) can be proved by some other methods, based on calculus. We illustrate this in one case. Take $f(x) = x^3 + 7x - 1$. Then (1) becomes

$$x^3 + 3x - 1 = 0 \tag{2}$$

This is a cubic equation and there is no obvious real root. But since $g(0) = -1 < 0$ while $g(1) = 3 > 0$, by the Intermediate Value Property, $g(x)$ has at least one root in the interval $(0, 1)$. Hence the two curves C_1 and C_2 intersect at at least two points. We note however, that $g'(x) = 3x^2 + 3 > 0$ for all x and so $g(x)$ is strictly increasing on the entire real line. So, it cannot have more than one root. Thus in this case the two curves intersect precisely at the two points $(x_0, \pm 2\sqrt{x_0})$ where x_0 is the unique real root of (1).

Thus we can settle the question of intersection of the two curves C_1 and C_2 without actually finding the points of intersection. Let us now see if the two curves touch each other at the point, say $P_0 = (x_0, y_0)$, of their intersection. For this to happen the slopes of the tangents to the curves at P_0 must be equal. By implicit differentiation of the equations $y^2 = 4x$ and $y^2 = f(x)$, these slopes are, respectively, $\dfrac{2}{y_0}$ and $\dfrac{f'(x_0)}{2y_0}$. For these to be equal, we must have $f'(x_0) = 4$, or $g'(x_0) = 0$ where $g(x) = f(x) - 4x$ as defined above. Put differently, x_0 must be a root of the equation

$$g'(x) = 0 \tag{3}$$

Since x_0 is already a root of $g(x)$, to say that it is also a root of $g'(x)$ is equivalent to saying that it is a multiple root of $g(x)$. In case $f(x)$ and hence $g(x)$ are polynomials then this is equivalent to saying that $(x - x_0)^2$ is a factor of $g(x)$. We already saw that in our given problem, $g(x) = x^2 - 2x + 1$ and this does have $(x-1)^2$ as a factor. (In fact, it equals $(x-1)^2$.) As another example, suppose $f(x) = x^3 + x - 1$. In this case, $g(x) = x^3 - 3x - 1$. Once again this is a cubic whose roots cannot be found by inspection. However, in this case we note that $g'(x) = 3x^2 - 3$ has two roots, viz. -1 and 1. As neither of these is a root of $g(x)$, we conclude that $g(x)$ has no multiple roots. Therefore the two curves $y^2 = 4x$ and $y^2 = x^3 + x - 1$ cannot touch each other at any point. They do intersect, however precisely at two points because from the observations $g(-1) = 1 > 0, g(0) = -1 < 0$ and $g(2) = 1 > 0$ and the Intermediate Value Property we see that $g(x)$ has precisely one positive real root (the other two roots being negative, one lying in $(-\infty, -1)$ and the other in $(-1, 0)$).

The same reasoning can be applied to a still more general situation. Suppose $C_1 : y^2 = f_1(x)$ and $C_2 : y^2 = f_2(x)$ are two curves, where $f_1(x)$ and $f_2(x)$ are differentiable functions of x. We assume $f_1(x)$ and $f_2(x)$ have no common zero, so that the two curves do not meet on the x-axis. We now let $g(x) = f_1(x) - f_2(x)$. Then the points of intersection of C_1 and C_2 are of the form $(x_0, \pm\sqrt{f_1(x_0)})$ where x_0 is a root of $g(x)$ for which $f_1(x_0)$ (and hence also $f_2(x_0)$) is positive. Further, the curves C_1 and C_2 touch each other at such a point if and only if x_0 is a multiple root of $g(x)$.

The given problem is a trivial special case of this where $f_1(x) = 4x$ and $f_2(x) = 6x - x^2 - 1$. Non-trivial special cases can be constructed by choosing $f_1(x)$ and $f_2(x)$ suitably. Of course in that case the credit (and hence the proportionate time) allocated to the question would be far too inadequate. If the paper-setters were not constrained to make all questions of the same degree of difficulty and had the freedom to allot marks proportionate to the degree of difficulty, the more challenging question could have been asked. Yet another casualty of the non-academic constraints faced by the paper-setters.

Q.2 If $0 < x < 1$, then $\sqrt{1+x^2}\left[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1\right]^{1/2}$ is equal to

$$\text{(A) } \frac{x}{\sqrt{1+x^2}} \quad \text{(B) } x \quad \text{(C) } x\sqrt{1+x^2} \quad \text{(D) } \sqrt{1+x^2}$$

Answer and Comments: (C). A very straightforward problem about the inverse trigonometric functions. Let $\theta = \cot^{-1} x$. Then $x = \cot\theta$ and hence $\tan\theta = \dfrac{1}{x}$. Also, $\pi/4 < \theta < \pi/2$ since it is given that $0 < x < 1$. For this range of values of θ, we have $\sin\theta = \dfrac{1}{\sqrt{1+x^2}}$ and $\cos\theta = \dfrac{x}{\sqrt{1+x^2}}$. So the given expression becomes $\sqrt{1+x^2}\left[\left(\frac{x^2}{\sqrt{1+x^2}} + \dfrac{1}{\sqrt{1+x^2}}\right)^2 - 1\right]^{1/2}$ which simplifies to $\sqrt{1+x^2}(1+x^2-1)^{1/2}$ which equals $x\sqrt{1+x^2}$ since $x > 0$.

The derivation is too straightforward to tempt anybody to take short cuts, which are sometimes possible in multiple choice questions where the wrong alternatives can be eliminated by working out the problem in special cases. In the given range of x, viz. from 0 to 1, the only value of x for which $\cot^{-1} x$ is a familiar angle is $x = \frac{1}{\sqrt{3}}$ (with the corresponding θ equaling $\pi/3$). But the computation of the given expression for this particular value takes almost the same work as for the general value. A clever student can still try to salvage the situation by observing that the given expression as well as all the four possible alternatives are continuous functions of x on the interval $[0, 1]$. As a result, if the given expression agrees with one of the alternatives on the open interval $(0, 1)$, then by continuity they will also have to agree at the end points 0 and 1. So, even if x is given to lie only in the open interval $(0, 1)$, we can give x the values 0 and 1. For these values of x, the expression in the statement of the problem is indeed easy to evaluate and comes out to be 0 and $\sqrt{2}$ respectively. For $x = 0$, the choices (A), (B) and (C) all reduce to 0, while for $x = 1$, both (C) and (D) reduce to $\sqrt{2}$. As a result, neither $x = 0$ nor $x = 1$ will alone betray the right alternative. It is only if a candidate has worked out *both* these values then he can identify (C) as the right answer. The paper-setters deserve to be commended for ensuring that the alternatives given do not sneakily give away the the right choice.

Q.3 The edges of a parallelopiped are of unit length and are parallel to non-coplanar unit vectors $\hat{a}, \hat{b}, \hat{c}$ such that $\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{c} = \hat{c}\cdot\hat{c} = 1/2$. Then the volume of the parallelopiped is

$$\text{(A) } \frac{1}{\sqrt{2}} \quad \text{(B) } \frac{1}{2\sqrt{2}} \quad \text{(C) } \frac{\sqrt{3}}{2} \quad \text{(D) } \frac{1}{\sqrt{3}}$$

Answer and Comments: (A). The desired volume is the scalar triple product $\hat{a}\cdot(\hat{b}\times\hat{c})$. The question is how to express this in terms of the data, where we are given the dot products of every two of the vectors $\hat{a}, \hat{b}$ and $\hat{c}$. There is one identity which does this job most directly, viz.

$$(\hat{a}\cdot(\hat{b}\times\hat{c}))^2 = \begin{vmatrix} \hat{a}\cdot\hat{a} & \hat{a}\cdot\hat{b} & \hat{a}\cdot\hat{c} \\ \hat{b}\cdot\hat{a} & \hat{b}\cdot\hat{b} & \hat{b}\cdot\hat{c} \\ \hat{c}\cdot\hat{a} & \hat{c}\cdot\hat{b} & \hat{c}\cdot\hat{c} \end{vmatrix} \tag{1}$$

The data says that all the diagonal entries equal 1 while all others equal 1/2 each. So, the R.H.S. reduces to the determinant $\begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix}$. A direct evaluation gives the value of the determinant as 1/2 and hence that of the box product as $\dfrac{1}{\sqrt{2}}$.

The identity (1) is not among the standard identities about the scalar triple product. Those who do not know it can proceed by directly expressing $(\hat{a}\cdot(\hat{b}\times\hat{c}))^2$ using some of the more well-known identities about the dot and the cross products. For example, it is quite well-known (and easy to prove) that

$$(\vec{u}\cdot\vec{v})^2 = |\vec{u}|^2|\vec{v}|^2 - |\vec{u}\times\vec{v}|^2 \tag{2}$$

for any two vectors $\vec{u}$ and $\vec{v}$. Taking $\vec{u} = \hat{a}$ and $\vec{v} = \hat{b}\times\hat{c}$, we get

$$(\hat{a}\cdot(\hat{b}\times\hat{c}))^2 = |\hat{a}|^2|\hat{b}\times\hat{c}|^2 - |\hat{a}\times(\hat{b}\times\hat{c})|^2 \tag{3}$$

We now use (2) again to rewrite $|\hat{b}\times\hat{c}|^2$ as

$$|\hat{b}\times\hat{c}|^2 = |\hat{b}|^2|\hat{c}|^2 - (\hat{b}\cdot\hat{c})^2 \tag{4}$$

We also use the well-known identity for the vector triple product to get

$$\hat{a}\times(\hat{b}\times\hat{c}) = (\hat{a}\cdot\hat{c})\hat{b} - (\hat{a}\cdot\hat{b})\hat{c} \tag{5}$$

and further,

$$\begin{aligned} |\hat{a}\times(\hat{b}\times\hat{c})^2|^2 &= ((\hat{a}\cdot\hat{c})\hat{b} - (\hat{a}\cdot\hat{b})\hat{c})\cdot((\hat{a}\cdot\hat{c})\hat{b} - (\hat{a}\cdot\hat{b})\hat{c}) \\ &= (\hat{a}\cdot\hat{c})^2(\hat{b}\cdot\hat{b})^2 - 2(\hat{a}\cdot\hat{c})(\hat{a}\cdot\hat{b})(\hat{b}\cdot\hat{c}) \\ &\quad + (\hat{a}\cdot\hat{b})^2(\hat{c}\cdot\hat{c})^2 \end{aligned} \tag{6}$$

If we substitute (4) and (6) into (3), the terms correspond exactly to those in the expansion of the determinant on the R.H.S. of (1) and so we get a proof of (1). In the present problem, however, (1) is only a means and not the goal. So, we may as well substitute the numerical values of the various dot products into (4) and (6) and get the answer from (3).

There is also a purely geometric way of doing the problem. Let $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ be the three vectors $\hat{a}, \hat{b}$ and $\hat{c}$ respectively. Then the volume of the parallelopiped is 6 times the volume of the tetrahedron $OABC$. From the data of the problem, this tetrahedron is regular with each edge of unit length. So the area of the base triangle OBC is $\frac{1}{2}\sin 60^\circ = \dfrac{\sqrt{3}}{4}$. To determine the altitude, note that the perpendicular from the vertex A to the base must fall, by regularity, on the centroid, say G of the base triangle OBC. (See the figure in Comment No. 18 of Chapter 8, where the notations are slightly different and each side of the tetrahedron is a rather than of unit length.) The distance OG is $2/3$ of the length of the median through O. As the median of an equilateral triangle with unit side is $\dfrac{\sqrt{3}}{2}$, we have $OG = \dfrac{1}{\sqrt{3}}$. From the right-angled triangle OGA, we have $AG^2 = OA^2 - OG^2 = 1 - \dfrac{1}{3} = \dfrac{2}{3}$. So the altitude AG has length $\dfrac{\sqrt{2}}{\sqrt{3}}$ units. Therefore the volume of the tetrahedron is $\dfrac{1}{3} \times \dfrac{\sqrt{3}}{4} \times \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{2}}{12}$ cubic units. Hence the volume of the parallelopiped is $\dfrac{\sqrt{2}}{2}$ i.e. $\dfrac{1}{\sqrt{2}}$ cubic units.

For those who know the identity (1), the problem is too straightforward. For others it is an interesting problem but demands a rather heavy price in terms of time. While there is basically nothing wrong in rewarding a candidate who knows more, the multiple choice format makes the disparity unfair, because a candidate who knows (1) but cannot prove it can't be distinguished from someone who can prove it. It is also questionable whether it is a good idea to ask a problem whose answer can be obtained by merely plugging in the given values into some formula. In the present case, for example, instead of giving the dot products of the vectors directly, it could have been given that every two of them are inclined at an angle of 60°. In fact, as the problem is from geometry, there was no need to mention vectors in its statement. Vectors are a convenient but not an indispensable tool and it should have been left to the candidate to paraphrase the problem so that this tool could be applicable. Also, instead of asking for the volume of the parallelopiped, the question could have asked for its height. In that case, the answer will not come directly from a single formula. Some more work will have to be done, viz. dividing the volume by the area of one of the surfaces. In fact, this modification would reward those who take the geometric approach because in this approach, the height is found before finding the volume, as we saw above.)

Q.4 Let a and b be non-zero real numbers. Then the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents

(A) four straight lines, when $c = 0$ and a, b are of the same sign
(B) two straight lines and a circle, when $a = b$, and c is of sign opposite to that of a
(C) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a
(D) a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a

Answer and Comments: (B). Here the key idea is very simple, viz. a product of expressions vanishes if and only if at least one of the factors vanishes. So the given equation represents the union of the curves obtained by equating each factor to 0. The second factor of the given expression further factorises into two linear factors and hence represents the pair of straight line $x - 2y =$ and $x - 3y = 0$. So the correct answer has to be (A), (B) or (C) depending on the signs of the constants a, b, c. If all three have the same sign, the first factor will represent the empty set. In all the three options, a, b have the same sign and c has the opposite sign. So the first factor will represent a circle if $a = b$ and an ellipse otherwise. So, out of the given options, (B) holds.

This is a good conceptual problem, which requires very little computation.

Q.5 Let $g(x) = \dfrac{(x-1)^n}{\log \cos^m (x-1)}$; $0 < x < 2$, m, n are integers, $m \neq 0$, $n > 0$ and let p be the left hand derivative of $|x - 1|$ at $x = 1$. If $\lim\limits_{x \to 1^+} g(x) = p$, then

(A) $n = 1, m = 1$
(B) $n = 1, m = -1$
(C) $n = 2, m = 2$
(D) $n > 2, m = n$

Answer and Comments: (C). This problem is a hotchpotch of two completely unrelated problems. Instead of specifying p directly, it is given as the left hand derivative of the function $|x - 1|$ at the point $x = 1$. The graph of this function is obtained by shifting the graph of the function $|x|$ by one unit to the right. The latter graph is well-known and has 0 as a point where the function is continuous but not differentiable, with the left and right hand derivatives being -1 and 1 respectively. The same holds for the function $|x - 1|$ at the point $x = 0$. So we get $p = -1$.

The main problem begins now. We are given that

$$\lim_{x \to 1^+} \frac{(x-1)^n}{\log \cos^m (x-1)} = -1 \tag{1}$$

With the substitution $h = x - 1$ (which again amounts to shifting the origin to the right by one unit), this means

$$\lim_{h \to 0^+} \frac{h^n}{m \log \cos h} = -1 \tag{2}$$

As we are given that n is a positive integer, it follows that the numerator h^n tends to 0 as h tends to 0. If the denominator $m \log \cos h$ tends to 0 more rapidly than the numerator then the ratio will tend to ∞. But if it tends to 0 less rapidly than h^n then the ratio will tend to 0. When the two tend to 0 at comparable rates then the ratio will tend to a finite non-zero limit. So the problem is of the same spirit as the JEE 2002 problem given in Comment No. 13 of Chapter 16, where we have to find which power of x is comparable to $(\cos x - 1)(\cos x - e^x)$ as $x \to 0$. And, as commented there, such problems can be done using either the l'Hôpital's rule, or the Taylor expansion (or Taylor series) of the given function near 0.

Before doing so, let us recast the problem slightly. Since $\frac{\sin h}{h} \to 1$ as $h \to 0$, we might as well replace the numerator h^n of the L.H.S. of (2) by $\sin^n h$. This may seem to complicate the problem. But the advantage is that the denominator can be expressed in terms of $\sin(h/2)$ if we use the trigonometric identity $\cos h = 1 - 2\sin^2(\frac{h}{2})$. Doing so and calling $\sin(h/2)$ as u we can rewrite (2) as

$$\lim_{u \to 0^+} \frac{2^n u^n}{m \log(1 - 2u^2)} = -1 \tag{3}$$

Let us now expand $\log(1 - 2u^2)$ near $u = 0$. This can be done either by repeated differentiation of the function or by putting $x = -2u^2$ in the expansion of $\log(1 + x)$ near $x = 0$, which is fairly standard. Either way, the first few terms are

$$\log(1 - 2u^2) = -2u^2 - \frac{4u^4}{2} - \frac{8u^6}{3} + \ldots \tag{4}$$

So, near $u = 0$, $\log(1 - 2u^2)$ is comparable to $-2u^2$. Hence the exponent n in the numerator must also equal 2 if the ratio is to tend to a finite non-zero limit. For this value of n, the ratio becomes $\frac{4u^2}{m \log(1 - 2u^2)}$, which, in view of (4), behaves the same way as the ratio $\frac{4u^2}{-2mu^2}$ as $u \to 0$. So we now get an equation for m, viz. $\frac{2}{-m} = -1$ which determines m as 2.

Instead of Taylor expansion, we could have used l'Hôpital's rule to evaluate the limit in (3) which is of the indeterminate form $\frac{0}{0}$. Differentiation reduces the limit to $\lim_{u \to 0^+} \frac{2^n n u^{n-1}(1 - 2u^2)}{-4mu}$ which is still of the $\frac{0}{0}$ form. But we cancel u from both the numerator and the denominator to get the limit as $\lim_{u \to 0^+} \frac{2^n n u^{n-2}}{-4m}$. If this limit is to equal -1 then the exponent of u in the numerator must vanish, which gives $n = 2$. The subsequent determination of m is similar to the one above.

The essential idea in this problem is to determine the rate at which a function tends to 0. The introduction of m and p is totally unrelated to the basic idea and only serves to complicate the problem. Even without them there is plenty of thinking in the problem, viz. replacing h by $\sin h$ and $\cos h$ by $1-2\sin^2(h/2)$. In fact, without m and p, it would have been a better problem. It is also not clear what purpose is served by giving only the right hand limit as x tends to 1. It could as well have been the two sided limit. And, finally, there was no need to pose the problem in terms of $x-1$ and letting $x \to 1$, instead of posing it in terms of x and letting $x \to 0$. So here is an example of a basically good problem marred by unnecessary appendages. Perhaps they have been added as otherwise the problem would have looked too similar to the JEE 2002 problem mentioned above.

Q.6 The total number of local maxima and local minima of the function $f(x)$ defined by $f(x) = \begin{cases} (2+x)^3, & -3 < x \leq -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$ is

(A) 0 (B) 1 (C) 2 (D) 3

Answer and Comments: (C). Whenever a function is specified by a formula, it is often a good idea to draw at least a qualitative graph of the function. Such a graph not only helps you understand the problem more easily, it often allows you to answer some simple questions about the function merely by inspection. This is a sound piece of advice when the function is a combination of some standard functions whose graphs are very familiar. The trouble with other functions is that in order to draw a reasonably good graph, you often have to put in a lot of work. In particular, you have to identify the intervals over which the function is increasing or decreasing, the intervals over which it is concave upward or downward and so on. If the question requires only some of this work, then it is better to answer it directly without drawing the graph.

That is exactly what happens in the present problem. We first observe that the function $f(x)$ is continuous at $x = -1$ because both the right and left handed limits at $x = -1$ equal 1 which is also the value of $f(-1)$. Continuity at other points is no problem. We are only asked to find the local maxima and the local minima of the function. For a continuous function, these are, respectively, the points at which the function changes its behaviour from increasing to decreasing and from decreasing to increasing. So, we shall be through if we identify the intervals over which the given function $f(x)$ is increasing and those where it is decreasing. And, instead of doing this mechanically by using derivatives, there is a simpler way in the present problem. We simply note that as x increases, so does $2+x$. The cube is a strictly increasing function and so the expression $(2+x)^3$ is increasing over the entire real line, and hence in particular on the interval $(-3,-1]$. As for the behaviour of $f(x)$ on the interval $[-1, 2)$, we note that for these values, it simply is the cube root of x^2. Once again, the cube root is throughout increasing. So the increasing/decreasing behaviour of $f(x)$ on $[-1, 2)$ is the same as that of x^2 on that interval. This is a very familiar function which decreases upto 0 and increases thereafter. Putting all these pieces of information together, we now see that $f(x)$ is increasing on $(-3, -1]$, decreasing on $[-1, 0]$ and then again increasing on $[0, 2)$. So, it has a local maximum at $x = -1$ and a local minimum at $x = 0$. (Note that the end-points -1 and 2 are not in the domain of the function. If they were, then the function $f(x)$ would have a global minimum at -3 since $f(-3) = -1 < 0 = f(0)$. Similarly the global maximum would be at the end point 2. As the question stands, there is neither a global maximum nor a global minimum over the domain of the function.)

This is a simple problem once the candidate correctly analyses what is asked and what is needed. So, it is a good test of the candidate's ability to focus on the essential part.

SECTION II

Multiple Correct Answers Type

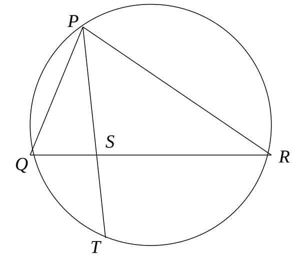
This section contains 4 multiple correct answer(s) type questions. Each question has 4 choices (A), (B), (C) and (D) out of which **ONE OR MORE** is/are correct. There are 4 marks for a completely correct answer (i.e. all the correct and only the correct options are marked), and zero mark in all other cases.

Q.7 A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then

(A) $\frac{1}{PS} + \frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$ (B) $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$

(C) $\frac{1}{PS} + \frac{1}{ST} < \frac{4}{QR}$ (D) $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$

Answer and Comments: (D). This is a problem from pure geometry. But usually in pure geometry problems you are asked to prove some equality. (Problems asking concurrency, collinearity or concyclicity can also be reduced to proving some equalities.) Inequalities are rare in pure geometry. So in that sense this is an unusual problem.



This unusual feature of the problem suggests that the solution will be based more on some inequalities rather than on geometry. If we recast the first two given answers by dividing the L.H.S. by 2, the L.H.S. becomes the A.M. of the quantities $\frac{1}{PS}$ and $\frac{1}{ST}$. So the A.M.-G.M. inequality gives

$$\frac{1}{PS} + \frac{1}{ST} \geq \frac{2}{\sqrt{PS.ST}} \tag{1}$$

We now use a fact from pure geometry which gives

$$PS.ST = QS.SR \tag{2}$$

(A proof of this can be given using the similarity of the triangles ΔPSQ and ΔRST.) If we substitute (2) into (1) we get

$$\frac{1}{PS} + \frac{1}{ST} \geq \frac{2}{\sqrt{QS.SR}} \tag{3}$$

It is now tempting to think that (B) is a correct choice. But this need not be so. In (B), strict inequality is to hold, whereas in (1) and hence in (3) equality is possible when $PS = ST$, i.e. when S is the mid-point of the chord PT. There is nothing in the data of the problem which says that this cannot happen. So (B) need not necessarily hold. All we can say for sure is that (A) cannot hold.

Let us now move on to the other two statements, viz. (C) and (D) which are similar to each other, except for the reversal of the inequality sign. We have $QS + SR = QR$. So, once again, the A.M.-G.M. inequality gives

$$\frac{QR}{2} = \frac{QS + SR}{2} \geq \sqrt{QS.SR} \tag{4}$$

Taking reciprocals of the first and the last terms and multiplying by 2,

$$\frac{2}{\sqrt{QS.SR}} \geq \frac{4}{QR} \tag{5}$$

If we combine (3) and (5) we get

$$\frac{1}{PS} + \frac{1}{ST} \geq \frac{4}{\sqrt{QR.RS}} \tag{6}$$

Once again, it is tempting to conclude that (D) holds. But we shall have to rule out the possibility of equality. Note that the inequality (6) was derived from the inequalities (2) and (4) and so equality will hold in (6) if and only if it holds in both (2) and (4). We already saw that it holds in (2) if S is the midpoint of PT. By a similar reasoning, equality holds in (4) if and only if S is the midpoint of QR. So, if both the equalities are to hold then both the chords PT and QR will be bisected at the point S. But by (3) this would also mean $PS = QS$. Thus the point S will be equidistant from all the four points P, Q, R and T and hence will be the centre of the circumcircle. But the statement of the problem specifically excludes this possibility. So we conclude that strict inequality must hold either in (3) or in (5) (or both). In any case it holds in (6) and so (D) is always true.

The only geometric fact needed in the solution is (2). The inequality needed is also a very standard one, viz. the A.M.-G.M. inequality. But the solution requires a careful analysis of when equality holds. Those who hastily conclude (B) will be penalised. So this is a very well designed problem. Sneaky short cuts by working out some special simple cases are also precluded. For example, it is tempting to take the triangle PQR to be an equilateral triangle inscribed in a unit circle and S to be the midpoint of the side QR. In that case PT will be a diameter and the lengths of the various segments can be calculated easily using trigonometry (or coordinates). But in this case S will not be the mid-point of PT and so (B) will be a correct answer. It is only if a candidate chooses to work with a special case in which S is the midpoint of PT that he will see that (B) is not a correct choice. But it is very unlikely that anybody will choose such a weird simple case. (It is not clear, however, if even the paper-setters had all this in mind when they designed the question, because in the answers published on the official websites of the IITs, both (B) and (D) are given as correct alternatives. Apparently, the paper-setters themselves missed the subtlety of the hypothesis that S is not the circumcentre of the triangle PQR. They probably mistook it to mean that neither the chord PT nor the chord QR is bisected at S. But, as we saw above, the only correct inference that can be drawn from the hypothesis is that at least one of the two chords is not bisected at S. Usually, the JEE answers are officially displayed after keen scrutiny by experts. It is shocking indeed that such a mistake went unnoticed by the experts.)

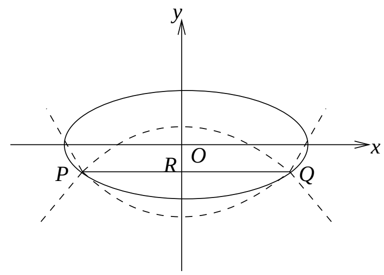
Q.8 Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ with $y_1 < 0$ and $y_2 < 0$ be the end-points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. Then the equations of the parabola with latus rectum PQ are

(A) $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$ (B) $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$
(C) $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$ (D) $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

Answer and Comments: (B,C). The solution depends crucially on the concept of a latus rectum. For a parabola, it is fairly well-known. However, for an ellipse and a hyperbola it is not so commonly used. For all the three conics, it is a chord through a focus which is perpendicular to the axis. Let us first cast the equation of the ellipse in a standard form, viz.

$$\frac{x^2}{4} + y^2 = 1 \tag{1}$$

So with usual notations, $a = 2$ and $b = 1$ for this ellipse. Hence its eccentricity e, given by the formula $b^2 = a^2(1 - e^2)$ comes out as $\frac{\sqrt{3}}{2}$. Therefore the foci of the ellipse are located at $(\pm\sqrt{3}, 0)$. Hence the length of each latus rectum is $2\sqrt{3}$. There are two latus recti, one above and the other below the x-axis. The problem deals with the latter. Its end-points are $P = (-\sqrt{3}, -\frac{1}{2})$ and $Q = (\sqrt{3}, -\frac{1}{2})$. Its midpoint R is $(0, -\frac{1}{2})$.



The problem now deals with a parabola whose latus rectum is the segment PQ. There are two such parabolas, one vertically upwards and the other vertically downwards, as shown by dotted lines in the figure. Their vertices will lie on the y-axis at a distance $\frac{1}{4}PQ$, i.e. at a distance $\frac{\sqrt{3}}{2}$ from the midpoint R of PQ. So the two positions for the vertices are $\left(0, -\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$ and $\left(0, -\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$. The equation of the first parabola is

$$x^2 = -2\sqrt{3}(y + \frac{1}{2} - \frac{\sqrt{3}}{2}) \tag{2}$$

while that of the other parabola is

$$x^2 = 2\sqrt{3}(y + \frac{1}{2} + \frac{\sqrt{3}}{2}) \tag{3}$$

which reduce to (C) and (B) respectively.

This is a straightforward, computational problem once you understand the meanings of the terms involved.

Q.9 Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}$, for $n = 1, 2, 3, \ldots$. Then

(A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$ (C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$

Answer and Comments: (A, D). It is tempting to start by evaluating both the sums in a closed form. But, as it happens with most sums, getting a closed form expression for such sums is usually difficult, if not impossible. Nor is it necessary. We are *not* interested in the exact value of either sum *per se*. We merely want to compare them with a fixed real number viz., $\frac{\pi}{3\sqrt{3}}$, and this may be possible if there is a common genesis for the sums and this peculiar real number.

Once this idea strikes, we look for something where sums of this type arise naturally. The general term of both the sums is the same, viz. $\frac{n}{n^2+nk+k^2}$. If we recast this as $\frac{1}{n}\frac{1}{1+\frac{k}{n}+(\frac{k}{n})^2}$, then we recognise that this is the k-th term of a Riemann sum of the function $f(x) = \frac{1}{1+x+x^2}$ over the interval $[0,1]$ partitioned into n equal parts. So, as $n \to \infty$, both S_n and T_n tend to a common limit, viz. $\int_0^1 \frac{dx}{1+x+x^2}$. If we write the denominator as $(x+1/2)^2 + (\sqrt{3}/2)^2$ we see that an antiderivative for the integrand is $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2(x+1/2)}{\sqrt{3}}\right)$. Hence the integral comes out to be $\frac{2}{\sqrt{3}}(\tan^{-1}(\sqrt{3}) - \tan^{-1}(\frac{1}{\sqrt{3}})) = \frac{2}{\sqrt{3}}(\frac{\pi}{3} - \frac{\pi}{6}) = \frac{\pi}{3\sqrt{3}}$.

Thus we see that the magic number $\frac{\pi}{3\sqrt{3}}$ is simply the integral of the function $f(x) = \frac{1}{1+x+x^2}$ over the interval $[0,1]$. The problem asks us to compare it with the Riemann sums S_n and T_n. For arbitrary Riemann sums this is usually not easy since although they are close to the integral, they can be either bigger or smaller than the integral. But in the present case, the integrand is a strictly decreasing function of x. So, when the interval $[0,1]$ is partitioned into n equal parts, the maximum and the minimum of the function $f(x)$ over the k-th subinterval $[\frac{k-1}{n}, \frac{k}{n}]$ occur at the left and the right end points respectively. As a result, T_n is the upper Riemann sum and S_n is the lower Riemann sum of $f(x)$ for this partition. Therefore we have

$$T_n > \int_0^1 f(x)\,dx > S_n \tag{1}$$

As the middle term has already been evaluated as $\frac{\pi}{3\sqrt{3}}$, our question is answered completely. The problem is simple once the key idea, viz. that of a definite integral strikes. But there is nothing to suggest in the statement of the problem that integrals are involved. So this is a very good problem provided the candidate knows the concept of upper and lower Riemann sums. These sums are not explicitly mentioned in the JEE syllabus, where integration is studied as the inverse process of differentiation. Even then,

(1) can be derived by an alternate argument as follow. We split the interval $[0,1]$ into n equal parts. Then the integral in (1) also gets split into n parts. Specifically,

$$\int_0^1 f(x)\, dx = \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} f(x)\, dx \qquad (2)$$

We now observe that as the function $f(x)$ is strictly decreasing over the interval $[\frac{k-1}{n}, \frac{k}{n}]$, we have

$$f(\frac{k-1}{n}) > f(x) > f(\frac{k}{n}) \qquad (3)$$

for all $x \in (\frac{k-1}{n}, \frac{k}{n})$. Keeping in mind that the length of this subinterval is $\frac{1}{n}$, integration of each of the three terms of this inequality gives

$$\frac{f(\frac{k-1)}{n})}{n} > \int_{(k-1)/n}^{k/n} f(x)\, dx > \frac{f(\frac{k}{n})}{n} \qquad (4)$$

If we calculate $f(\frac{k-1}{n})$ and $f(\frac{k}{n})$, we get

$$\frac{1}{n^2 + n(k-1) + (k-1)^2} > \int_{(k-1)/n}^{k/n} f(x)\, dx > \frac{1}{n^2 + kn + k^2} \qquad (5)$$

for $k = 1, 2, \ldots, n$. Summing over for these values, we get

$$\sum_{k=1}^{n} \frac{1}{n^2 + n(k-1) + (k-1)^2} > \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} f(x)\, dx > \sum_{k=1}^{n} \frac{1}{n^2 + kn + k^2} \quad (6)$$

By (2), the middle sum is $\int_0^1 f(x)\, dx$, while the leftmost and the rightmost sums are precisely T_n and S_n respectively. So, we get an alternate derivation of (1).

Summing up, once a candidate realises that the problem involves the integral of the function $\frac{1}{1+x+x^2}$, it is a manageable problem whether he knows upper and lower Riemann sums or not. But once again, because of the multiple choice format, the sneakers have a field day. All they need to do is to evaluate the sums S_n and T_n for just one value of n. For $n = 1$, there is just one term in each summation and so we get very directly that $T_1 = 1$ and $S_1 = \frac{1}{3}$. From any crude approximate values of π and $\sqrt{3}$ (e.g. $\pi = 3.2$ and $\sqrt{3} = 1.7$) we see easily that $\frac{\pi}{3\sqrt{3}}$ is approximately $\frac{2}{3}$ and so we get $T_1 > \frac{\pi}{3\sqrt{3}} > S_1$. In other words, it follows that the inequality

$$T_n \;>\; \frac{\pi}{3\sqrt{3}} \;>\; S_n \qquad (7)$$

is true for $n = 1$. Of course, this does not mean that (7) is true for *all* n. But it does eliminate both (B) and (C) as the right answers and suggests that (A) and (D) might be the right answers. A rigorous proof of this is a far cry. But what does it matter to cheaters, especially when there is no negative credit. The time they save by not bothering about such a proof will fetch them a lot of credit elsewhere. Yet another sad testimony how the multiple choice format (with no need to prove that the choice is correct) punishes scruples.

However, even as the things stand, the paper-setters could have done something to disallow such a blatantly sneaky answer. Replace the number $\frac{\pi}{3\sqrt{3}}$ by some slightly smaller number, say A, e.g. $A = \frac{\pi}{6}$. And the four options could have been as follows :

(A$'$) $T_n > A$ for all $n = 1, 2, 3 \ldots$

(B$'$) $T_n > A$ for some but not for all n

(C$'$) $S_n < A$ for all $n = 1, 2, 3, \ldots$

(D$'$) $S_n < A$ for some but not for all n.

In this modified set-up, the honest candidate will do essentially the same work. But the added feature will be that since A is less than the integral $\int_0^1 f(x)\,dx$, all the upper Riemann sums will continue to be bigger than A. But as for the lower Riemann sums, they will eventually exceed A. So the correct alternatives are (A$'$) and (D$'$). Had we taken A to be slightly bigger than $\frac{\pi}{3\sqrt{3}}$, the situation would have reversed and the correct alternatives would have been (B$'$) and (C$'$). The point is that the wrong alternatives cannot be eliminated merely by calculating S_n and T_n for some lower values of n. If we want to stress the importance of the role of the number $\frac{\pi}{3\sqrt{3}}$ in this problem, then A could be given as $\frac{\pi}{3\sqrt{3}} - \frac{1}{100}$. The perceptive candidate will take the hint that there is really nothing so great about $\frac{1}{100}$ and that the only role it has is to make the number slightly less than $\frac{\pi}{3\sqrt{3}}$. He will then realise that this number $\frac{\pi}{3\sqrt{3}}$ must have a special role in the problem and when his integral comes out to be precisely this number, he would know that he is on the right track.

So, with a little more imagination and work, the paper-setters can save the problem from cheaters.

Q.10 Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1-x)$ and $f'(\frac{1}{4}) = 0$. Then,

(A) $f''(x)$ vanishes at least twice on $[0, 1]$

(B) $f'(\frac{1}{2}) = 0$

(C) $\displaystyle\int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right)\sin x\, dx = 0$

(D) $\displaystyle\int_0^{1/2} f(t)e^{\sin\pi t}dt = \int_{1/2}^{1} f(1-t)e^{\sin\pi t}dt.$

Answer and Comments: (A,B,C,D). The function $f(x)$ is not given explicitly. Apart from twice differentiability and the derivative at one point all that is given about the function is a functional equation satisfied by it, viz.

$$f(x) = f(1-x) \tag{1}$$

for all $x \in I\!R$. This condition is far from sufficient to determine the function $f(x)$ uniquely. However, by its very nature, condition (1) is similar to the definition of an even function, say $g(x)$, viz.

$$g(x) = g(-x) \tag{2}$$

for all x. Note that in (2), the points x and $-x$ are symmetric about the origin. In (1), the points x and $1-x$ are symmetrically located w.r.t. the point $\frac{1}{2}$. Keeping this in mind, (1) can be rewritten as

$$f(\frac{1}{2}+x) = f(\frac{1}{2}-x) \tag{3}$$

for all $x \in I\!R$. (For a formal proof, replace x in (1) by $x+\frac{1}{2}$ which is permissible since (1) is given to hold for *all* values of x.) It follows that if we define a new function $g(x)$ by

$$g(x) = f(x+\frac{1}{2}) \tag{4}$$

for $x \in I\!R$, then this new function satisfies (2) and hence is an even function. Once this key idea strikes, the question can be answered using some very standard properties of even functions. Let us note first that by the chain rule

$$g'(x) = f'(x+\frac{1}{2}) \tag{5}$$

for all $x \in I\!R$. In particular, $f'(\frac{1}{2}) = g'(0)$. As $g(x)$ is an even function and is given to be differentiable everywhere, we have

$$g'(-x) = -g'(x) \tag{6}$$

for all $x \in I\!R$. In view of (5) this translates as

$$f'(-x+\frac{1}{2}) = -f'(x+\frac{1}{2}) \tag{7}$$

for all $x \in I\!R$. In particular, putting $x = 0$ we get that $f'(\frac{1}{2}) = -f'(\frac{1}{2})$ which means $f'(\frac{1}{2}) = 0$. Thus (B) holds. As for (A), we are given that $f'(\frac{1}{4}) = 0$. So, putting $x = \frac{1}{4}$ in (7) we get that $f'(\frac{3}{4}) = 0$. Thus we have shown that $f'(x)$ has at least three distinct zeros in the interval $[0, 1]$, viz. $\frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$. So by Rolle's theorem, its derivative, viz. $f''(x)$ has at least two zeros in $(0, 1)$. (More specifically $f''(x)$ has at least one zero in $(\frac{1}{4}, \frac{1}{2})$ and at least one zero in $(\frac{1}{2}, \frac{3}{4})$.) Hence (A) also holds.

The remaining two alternatives deal with the integrals of even (and of odd) functions. In view of (4), the integral in (C) is simply $\displaystyle\int_{-1/2}^{1/2} g(x)\sin x\,dx$. Since $g(x)$ is an even function while $\sin x$ is an odd function, the integrand $g(x)\sin x$ is an odd function of x. As the interval of integration is symmetric about the origin, we get that the integral must vanish. So (C) is also true. As for (D), denote the integrals on the left and on the right by I_1 and I_2. By (4), we have

$$\begin{aligned} I_1 &= \int_0^{1/2} f(t)e^{\sin \pi t}\,dt \\ &= \int_0^{1/2} g(t-\frac{1}{2})e^{\sin \pi t}\,dt \qquad (8) \\ &= \int_{-1/2}^{0} g(u)e^{\sin \pi(u+1/2)}\,du \qquad (9) \\ &= \int_{-1/2}^{0} g(u)e^{\cos \pi u}\,du \qquad (10) \end{aligned}$$

where the transition from (8) to (9) was obtained through the substitution $u = t - \frac{1}{2}$. In a similar manner we transform the integral I_2 as

$$\begin{aligned} I_2 &= \int_{1/2}^{1} f(1-t)e^{\sin \pi t}\,dt \\ &= \int_{1/2}^{1} g(t-\frac{1}{2})e^{\sin \pi t}\,dt \\ &= \int_0^{1/2} g(v)e^{\sin \pi(v+1/2)}\,dv \\ &= \int_0^{1/2} g(v)e^{\cos \pi v}\,dv \qquad (11) \\ &= \int_{-1/2}^{0} g(v)e^{\cos \pi v}\,dv \qquad (12) \end{aligned}$$

where the transition from (11) to (12) is valid because the integrand is an even function of v. The integrals in (10) and (12) are identical (since u and v are only dummy variables of integration). Thus we see that $I_1 = I_2$, which means (D) is also true.

The problem is a good testing of the properties of even and odd functions. But before these properties can be applied, one has to recognise the symmetry of the given function about the point $\frac{1}{2}$. So, in this problem you first need some alertness, and then a lot of computational drill. You also need Rolle's theorem. And, as if all this was not enough, some trigonometric functions are also thrown in. So, although it is a good problem, the credit given is way too inadequate as compared to the work required.

SECTION III

Reasoning Type

This section contains four reasoning type questions. Each question has four choices out of which **ONLY ONE** is correct. The marking scheme is as in Section I.

Q.11 Consider the system of equations $ax+by=0,\ cx+dy=0$ where $a,b,c,d \in \{0,1\}$.

STATEMENT - 1: The probability that the system of equations has a unique solution is 3/8.

and

STATEMENT - 2: The probability that the system of equations has a solution is 1.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (B) The necessary and sufficient condition for a unique solution is that the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ should be non-zero, or in other words,

$$ad - bc \neq 0 \tag{1}$$

In the present case each of the variables a,b,c,d can take two possible values independently of each other. So, in all there are 16 possibilities. Because the only possible values of these variables are 0 and 1, (1) can hold only in two cases, viz. (i) $ad = 1, bc = 0$ or (ii) $ad = 0, bc = 1$. For (i) to hold, we must have $a = d = 1$ and at least one of b and c is 0. So (i) holds in 3 out of the 16 possible cases. A similar argument shows that (ii) holds in 3 cases. As (i) and (ii) are mutually exclusive, we see that (1) holds in 6 out of the 16 cases. So the probability of (1) is $6/16 = 3/8$. Thus the first statement is true. As for the second statement, being a homogeneous system, the equations always have a trivial solution, viz. $x = y = 0$. So Statement-2 is also true. But since existence of a solution says nothing about its uniqueness, Statement-2 is not a correct explanation of Statement-1.

Q.12 Consider the system of equations

$$\begin{aligned} x - 2y + 3z &= -1 \\ -x + y - 2z &= k \\ x - 3y + 4z &= 1 \end{aligned}$$

STATEMENT - 1 : The system of equations has no solution for $k \neq 3$.

and

STATEMENT - 2: The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, for $k \neq 3$.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (A). The problem deals with a non-homogeneous system of three linear equations in three unknowns. The homogeneous system always has at least one solution (viz. the trivial solution). But the condition for existence of solutions for a non-homogeneous system is a complicated one involving the concept of the rank of a matrix, which is beyond the JEE syllabus. Nevertheless, we begin by writing the system in a matrix form as

$$\begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ k \\ 1 \end{bmatrix} \tag{1}$$

We let A be the coefficient matrix appearing on the L.H.S. If A is non-singular, then the system has a unique solution regardless of the column vector on the R.H.S. In the present problem, however, the determinant of A is 0 by a direct calculation (or by observing that its third column is obtained by subtracting the second column from the first) and so A is singular. In this case a necessary condition for the existence of a solution is that all the three matrices obtained by replacing one column of A at a time by the column vector on the right of (1) should also be singular. In other words, if a solution is to exist then all the three determinants $\begin{vmatrix} -1 & -2 & 3 \\ k & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$, $\begin{vmatrix} 1 & -1 & 3 \\ -1 & k & -2 \\ 1 & 1 & 4 \end{vmatrix}$ and $\begin{vmatrix} 1 & -2 & -1 \\ -1 & 1 & k \\ 1 & -3 & 1 \end{vmatrix}$ must vanish. Note that the second of these three determinants is the same as the determinant given in Statement-2 except for an interchange of the second and the third column. So, if the system is to have a solution then, in particular, we must have

$$\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} = 0 \tag{2}$$

By a direct expansion this determinant comes out to be $-k+3$. So, for $k \neq 3$ it is non-zero. Hence Statement-2 is true and, in view of what we have said, it also implies that Statement-1 is true. It is, therefore, a correct explanation of Statement-1.

Note that the vanishing of the three determinants above is only a necessary condition for the system to have a solution. In the present problem it also turns out to be sufficient, because in that case $k = 3$ and the system has an infinite number of solutions of the form $x = -t-5, y = t-2, z = t$ where t is any real number. But examples can be given where the condition is not sufficient. A trivial example is to take a system in which the first two equations are the same and the third one is inconsistent with them. (For example, take the first two equations as $x+y+z = 1$ and the third as $x+y+z = 2$.) Here the coefficient matrix A as well as the three matrices obtained from it by column replacement are all singular (because the first two rows of each one of them are equal). Still there is no solution for the system.

Those who are not familiar with the necessary condition given above for the existence of a solution can work directly in terms of the given system of equations. Suppose the system has a solution, i.e. there exist x_0, y_0, z_0 such that

$$x_0 - 2y_0 + 3z_0 = -1 \tag{3}$$
$$-x_0 + y_0 - 2z_0 = k \tag{4}$$
$$x_0 - 3y_0 + 4z_0 = 1 \tag{5}$$

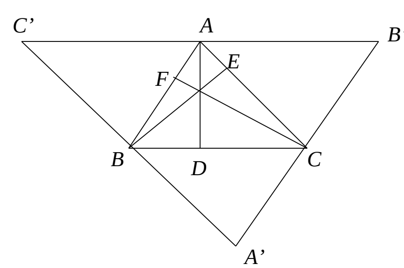
If we multiply the first equation by 2 throughout and add the second one to it we get

$$x_0 - 3y_0 + 4z_0 = k - 2 \tag{6}$$

Note that the left hand sides of (5) and (6) are identical. So, they can hold together if and only if $k-2 = 1$, i.e. $k = 3$. It follows that for

$k \neq 3$, the system has no solution. Hence Statement-1 is correct. By a direct computation of the determinant given in Statement-2, we see that Statement-2 is also correct. But in this method, it is difficult to say that Statement-2 is a correct explanation of Statement-1, because we have proved Statement-1 without even referring to any determinants. This point has to be kept in mind while answering questions of this type. Such questions are not answered completely by proving that both Statement-1 and Statement-2 are true. One still has to decide if Statement-2 is *a* correct explanation of Statement-1. This amounts to asking whether Statement-1 can be *deduced* from Statement-2. It often happens in mathematics that the same statement can be proved in many different ways. This is especially true in pure geometry. Now, some of these proofs may use a certain idea or fact while the others are completely independent of it. When we are asked to see if one statement can be deduced from another, we have to see if there is *some* proof of the former which uses the latter.

Suppose for example, that Statement-1 says that the three altitudes of a triangle are concurrent while Statement-2 says that the perpendicular bisectors of the three sides of a triangle are concurrent. Now, there are many proofs of the concurrency of the altitudes of any triangle using a variety of methods. (See Comment No. 3 of Chapter 8 for a proof using coordinates and Comment No. 1 of Chapter 11 for a proof using Ceva's theorem and a little trigonometry.) A proof using vectors (or complex numbers) is probably the shortest. There are also proofs from pure geometry, e.g. a proof based on properties of cyclic quadrilaterals. But these proofs are of little relevance when we are asked to see if the concurrency of the altitudes of a triangle can be *deduced* from that of the perpendicular bisectors of the sides of a (possibly different) triangle. And this indeed can be done. Suppose ABC is a triangle with altitudes AD, BE, CF. Through the vertices A, B, C draw lines parallel to the respective opposite sides so as to form a new triangle $A'B'C'$ as shown in the figure. It is easily seen that AD, BE and CF are the perpendicular bisectors of the sides of the triangle $A'B'C'$. Therefore they are concurrent. So concurrency of altitudes can be deduced from that of the perpendicular bisectors of the sides.



(It may be objected that we are deducing the concurrency of the altitudes of the triangle ABC from the concurrency of the perpendicular bisectors of the sides of a *different* triangle, viz. $A'B'C'$. But this is perfectly all right. Both Statement-1 and Statement-2 are statements about the class of *all* triangles. The expression 'a triangle' used in them really means any typical member of that class. The argument above shows that if Statement-2 holds for *every* triangle then so does Statement-1, and that is all that is intended.)

Q.13 Consider the planes

$$\begin{aligned} P_1: &\quad x - y + z = 1 \\ P_2: &\quad x + y - z = -1 \\ P_3: &\quad x - 3y + 3z = 2 \end{aligned}$$

Let L_1, L_2, L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1 and P_1 and P_2 respectively.

STATEMENT - 1 : At least two of the lines L_1, L_2, L_3 are non-parallel

and

STATEMENT - 2 : The three planes do not have a common point.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (D). Statement-2 amounts to saying that the system of the equations of the three planes has no solution. This can be verified using the determinant criterion given in the solution of Q.13. But it is much easier to do it directly. Adding the first two equations we get $x = 0$ and $y = z - 1$. But with these, the third equation does not hold. So the system is not consistent.

As for Statement-1, let us first find vectors, say $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$ parallel to the lines L_1, L_2 and L_3 respectively. From the coefficients of the variables x, y, z in the equations of the planes P_1, P_2 and P_3, we see that the vectors $\mathbf{v}_1 = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v}_2 = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{v}_3 = \mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ are respectively perpendicular to them. As the line L_1 lies in both P_2 and P_3, it is perpendicular to both $\mathbf{v}_2$ and $\mathbf{v}_3$ and hence parallel to $\mathbf{v}_2 \times \mathbf{v}_3$. So we may take

$$\mathbf{u}_1 = \mathbf{v}_2 \times \mathbf{v}_3 = -4\mathbf{j} - 4\mathbf{k} \tag{1}$$

and similarly,

$$\mathbf{u}_2 = \mathbf{v}_3 \times \mathbf{v}_1 = 2\mathbf{j} + 2\mathbf{k} \tag{2}$$

$$\text{and } \mathbf{u}_3 = \mathbf{v}_1 \times \mathbf{v}_2 = 2\mathbf{j} + 2\mathbf{k} \tag{3}$$

Thus we see that all the three lines are parallel to the vector $\mathbf{j} + \mathbf{k}$. Therefore they are all parallel to each other, except when two of them coincide.

But if that happens then the points on this common line will lie on all the three planes and we have already ruled this out. So, Statement-1 is false.

Q.14 Let f and g be two real valued functions defined on the interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$ anf $f(x) = g(x)\sin x$.

STATEMENT - 1 : $\lim\limits_{x\to 0}[g(x)\cot x - g(0)\text{cosec } x] = f''(0)$.

and

STATEMENT - 2: $f'(0) = g(0)$.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (A). Since $f(x) = g(x)\sin x$, a straightforward application of the Leibnitz rule gives

$$\begin{aligned} f'(x) &= g'(x)\sin x + g(x)\cos x & (1)\\ \text{and} \quad f''(x) &= g''(x)\sin x + 2g'(x)\cos x - g(x)\sin x & (2) \end{aligned}$$

from which we immediately get

$$\begin{aligned} f'(0) &= g(0) & (3)\\ \text{and} \quad f''(0) &= 2g'(0) = 0 & (4) \end{aligned}$$

From (3) we see that Statement-2 is true. Now, for Statement-1, the limit on the L.H.S. can be evaluated using l'Hôpital's rule as

$$\begin{aligned} \lim_{x\to 0}[g(x)\cot x - g(0)\text{cosec } x] &= \lim_{x\to 0}\frac{g(x)\cos x - g(0)}{\sin x}\\ &= \lim_{x\to 0}\frac{g'(x)\cos x - g(x)\sin x}{\cos x}\\ &= g'(0) = 0 \qquad (5) \end{aligned}$$

From (4) and (5) we see that Statement-1 is also true. But (3) was hardly used in its derivation. In this derivation, it is difficult to say that Statement-2 is a correct explanation of Statement-1. Hence (B) would seem to be the correct answer. is (B).

But as commented after the solution to Q.12, in questions of this type, what really matters is whether there is *some* way to *deduce* the truth of Statement-1 from that of Statement-2. In the present case, such a deduction is not obvious. But a hint is obtained if we rewrite Statement-1 as

$$f''(0) = \lim_{x\to 0}[g(x)\cot x - g(0)\operatorname{cosec} x] \tag{6}$$

All we have done is to interchange the L.H.S. and the R.H.S. of the equation in Statement-1. If A and B are any two expressions then the equalities $A = B$ and $B = A$ are mathematically equivalent. But their connotations can be different. Suppose A is a relatively complicated expression and B is a relatively simple expression. Then $A = B$ means that the expression A can, after some manipulations, be reduced to B, or, in some sense, that B is the value of A. On the other hand, when you write $A = B$ it means that you are recasting A as B. Such a recasting may be useful as a part of some proof. For example, consider the most basic trigonometric identity

$$\sin^2\theta + \cos^2\theta = 1 \tag{7}$$

The natural interpretation of this is that the L.H.S. reduces to the R.H.S. after some work such as an application of the Pythagoras theorem. And this identity is used innumerably many times in simplifying trigonometric expressions. Nobody will write (7) as

$$1 = \sin^2\theta + \cos^2\theta \tag{8}$$

The number 1 is a very common number and equals so many different unrelated things such as $\sin^2\theta + \cos^2\theta$, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$, $\displaystyle\int_0^1 3x^2\,dx$ or $\displaystyle\lim_{x\to 0}\frac{\tan x}{x}$, that there is hardly any reason in general to equate it with any one such expression. But in special circumstances, recasting the number 1 using (7) may be the key idea in the proof of some trigonometric identity. (As an actual example, see the uncanny solution to the problem in Comment No. 14 of Chapter 7 where the term 1 is recast as $(\sin^2 A+\cos^2 A)(\sin^2 B+\cos^2 B)$ and hence as $\sin^2 A\sin^2 B + \sin^2 A\cos^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B$.)

Returning to our problem, the key idea lies in recasting Statement-1 as (6). The problem now is not so much to evaluate the limit on the R.H.S. bu rather to show that $f''(0)$ can be recast as that limit. This can be done as follows. By very definition,

$$f''(0) = \lim_{x\to 0}\frac{f'(x) - f'(0)}{x} \tag{9}$$

Near $x = 0$, $\sin x$ behaves like x and so we may rewrite (9) as

$$f''(0) = \lim_{x\to 0}\frac{f'(x) - f'(0)}{\sin x} \tag{10}$$

(For a formal proof multiply the fraction on the R.H.S. of (9) by the fraction $\dfrac{x}{\sin x}$ which tends to 1 as $x \to 0$.)

We now use (1) and (3) to rewrite this as

$$f''(0) = \lim_{x\to 0} \frac{g'(x)\sin x + g(x)\cos x - g(0)}{\sin x} \tag{11}$$

and hence further as

$$f''(0) \quad = \quad \lim_{x\to 0} g'(x) + \lim_{x\to 0}[g(x)\cot x - g(0)\ \text{cosec}\ x] \tag{12}$$

The first limit is $g'(0)$ since g' is given to be continuous (in fact, differentiable) at 0. But $g'(0)$ is given to be 0. So we finally have

$$f''(0) = \lim_{x\to 0}[g(x)\cot x - g(0)\ \text{cosec}\ x] \tag{13}$$

which is precisely Statement-1. In this derivation we have crucially used (3), i.e. Statement-2. So now we can say that Statement-2 is a correct explanation of Statement-1.

As said before, a normal interpretation of Statement-1 would be to compute the limit on its L.H.S. and show that it equals $f''(0)$. In the present problem, however, what is really needed is to be able to recast $f''(0)$ as this limit. This is a very unusual demand. The only hint provided is to rewrite the coefficient $g(0)$ as $f'(0)$. But this hint is too faint. It would have been far better if Statement-1 had been stated in the form (6). Also, it is not clear why $g(0)$ and $g''(0)$ are given to be non-zero. This fact is never needed in the proof of either Statement-1 or Statement-2. Such redundant pieces of data only serve to confuse a candidate.

SECTION IV

Linked Comprehension Type

This section contains 3 paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices out of which **ONLY ONE** is correct. There are 4 marks for a correct answer, 0 marks if the question is not answered and -1 mark in all other cases.

Paragraph for Question No. 15 to 17

A circle of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of the circles with the sides PQ, QR, RP are D, E, F respectively. The line PQ is given by the equation $\sqrt{3}x - y + 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further it is given that the origin and the centre of C are on the same side of the line PQ.

Q.15 The equation of the circle C is

(A) $(x-2\sqrt{3})^2+(y-1)^2=1$ (B) $(x-2\sqrt{3})^2+(y+\frac{1}{2})^2=1$
(C) $(x-\sqrt{3})^2+(y+1)^2=1$ (D) $(x-\sqrt{3})^2+(y-1)^2=1$

Answer and Comments: (D). The problem amounts to locating the centre, say M, of the circle C. This can be done in a purely algebraic manner. Let $M=(h,k)$. We need a system of two equations in the two unknowns h and k. From the data of the problem, the distance MD and also the perpendicular distance of M from the line PQ are both equal to 1. As we are given D as $\left(\frac{3\sqrt{3}}{2},\frac{3}{2}\right)$ and the equation of the line PQ as $\sqrt{3}x-y+6=0$, we get

$$(h-\frac{3\sqrt{3}}{2})^2+(k-\frac{3}{2})^2 \quad = \quad 1 \qquad (1)$$

and

$$\pm\frac{\sqrt{3}h+k-6}{2} \quad = \quad 1 \qquad (2)$$

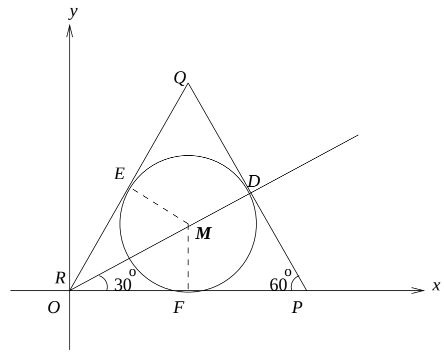
Solving this system will be rather complicated, especially because of the two cases into which (2) falls because of the $\pm$ sign. In each case, by eliminating one of the two variable, we get a quadratic in the other variable. But the two roots of the quadratic will be the same, because in each case (2) represents a straight line parallel to PQ which must touch the circle (1). So even though superficially the system above has four solutions, it will actually have only two solutions. So there will be two possible locations for the centre M, one on each side of the line PQ and then we shall have to choose the one which lies on the same side of PQ as the origin does. It is a little clumsy to make this choice purely algebraically.

A better approach is to note that the lines MD and PQ are perpendicular to each other. Hence the product of their slopes is -1. As the equation of PQ is $-\sqrt{3}x+y-6=0$, its slope is $\sqrt{3}$ and so the slope of MD is $\frac{1}{\sqrt{3}}$. So we get

$$k-\frac{3}{2}=\frac{1}{\sqrt{3}}\left(h-\frac{3\sqrt{3}}{2}\right) \qquad (3)$$

Instead of solving (1) and (2) together, we might as well solve (2) and (3) together, which is much easier since both the equations are linear. Again we shall have to make two cases because of the sign $\pm$ in (2). This will give two possible locations for M and the choice will have to be made as before, using the fact that M lies on the same side as the origin.

Thus we see that even in algebraic approaches, one can be superior to the other. In the present problem, even the second approach can be improved slightly by looking at a well-drawn diagram. Drawing a diagram is a good idea in any problem in general. But in the present case it is especially rewarding since the same diagram would be useful in three problems.



Note that we are *not* given the points P, Q, R. But we are given the equation of the line PQ as $\sqrt{3}x + y - 6 = 0$, whence its slope is $-\sqrt{3}$. As MD is perpendicular to PQ we get the slope of MD as $\frac{1}{\sqrt{3}}$ as calculated above. So, MD makes an angle of $30°$ with the positive x-axis. As MD is given to be of unit length, its horizontal and vertical projections are $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$ respectively. Since D is given as $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$, we get that the two possible locations for M are $\left(\frac{3\sqrt{3} \pm \sqrt{3}}{2}, \frac{3 \pm 1}{2}\right)$ where the same sign is to hold for both the occurrences of $\pm$. So M is either $(2\sqrt{3}, 2)$ or $(\sqrt{3}, 1)$. (We would have gotten the same answer had we solved (2) and (3) simultaneously. But now we got it visually.) To decide which possibility holds, we use the fact that M and the origin O lie on the same side of the line $\sqrt{3}x + y - 6 = 0$. This means that the value of the expression $\sqrt{3}x + y - 6$ when x, y are replaced by the coordinates of M is of the same sign as the value obtained by replacing them by $0, 0$. The latter value is -6 which is negative. Out of the two possibilities for M, the second one satisfies this condition. So we get $M = (\sqrt{3}, 1)$. Hence the equation of the circle is as given in (D).

Q.16 The points E, F are given by

(A) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$, $(\sqrt{3}, 0)$ (B) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $(\sqrt{3}, 0)$

(D) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$, $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (D) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Answer and Comments: (A). As in the last question we can determine the points E, F purely algebraically by writing down suitable systems of equations for their coordinates. But now that we have drawn the diagram already, let us make full use of it. (Note that in the diagram above, the vertex R is shown to coincide with the origin. But we have not proved this yet and so it is not yet a good idea to rely on it. Such unwarranted inferences from diagrams can be misleading because unless the diagram is drawn to scale, which is often not worth the time. The best thing is to keep in mind that a diagram is an aid to understanding rather than a substitute for a proof.)

Nevertheless, even without bothering to determine the coordinates of the three vertices P, Q, R. We already have enough information to determine those of E and F. As the triangle PQR is equilateral and M is its centre, the vectors $\overrightarrow{ME}$ and $\overrightarrow{MF}$ are obtained from the vector $\overrightarrow{MD}$ through rotations of 120° each, the former counterclockwise and the latter clockwise. We already know that $\overrightarrow{MD}$ is a unit vector which makes an angle of 30° with the positive x-axis. So it follows that $\overrightarrow{ME}$ and $\overrightarrow{MF}$ are also unit vectors making angles of 150° and −90° with the positive x-axis. Since $\cos 150° = -\frac{\sqrt{3}}{2}$ and $\sin 150° = \frac{1}{2}$ we get the coordinates of E by adding these numbers to the coordinates of M, which we already know. So

$$\begin{aligned} E &= \left(\sqrt{3} - \frac{\sqrt{3}}{2}, 1 + \frac{1}{2}\right) \\ &= \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right) \end{aligned} \tag{4}$$

and similarly,

$$\begin{aligned} E &= \left(\sqrt{3} + \cos(-90°), 1 + \sin(-90°)\right) \\ &= (\sqrt{3}, 0) \end{aligned} \tag{5}$$

In the diagram above we have the points P, Q, R are located counterclockwise on the circumcircle of the triangle PQR. But there is nothing in the data of the problem to imply this. So we could have as well interchanged P and Q and still the data will hold. But in that case, the points E and F will also be interchanged. This may confuse a candidate. The paper-setters should have thought of this possibility and avoided the confusion by wording the question as 'The points E, F are given (not necessarily in that order) by'.

Q.17 The equations of the sides QR RP are

(A) $y = \frac{2}{\sqrt{3}}x + 1, y = -\frac{2}{\sqrt{3}}x - 1$ (B) $y = \frac{1}{\sqrt{3}}x, y = 0$

(C) $y = \frac{\sqrt{3}}{2}x + 1, y = -\frac{\sqrt{3}}{2}x - 1$ (D) $y = \sqrt{3}x, y = 0$

Answer and Comments: (D). Once again, those who equate R with O from the diagram will be rewarded because then the lines QR, PR

are the same as the lines OE, OF respectively. As we already know the points E and F, we write their equations by the two-point formula. But a cleaner way is to use the point-slope formula after determining the slopes of QR and PR. As QR is perpendicular to ME whose slope is $\tan(150°)$, i.e. $-\frac{1}{\sqrt{3}}$, the slope of QR comes out as $\sqrt{3}$. As for RP, we know it is horizontal since MF is vertical. So, the equations are as in (D). (It is only now that we know that the origin lies on both QR and RP and hence must coincide with R.) As in the last question, if we interchange the points P and Q, the equations in (D) are correct but in the wrong order.

This is a straightforward bunch of problems based on elementary coordinate geometry and a little trigonometry. (We have used vectors in the solution to Q.16. But they were used only superficially. The same work could have been done directly in terms of the angles which the lines RQ, RP make with the positive x-axis.) The three questions are well related as each of the first two is needed in the next.

Paragraph for Question No.s 18 to 20

Consider the function defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defined a unique real-valued differentiable function $y = f(x)$. If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

Q.18 If $f(-10\sqrt{2}) = 2\sqrt{2}$ then $f''(-10\sqrt{2}) =$

(A) $\frac{4\sqrt{2}}{7^3 3^2}$ (B) $-\frac{4\sqrt{2}}{7^3 3^2}$ (C) $\frac{4\sqrt{2}}{7^3 3}$ (D) $-\frac{4\sqrt{2}}{7^3 3}$

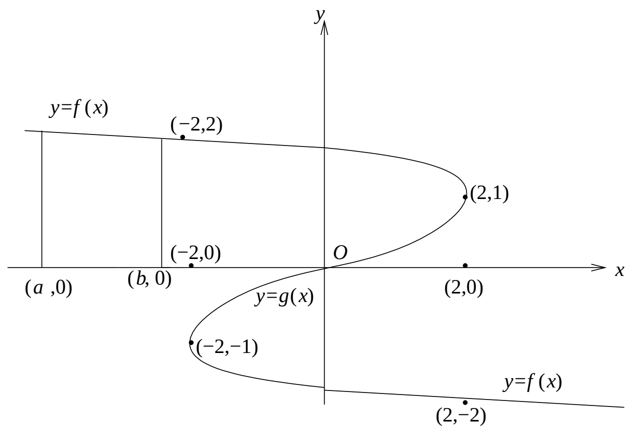
Answer and Comments: (B). Although not asked explicitly, it is a good idea to draw a graph of the given relation. Rewrite it as

$$x = 3y - y^3 \tag{1}$$

Drawing the graph of this curve is simplified if we first interchange x and y to get

$$y = 3x - x^3 \tag{2}$$

The graphs of (1) and (2) are the reflections of each other in the line $y = x$. The graph of (2) is a familiar one (cf. Comment No. 13 of Chapter 13) as the function on the R.H.S. is a cubic in x with a negative leading coefficient. Taking derivatives we see that there is a local minimum at $x = -1$ and a local maximum at $x = 1$ with a point of inflection at $x = 0$. We do not show this graph. But if we take its reflection in the line $y = x$, we get the graph in the figure below which is the graph of (1).



Coming to the question asked, since $-10\sqrt{2} \in (\infty, -2)$, in a neighbourhood of $-10\sqrt{2}$ the function $y = f(x)$ is defined implicitly by (1) and our task is to find f'' or equivalently, $\dfrac{d^2y}{dx^2}$ at this point. Differentiating (1) w.r.t. x we get

$$1 = 3y' - 3y^2y' \tag{3}$$

which gives

$$y' = \frac{1}{3(1-y^2)} \tag{4}$$

One more differentiation gives

$$y'' = \frac{2yy'}{3(1-y^2)^2} \tag{5}$$

To find $y''(-10\sqrt{2})$ we need the values of y and y' at $x = -10\sqrt{2}$. The former will have to be found by solving the cubic

$$y^3 - 3y - 10\sqrt{2} = 0 \tag{6}$$

In general, solving a cubic is not an easy task. Fortunately, we are spared of it because the statement of the question already gives $f(-10\sqrt{2}) = 2\sqrt{2}$ which amounts to saying that $y = 2\sqrt{2}$ when $x = -10\sqrt{2}$. (It is easy to verify that this value of y does satisfy (6). But that is not needed in the solution.) Putting $y = 2\sqrt{2}$ in (4) gives $y' = -\dfrac{1}{21}$. Finally, (5) gives $y'' = -\dfrac{4\sqrt{2}}{3 \times 21 \times 49} = -\dfrac{4\sqrt{2}}{7^3 3^2}$.

Q.19 The area of the region bounded by the curves $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$ is

(A) $\displaystyle\int_a^b \frac{x}{3\left((f(x))^2-1\right)}dx + bf(b) - af(a)$

(B) $\displaystyle -\int_a^b \frac{x}{3\left((f(x))^2-1\right)}dx + bf(b) - af(a)$

(C) $\displaystyle\int_a^b \frac{x}{3\left((f(x))^2-1\right)}dx - bf(b) + af(a)$

(D) $\displaystyle -\int_a^b \frac{x}{3\left((f(x))^2-1\right)}dx - bf(b) + af(a)$

Answer and Comments: (A). In principle, this is an extremely routine problem of determining the area under the graph of a function $y = f(x)$ and the answer is $\displaystyle\int_a^b f(x)\,dx$. The catch is that here we do not know the function $f(x)$ explicitly. As a result, we cannot evaluate the integral explicitly either. Nor does the problem require it. Rather, the problem asks you to express this integral in terms of some other integral. The presence of the expression $bf(b)-af(a)$ which occurs in all the alternatives gives a hint. We rewrite this expression as $xf(x)\big|_a^b$. This suggests that we can try integration by parts. Doing so, we get (after denoting the given area by A)

$$\begin{aligned} A = \int_a^b f(x)\,dx &= \int_a^b xf(x)\,dx - \int_a^b xf'(x)\,dx \\ &= bf(b) - af(a) - \int_a^b xf'(x)\,dx \qquad (7) \end{aligned}$$

The problem now reduces to finding an expression for $f'(x)$. This was already done in (4) above. So a straight substitution from (4) (with y, y' replaced by $f(x), f'(x)$ respectively) gives (A) as the correct answer.

Q.20 $\displaystyle\int_{-1}^1 g'(x)\,dx =$

(A) $2g(-1)$ (B) 0 (C) $-2g(1)$ (D) $2g(1)$

Answer and Comments: (D). The interval of integration here is $[-1, 1]$ which is contained in $(-2, 2)$. For $-2 < x < 2$, there are three values of y which satisfy (1) and one of these is $g(x)$. For $x = 0$, the three possible values are $-\sqrt{3}, 0$ and $\sqrt{3}$. Out of these, the middle value, viz. 0 is given as the value of $g(0)$. By continuity of $g(x)$ on $(-2, 2)$, for every $x \in (-2, 2)$, we must also take the middle of the three values of y satisfying (1) as the (unique) value of $g(x)$. If we do so, then $g(x)$ is a continuously differentiable function on $(-2, 2)$ and so, by the fundamental theorem of calculus,

$$\int_{-1}^1 g'(x)\,dx = g(1) - g(-1) \qquad (8)$$

Note that the problem does not ask to evaluate the integral, but only to see which of the given alternatives equals $g(1)-g(-1)$. From the diagram above it is obvious that $g(x)$ is an odd function of x. A rigorous proof of this is to note that as both the sides of (1) involve only odd powers of the variables x and y, whenever a point (x, y) satisfies (1), so does $(-x, -y)$. As a result, if for a given $x \in (-2, 2)$, y_1, y_2, y_3 are the three roots of (1), then $-y_1, -y_2, -y_3$ are the roots of

$$-x = 3y - y^3 \tag{9}$$

Hence if we take $y_1 < y_2 < y_3$ so that the middle root is y_2, then we get $-y_3 < -y_2 < -y_1$ and therefore $-y_2$ is the middle value of y which satisfies (9). So we conclude that $g(-x) = -g(x)$, or in other words that g is an odd function of x on $(-2, 2)$. In particular $g(-1) = -g(1)$ and so the R.H.S. of (8) becomes $2g(1)$.

The JEE questions about implicit functions are often mechanical. This paragraph is a good test of a candidate's understanding of implicit functions. The cubic (1) cannot be solved explicitly except for some special values of x. So, the candidate has to resort to reasoning. In particular, recognising that $g(x)$ is an odd function of x requires good thinking on the part of the candidate. So this is well-designed bunch. Unfortunately, the multiple choice format masks the difference between those who merely realise that $g(x)$ is odd and those who can actually prove it (as outlined above). There is also some duplication of ideas with Q.10 which was also crucially based on properties of even and odd functions.

Paragraph for Question No.s 21 to 23

Let A, B, C be three sets of complex numbers as defined below

$$\begin{aligned} A &= \{z : \text{ Im } z \geq 1\} \\ B &= \{z : |z-2-i| = 3\} \\ C &= \{z : \text{ Re } ((1-i)z) = \sqrt{2}\} \end{aligned}$$

Q.21 The number of elements in the set $A \cap B \cap C$ is

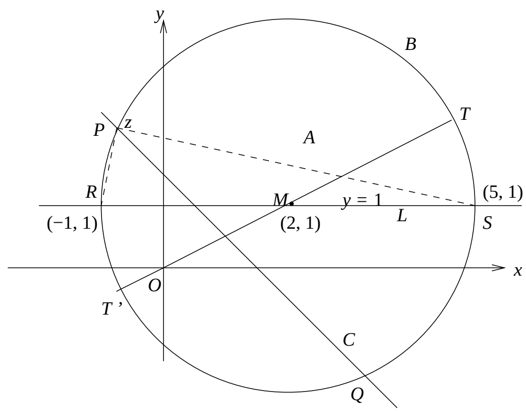
(A) 0 (B) 1 (C) 2 (D) ∞

Answer and Comments: (B). Yet another question where a good diagram is half (or more than half) the solution. In the present problem, we translate the three sets in terms of the real coordinates and show them in an Argand diagram. If $z = x + iy$, then Im $z = y$ while Re $((1-i)z)$ = Re $((1-i)(x+iy)) = x + y$. Also $|z - 2 - i| = |(x-2) + i(y-1)| = \sqrt{(x-2)^2 + (y-1)^2}$. We can now identify the sets A, B, C in terms of the real coordinates of their points as follows.

$$A = \{(x, y) : y \geq 1\} \tag{1}$$

$$B = \{(x, y) : (x-2)^2 + (y-1)^2 = 9\} \quad (2)$$
$$C = \{(x, y) : x + y = \sqrt{2}\} \quad (3)$$

Clearly A is the set of all points lying on or above the horizontal line, say L whose equation is $y = 1$, B is a circle of radius 3 centred at $(2, 1)$ while C is the straight line $x + y = \sqrt{2}$. All the three sets are shown in the diagram below.



The question deals with $A \cap B \cap C$, i.e. with the intersection of these three sets. As B and C are a circle and a straight line respectively, their intersection consists of at most two points. We could actually determine it by solving the equations of B and C simultaneously. But the question only asks the number of points in $A \cap B \cap C$ and does not ask to identify them. From the diagram it is very clear that B and C intersect only at the points P and Q shown in the figure. Out of these only one, viz. P lies on or above the line $y = 1$ and hence is in A. So we conclude that there is only one point, viz. P in $A \cap B \cap C$.

Q.22 Let z be any point in $A \cap B \cap C$. Then $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(A) 25 and 29 (B) 30 and 34 (C) 35 and 39 (D) 40 and 44

Answer and Comments: (C). We already know that there is only one point in $A \cap B \cap C$ viz. P. So, a straightforward way to answer this question would be to identify P, i.e. its coordinates, and actually compute the expression in the statement of the question. But there is a better way out if we notice that the expression is the sum of the squares of the distances of P from the points $R = (-1, 1)$ and $S = (5, 1)$. These points are the extremities of a diameter of the circle B. As P is a point on the circle B, PR and PS are at right angles to each other and so we have $PR^2 + PS^2 = RS^2 = 36$. This lies between 35 and 39. Note that the expression would have the same value if z were any point of B. So, the data that it also lies in A and C is irrelevant.

Q.23 Let z be any point in $A \cap B \cap C$ and w be any point satisfying $|w-2-i| < 3$. The $|z| - |w| + 3$ lies between

(A) -6 and 3 (B) -3 and 6 (C) -6 and 6 (D) -3 and 9

Answer and Comments: (B, C, D). Once again, there is only one point, viz., P in $A \cap B \cap B \cap C$. So in the expression $|z| - |w| + 3$, $|z|$ is a fixed number. We have not yet calculated it. But even without doing it we see that the range for the expression $|z| - |w| + 3$ depends solely on the range for the values of $|w|$. The condition on w, viz. $|w - 2 - i| < 3$ amounts to saying that the point w lies inside the circle B. Geometrically, $|w|$ is simply the distance of w from the origin. As the origin also lies inside this circle, the minimum possible value of $|w|$ is 0. To get an upper bound on $|w|$, let $M = (2, 1)$ be the centre of B. Draw the diameter through O and let T be its end lying on the opposite side of the centre as the origin as shown in the figure. Then T is the point on the circle B which is farthest from the origin. Further,

$$OT = OM + MT = 3 + \sqrt{5} \tag{4}$$

It is clear that if w lies inside B, then $|w|$ cannot exceed $3 + \sqrt{5}$, although it can come as close to $3 + \sqrt{5}$ as we like if w is sufficiently close to T. (Technically, we say that $3 + \sqrt{5}$ is a supremum for the set of values of $|w|$ as w ranges over the inside of the circle B. But this supremum is not a maximum because the point T at which $|w|$ is maximum lies on the circle B but not inside B.)

Summing up, for every w which satisfies $|w - 2 - i| < 3$, we have

$$0 \ \leq \ |w| \ < \ 3 + \sqrt{5} \tag{5}$$

Multiplying throughout by -1 will reverse the inequalities. Doing so and adding $|z| + 3$ (which is a constant) to all the three terms, we get

$$|z| - \sqrt{5} \ < \ |z| - |w| + 3 \ \leq \ |z| + 3 \tag{6}$$

or in other words, the expression $|z| - |w| + 3$ lies between $|z| - \sqrt{5}$ and $|z| + 3$ where $|z|$ is the distance OP. To calculate it we shall have to determine the coordinates of the point P by solving the system

$$x + y = \sqrt{2} \tag{7}$$
$$(x-2)^2 + (y-1)^2 = 9 \tag{8}$$

simultaneously. As the second equation is a quadratic, there will be two solutions and we have to take the one for which $y \geq 1$. The other solution corresponds to the point Q.

To solve this system we can put $x = \sqrt{2} - y$ into (8) to get

$$(\sqrt{2} - 2 - y)^2 + (y - 1)^2 = 9 \tag{9}$$

This can be solved by the quadratic formula. But because of the radicals involved, the expression for y will be fairly complicated. After getting y, we shall have to take $x = \sqrt{2} - y$ and then compute $\sqrt{x^2 + y^2}$ as the value of $|z|$. This will be very cumbersome and time consuming. Fortunately we are spared of it. The problem merely asks you to identify which of the given four ranges contains the semi-open interval $(|z| - \sqrt{5}, |z| + 3]$ and to answer this, some crude estimates on $|z|$ will suffice. Such estimates can be obtained very easily by looking at the diagram. Clearly the left-most point on the circle B is the point $R = (-1, 1)$. So P lies on the right of R. At the same time P also lies in the second quadrant. So we get an inequality for its x-coordinate, viz.

$$-1 \; < \; x \; < \; 0 \tag{10}$$

The first inequality, coupled with $x + y = \sqrt{2}$ gives $y < \sqrt{2} + 1$. At the same time, as P lies above the line $y = 1$, we also have $y > 1$. Therefore

$$1 \; < \; y \; < \; \sqrt{2} + 1 \tag{11}$$

The double inequalities (10) and (11) give an upper bound on $|z|$, viz.

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ &< \sqrt{1 + (\sqrt{2} + 1)^2} \\ &= \sqrt{4 + 2\sqrt{2}} \end{aligned} \tag{12}$$

As 0 is a trivial lower bound on $|z|$, from (6) and (12) together we get

$$-\sqrt{5} \; < \; |z| - |w| + 3 \; < \; 3 + \sqrt{4 + 2\sqrt{2}} \tag{13}$$

Once again, our interest is not in finding the exact values of these bounds but only in deciding which of the given four alternatives are implied by them. Since $\sqrt{5} < 3$ we see that the lower bound $-\sqrt{5}$ is bigger than the left end points of all the four given intervals viz. $[-6, 3], [-3, 6], [-6, 6]$ and $[-3, 9]$. The question now left is which of the right end points of these intervals exceed the upper bound $3 + \sqrt{4 + 2\sqrt{2}}$ in (13). Obviously, the first interval is ruled out. But since $\sqrt{2} < 1.5$, we get $4 + 2\sqrt{2} < 7$. So $\sqrt{4 + 2\sqrt{2}} < \sqrt{7} < 3$. Therefore the upper bound in (13) is less than 6. As a result, (B), (C) and (D) are all correct alternatives. This is contrary to the instruction that only one of the alternatives is correct. So, obviously there is something wrong in the question.

One possibility is that the intended expression in the statement of the question is not $|z| - |w| + 3$ but $|z - w| + 3$. With this interpretation the problem amounts to determining the minimum and the maximum distance between z and w where z lies on the circle B and w lies inside B. Obviously, w can come as close to z as we like. So, $|z - w|$ is as close to 0 as we like. (Once again, 0 is an infimum but not the minimum for $|z - w|$.) As for the maximum, taking w very close to the point on B diametrically opposite to z, we see that the supremum (but not the maximum) of $|z - w|$ is the diameter of the circle B, viz. 6. So the range of the expression $|z - w|$ is the open interval $(0, 6)$ and hence the range of the expression $|z - w| + 3$ is the interval $(3, 9)$. Therefore (D) would be the (only) correct answer. Moreover, now the problem is reasonably simple and does not demand messy calculations.

However, this guess at the intended formulation of the question is open to some questioning. If the question is really to get lower and upper bounds on the value of $|z - w| + 3$, the answer is 3 and 9 as we just saw. But the bounds given in (D) are -3 and 9. Of course, this still makes (D) as the only correct answer. But one wonders why the lower bound given is -3 rather than 3. Put differently, even though the interval given in (D), viz., $(-3, 9)$ does contain all the possible values of the expression $|z - w| + 3$, it is extravagantly large, since a much smaller interval, viz. $(3, 9)$ also does the same job.

This suggests another possibility, viz. that the question indeed was about the bounds for $|z| - |w| + 3$, but the reasoning applied by the paper-setters was such that only (D) would come out to be the correct answer. One such reasoning is based on the inequality

$$\big||z| - |w|\big| \;\leq\; |z - w| \tag{14}$$

which holds for any two complex numbers z and w. This inequality is a less frequently stated form of the triangle inequality and can, in fact, be derived from the latter by separately proving (i) $|z| \leq |w| + |z - w|$ and (ii) $|w| \leq |z| + |w - z|$ which together imply that $|z - w| \geq \pm(|z| - |w|)$ regardless of which sign is taken in $\pm$. If we apply this inequality, we get

$$-|z - w| \;\leq\; |z| - |w| \;\leq\; |z - w| \tag{15}$$

for any two complex numbers z and w. (As a matter of fact, as shown above, we first prove (15) and then derive (14) from it.) In the present problem, z lies on the circle B of radius 3 while w lies inside it. We already observed that in this case

$$0 \;<\; |z - w| \;<\; 6 \tag{16}$$

The second inequality in (16), coupled with (15), together imply

$$-6 \;<\; |z| - |w| \;<\; 6 \tag{17}$$

Adding 3 to all the terms this gives

$$-3 \;<\; |z| - |w| + 3 \;<\; 9 \tag{18}$$

This gives another proof that (D) is a correct answer. Moreover, with this reasoning it would appear as if it is the only correct answer. But that is not the case as we already saw, So here is an instance where the paper-setters' minds were channelised in a particular line of thinking. Apparently they wanted to test whether a candidate knows the inequality (14). In a conventional type examination they could have simply asked a proof of (14). But as they are forced to ask a multiple choice question, they designed a numerical question where the use of the inequality (14) would lead to a certain answer, viz. (D). What they failed to realise was that there is considerable extravagance in the inequality (17). If we cut down on it, then there are other ways to find bounds on the expression $|z| - |w| + 3$ besides using (15) (which is equivalent to (14)) and, in fact, they give sharper bounds which fit into the intervals given by (B) and (C) too. It may be noted that even if z is allowed to vary all over the circle B (instead of being fixed as the unique point in $A \cap B \cap C$), better bounds for $|z| - |w|$ are still possible. If we let T' be the point on B diametrically opposite to T, then the variation of $|z| - |w|$ is from $OT' - OT$ to $OT - OT'$, i.e. from $-2\sqrt{5}$ to $2\sqrt{5}$, which is a considerable improvement over (17). The reason is that here the origin lies inside the circle B. If it lay on or outside B, then (17) would indeed be the best inequality for $|z| - |w|$.

Whatever be the cause of the mistake on the part of the paper-setters, the effect is a disaster for the sincere candidate. Since z is a constant and the problem involves $|z|$ it is but natural for him to spend considerable time in determining the exact value of $|z|$. A more intelligent candidate will bypass these calculations with suitable bounds on $|z|$ as shown above. But no matter which approach a candidate takes, when he sees that all the three answers (B), (C) and (D) are correct, he will be thoroughly confused. He will then recheck his calculations spending valuable time.

Apparently, this mistake in the question paper was noticed later, because in the official model answers, a provision is made to give the candidate full credit if he marks any one of (B), (C), (D) as the correct choice. As a result, many students must have scored on this question, some of them undeservedly. It is also not clear whether a candidate who ticks more than one out of (B), (C) and (D) was given any credit or whether he was given a negative credit. A negative credit in such a case would amount to a mockery of the perseverance and/or the intelligence of the candidate.

The addition of the constant 3 to the expression $|z| - |w|$ serves absolutely no purpose. It is a nuisance which has to be carried over in every step of the calculations. A candidate who misses to do so in one of these steps may get the final answer wrong even though his reasoning is correct otherwise. In a conventional examination, the candidate has to show his work and so an examiner has a chance to notice that the omission is a slip of hand rather than a slip of reasoning. He can then ignore the mistake completely, or give some partial credit to the candidate.

> In a multiple choice type examination, this is impossible and so, in effect, such useless appendages only serve to give an unduly heavy penalty for a relatively minor mistake. It is almost like rejecting an otherwise well written essay about Mahatma Gandhi on the ground that the year of his birth was given as 1872 instead of 1869.

Although ostensibly this paragraph is on complex numbers, they are involved only peripherally. In essence all the three problems are about circles in a plane. It is not clear what is achieved by restricting z to $A \cap B \cap C$ in the last two questions. The answers would have been the same if z was allowed to vary over the circle B and, in fact, the problems would have been more interesting. The mistake in the third problem mars the simplicity of the bunch.

Out of the three paragraphs, only the middle one (about an implicitly defined function) tests some understanding or 'comprehension'. The other two bunches are collections of inter-related questions on familiar, common themes. The only bit of 'comprehension' needed to answer them is the ability to correctly understand the question. But this is something which you need on most mathematics questions anyway. Even the paragraph on implicitly defined functions does not really test the ability to understand any new concept.

One of the criticisms often voiced against the JEE is that it tests the ability to solve problems rather than the ability to understand concepts. In principle, this criticism is not very valid because a candidate who can correctly solve a problem, needs to understand it correctly in the first place. A candidate who correctly understands a new, subtly designed problem entirely on his own certainly has a good power of comprehension. But the catch is that nowadays most candidates are exposed to huge collections of solved problems. As a result, when they see a problem, they hardly need to think fresh about it. Moreover, in preparing for the JEE, the over-emphasis on problem solving often leads to simply skipping any theory with the assurance that no 'theory' questions would be asked.

Comprehension questions were introduced apparently to counter this criticism. But in that case they ought to give some paragraph where some text which is simple but not covered in the syllabus is given and questions are asked which can be answered only if that text is correctly understood.

As an example, consider the following paragraph.

A **binary sequence** of length n is defined as an ordered n-tuple, say $\vec{x} = (x_1, x_2, \ldots, x_n)$ where each x_i takes two possible values, say 0 and 1. If $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$ are two binary sequences of the same length, we define the distance between them, denoted by $d(\vec{x}, \vec{y})$ to be the number of entries where they differ. For example, with $n = 6$, the distance between $(1, 0, 1, 0, 1, 1)$ and $(1, 1, 1, 0, 0, 1)$ is 2 because they differ in their second and fifth entries. A **ternary sequence** is defined similarly, except that each term can take 3 possible values, say 0, 1 and 2. The distance between two ternary sequences of length n is also defined similarly, viz. the number of entries in which they differ.

Based on this paragraph, the following questions could be asked.

(a) How many binary and ternary sequences of length 10 are there?

(b) Given a particular binary sequence of length 10, how many binary sequences are at a distance 2 from it?

(c) What will be the answer to (b) for ternary sequences?

While teaching first year courses at the IITs it is often found that many students cannot answer these questions without some prodding. This is a clear indication that their ability to comprehend written mathematical text has not been tested.

Comprehension questions can also be asked in several other forms. For example, the given paragraph gives the proof of some (unfamiliar but simple) theorem. The candidates can then be asked to identify exactly where in the proof a particular part of the hypothesis of the theorem was used. Occasionally, a false argument can be given and the candidates can be asked to identify where the fallacy lies.

It is only if such innovative questions are asked that there would be a true testing of the ability to 'comprehend'. The way the things stand now, the so-called comprehension questions are hardly different from the remaining questions.

PAPER 2

Contents

SECTION I

Straight Objective Type

This section contains 9 multiple choice questions. Each question has 4 choices out of which **ONLY ONE** is correct. There are 3 marks for a correct answer, 0 marks if the question is not answered and -1 mark in all other cases.

Q.1 A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from the origin by 5 units and then vertically away from the origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\vec{i} + \vec{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at the origin, to reach a point z_2. The point z_2 is given by

(A) $6 + 7i$ (B) $-7 + 6i$ (C) $7 + 6i$ (D) $-6 + 7i$

Answer and Comments: (D). The motion of the particle consists of two parts, from z_0 to z_1 and from z_1 to z_2. But each part itself can be broken down into two parts. So we introduce two more complex numbers, say w_1 and w_2 for the intermediate positions occupied by P. Now the journey consists of four parts :

(i) from z_0 to w_1, horizontally away from the origin by 5 units,

(ii) from w_1 to z_1 vertically away from the origin by 3 units,

(iii) from z_1 to w_2 $\sqrt{2}$ units in the direction of the vector $\vec{i} + \vec{j}$, and

(iv) from w_2 to z_2 along a circle centred at O anticlockwise through an angle $\frac{\pi}{2}$

We are given z_0 as $1 + 2i$. We shall successively determine w_1, z_1, w_2 and finally z_2 from the descriptions of the four parts of the journey.

In (i), as the origin is on the left side of z_0, the real part of z_0 increases by 5 units while the imaginary part remains the same. This gives

$$w_1 = z_0 + 5 = (1 + 2i) + 5 = 6 + 2i \tag{1}$$

Similarly, in (ii) the imaginary part increases by 3 units and we get

$$z_1 = w_1 + 3i = 6 + 2i + 3i = 6 + 5i \tag{2}$$

In the (iii) the motion is along the vector which makes an angle of 45° with the x-axis. So the horizontal and vertical parts of the motion are $\sqrt{2}\cos 45°$ and $\sqrt{2}\sin 45°$, i.e. 1 and 1 respectively. Therefore

$$w_2 = z_1 + (1 + i) = 6 + 5i + (1 + i) = 7 + 6i \tag{3}$$

Finally, in going from w_2 to z_2, the particle moves along a circle centred at O through an angle $\frac{\pi}{2}$. This amounts to multiplying w_2 by the complex number $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$. Therefore,

$$z_2 = iw_2 = i(7 + 6i) = -6 + 7i \tag{4}$$

The problem is a simple testing of the ability to convert motions along lines and circles in terms of complex numbers. The last part could have been made a little more challenging by describing the motion as an anticlockwise rotation through some given angle, say θ, along a circle centred at a given point, say w_0. (In the present problem $\theta = \frac{\pi}{2}$ and $w_0 = 0$.) Then instead of (4) we would have gotten

$$z_2 - w_0 = e^{i\theta}(w_2 - w_0) \tag{5}$$

Had the journey been clockwise, θ should be replaced by $-\theta$.

Q.2 Let the function $g : (-\infty, \infty) \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is

(A) even and is strictly increasing in $(0, \infty)$

(B) odd and is strictly decreasing in $(-\infty, \infty)$

(C) odd and is strictly increasing in $(-\infty, \infty)$

(D) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

Answer and Comments: (C). The given function is a composite of several functions, first the exponential function and then the arctan function. As both these functions are strictly increasing in their domains, so is the composite $\tan^{-1}(e^u)$. Multiplying by the positive constant 2 and subtracting the constant $\frac{\pi}{2}$ does not affect the increasing/decreasing behaviour of a function. So, we conclude that the function $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$ is strictly increasing on $(-\infty, \infty)$. Note that it was not necessary to take any derivatives. Of course, those who want to apply derivatives can aslo get the answer easily since $g'(u) = \frac{2e^u}{1+e^{2u}}$ is positive everywhere.

Now, to determine if g is an even or odd function or neither, we note that $e^{-u} = \frac{1}{e^u}$ for all $u \in I\!R$. Moreover, regardless of the sign of u, the expression e^u is always positive. Therefore

$$\tan^{-1}(e^{-u}) = \tan^{-1}\left(\frac{1}{e^u}\right) = \frac{\pi}{2} - \tan^{-1}(e^u) \tag{1}$$

from which we immediately get

$$\begin{aligned} g(-u) &= 2\tan^{-1}(e^{-u}) - \frac{\pi}{2} \\ &= 2\left(\frac{\pi}{2} - \tan^{-1}(e^u)\right) - \frac{\pi}{2} \\ &= -2\tan^{-1}(e^u)) + \frac{\pi}{2} \\ &= -g(u) \end{aligned} \tag{2}$$

for all $u \in I\!R$, which proves that $g(u)$ is an odd function of u.

A simple problem requiring little more than some very basic identities about the exponential function and the inverse trigonometric functions. The paper-setters have been a little careless in giving the interval in (A) as $(0, \infty)$ instead of $(-\infty, \infty)$. The concept of an even (or an odd) function makes sense only when the domain is symmetric about the origin. So an alert candidate can weed out (A) on this ground alone. That is not much of a help, because there is no further short cut. The examiners probably did not bother because (A) is a wrong alternative anyway.

Note that the constant $-\frac{\pi}{2}$ in the definition of the function g is very crucial. If it is dropped or replaced by some other constant, then the new function will be neither even nor odd. This is in sharp contrast with the constant 3 appearing in the statement of Q. 23 of Paper 1, which has absolutely no role there.

Q.3 Consider a branch of the hyperbola

$$x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$$

with vertex at the point A. Let B be one of the end-points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is

(A) $1 - \sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}} - 1$ (C) $1 + \sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}} - 1$

Answer and Comments: (B) Yet another question based on the concept of a latus rectum of a conic. By definition, this is a focal chord (i.e. a chord passing through a focus) and perpendicular to the axis of the conic. To determine it in the present problem, we must first cast the equation in its standard form. This would have been very complicated if the xy-term were present in the equation because in that case the axes of the hyperbola would not be parallel to the coordinate axes and to make them so, a rotation of the coordinate axes (possibly in addition to a translation of the origin) would be needed. And in that case the calculations needed would be fairly complicated.

Fortunately, we are spared of this because of the absence of the xy-term. All we have to do now is to complete the squares. Doing so we get the equation of the hyperbola as

$$(x-\sqrt{2})^2 - 2(y+\sqrt{2})^2 = 4 \tag{1}$$

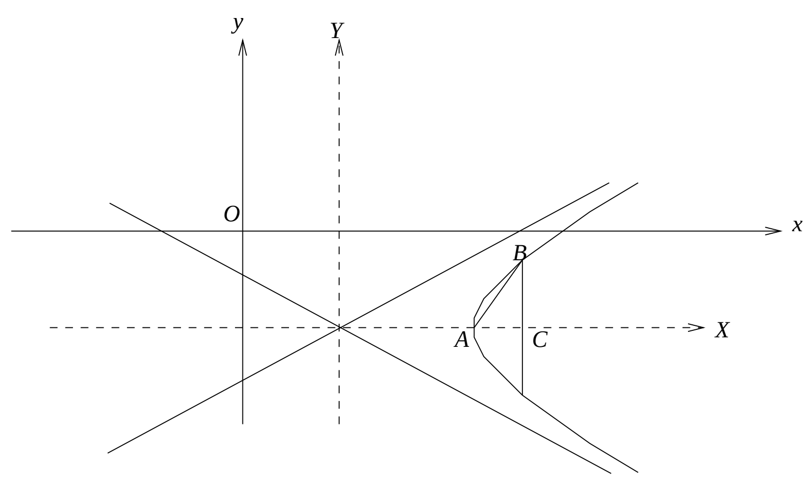
To bring it to the standard form we first divide by 4 and get

$$\frac{(x-\sqrt{2})^2}{4} - \frac{(y+\sqrt{2})^2}{2} = 1 \tag{2}$$

We now have to shift the origin to the point $(\sqrt{2}, -\sqrt{2})$ and work in terms of the new coordinates, say $X = x - \sqrt{2}$ and $Y = y + \sqrt{2}$. Thus we get

$$\frac{X^2}{4} - \frac{Y^2}{2} = 1 \tag{3}$$

which is the equation of the hyperbola in the standard form with $a = 2$ and $b = \sqrt{2}$. From these values and the formula $b^2 = a^2(e^2-1)$, the eccentricity e comes out as $\sqrt{\frac{3}{2}}$. So the two vertices of the hyperbola are at $(\pm 2, 0)$ and the two foci are at $(\pm\sqrt{6}, 0)$ (all w.r.t. the new coordinates).



Note that the statement of the problem does not tell us which branch of this parabola is to be considered. That is upto us. But once the branch is chosen, rest of the data depends on the choice. But because of the symmetry of the hyperbola around its major as well as minor axes, the answer would be independent of it. For simplicity we choose the right branch of the hyperbola. Then the vertex A is at the point $(2, 0)$ and the corresponding focus C is at $(\sqrt{6}, 0)$. The end-points of the latus rectum are $(\sqrt{6}, \pm 1)$. Again, the problem does not specify which of these is to be taken as A and the answer does not depend on the choice. We take B as $(\sqrt{6}, 1)$. Having determined all the three vertices of the triangle ABC it is now a mechanical task to find its area from the formula. But even that is not necessary. As the triangle is right-angled at B, its area is simply $\frac{1}{2} \times AC \times BC$ i.e. $\frac{1}{2} \times (\sqrt{6} - 2) \times 1 = \sqrt{\frac{3}{2}} - 1$ square units.

Note that Equation (3) was not very vital. Since the points A, B, C are described directly in terms of the hyperbola in a purely geometric manner, even through their coordinates will change as the frame of reference changes, what matters for the area of the triangle is only the lengths of its sides and they remain unchanged under any change of coordinates. The side AC is the distance between the vertex A and the corresponding focus C and therefore equals $ae - a = a(e - 1)$, which in the present case comes to $2(\sqrt{\frac{3}{2}} - 1)$. Similarly, BC is half the latus rectum and there is a ready made formula for it. Those who know it will have an easier time. Q. 8 of Paper 1 was also based on the latus rectum, except that there the conic was an ellipse. Considering that the latus rectum, although very common in old books, is not such an important concept, two questions based on it in the same JEE is a bit too much.

Q.4 The area of the region between the curves $y = \sqrt{\frac{1 + \sin x}{\cos x}}$ and $y = \sqrt{\frac{1 - \sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \frac{\pi}{4}$ is

(A) $\displaystyle\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}} dt$ (B) $\displaystyle\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}} dt$

(C) $\displaystyle\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}} dt$ (D) $\displaystyle\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}} dt$

Answer and Comments: (B). A typical problem about finding the area between two curves using integration. However, we do not have to actually evaluate the area, but only to express it as a definite integral in a certain way. For $0 \leq x \leq \frac{\pi}{4}$, evidently the first curve is above the second at all points and so, the given area equals the integral

$$I = \int_0^{\pi/4} \left(\sqrt{\frac{1+\sin x}{\cos x}} - \sqrt{\frac{1+\sin x}{\cos x}} \right) \, dx \tag{1}$$

We now have to transform this integral to one of the four given ones. An alert student will realise that in all the four given alternatives, because of the radical in the denominator, the variable t can assume only values between -1 and 1. Since in (C) and (D), the upper limit of integration, viz. $\sqrt{2}+1$ is bigger than 1, they are eliminated on this ground alone.

The most standard substitution for transforming integrals such as (1) is to put $t = \tan(x/2)$. In the present case, however, the variable t is already used in the statement of the problem. So, we use a different one, say $u = \tan(x/2)$. (Technically, t is only a dummy variable of integration and so there is no harm in taking $\tan(x/2)$ as t. But we do not know beforehand if this substitution will transform I to any of the four alternatives. As it sometimes happens in integration by substitution, one more substitution may be needed and we shall have to introduce a new variable for it. So it is better to start from $u = \tan(x/2)$ rather than $t = \tan(x/2)$.)

Taking $u = \tan(x/2)$, we see that upper and the lower limits of integration become $\tan(\pi/8)$ and 0 respectively. The value of $\tan(\pi/8)$ can be determined from the quadratic

$$1 = \tan\frac{\pi}{4} = \frac{2\tan(\pi/8)}{1-\tan^2(\pi/8)} \tag{2}$$

which, after putting $y = \tan(\pi/8)$ becomes

$$2y = 1 - y^2 \tag{3}$$

whose solutions are $y = \pm\sqrt{2} - 1$. Obviously the + sign holds since $\tan(\pi/8)$ is positive. So we get that the upper and the lower limits of the transformed integral are $\sqrt{2} - 1$ and 0.

The identities $\sin x = \dfrac{2\tan(x/2)}{1+\tan^2(x/2)}$ and $\cos x = \dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$ transform (1) into

$$\begin{aligned} I &= \int_0^{\sqrt{2}-1} \left(\frac{1+u}{\sqrt{1-u^2}} - \frac{1-u}{\sqrt{1-u^2}} \right) \frac{2}{1+u^2} \, du && (4) \\ &= \int_0^{\sqrt{2}-1} \frac{4u}{(1+u^2)\sqrt{1-u^2}} \, du && (5) \end{aligned}$$

where, in deriving (4) the signs of the radicals have been chosen to make the expressions positive. Thus we see that no further substitution is necessary to equate this with the integral in (B). The problem is really not about evaluating an integral but only about transforming it. As the answer comes in a single substitution which is very standard, the problem is a straightforward one. Note that the integral cannot be evaluated further in a closed form. The problem would have been more interesting if the nature of the integrand was such that it could be further evaluated, or if some further substitution was necessary. For example, we could put $u^2 = v$. Then (5) would become

$$\int_0^{3-2\sqrt{2}} \frac{2dv}{(1+v)\sqrt{1-v}} \tag{6}$$

But perhaps the paper-setters decided that for 3 marks the work required to reach (5) is all that can be expected.

Q.5 Consider three points $P = (-\sin(\beta - \alpha), -\cos\beta), Q = (\cos(\beta - \alpha), \sin\beta)$ and $R = (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$, where $0 < \alpha, \beta, \theta < \frac{\pi}{4}$. Then,

(A) P lies on the line segment RQ

(B) Q lies on the line segment PR

(C) R lies on the line segment QP

(D) P, Q, R are non-collinear.

Answer and Comments: (D). Suppose $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$ are three distinct points in the xy-plane. Then the point C lies on the line AB if and only if there exists some real number λ such that

$$x_3 = \lambda x_2 + (1-\lambda)x_1 \quad \text{and} \quad y_3 = \lambda y_2 + (1-\lambda)y_1 \tag{1}$$

(See the beginning of Comment No. 16 of Chapter 8. Actually, this holds not only for points in the plane but for points in the n-dimensional space $\mathbb{R}^n$ as well.) If such a real number λ exists, it is unique. As λ moves from $-\infty$ to ∞, the point C moves from one end to the other end of the line AB. Depending upon the values of λ, we can decide where C lies. Specifically,

(i) If $0 < \lambda < 1$, then C lies on the segment AB

(ii) If $\lambda < 0$, then A lies on the segment CB

(iii) If $\lambda > 1$, then B lies on the segment AC.

(Note that for $\lambda = 0$, C coincides with A while for $\lambda = 1$ it coincides with B.) No matter which of these cases holds, we have

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \tag{2}$$

which follows by writing the third row as a linear combination of the first two rows, or simply by equating the area of the triangle ABC to 0.

Now, coming to the given problem, let us first check if (2) holds. Here we have

$$\Delta = \begin{vmatrix} -\sin(\beta-\alpha) & -\cos\beta & 1 \\ \cos(\beta-\alpha) & \sin\beta & 1 \\ \cos(\beta-\alpha+\theta) & \sin(\beta-\theta) & 1 \end{vmatrix} \tag{3}$$

To see if this determinant vanishes we try to write its third row as a linear combination of the first two rows. This suggests that we first expand the entries in the third column and rewrite Δ as

$$\Delta = \begin{vmatrix} -\sin(\beta-\alpha) & -\cos\beta & 1 \\ \cos(\beta-\alpha) & \sin\beta & 1 \\ \cos(\beta-\alpha)\cos\theta - \sin(\beta-\alpha)\sin\theta & \sin\beta\cos\theta - \cos\beta\sin\theta & 1 \end{vmatrix} \tag{4}$$

Therefore subtracting $\sin\theta$ times the first row and $\cos\theta$ times the second row from the third and expanding w.r.t. the third column, we get

$$\begin{aligned} \Delta &= \begin{vmatrix} -\sin(\beta-\alpha) & -\cos\beta & 1 \\ \cos(\beta-\alpha) & \sin\beta & 1 \\ 0 & 0 & 1-\sin\theta-\cos\theta \end{vmatrix} \\ &= (1-\sin\theta-\cos\theta)(\cos\beta\cos(\beta-\alpha) - \sin\beta\sin(\beta-\alpha)) \\ &= (1-\sin\theta-\cos\theta)(\cos(2\beta-\alpha)) \qquad (5) \end{aligned}$$

There is nothing in the data to conclude that this determinant necessarily vanishes. Hence the points $P, Q.R$ are not collinear in general. They will be so if and only if one of the two factors in the last expression in (5) vanishes. This means either $\sin\theta + \cos\theta = 1$ or $\cos(2\beta - \alpha) = 0$. The first possibility would give $\sin(\theta + \frac{\pi}{4}) = \dfrac{1}{\sqrt{2}}$ which can never hold for the given range of θ, viz. $0 < \theta < \dfrac{\pi}{4}$. As α, β are also in the same range, the second possibility cannot hold either.

The non-vanishing of the determinant Δ puts an abrupt end to the solution. If it had vanished, then there would have been a unique real number λ such that $R_3 = (1-\lambda)R_1 + \lambda R_2$ where R_1, R_2, R_3 are the rows of the determinant Δ. And then depending upon which of the intervals $(-\infty, 0), (0, 1)$ and $(1, \infty)$ contains λ, one of the possibilities (A), (B) and (C) would have been true. The problem would have been more interesting in that case. As it stands, the purpose of including these three alternatives is unclear. Perhaps it is just to make up the required number of fake alternatives.

Q.6 An experiment has 10 equally likely outcomes. Let A and B be two non-empty events of the experiment. If A consists of 4 outcomes, the number of outcomes that B must have so that A and B are independent is

(A) 2, 4 or 8 (B) 3, 6 or 9 (C) 4 or 8 (D) 5 or 10

Answer and Comments: (D). As all the ten outcomes are equally likely, the probability of each is $\frac{1}{10}$. So, regardless of which four outcomes constitute A, we have

$$P(A) = \frac{4}{10} = \frac{2}{5} \tag{1}$$

Similarly, if we let n be the number of elements in the event B, then we get

$$P(B) = \frac{n}{10} \tag{2}$$

Let m be the number of elements in the intersection $A \cap B$. Note that m is not uniquely determined by n. All we can say for sure is that $m \leq n$. Then the probability of the joint occurrence of the two events (also called the probability of their conjunction) is given by

$$P(A \cap B) = \frac{m}{10} \tag{3}$$

Now, by definition, the events A and B are (mutually) independent if and only if

$$P(A \cap B) = P(A)P(B) \tag{4}$$

which, in view of (1), (2) and (3) becomes

$$5m = 2n \tag{5}$$

Here m, n are integers (being the numbers of elements in some sets). As the L.H.S. is divisible by 5, so is the R.H.S. But 5 is relatively prime to 2. So 5 must divide n. Now n cannot exceed 10. Also, since the event B is non-empty, $n > 0$. Therefore the only possible values of n are 5 or 10. Hence the answer is (D).

Although the solution is complete as far as the given problem is concerned, it needs to be noted that $n = 5$ or 10 is only a *necessary* condition for the independence of the events. If $n = 10$, then of course, B is the entire set of the 10 outcomes. In this case, $P(B) = 1$, $A \cap B = A$ and so the events A and B are trivially independent of each other. But if $n = 5$, then all we know is that B consists of 5 outcomes. In that case, (5) will hold only if $m = 2$, i.e. if $A \cap B$ has exactly 2 elements. An alert candidate who notices this is likely to get slightly confused because although technically necessity of a condition is different from its sufficiency, in most problems at the JEE level, the two are the same. Thus for example, Equation (2) in the solution to the last problem is necessary as well as sufficient for the collinearity of the three points.

The problem is a combination of probability and a little bit of number theory. The latter was not at all tested in Paper 1. The wording of the problem is too technical. Many probability problems can be posed as real life problems in a delightfully rich variety of ways. In the present problem, for example, instead of the set of ten possible outcomes of an experiment, we could have taken a class of 10 students. The event A could have been described by saying that 4 of these students are girls. Instead of the event B we could have said that exactly n of these 10 students are intelligent. Now the question would amount to asking to determine for which values of n it may be concluded that in this class being intelligent is independent of sex and, further, when this is the case, to determine the number of intelligent girls. In that case, the answer would have been $n = 0, 5$ or 10 and correspondingly, the number of intelligent girls must be 0, 2 and 4 respectively. In this formulation the problem is more palpable and less confusing.

In fact, this reformulation of the problem suggests an interesting extension. In the example above, the sex of a student can have only two possibilities. But there are many other attributes such as religion, language etc. which run into much wider spectra of possibilities. Suppose, for example, that some population is divided into k religions, say $R_1, R_2, \ldots, R_k$. Once again, we let B be the event that a person is intelligent. Then to say that being intelligence is independent of a person's religion means that for every $i = 1, 2, \ldots, k$ the conditional probability of the event B given R_i (often denoted by $P(B|R_i)$) is the same, i.e. independent of i. Using the formula for the conditional probability, this means that the ratio $\dfrac{P(R_i \cap B)}{P(R_i)}$ is independent of i. Using the fact, that $\sum_{i=1}^{k} P(R_i) = 1$, it is easy to show that this common value is simply $P(B)$. Thus, to say that being intelligent is independent of one's religion means that for every i, $P(R_i \cap B) = P(B)P(R_i)$.

Q.7 Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time t the position vector $\overline{OP}$ (where O is the origin) is given by $\hat{a}\cos t + \hat{b}\sin t$. When P is farthest from the origin O, let M be the length of $\overline{OP}$ and let $\hat{u}$ be the unit vector along OP. Then,

(A) $\hat{u} = \dfrac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M = (1 + \hat{a}\cdot\hat{b})^{1/2}$

(B) $\hat{u} = \dfrac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M = (1 + \hat{a}\cdot\hat{b})^{1/2}$

(C) $\hat{u} = \dfrac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M = (1 + 2\hat{a}\cdot\hat{b})^{1/2}$

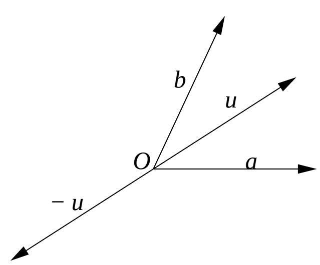
(D) $\hat{u} = \dfrac{\hat{a} - \hat{b}}{|\hat{a} - \hat{b}|}$ and $M = (1 + 2\hat{a} \cdot \hat{b})^{1/2}$

Answer and Comments: (A). A straightforward problem about the dot products and lengths of vectors. Let $\vec{u}(t) = \hat{a}\cos t + \hat{b}\sin t$. Then the length of $\overline{OP}$ is simply the length $||\vec{u}(t)||$ of this vector. By a direct computation,

$$\begin{aligned} ||\vec{u}(t)||^2 &= \vec{u}(t) \cdot \vec{u}(t) \\ &= (\hat{a}\cos t + \hat{b}\sin t) \cdot (\hat{a}\cos t + \hat{b}\sin t) \\ &= ||\hat{a}||^2 \cos^2 t + ||\hat{b}||^2 \sin^2 t + (\hat{a} \cdot \hat{b}) \sin 2t \qquad (1) \\ &= 1 + (\hat{a} \cdot \hat{b}) \sin 2t \qquad (2) \end{aligned}$$

where in going from (1) to (2) we have used that $\hat{a}, \hat{b}$ are unit vectors. We are further given that $\hat{a}$ and $\hat{b}$ are inclined at an acute angle with each other. Therefore $\hat{a} \cdot \hat{b}$ is positive. Hence the expression in (2) will be maximum when $\sin 2t = 1$ and the maximum value will be $1 + (\hat{a} \cdot \hat{b})$. Therefore M is the square root of this, i.e. $(1 + \hat{a} \cdot \hat{b})^{1/2}$.

To complete the solution, we have to find the unit vector along $\vec{u}(t)$ when $\sin 2t = 1$. This has infinitely many solutions. But because of the periodicity of the sine and the cosine functions, and hence also of the vector $\vec{u}(t)$, we need only consider those values of t which lie in, say $[0, 2\pi]$. These values are $t = \dfrac{\pi}{4}$ and $t = \dfrac{5\pi}{4}$. The corresponding vectors $\vec{u}(t)$ are $\dfrac{1}{\sqrt{2}}(\hat{a} + \hat{b})$ and $-\dfrac{1}{\sqrt{2}}(\hat{a} + \hat{b})$. Therefore the corresponding unit vectors $\hat{u}$ are $\pm\dfrac{\hat{a} + \hat{b}}{|\hat{a} + \hat{b}|}$. These two vectors are oppositely directed to each other. The $+$ sign gives the answer in (A). But the $-$ sign is also a valid answer which is not included in any of the alternatives. The alternative (A) ought to have included the $\pm$ sign. So, like Q.23 of Paper 1, this problem is also erroneously set. But the mistake here is not as disastrous as that in Q.23 of Paper 1 because, although strictly speaking none of the given four alternatives is *necessarily* correct, the alternative (A) is the closest to the correct answer. (It is to be noted, however, that, in the past, mistakes of this kind, when done by the candidates, were penalised.) Apparently, the mistake arose because the paper-setters took $\hat{u}$ to lie only in the angle spanned by the vectors $\hat{a}$ and $\hat{b}$ and missed that its opposite vector is also equally long.



Note that the transition from (1) to (2) and the subsequent work was considerably simplified by the assumption that both $\hat{a}$ and $\hat{b}$ are unit vectors. The reasoning would be essentially the same if instead, we had started with two vectors $\vec{a}$ and $\vec{b}$ of equal lengths. But if their lengths were different then maximising the R.H.S. of (1) would be fairly complicated. It can be done either by using calculus or by rewriting the expression as

$$\begin{aligned} ||\vec{u}(t)||^2 &= ||\vec{a}||^2\frac{1+\cos 2t}{2} + ||\vec{b}||^2\frac{1-\cos 2t}{2} + (\vec{a}\cdot\vec{b})\sin 2t \\ &= \frac{||\vec{a}||^2+||\vec{b}||^2}{2} + \left[\left(\frac{||\vec{a}||^2-||\vec{b}||^2}{2}\right)\cos 2t + (\vec{a}\cdot\vec{b})\sin 2t\right] \quad (3) \end{aligned}$$

and then writing the bracketed expression as

$$L\cos(\theta - 2t) \quad (4)$$

where

$$L = \sqrt{\left(\frac{||\vec{a}||^2-||\vec{b}||^2}{2}\right)^2 + (\vec{a}\cdot\vec{b})^2} \quad (5)$$

and θ is so chosen that (4) holds true. It is clear that the maximum value of $||\vec{u}(t)||^2$ now is $\frac{||\vec{a}||^2+||\vec{b}||^2}{2} + L$.

Q.8 Let $I = \int \frac{e^x}{e^{4x}+e^{2x}+1}dx$ and $J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$. Then, for an arbitrary constant C, the value of $J - I$ equals

(A) $\frac{1}{2}\log\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+C$ (B) $\frac{1}{2}\log\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+C$

(C) $\frac{1}{2}\log\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+C$ (D) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+C$

Answer and Comments: (C). The most straightforward method is to begin by evaluating the two integrals separately. The substitutions $u = e^x$ for I and $v = e^{-x}$ for J suggest themselves. So, we have

$$\begin{aligned} I &= \int \frac{e^x}{e^{4x}+e^{2x}+1}dx \\ &= \int \frac{du}{u^4+u^2+1} \quad (1) \end{aligned}$$

and similarly,

$$\begin{aligned} J &= \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx \\ &= \int -\frac{dv}{v^4+v^2+1} \quad (2) \end{aligned}$$

It is clear from (1) and (2) that the subsequent work involved in both the integrals is identical. It consists of resolving the integrand into partial fractions and then integrating each term. This is rather cumbersome. An alert student would realise that if the idea behind the question is to test integration by partial fractions, the question could have been to evaluate either one of the two integrals I and J. Instead, the question is to evaluate the difference $J - I$ and this must be so with some purpose. As illustrated in Comment No. 14 of Chapter 18 or the last problem of Comment No. 24 of Chapter 17, it is sometimes easier to find the sum (or the difference) of two things than to find those things individually. This is so because the troublesome parts of the two things cancel each other or add up to something much simpler. (As a more pointed example, integrating $\sin^2 x + \cos^2 x$ is certainly a lot easier than integrating $\sin^2 x$ and $\cos^2 x$ separately and adding.)

We adopt this strategy in the present problem. So, we continue with the substitution $u = e^x$, but use it not only for I but also for J and hence for the difference $J - I$. So, instead of (1) and (2) we now have

$$\begin{aligned}
J - I &= \int \left(\frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} - \frac{e^x}{e^{4x} + e^{2x} + 1} \right) dx \\
&= \int \left(\frac{e^{3x}}{1 + e^{2x} + e^{4x}} - \frac{e^x}{e^{4x} + e^{2x} + 1} \right) dx \\
&= \int \frac{e^{3x} - e^x}{e^{4x} + e^{2x} + 1} dx \qquad (3) \\
&= \int \frac{u^2 - 1}{u^4 + u^2 + 1} du \qquad (4)
\end{aligned}$$

The trick now is to divide both the numerator and the denominator of the integrand in (4) by u^2 so that the numerator becomes $1 - \frac{1}{u^2}$ which is precisely the derivative of $u + \frac{1}{u}$. Doing so and using the substitution $y = u - \frac{1}{u}$, we get

$$\begin{aligned}
J - I &= \int \frac{1 - \frac{1}{u^2}}{u^2 + 1 + \frac{1}{u^2}} du \\
&= \int \frac{1 - \frac{1}{u^2}}{\left(u + \frac{1}{u}\right)^2 - 1} du \\
&= \int \frac{1}{y^2 - 1} dy \\
&= \frac{1}{2} \log \left(\frac{y - 1}{y + 1} \right) \\
&= \frac{1}{2} \log \left(\frac{u^2 - u + 1}{u^2 + u + 1} \right) \qquad (5)
\end{aligned}$$

Putting back $u = e^x$ and adding an arbitrary constant C gives (C) as the answer.

The problem is a good test of the mental alertness of a candidate. Those who try to do it by separately evaluating I and J will pay a heavy price in terms of time. But, once again, the multiple choice format kills the beauty of the problem. A candidate who is extra smart will realise that there is no need to use any tricks. The answer can be obtained simply by differentiating each of the given alternatives and seeing which of the derivatives coincides with the difference of the integrands of J and I, i.e. with the integrand in (3). This is not as prohibitively laborious as it appears. In all the four alternatives we have expressions of the form $\log\left(\dfrac{f(x)}{g(x)}\right)$ whose derivative is simply $\dfrac{f'(x)}{f(x)} - \dfrac{g'(x)}{g(x)} = \dfrac{f'(x)g(x) - f(x)g'(x)}{f(x)g(x)}$. From the nature of the functions $f(x)$ and $g(x)$ in (A) and (D), we see that $f(x)g(x)$ cannot possibly match the denominator of (3) in the case of (A) and (D). So, the right alternative is confined either to (B) or to (C). The former can be eliminated by actual trial.

Q.9 Let $g(x) = \log f(x)$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x+1) = xf(x)$. Then for $N = 1, 2, 3, \ldots$,

$$g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(A) $-4\left\{1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots + \dfrac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots + \dfrac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots + \dfrac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots + \dfrac{1}{(2N+1)^2}\right\}$

Answer and Comments: (A). This is a problem on functional equations. We are not given the function $f(x)$ explicitly. All we are given about it (apart from its being positive and twice differentiable) is the functional relation

$$f(x+1)xf(x) \tag{1}$$

for all $x > 0$. Since $g(x) = \log f(x)$ this translates as

$$g(x+1) = \ln x + g(x) \tag{2}$$

Differentiating twice,

$$g''(x+1) = g''(x) - \frac{1}{x^2} \tag{3}$$

for all $x > 0$. Verbally, this gives a formula for the difference in the values of g'' at two points which differ by 1. The key idea of the problem is that if we apply this repeatedly, then we shall get the difference between the values of g'' at two points that differ by an integer. That is, we apply (3) with x replaced successively by $x+1, x+2, \ldots, x+k-1$ to get

$$\begin{aligned} g''(x+1) &= g''(x) - \frac{1}{x^2} \\ g''(x+2) &= g''(x+1) - \frac{1}{(x+1)^2} \\ \ldots\ldots &= \ldots\ldots\ldots \\ g''(x+k) &= g''(x+k-1) - \frac{1}{(x+k-1)^2} \end{aligned}$$

for every $x > 0$. If we add these equations and cancel the terms appearing on both the sides we get

$$g''(x+k) - g''(x) = -\left\{\frac{1}{x^2} + \frac{1}{(x+1)^2} + \ldots + \frac{1}{(x+k-1)^2}\right\} \qquad (4)$$

for every $x > 0$ and every positive integer k. If we put $x = \frac{1}{2}$ and $k = N$, we have

$$g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N-1)^2}\right\} \qquad (5)$$

As far as this problem is concerned, it does not matter if there is indeed any function which satisfies the functional equation given in the statement of the problem. Actually, there is no dearth of such functions. All one has to do is to start with any function $f : (0,1] \longrightarrow I\!R$ and then extend it to the intervals $(1,2], (2,3], \ldots$ recursively by the formula $f(x) = (x-1)f(x-1)$. But among all such functions, by far the most important one is the **gamma function**, defined by means of an improper integral. Specifically,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \qquad (6)$$

One has to prove first that this improper integral really converges for all $x > 0$. This can be done by observing that because of the factor e^{-t}, as t tends to ∞, the integrand tends to 0 very rapidly, no matter how large x is. The functional relation can be proved by integrating by parts as follows:

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty e^{-t} t^x\, dt \\ &= -e^{-t} t^x \Big|_0^\infty + \int_0^\infty e^{-t} x t^{x-1}\, dt \qquad (7) \end{aligned}$$

$$\begin{aligned} &= 0 + x \int_0^\infty e^{-t} t^{x-1} \, dt \\ &= x\Gamma(x) \end{aligned} \tag{8}$$

where to evaluate the first term in (7) we have again used the fact that for any fixed $x > 0$, $e^{-t}t^x \to 0$ as $t \to \infty$.

The gamma function occurs very frequently in applied mathematics and numerous identities are known about it. It is easy to show by direct computation that $\Gamma(1) = 1$. This fact and the functional equation (8) applied repeatedly imply that

$$\Gamma(n) = (n-1)! \tag{9}$$

for every positive integer n. Note that in (9), the variable n is discrete. But in the definition (6) of the gamma function, x is a continuous variable. So we may very well say that the gamma function is a continuous analogue of the familiar factorial function. Using improper double integrals it can be shown that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Usinng this and (8) repeatedly, one can compute $\Gamma(n + \frac{1}{2})$ for every positive integer n. Other values of the gamma function are not easy to obtain exactly. But extensive tables are available for its approximate values.

SECTION II

Reasoning Type

This section contains four reasoning type questions. Each question has four choices out of which **ONLY ONE** is correct. The marking scheme is as in Section I.

Q.10 Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1, b_2 = b_1 + a_2, b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

STATEMENT-1 : The numbers b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P.

and

STATEMENT-2 : The numbers b_1, b_2, b_3, b_4 are in H.P.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (C). Any type of progression is unaffected if we divide all its terms by some positive constant. As a result, without loss of generality, we take a_1, a_2, a_3, a_4 as $1, r, r^2, r^3$ respectively, where $r > 0$ and $r \neq 1$ (as the numbers are all distinct). Then we have

$$b_1 = 1, \;\; b_2 = 1 + r, \;\; b_3 = 1 + r + r^2, \;\; b_4 = 1 + r + r^2 + r^3$$

which are definitely not in an A.P. and not in G.P. either since, for example $b_2^2 = (1+r)^2 = 1 + 2r + r^2 \neq 1 + r + r^2 = b_1 b_3$. Hence Statement-1 is true. As for Statement-2, if the numbers b_1, b_2, b_3, b_4 are in H.P. then we must have, in particular,

$$\frac{2}{b_2} = \frac{1}{b_1} + \frac{1}{b_3} \tag{1}$$

which is not true because the L.H.S. equals $\frac{2}{1+r}$ while the R.H.S. equals $1 + \frac{1}{1+r+r^2} = \frac{2+r+r^2}{1+r+r^2}$ and in general the two are not equal. So, Statement-2 is false.

The purpose of the problem is not clear. Probably the idea is just to penalise the false notion that if some numbers are neither in A.P. nor in G.P. then they must be in H.P.

Q.11 Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and $\alpha, \frac{1}{\beta}$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$.

STATEMENT-1 : $(p^2 - q)(b^2 - ac) \geq 0$

and

STATEMENT 2: $b \neq pa$ or $c \neq qa$

- (A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
- (B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
- (C) Statement-1 is True, Statement-2 is False
- (D) Statement-1 is False, Statement-2 is True

Answer and Comments: (B). The factors of the expression that appears in Statement-1 are precisely the discriminants of the two quadratics. Now, for a quadratic with real coefficients, either both the roots are real or both are complex and further they are complex conjugates of each other. If α, β are both real then both the quadratics have real roots and hence both the discriminants are non-negative and therefore so is their product. If α, β are complex, then we have

$$\beta = \overline{\alpha} \tag{1}$$

But since α and $\frac{1}{\beta}$ are the roots of the second quadratic (whose coefficients are real too) we must also have

$$\frac{1}{\beta} = \overline{\alpha} \tag{2}$$

(1) and (2) together imply that $\beta = \frac{1}{\beta}$, i.e. $\beta^2 = 1$, which contradicts the data. So we conclude that Statement-1 is true.

As for Statement-2, if both the equalities $b = pa$ and $c = qa$ hold, then the second quadratic would be simply $a(x^2 + 2px + q) = 0$ which is essentially the same as the first quadratic. (It is inherent in the definition of a quadratic that the leading coefficient is non-zero. Otherwise it degenerates into a linear equation.) But then, both the given quadratics would have the same roots. We already know that one of their roots, viz. α is common. t the other roots are also equal to each other, then we must have $\beta = \frac{1}{\beta}$ which would again mean $\beta^2 = 1$, contradicting the hypothesis. So, Statement-2 is also true. However, Statement-2 merely amounts to saying that the two quadratics are not the same. And from this fact alone, it is impossible to deduce anything about the nature of their roots. So, even though both the statements are true, Statement-2 is not a correct explanation of Statement-1.

The problem is a good testing of the ability to translate the statements into conditions about the quadratics. Once this is realised, the computations needed are very elementary. It is not clear what purpose is served by specifying that $\beta^2 \neq -1$. It would have been sufficient to say merely that $\beta^2 \notin \{0, 1\}$. These superfluous stipulations only serve to confuse a candidate. (Probably, the paper-setters intended to say $\beta \notin \{-1, 0, 1\}$, which would be quite sufficient and β^2 appeared in place of β by a slip.)

Q.12 Consider
$L_1 : 2x + 3y + p - 3 = 0$
$L_2 : 2x + 3y = p + 3 = 0$
where p is a real number, and $C : x^2 + y^2 + 6x - 10y + 30 = 0$.

STATEMENT - 1: If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C.

and

STATEMENT - 2: If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C.

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (C). The value of the parameter p is not specified and for different values of it, we get different lines. But regardless of the value of p, all these lines have slope $-\frac{2}{3}$. Therefore they belong to a family of parallel lines. Exactly one member of such a family will pass through the centre of the circle and hence will be a diameter of it. Lines which are parallel to this diameter and whose parallel distance from it is less than the radius of the circle will be chords. There are exactly two lines whose distance from the diameter will equal the radius of the circle. Both these lines will touch the circle. The remaining lines in the family will not intersect the circle as they will be too far away from its centre.

This qualitative reasoning is sufficient to prove Statement-1. There are infinitely many values of p for which L_1 will be a chord of the circle C. But there is only one value of p for which L_2 will be a diameter of C. So clearly, Statement-1 is true. For Statement-2, however, we shall have to do some quantitative work because if L_1 is a diameter of C (which will happen for a unique value of p, say p_0), then whether L_2 is a chord of C for this value of p_0 will depend on the perpendicular distance between L_1 and L_2 for this particular value of p. The interesting part of the problem is that this perpendicular distance is independent of p. In fact, no matter what p is, the perpendicular distance, say d, between L_1 and L_2 is given by

$$d = \frac{|(p-3)-(p+3)|}{\sqrt{2^2+3^2}} = \frac{6}{\sqrt{13}} \tag{1}$$

By completing the squares, we rewrite the equation of the circle C as

$$(x+3)^2 + (y-5)^2 = 4 \tag{2}$$

which means that the radius, say r, of the circle is 2. By easy comparison, we have

$$d < r \tag{3}$$

So, when L_1 is a diameter, L_2 is a chord of C. So Statement-2 is false.

Normally coordinate geometry problems run into computations. The present problem too, would be so, if attempted using the condition for a line to be a chord of a circle. But, once the key idea strikes, the problem is more conceptual than computational and so it is a good problem. The key idea is similar to that behind the 1998 JEE problem at the beginning of Comment No.2 of Chapter 9.

Q.13 Let a solution $y = y(x)$ of the differential equation

$$x\sqrt{x^2-1}dy - y\sqrt{y^2-1}dx = 0$$

satisfy $y(2) = \frac{2}{\sqrt{3}}$.

STATEMENT-1 : $y(x) = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$

and

STATEMENT-2 : $y(x)$ is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$

(A) Statement-1 is True, Statement-2 is True and Statement-2 **is** a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True and Statement-2 **is NOT** a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

Answer and Comments: (C). Separating the variables, the given differential equation can be recast as

$$\frac{dx}{x\sqrt{x^2-1}} = \frac{dy}{y\sqrt{y^2-1}} \tag{1}$$

The work needed in integrating both the sides is identical. The substitution $x = \sec\theta$ reduces the integral $\int \frac{dx}{x\sqrt{x^2-1}}$ to $\int 1\, d\theta$ which equals simply θ, i.e. $\sec^{-1} x$. We integrate the R.H.S. similarly and get

$$\sec^{-1} x = \sec^{-1} y + C \tag{2}$$

as the general solution of (1). The initial condition $y(2) = \frac{2}{\sqrt{3}}$ gives

$$\sec^{-1} 2 = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) + C \tag{3}$$

Since $\sec^{-1} 2 = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ and $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$, we get

$$\frac{\pi}{3} = \frac{\pi}{6} + C \tag{4}$$

which determines C as $\frac{\pi}{6}$. So the particular solution of our interest is

$$\sec^{-1} x = \sec^{-1} y + \frac{\pi}{6} \tag{5}$$

Subtracting $\frac{\pi}{6}$ from both the sides and taking their secants,

$$y = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right) \tag{6}$$

which proves Statement-1. To see if Statement-2 is true, we recast (5) using the relation $\sec^{-1} t = \cos^{-1}(\frac{1}{t})$.

$$\cos^{-1}\left(\frac{1}{x}\right) = \cos^{-1}\left(\frac{1}{y}\right) + \frac{\pi}{6} \tag{7}$$

Again subtracting $\frac{\pi}{6}$ from both the sides and taking their cosines, we have

$$\begin{aligned} \frac{1}{y} &= \cos\left(\cos^{-1}(\frac{1}{y})\right) \\ &= \cos\left(\cos^{1}(\frac{1}{x}) - \frac{\pi}{6}\right) \\ &= \frac{1}{x}\cos(\frac{\pi}{6}) + \sqrt{1 - \frac{1}{x^2}}\sin(\frac{\pi}{6}) \\ &= \frac{\sqrt{3}}{2x} + \frac{1}{2}\sqrt{1 - \frac{1}{x^2}} \end{aligned} \tag{8}$$

which shows that Statement-2 is false.

This is one of those problems where the form of the alternatives gives a clue to solution. Although the substitution needed in integrating (1) is fairly obvious by itself, in case some candidate cannot think of it, he can take a clue from Statement-1 that somewhere secants are involved. After that the problem is more on inverse trigonometric functions than on differential equations. It is a little surprising that the falsity of Statement-2 has to be proved using a numerical mismatch rather than any reasoning. If the paper-setters had given (8) as Statement-2 then both the statements would have been correct. But then a controversy would have arisen whether Statement-2 is a correct explanation of Statement-1. That is, whether there is any way to deduce Statement-1 from Statement-2. In the solution given above, we derived Statement-2 from Statement-1 (or rather from (5) which is essentially the same as Statement-1). So, in this derivation, Statement-2 is certainly not a correct explanation of Statement-1. But the point to note is that the equations (5) and (7) (which are effectively Statement-1 and Statement-2, in its new form, respectively) are basically the same statements. Either can be derived from the other. So, if somebody first derives (7) from the differential equation then from (7) he will be able to deduce (5) and in that case, Statement-1 would indeed be a correct explanation of Statement-1.

When two statements are such that each one implies the other, the question as to which of them is a correct explanation of the other ultimately boils down to which of the two statements is normally proved first. For example, everybody will agree that the Pythagoras theorem is a correct explanation of the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, because that is how the identity is derived (at least when θ is an acute angle). It will be ridiculous to say that the Pythagoras theorem can be deduced from this identity (unless, of course, one starts with purely analytic definitions of sine and cosine and uses them to define the concept of an angle and then a right angle).

But in the present problem, even though most people would solve the d.e. to get (5) first and then derive (7) from it, the other way is also not entirely unthinkable. We obtained (5) by integrating (1) using the substitution $x = \sec\theta$. But we could have first rewritten (1) as

$$\frac{dx}{x^2\sqrt{1-(\frac{1}{x})^2}} = \frac{dy}{y^2\sqrt{1-(\frac{1}{y})^2}} \tag{9}$$

Now, if we use the substitutions $u = \dfrac{1}{x}$ and $v = \dfrac{1}{y}$, then this becomes

$$\frac{du}{\sqrt{1-u^2}} = \frac{dv}{\sqrt{1-v^2}} \tag{10}$$

which can be integrated to give

$$\cos^{-1} u = \cos^{-1} v + C \tag{11}$$

and hence

$$\cos^{-1}\left(\frac{1}{x}\right) = \cos^{-1}\left(\frac{1}{y}\right) + C \tag{12}$$

where the arbitrary constant C can be determined as $\dfrac{\pi}{6}$ from the initial condition given in the statement of the problem. This would give (7), from which (5) can be deduced. So, in this approach Statement-2 would indeed have been a correct explanation of Statement-1.

When the correctness of the answer to a question depends on which of two approaches one takes, it usually leads to an endless controversy. It is quite possible that originally the paper-setters wanted to make both the statements true, but then backed out for fear of such a controversy. A simple way out was to make the second statement false.

SECTION III

Linked Comprehension Type

This section contains 2 paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices out of which **ONLY ONE** is correct. There are 4 marks for a correct answer, 0 marks if the question is not answered and -1 mark in all other cases.

Paragraph for Question Nos. 14 to 16

Consider the function $f : (-\infty, \infty) \longrightarrow (-\infty, \infty)$ defined by $f(x) = \dfrac{x^2 - ax + 1}{x^2 + ax + 1}$, $0 < a < 2$.

Q.14 Which of the following is true?

(A) $(2+a)^2 f''(1) + (2-a)^2 f''(-1) = 0$
(B) $(2+a)^2 f''(1) - (2-a)^2 f''(-1) = 0$
(C) $f'(1)f'(-1) = (2-a)^2$
(D) $f'(1)f'(-1) = -(2+a)^2$

Answer and Comments: (A). Before doing anything else with the given function $f(x)$, it is good to check that it is indeed defined on the entire real line which is given to be its domain. This amounts to ensuring that the denominator of $f(x)$ has no real roots. This is true because the discriminant $a^2 - 4$ cannot vanish since a is given to lie only in the interval $(0, 2)$. Checks like this are not needed in the solution of the problem and are therefore a waste of time from the point of view of quickly scoring in an examination. But when you are studying a problem in a relaxed manner, such scruples help raise your level of maturity. Even in an examination, such a thinking can save you time by warning you that certain tempting short cuts are not going to work. For example, if we could resolve the function $f(x)$ into partial fractions then computation of its derivatives will be considerably simplified. But what we just said tells us that this approach is not going to work because the denominator cannot be factorised into two real linear factors.

Still, everything is not lost. If we recast $f(x)$ as

$$f(x) = 1 - \frac{2ax}{x^2 + ax + 1} \qquad (1)$$

we see that the differentiation is slightly simplified. Doing it we get

$$\begin{aligned} f'(x) &= -\frac{2a(x^2 + ax + 1) - 2ax(2x + a)}{(x^2 + ax + 1)^2} \\ &= \frac{2a(x^2 - 1)}{(x^2 + ax + 1)^2} \qquad (2) \end{aligned}$$

Before rushing to differentiate once more, let us pause and think a little. Only the alternatives (A) and (B) involve $f''(x)$. The remaining two involve only the first derivative of fx). If one of them is true then there is no need to check if (A) or (B) is true, as the instruction specifically says that only one of the alternatives is correct. In that case, the work in finding $f''(x)$ will be a waste.

So, let us first see if any of the alternatives (C) and (D) is true. By direct substitution, we have $f'(1) = 0$. So without doing anything further we see that both (C) and (D) are false. Now we have no option but to find $f''(x)$. We get this by differentiating (2).

$$f''(x) = \frac{4ax(x^2+ax+1) - 2(2ax^2-2a)(2x+a)}{(x^2+ax+1)^3} \tag{3}$$

A straightforward substitution gives

$$\begin{aligned} f''(1) &= \frac{4a(2+a)}{(2+a)^3} = \frac{4a}{(2+a)^2} && (4) \\ \text{and } f''(-1) &= \frac{-4a(2-a)}{(2-a)^3} = \frac{-4a}{(2-a)^2} && (5) \end{aligned}$$

With these values we see that (A) is true.

There is also a clever way of arriving at (A) by eliminating (B) without actually finding the second derivative. From (2) it is clear that the points ± 1 are the only critical points of $f(x)$. Further $f(x) \to 1$ as $x \to \pm\infty$. So $f(x)$ is a bounded, continuously differentiable function on $I\!R$. As the function has only two critical points one of them is a local maximum and the other a local minimum. So the second derivative at one of them is non-negative while that at the other is non-positive. If one of them is positive and the other negative, then (B) cannot hold because the coefficients $(2+a)^2$ and $(2-a)^2$ are both positive. Even if one of them is 0, (B) cannot hold, because it will imply that the other one must also vanish, in which case (A) will also be true, contrary to the instruction that only one of the alternatives is correct. So, if at all one of the alternatives is correct, it must be (A). Note that we have *not* actually proved (A). We have only shown that none of the remaining three alternatives can be true.

It is not clear if the paper-setters had thought about this sneaky solution. Apparently, it has not occurred to them. For if it did and they wanted to preclude such a sneaky path, then in (B) they could have made the coefficient of $f''(-1)$ some positive number other than $(2-a)^2$. On the other hand, if they wanted to reward a candidate who could come up with a clever thinking like this (which, in the present problem, reveals a high level of maturity) then this question ought to have been asked after the next question.

Q.15 Which of the following is true?

(A) $f(x)$ is decreasing on $(-1,1)$ and has a local minimum at $x=1$

(B) $f(x)$ is increasing on $(-1,1)$ and has a local maximum at $x=1$

(C) $f(x)$ is increasing on $(-1,1)$ but has neither a local maximum nor a local minimum at $x=1$

(D) $f(x)$ is decreasing on $(-1,1)$ but has neither a local maximum nor a local minimum at $x=1$

Answer and Comments: (A). From Equation (2), we see that $f'(x)$ is positive for $x^2 > 1$ and negative for $x \in (-1,1)$. So, $f(x)$ is strictly increasing in $(-\infty,-1)$, strictly decreasing in $(-1,1)$ and then strictly increasing again in $(1,\infty)$. So there is a local minimum (in fact, a strict local minimum) at $x=1$. Therefore (A) is true.

This data also tells us that $f(x)$ has a local maximum at $x=-1$. Although that is not needed as far as the present question is concerned, as commented in the answer to the last question, that question could have been answered more elegantly using the knowledge of the sign of the second derivative at points of local maximum and local minimum of $f(x)$. That is why, it would have been better if the present question had come before the last one. Candidates normally attempt the questions in a serial order. This is especially true for questions that appear in the same bunch. As the order of the questions now stands, a candidate who did the last problem by actually calculating $f''(1)$ and $f''(-1)$ may realise while doing the present problem that all that donkey work was really not necessary. But this realisation is now too late. So he may get a feeling that the paper-setters have cheated him.

Q.16 Let $g(x) = \displaystyle\int_0^{e^x} \frac{f'(t)}{1+t^2}dt$.
Which of the following is true?

(A) $g'(x)$ is positive on $(-\infty,0)$ and negative on $(0,\infty)$

(B) $g'(x)$ is negative on $(-\infty,0)$ and positive on $(0,\infty)$

(C) $g'(x)$ changes sign on both $(-\infty,0)$ and $(0,\infty)$

(D) $g'(x)$ does not change sign on $(-\infty,\infty)$

Answer and Comments: (B). The function $g(x)$ is an example of a function defined by an integral. The derivatives of such functions are obtained using the second form of the Fundamental Theorem of Calculus. In the present problem, first put $u = e^x$ and let $h(u)$ be the function

$$h(u) = \int_0^u \frac{f'(t)}{1+t^2}dt \qquad (6)$$

So, by the Fundamental Theorem of Calculus,

$$h'(u) = \frac{f'(u)}{1+u^2} \tag{7}$$

Therefore, by the chain rule,

$$\begin{aligned} g'(x) &= h'(e^x)\frac{du}{dx} \\ &= \frac{f'(e^x)}{1+e^{2x}}e^x \end{aligned} \tag{8}$$

Since the factor e^x and the denominator $1+e^{2x}$ are positive for all x, the sign of $g'(x)$ is the same as the sign of $f'(e^x)$. Note that as x varies from $-\infty$ to 0, e^x varies from 0 to 1 and as x varies from 0 to ∞, e^x varies from 1 to ∞. We already know from the last problem that f' changes sign at 1 from negative to positive. Therefore (B) is the correct answer.

The problem is a good combination of the second form of the Fundamental Theorem of Calculus and the behaviour of the exponential function. Those who apply the Fundamental Theorem hastily are likely to miss the factor e^x appearing in (8). Luckily, as this factor turns out to be positive everywhere, its omission does not affect the rest of the work. As a result candidates who apply the theorem incorrectly get an undeserved clemency. This could have been avoided by replacing e^x by, say e^{-x}, in the definition of the function $g(x)$ and then changing the alternatives suitably.

The present bunch of problems is a total mockery of the concept of a 'paragraph'. The so-called 'paragraph' for these three questions consists of just one line, which merely gives the formula for the function $f(x)$. Many other single problems have far longer 'paragraphs' which have to be correctly understood before even attempting the problem. In the past when heights and distances were a part of the JEE mathematics syllabus, a correct visualisation of the data was a far better test of a candidate's comprehension, than any of the so-called comprehension type questions in the present JEE.

Paragraph for Question Nos. 17 to 19

Consider the lines

$$L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2} L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$$

Q.17 The unit vector perpendicular to both L_1 and L_2 is

(A) $\dfrac{-\hat{\imath}+7\hat{\jmath}+7\hat{k}}{\sqrt{99}}$ (B) $\dfrac{-\hat{\imath}-7\hat{\jmath}+5\hat{k}}{5\sqrt{3}}$

(C) $\dfrac{-\hat{\imath}+7\hat{\jmath}+5\hat{k}}{5\sqrt{3}}$ (D) $\dfrac{7\hat{\imath}-7\hat{\jmath}-\hat{k}}{\sqrt{99}}$

Answer and Comments: (B). Yet another very straightforward problem. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be vectors parallel to the lines L_1, L_2 respectively. From the given equations of the lines we simply read them off from the denominators of the three expressions to get

$$\begin{aligned} \mathbf{u}_1 &= 3\hat{\imath} + \hat{\jmath} + 2\hat{k} & (1)\\ \text{and} \quad \mathbf{u}_2 &= \hat{\imath} + 2\hat{\jmath} + 3\hat{k} & (2) \end{aligned}$$

The vector $\mathbf{u}_1 \times \mathbf{u}_2$ is perpendicular to both L_1 and L_2. From (1) and (2), we have

$$\begin{aligned} \mathbf{u}_1 \times \mathbf{u}_2 &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} \\ &= -\hat{\imath} - 7\hat{\jmath} + 5\hat{k} & (3) \end{aligned}$$

The length of this vector is $\sqrt{1 + 49 + 25} = \sqrt{75} + 5\sqrt{3}$. Therefore a unit vector perpendicular to both the planes is

$$\pm \frac{\mathbf{u}_1 \times \mathbf{u}_2}{|\mathbf{u}_1 \times \mathbf{u}_2|} = \pm \frac{-\hat{\imath} - 7\hat{\jmath} + 5\hat{k}}{5\sqrt{3}} \qquad (4)$$

Technically, there are two unit vectors that are perpendicular to a plane. So strictly speaking, the wording 'The unit vector' in the statement of the problem is wrong. However, these two vectors are the negatives of each other. So, the mistake in this problem is of the same type as that in Q.7 of the present paper. (Interestingly, both the questions are on vectors.) If we ignore the $\pm$ sign then (B) is the right answer.

Q.18 The shortest distance between L_1 and L_2 is

(A) 0 (B) $\frac{17}{\sqrt{3}}$ (C) $\frac{41}{5\sqrt{3}}$ (D) $\frac{17}{5\sqrt{3}}$

Answer and Comments: (D). We note that the lines L_1 and L_2 are already given in parametric forms. The shortest distance, say d, between them is the minimum distance between two points, say P and Q moving on L_1 and L_2 respectively. Since both P and Q are specified by some real parameters, say r and s, the distance between them will be a function of two variables, s and t and determining the minimum of such a function by calculus is beyond the JEE level. But there is a better way out. Suppose that the shortest distance occurs when P is at P_0 and Q is at Q_0. If the two lines intersect then $P_0 = Q_0$ and the shortest distance is 0. When they do not intersect, it is easy to see that the line P_0Q_0 is perpendicular to L_1, as otherwise there will be some points on L_1 which are closer to Q_0 than P_0 is. Similarly, P_0Q_0 is perpendicular to the line L_2. Put together, the vector $\overrightarrow{P_0Q_0}$ is a vector parallel to the unit vector $\hat{u}$ (say) which is perpendicular to both L_1 and L_2. We already obtained $\hat{u}$ in Equation (4) of the solution to the last problem.

We can now find an easy formula for the shortest distance $d = P_0Q_0$ as follows. Let P and Q be any two points on L_1 and L_2 respectively. Then

$$\begin{aligned} d &= P_0Q_0 \\ &= \hat{u}\cdot \overrightarrow{P_0Q_0} \\ &= \hat{u}\cdot (\overrightarrow{P_0P} + \overrightarrow{PQ} + \overrightarrow{QQ_0}) \\ &= \hat{u}\cdot \overrightarrow{P_0P} + \hat{u}\cdot \overrightarrow{PQ} + \hat{u}\cdot \overrightarrow{QQ_0} \\ &= 0 + \hat{u}\cdot \overrightarrow{PQ} + 0 \qquad (5) \\ &= \hat{u}\cdot \overrightarrow{PQ} \qquad (6) \end{aligned}$$

where in (5) we have used that $\hat{u}$ is perpendicular to both L_1 and L_2 which contain the segments P_0P and Q_0Q respectively.

Although in the derivation above we assumed that L_1 and L_2 do not intersect each other, the formula (6) also holds true even if they do, because in that case L_1 and L_2 are coplanar, $\hat{u}$ is a vector perpendicular to their common plane and PQ is a vector which lies in that plane. So (6) is a handy formula for the shortest distance between two lines if we have already identified a unit vector perpendicular to both of them. Note that in this formula it does not matter which points P and Q we take as long as they lie on the lines L_1 and L_2 respectively.

Returning to the problem, since $\hat{u}$ is already given by Equation (4), all we now need is some point on L_1 and some point on L_2. As the lines are given in parametric forms, we can take $P = (-1, -2, -1)$ and $Q = (2, -2, 3)$. Then $\overrightarrow{PQ} = -3\hat{\imath} - 4\hat{k}$ and

$$\begin{aligned} d &= \frac{-\hat{\imath} - 7\hat{\jmath} + 5\hat{k}}{5\sqrt{3}} \cdot (-3\hat{\imath} - 4\hat{k}) \\ &= -\frac{17}{5\sqrt{3}} \qquad (7) \end{aligned}$$

As the distance is positive we ignore the negative sign and get (D) as the answer. (In effect, we are taking the absolute value of the dot product $\hat{u}\cdot \overrightarrow{PQ}$.)

The formula (6) for the shortest distance between two lines is quite well-known and for those who know it, the problem is an extremely straightforward computation. If instead of giving both the lines in parametric form, at least one had been given as the intersection of two planes, then some work would have been needed to cast it in the parametric form. Although this work is also very standard, at least the candidates would not have got the answers by merely plugging values into well-known formulas. Perhaps, the idea was to enable good students to save some time on these questions and utilise it on some other questions which required considerable work.

Q.19 The distance of the point $(1, 1, 1)$ from the plane passing through the point $(-1, -2, -1)$ and whose normal is perpendicular to both the lines L_1 and L_2 is

(A) $\frac{2}{\sqrt{75}}$ (B) $\frac{7}{\sqrt{75}}$ (C) $\frac{13}{\sqrt{75}}$ (D) $\frac{23}{\sqrt{75}}$

Answer and Comments: (C). Yet another straightforward problem. We already identified a vector, viz. $-i - 7j + 5k$, which is perpendicular to the plane in question. If this plane is to pass through the point $(-1, -2, -1)$, then its equation is given by

$$((x+1)\hat{i} + (y+2)\hat{j} + (z+1)\hat{k}) \cdot (-\hat{i} - 7\hat{j} + 5\hat{k}) = 0 \quad (8)$$

i.e.

$$-x - 7y + 5z = 10 \quad (9)$$

The perpendicular distance of the point $(1, 1, 1)$ from this plane is given by another standard formula and comes out as $\frac{-1-7+5-10}{1+49+25} = -\frac{13}{\sqrt{75}}$. The negative sign indicates that this point lies on the the opposite side of the plane as the normal vector we have chosen. If we had taken an oppositely directed normal vector, we would have gotten $\frac{13}{\sqrt{75}}$. These are the algebraic distances. The problem asks for the geometric distance, which is always non-negative. So, we take the absolute value and get (C) as the answer.

Once again, it is a gross mockery to say that the three questions in this paragraph are a test of the ability to comprehend. There is absolutely nothing to 'comprehend' in any of the three problems. The formulas needed for all the three are very standard. Similar questions are often asked in examinations and have been asked in JEE itself. So these three problems are a godsend for the mediocre candidates who are good at remembering formulas and quick at computations. While these qualities have their own importance, it will be a pity if in the selection process they dominate over the ability to think analytically. Such questions deserve to be asked only in an elimination round but not in the final round. But that is the sad state of affairs after the JEE was made a single round examination since 2006.

SECTION IV

Matrix-Match Type

This section contains 3 questions. In each question, Statements/Expressions in **Column I** are to be matched with the Statements/Expressions in **Column II** and the matchings are to be indicated by ordered pairs of the form (X,y) where X is an entry in Column I and y is an entry in Column II. It is possible that some entries in either column have none or more than one matches in the other column. There are 6 marks if all the correct matchings and nothing else are shown, or 1 mark each for indicating all the correct matchings (and no others) for each entry in Column I. There are no negative marks for any incorrect matches.

Q.20 Consider the lines given by

$$L_1 : x + 3y - 5 = 0$$
$$L_2 : 3x - ky - 1 = 0$$
$$L_3 : 5x + 2y - 12 = 0$$

	Column I		**Column II**
(A)	L_1, L_2, L_3 are concurrent, if	(p)	$k = -9$
(B)	One of L_1, L_2, L_3 is parallel to at least one of the other two, if	(q)	$k = -\dfrac{6}{5}$
(C)	L_1, L_2, L_3 form a triangle, if	(r)	$k = \dfrac{5}{6}$
(D)	L_1, L_2, L_3 do not form a triangle, if	(r)	$k = 5$

Answer and Comments: (A,s), (B,p), (B,q), (C, r), (D,p), (D,q), (D,s).

The concurrency of the three lines can be tested by the determinant criterion, viz.

$$\begin{vmatrix} 1 & 3 & -5 \\ 3 & -k & -1 \\ 5 & 2 & -12 \end{vmatrix} = 0 \qquad (1)$$

After expanding the determinant, this comes out to be $12k + 2 - 3(-36 + 5) - 5(6 + 5k) = 0$, i.e. $-13k + 65 = 0$, which gives $k = 5$. So, this is the only value of k for which L_1, L_2, L_3 will be concurrent. Therefore, Statement (A) in Column I matches only with (p) in Column II.

As for Statement (B), we test the parallelism of the lines taken two at a time. There is a simple criterion for this, viz. that the coefficients of x and y in the equation of one of them be proportional to those in the other but the constant terms not be proportional by the same ratio (as otherwise the two lines will be identical). With this criterion, we see that L_1 and L_3 are not parallel to each other because

$$\frac{5}{1} \neq \frac{2}{3} \tag{2}$$

For L_1 and L_2 to be parallel we must have

$$\frac{3}{1} = \frac{-k}{3} \neq \frac{-1}{-5} \tag{3}$$

which holds for $k = -9$. Similarly, L_2 and L_3 will be parallel if and only if

$$\frac{3}{5} = \frac{-k}{2} \neq \frac{-1}{-12} \tag{4}$$

which is true if and only if $k = -\frac{6}{5}$. So, (B) matches with both (p) and (q).

It is hardly necessary to do any more work for the remaining statements in Column I. Three lines form a triangle if and only if they are not concurrent and no two of them are parallel. So (C) will hold whenever (A) and (B) both fail, i.e. only in case (r) from Column II and in all the remaining cases, viz. (p), (q) and (s), (D) will hold. (It may be noted that given three lines at random in a plane, in most of the cases they do form a triangle, because the conditions under which they do not form a triangle are degenerate and have probability 0 of occurrence, see Exercise (23.18). In view of this it may appear strange that in the present problem, out of the four instances given in Column II, the lines form a triangle only in one case. But one has to keep it in mind that here the sample space is not the set of all possible real values of k, but only of the four chosen values, viz. $-9, -\frac{6}{5}, \frac{5}{6}$ and 5.)

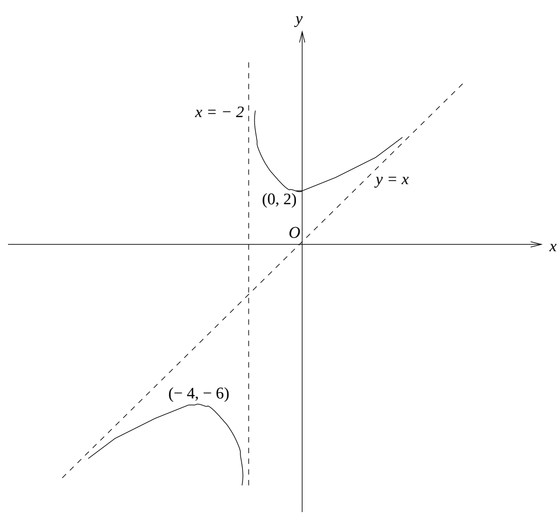
Q.21

	Column I		Column II
(A)	The minimum value of $\frac{x^2+2x+4}{x+2}$ is	(p)	0
(B)	Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew-symmetric and $(A+B)(A-B) = (A-B)(A+B)$. If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then the possible values of k are	(q)	1
(C)	Let $a = \log_3 \log_3 2$. An integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, must be less than	(r)	2
(D)	If $\sin\theta = \cos\phi$, then the possible values of $\frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right)$ are	(s)	3

Answer and Comments: (A,r), (B,q), (B,s), (C,r), (C,s), (D,p), (D,r). In the last question, the four statements in Column I were closely related to each other. In fact, after handling the first two, no further work was needed for the remaining two. In the present question, the situation is exactly the opposite. Here no two statements in Column I are even remotely related, as they come from totally different areas of mathematics. So, this question is like four separate, totally unrelated problems. We tackle them one-by-one.

Let $f(x) = \dfrac{x^2+2x+4}{x+2} = x + \dfrac{4}{x+2}$. So $f'(x) = 1 - \dfrac{4}{(x+2)^2}$ which vanishes precisely when $(x+2) = \pm 2$. So $x = 0$ and $x = -4$ are the only critical points of $f(x)$. Note that the function is not defined at $x = -2$. From the sign of the derivative we see that $f(x)$ is increasing on $(-\infty, -4)$, decreasing on $(-4, -2)$, again decreasing on $(-2, 0)$ and finally increasing on $(0, \infty)$. So, it follows that $f(x)$ has a local maximum at -4 and a local minimum at $x = 0$. (A more mechanical way to see this is to consider the second derivative $f''(x)$ which comes out as $\dfrac{8}{(x+2)^3}$ and note that $f''(0) = 1 > 0$ while $f''(-4) = -1 < 0$.)

By a direct calculation, $f(0) = 2$. We therefore have that the value of the local miminum of $f(x)$ is 2. It is therefore tempting to think that (A) in the Column I matches with (r) in Column II. But strictly speaking this is not correct. The Statement (A) is *not* about the local minimum but the *absolute* minimum of $f(x)$, and in the present problem the (absolute) minimum of $f(x)$ simply does not exist! This can be seen very easily by observing that $f(x) \to -\infty$ as $x \to -\infty$.



A further analysis of the behaviour of the function $f(x)$ can be made as follows. As already noted, it is not defined at $x = -2$. Moreover, as the numerator is always positive, $f(x) \to -\infty$ as $x \to -2^-$ and $f(x) \to \infty$ as $x \to -2^+$. Further $f(x) \to \infty$ as $x \to \infty$. The line $x = -2$ is what is called a **vertical asymptote** of the graph of $y = f(x)$. As $x \to \pm\infty$, the difference between $f(x)$ and y tends to 0. So the line $y = x$ is also an asymptote of the graph. A sketch of the graph is shown above.

Already in Paper 2, there have been two mistakes of omitting the $\pm$ sign from the correct answer, one in Q.7 and another in Q.17. But they were not very confusing. The mistake in the present problem will, however, confuse a discerning candidate. But a candidate who does not distinguish between a local minimum and a minimum and finds the former mechanically will not be disturbed. And *that* is really disturbing. What makes this even more so is that in the same paper, in the statement of Q.15, the phrase 'local minimum' has been used twice. So a discerning candidate is fully justified in thinking that in the present problem, 'minimum' means the absolute minimum, because if the paper-setters meant a local minimum they would have used that phrase as they did in Q.15. (There is, in fact, little point in asking this subquestion after already asking Q.15 as nothing new is involved.)

The problem can also be looked at slightly differently. The minimum value of a function $f(x)$ is the smallest element of its range. When the function is a ratio of two polynomials of degrees at most one each, its range can be determined purely algebraically, as shown in Comment No. 3 of Chapter 6. So, we let $f(x) = \dfrac{x^2 + 2x + 4}{x + 2}$. Then a real number y will be in the range of $f(x)$ if and only if there exists some real x such that

$$y = \frac{x^2 + 2x + 4}{x + 2} \tag{5}$$

or, equivalently, the quadratic equation in x

$$x^2 + (2 - y)x + (4 - 2y) = 0 \tag{6}$$

has at least one real root. From the discriminant criterion, this is so if and only if

$$(2 - y)^2 - 4(4 - 2y) \geq 0 \tag{7}$$

The L. H. S. can be rewritten as a quadratic expression in y to get

$$y^2 + 4y - 12 \geq 0 \tag{8}$$

The roots of the quadratic $y^2 + 4y - 12 = 0$ are -6 and 2. As the leading coefficient is positive, the expression $y^2+4y-12$ is negative for $y \in (-6, 2)$. For all other real values of y, (8) holds and therefore the equation (6) has at least one real root. Thus we see that the range of the given function $f(x)$ is the set $(-\infty, -6] \cup [2, \infty)$ which tallies perfectly with the graph of

$f(x)$. But note again that the range has no smallest element. The range is the union of two mutually disjoint intervals and the number 2 is the smallest element of one of these two intervals, but not that of the entire range set.

We now turn to Statement (B) in Column I. We are given two 3×3 matrices A and B which satisfy the condition

$$(A+B)(A-B) = (A-B)(A+B) \tag{9}$$

As matrix multiplication is not commutative in general, the L.H.S. is not $A^2 - B^2$, but $A^2 - AB + BA - B^2$. Similarly, the R.H.S. is $A^2 - BA + AB - B^2$. Canceling the common terms this gives $AB - BA = BA - AB$, which reduces to $2AB = 2BA$, i.e. to

$$AB = BA \tag{10}$$

Thus we see that (9) is nothing but a clumsy way of saying that the matrices A and B commute with each other. We are also given that

$$(AB)^t = (-1)^k AB \tag{11}$$

But, by a well-known property of transposes, we also have

$$(AB)^t = B^t A^t \tag{12}$$

So far we have not used that A is symmetric and B is skew-symmetric, which mean, respectively, that $A^t = A$ and $B^t = -B$. Putting these into (12) and applying (10) gives

$$(AB)^t = -BA = -AB \tag{13}$$

Combining this with (11) we have

$$(AB)^k = -AB \tag{14}$$

which will hold for all odd positive values of the integer k. Therefore (B) matches with (q) and (s) in Column II.

Statement (C) in Column I requires some standard properties of logarithms. Since we are given that $a = \log_3 \log_3 2$, we have, from very definition of logarithms, $3^a = \log_3 2$ and hence

$$3^{-a} = \frac{1}{\log_3 2} = \log_2 3 \tag{15}$$

where in the last step we have used that $\log_x y \log_y x = 1$ for all positive real numbers x and y. Therefore, we have

$$\begin{aligned} 2^{(-k+e^{-a})} &= 2^{-k} 2^{(3^{-a})} \\ &= 2^{-k} 2^{\log_2 3} \\ &= 2^{-k} 3 \end{aligned} \tag{16}$$

where the last step follows, once again from the very definition of logarithms. Therefore the inequality $1 < 2^{(-k+3^{-a})} < 2$ reduces to

$$\frac{1}{3} < 2^{-k} < \frac{2}{3} \tag{17}$$

or, by taking reciprocals to

$$\frac{3}{2} < 2^{k} < 3 \tag{18}$$

The only integral value of k which satisfies this double inequality is $k = 1$. So (C) matches with both (r) and (s) in Column II.

Finally, for the Statement (D), in Column I we have to solve a trigonometric equation in which the sine of one angle is equated with the cosine of another. There are standard formulas for the general solutions of equations where either the sines or the cosines of two angles are given to be equal. Specifically, they are

$$\sin\beta = \sin\alpha \text{ if and only if } \beta = n\pi + (-1)^n\alpha \tag{19}$$

$$\text{and} \quad \cos\beta = \cos\alpha \text{ if and only if } \beta = 2n\pi \pm \alpha \tag{20}$$

$$\text{for some integer } n$$

In the present problem, the sine of one angle is equated with the cosine of another. So neither of these two formulas can be applied directly. Here we follow a simple trick, also used in the solution to the JEE 1985 problem given in Comment No. 6 of Chapter 10, viz. to convert sines into cosines or the other way. So, we can rewrite the given equation $\sin\theta = \cos\phi$ either as $\sin\theta = \sin(\frac{\pi}{2} - \phi)$ and apply (19) or as $\cos(\frac{\pi}{2} - \theta) = \cos\phi$ and apply (20). Because the expression in the Statement (D) involves $\pm\phi$, the second alternative is preferable. Choosing it, the given equation becomes

$$\cos(\frac{\pi}{2} - \theta) = \cos\phi \tag{21}$$

the general solution of which is

$$\frac{\pi}{2} - \theta = 2n\pi \pm \phi \tag{22}$$

where n is any integer. Multiplying by $-\frac{1}{\pi}$, this becomes

$$\frac{1}{\pi}(\theta \pm \phi - \frac{\pi}{2}) = -2n \tag{23}$$

Thus we see that the L.H.S. is an even integer. The only even integers in Column II are 0 and 2. So (D) matches with (p) as well as (r).

Far more interesting (and challenging) trigonometric equations have appeared in the JEE in the past. But then in those days, they could be asked as full length questions, allowing the candidates as many as 8 to 10 minutes for solving (and also writing the solution). In the new objective type format, even if a candidate does not have to show his work, he still has to do it anyway and for a problem like this, that demands considerable time which is simply not available, because the question involves four statements for 6 marks and proportionately that means an average time of hardly over a minute for each statement. So, only such equations as can be solved very easily and quickly can be asked. This puts a severe constraint on the paper-setters.

Except for the confusing Statement (A), all other statements in Column I are simple. But they all require separate work. The entire question carries 6 marks. So, proportionately, a candidate gets barely 4 minutes, which is inadequate. Another nagging feature is that a candidate who notices that the function $f(x)$ in (A) has no minimum will probably say that (A) does not match with anything in Column II. In that case, even if his matchings for (B), (C), (D) are correct, he will get only 3 out of 6 marks. In other words, apart from the time he spends because of his confusion, he will also lose 3 marks for no fault of his. In a keenly competitive examination, this is very unfair.

Q.22 Consider all possible permutations of the letters of the word ENDEANOEL.

	Column I		**Column II**
(A)	The number of permutations containing the word ENDEA is	(p)	5!
(B)	The number of permutations in which the letter E occurs in the first and the last positions is	(q)	$2 \times 5!$
(C)	The number of permutations in which none of the letters D, L, N occurs in the last five positions is	(r)	$7 \times 5!$
(D)	The number of permutations in which the letters A, E, O occur only in odd positions is	(s)	$21 \times 5!$

Answer and Comments: (A, p), (B, s), (C, q), (D, q). All the entries in Column I ask for a unique numerical answer. So, the possibility of more than one matches for any of them is automatically precluded. For (A), we think of the word ENDEA as a single symbol, say X. Now the problem is equivalent to counting the number of all permutations of the five symbols X, N, O, E, L. As these are all distinct, the answer is simply 5!. In (B), let us first put an E at the first and the last positions, where we have no choice. The remaining 7 positions are to be filled, without any further restriction, by the remaining 7 symbols, viz. N, D, A, N, O, E, L. Here two symbols (viz. the two N's) are identical and the remaining all distinct. So, the number of such arrangements is $\frac{7!}{2!}$ which comes out to be $21 \times 5!$. As for (C), the letters D, L and both N's are allowed to occur only in the first four places. This can be done in $\frac{4!}{2!}$ i.e. in 12 ways. The remaining five symbols, viz. E, E, E, O and A can occupy the last five places without any further restriction, i.e. independently of the way the four places are filled. This can be done in $\frac{5!}{3!} = 20$ ways. So, the correct count in (C) is $12 \times 20 = 240$, which tallies with $2 \times 5!$. Finally, in (D), there are 5 odd numbered positions and the 5 symbols E, E, E, A and O can occur only in these positions. So, these 5 positions are exhausted by these 5 symbols of which 3 are identical and the remaining 2 are different and distinct. So, the five odd numbered positions can be filled in $\frac{5!}{3!}$ i.e. in 20 ways. The remaining four places have to be filled in by the remaining four symbols, viz. N, N, D and L. This can be done in $\frac{4!}{2!} = 12$ ways, independently of how the odd positions are filled. So, the correct count in (D) is 20×12 which again tallies with $2 \times 5!$.

Thus we see that both (C) and (D) have the same answer. This is hardly surprising because in both of them, the restrictions on the permutations are very similar, viz. that the five symbols E, E, E, A, O are to appear in 5 designated places and the remaining four symbols, viz. N, N, D, L are to appear in the remaining four places, independently of each other. The only difference is that while in (C), these five designated places are the last five places, in (D) they are the odd numbered positions. In fact, there is considerable duplication of ideas in the whole problem. The last three parts are based on counting the permutations of a multiset, i.e. a set in which some elements appear more than once. There is a very standard formula for this and anybody who knows it should have no difficulty getting (B), (C) and (D). It is only in (A), that something different figures in, viz. the idea of treating the block ENDEA as a single symbol. But questions based on this idea are also common in JEE.

The question could have been made more interesting by asking for the number of permutations where the restriction is in terms of the relative positions occupied by some of the symbols. For example, the problem could have asked for counting the number of permutations in which the symbols D and L do not appear next to each other (in either order). This can be done by complementary counting. A more challenging problem would be to find the number of those permutations in which no two E's appear adjacently. Here complementary counting is a little clumsy. But if we follow the trick used in the solution to the JEE 1983 problem at the beginning of Comment No. 5 of Chapter 1, then the answer can be obtained as $\frac{6!}{2!} \times \binom{7}{3}$ where the first factor is the number of arrangements of the six letters N, N, A, O, D and L and for each such arrangement, we are placing the 3 E's in 7 possible locations, 5 between two of these 6 letters appearing consecutively and the two positions at the two ends.

Interesting problems about permutations can also be based on the principle of inclusion and exclusion (cf. Exercise (1.43)). For example, to count the number of those arrangements in which no N appears in the first place, D does not appear in the fourth place and L does not appear in the sixth place.

As compared with the last question, the work involved in the present one is much easier as all the four parts deal with the same situation and have unique answers. In essence, the problem is about the permutations of 9 symbols, out of which 3 are of one type, 2 are of another type and the remaining four are of four distincr types, different from the two earlier ones. Mathematically it makes no difference which particular symbols one takes. In the present problem, they are the letters of the word ENDEA-NOEL. The choice of this word does not seem arbitray and calls for a non-mathematical comment because of its unmistakable resemblance with the word INDIANOIL which is the name of a famous public sector oil company in India. If you replace I by E, you get the word in the question.

If the similarity and the modification are intentional, several wild guesses are possible. One is that some paper-setter conceived the problem with the word INDIANOIL in mind but changed it to avoid any inference of possible connection to the company. This is a benign possibility. Another possibility is that an essentially similar problem with the word INDIANOIL appeared in some book or some other source and the change was made to conform to the restriction that the papersetters must not take a problem from any such source. The idea is that the candidates who might have seen the problem earlier should not get an unfair advantage over the others. There can be no two opinions about the inherent fairness of such a stipulation. But it poses some difficulties for the paper-setters, especially when they have to set a large number of questions. Many commercial coaching classes for JEE have huge problem banks and it is really a challenge to come up with a radically new problem. So, sometimes they have to resort to such modifications.

The purpose of matrix matching questions when the four parts to a question are totally independent of each other, as in Q.21, is far from clear. Why not replace the question by two ordinary questions? That would also enable the paper-setters to ask some relatively less trivial problems rather than those that are based on single ideas or formulas.

Perhaps, the matrix matching questions only serve to enable the paper-setters to accommodate some areas which were not represented anywhere else. For example, nowhere else in either of the two papers have matrices, logarithms and trigonometric equations appeared. So, they have been dumped in Q.21.

CONCLUDING REMARKS

The disadvantages of the multiple choice format, especailly in a subject like mathematics, have been discussed in the commentaries on the JEE Mathematics papers of the past two years. There is not much point in giving a consolidated count of the harm they have done in the present year. In commenting on the individual questions, we have already pointed out where sneaky short cuts were possible. Just as the real harm caused by an epidemic can be gauged not so much by considering the condition of the patients who are hospitalised, but rather by considering those who died before they could be hospitalised, the real harm wrought out by the multiple choice format cannot be seen from the questions asked, but from the many interesting questions which simply could not be asked, such as giving proofs, drawing figures, giving constructions, writing precisely and concisely, and, in general, *coming up* with a correct answer rather than by merely identifying it from a given bunch.

In several questions we have also pointed out how the questions could have been made more interesting and challenging even within the framework of a completely objective type testing. Of course, this would demand more time on the part of the candidates. So, it would be fair to give more credit to such questions. Apparently, the paper-setters do not have this freedom. As a result, what we have here is a large number of simple, single idea or single formula based questions, rather than a few really testing and well-chosen ones.

It is to the credit of the paper-setters that even within the limited scope permitted by these strong constraints, they have come up with some good, innovative problems, especailly in Paper 1. A special mention must be made of Q.9 (where Riemann sums are involved), the bunch of questions on a function implicitly defined by a cubic equation (Q.18 to Q.20), Q.4 where the problem asks to identify the type of the curve represented by a given equation without necessitating any computation and finally, Q.7 which is a nice combination of pure geometry and the A.M.-G.M. inequality. In fact, this question is easily the best question in both the papers. Comparatively, the second paper is disappointing. The only innovative question is Q.9 about a function which satisfies the functional equation $f(x+1) = xf(x)$. Q.12 is the only question on coordiante geometry where thought is more important than calculations. As pointed out in the individual comments, the questions on probability (Q.6) and permutaions (Q.22) could have been made more interesting.

In fact, one of the reasons for the dearth of interesting questions seems to be that the paper-setters have either simply omitted or only given a lip service to many topics where there is an opportunity to ask interesting problems. Shockingly absent are any questions on solution of triangles, the binomial theorem or binomial identities and conditional probability. Similarly, there is not a single question on surds, or more generally, on rational and irrational numbers.

Coming to areas where only a lip service is paid, the only question on differential equations, viz. Q.13 in Paper 2 is on differential equations only for the name's sake. The given differential equation can be easily cast in the separate variables form and so the real problem is that of finding antiderivatives, and later on, inverse trigonometric functions. Similarly, number theory has been barely touched upon in Q. 6 of Paper 2 and that too only as an incidental part of a question on probability. Q.1 in Paper 2 is the only question which is genuinely on complex numbers. The three questions (Q.21 to 23) in Paper 1, are only superficially about complex numbers. The solutions require that the various conditions be expressed in terms of the real and imagianry parts. As a result, all the three questions reduce to questions on coordinate geometry. (It may be argued that Q.23 can be done purely with the triangle inequality for complex numbers. But then it leads to an incomplete answer and the mistake in the question becomes unmistakably clear when the problem is attempted in terms of coordinates.)

As commented at the end of Q.21 in Paper 2, matrices, logarithms and trigonometric equations are dumped as subquestions of a matching pairs question, each thereby getting only 1.5 marks. The matrices especially deserve more representation. (They are also a realtively recent addition to the JEE syllabus and hence provide ample opportunity to set 'new' questions.) Also, although pure geometry and inequalities are clubbed together in Q.7 of Paper 1, surely they deserve more representation. There is nothing on trigonometric identities except the very standard identities for the sine and the cosine of the sum of two angles which are needed as an incidental part of the solution to Q.5 of Paper 2, which is basically a problem in coordiante geometry. On the other hand, inverse trigonometric functions have figured in as many as three problems, viz. Q.2 in each paper and Q.13 of Paper 2. Surely, this triplication could have been avoided to make room for other things.

In fact, there are many avoidable instances of such duplications. Q.8 in Paper 1 and Q.3 in Paper 2 are both based on the relativley unimportant concept of a latus rectum of a conic. The topic of even and odd functions, too, deserves at most one place in JEE. But here we have three questions dealing with them, viz. Q.10 and 20 in Paper 1 and Q.2 of Paper 2. Similarly there are three problems asking for the local maxima and minima of given functions, viz. Q.6 in Paper 1 and Q.15 and Q.21(A) of Paper 2. Surely, two of these could have been replaced by an application type problem where a candidate has to first identify (from the data) the function which is to be optimised. Systems of linear equations have appeared four times, thrice in Paper 1 (Q.11 to Q.13) and once in Paper 2 (Q.20).

These repetitions have made the papers heavily dominated by calculus and coordinate geometry. A rough count of marks shows that out of the combined 163 marks of the two papers, calculus takes more than 50 marks and coordinate geometry takes even more. These are the areas where the mediocre students are more comfortable. With nearly two thirds of the marks catering to these areas, mediocre students who can compute fast and without making numerical mistakes must have had an easier time.

It does not appear that the paper-setters took an over-all view of both the papers together. Had they done so, the duplicate and triplicate appearances of the same concepts would have been noticed and after dropping some of them, room could have been made for areas which have received little or no representation.

The greatest disappointment came from the so-called comprehension questions. In all there are five such paragraphs, three in Paper 1 and two in Paper 2. As commented at the end of the individual bunches, except possibly the bunch on implicitly defined questions (Q.18 to Q.20 of Paper 1), none of the bunches tests a candidate's ability to read and understand mathematical text. A hypothetical example of a good comprehension question was given at the end of the comments on Paper 1.

The questions of the reasoning type (Q.11 to 14 in Paper 1 and Q.10 to Q.13 in Paper 2) also call for a collective comment. In each such question, two statements, say p and q are given. When at least one of them is false, there is no problem and the question merely reduces to determining the truth of each. But when both are true, the question further asks whether the statement q is a correct explanation of statement p. This amounts to asking whether the statement p can be logically deduced from q. In the conventional type examinations, it was common to ask a question which first asks for a proof of some statement, say q, and then asks the candidates to deduce some other statement, say p, from q. In the multiple choice format there is no scope for such questions because there is no scope for asking the proof of *anything*. Apparently, the reasoning type questions are made to fill this gap.

But there is a drastic difference. In the old days, candidates were asked to deduce p from q. Now they are no longer asked to do this. Instead they are asked to answer whether *p can be deduced* from q in some way. This opens the flood gates for controversies because the answer can never be conclusively in the negative. In mathematics it is easier to show that something can be done. You simply do it and that is the best proof that it can be done. But it is usually very difficult to prove that something *cannot be* done. There is often a possibility that what appears impossible at one time can be done sometime later on, by a cleverer argument or an improved technique. For example, as shown in the comments on Q.14 in Paper 1, although both the given statments p and q were true, everybody will prove them separately, rather than try to deduce p from q. In fact, the way p was framed there, nobody would even think that it *could be* deduced from q. It is only after recasting p that such a deduction becomes possible.

Now the point is this. Suppose a person cannot deduce the statement p from the statement q. Then he cannot say that it is inherently impossible to do so for anybody at any time. Just because the link from q to p is not apparent to a person at a particular time does not automatically mean that no such link exists. It is possible to discover it later. In the comments on Q.12 in Paper 1, an example was given where the concurrency of the three altitudes of a triangle can be deduced from the concurrency of the perpendicular bisectors of the three sides of a triangle. But this link is not obvious and may even appear non-existenct when we try to deduce the concurrency of the altitudes of a given triangle from the concurrency of the perpendicualr bisectors of the three sides of the *same* triangle.

In advanced mathematics, the link may indeed be extremely difficult to believe. Take, for example, the well-known theorem of algebra that there is no formula for obtaining the roots of a general polynomial of degree 5 (even though there is such a formula for polynomials of lower degrees). In algebra, one also studies what are called solvable groups. It is not hard to prove that the group of permutations of n symbols is solvable for $n \leq 4$ but is not solvable for $n = 5$. The way solvavility of groups is defined, it is far from clear how it is linked to the solvability of a polynomial equation. So, until this link is established, nobody would even think that this result about groups is a correct explanation of the theorem about fifth degree polynomials. The two are so far apart that linking them together may sound as absurd as linking cotton prices with the spots on the sun! The astrologers do tell us that the kind of wife a man will get is governed by the position of Mars at the time of his birth. But this is a matter of unending controversy because although there is no known scientific justification for such a link, it is beyond science to prove that such a link will never be discovered.

To avoid such controversies in examinations, if at all the linked reasoning type questions with two statements p and q are to be asked, then either at least one of them should be false, or else it should be possible to deduce p from q. In other words, the option (B) given in these questions must never hold, because it is this option which is very controversial. But if the candidates know this beforehand, then there is little point in asking the question, because after verifying that both p and q are true, they will automatically tick option (A), even though they may themselves not be in a position to deduce p from q. (In the conventional examinations this difficulty did not arise because the question asked for a proof of such a deduction and not merely its existence.)

So, it is better to drop these questions altogether. If the idea in asking them is to test a candidate's ability to give logical reasoning, there are other ways to do it. Why not simply introduce logic and set theory in the JEE syllabus again? These topics are taught in senior high schools. But most students ignore them since they are not included in the JEE syllabus. As a result, many students who clear the JEE are not able to argue logically or to express themselves in terms of set theory which is the language of mathematics today. Such students therefore face difficulties even though some of them are good otherwise. In the past, pure geometry served to inculcate a certain sense of logical deduction. As coordinate geometry replaces pure geometry, reasoning is often taken over by manipulations of formulas. (Compare, for example, the proofs of the concurrency of the three altitudes of a triangle. A pure geometry proof consists of an ingenious application of properties of cyclic quadrilaterals or, as shown in the comments on Q. 12 of Paper 1, reducing it to the concurrency of the perpendicular bisectors of the sides of another triangle. A coordinate geometry proof consists of routinely writing down the equtions of the three altitudes and showing that they have a common solution.)

The topic of logic also provides an opportunity to devise questions in a real life setting which is often appealing, rather than the drab world of equations and exponential functions. As a sample, the following question could be asked on logic.

Question: Which is the correct logical negation of the statement 'Every man married to a rich woman is happy.'?

(A) Every man married to a rich woman is unhappy.

(B) There is a man who is married to a rich woman and is unhappy.

(C) There is a man who is married to a poor woman and is happy.

(D) Every man married to a poor woman is unhappy.

Similarly, a question such as the one below could be asked to test a candidate's ability to translate a real life data in set theoretic terms.

Question: Suppose that at a dance party there are m boys $b_1, b_2, \ldots, b_m$ and n girls $g_1, g_2, \ldots, g_n$. Let B and G be, respectively, the sets of the boys and the girls at the party. For each $i = 1, 2, \ldots, m$, let G_i be the set of girls who dance with the boy b_i and for every $j = 1, 2, \ldots, n$, let B_j be the set of boys who dance with the girl g_j. Which of the following statements amounts to saying that every boy dances with at least one girl but no girl dances with every boy.

(A) $\displaystyle\bigcup_{j=1}^{n} B_j = B$ and $\displaystyle\bigcap_{i=1}^{m} G_i = \emptyset$

(B) $\displaystyle\bigcup_{j=1}^{n} B_j = B$ or $\displaystyle\bigcap_{i=1}^{m} G_i = \emptyset$

(C) $\displaystyle\bigcap_{j=1}^{n} B_j = B$ and $\displaystyle\bigcup_{i=1}^{m} G_i = G$

(D) $\displaystyle\bigcup_{j=1}^{n} B_j = B$ or $\displaystyle\bigcap_{i=1}^{m} G_i \neq \emptyset$

Questions can also be asked to translate data expressed using sets to verbal sentences. Naturally, such questions could be asked only after the JEE Mathematics syllabus is revised suitably.

We conclude the commentary on JEE 2008 Matheamtics with a very unpleasant task, viz. that of analysing the mistakes in it. Every year, the JEE papers are set and the model answers prepared with great care. So, as compared to many other examinations of a similar type, JEE papers and model answers contain noticeably fewer errors. However, because of the inherent human fallibility, occasionally some mistakes do creep in. (Several examples of such mistakes in the past JEEs are pointed out in Comment No. 10 of Chapter 24.)

But even after making an allowance for such inherent human fallibility, the sad conclusion is that JEE 2008 stands out for the mistakes in it. We have already elaborated the mistakes in the question papers while commenting about the individual questions. Some of them are relatively minor. For example, in Q. 7 and Q. 17 of Paper 2, the vectors given in the answers ought to have been preceded by a $\pm$ sign. In the past, such mistakes, when committed by the candidates were often penalised. In the paragraph for Q.15 to Q.17 in Paper 1, the data should have specified if the vertices P, Q, R appear clockwise or anticlockwise. Because of the ambiguity created by the silence, a candidate's answers to Q.16 and Q.17 may tally with the given answers collectively but not in their order. Although the resulting confusion is not very serious, the ambiguity does reveal a certain lapse.

In some questions, some pieces of data are given which are not at all needed for the solution. For example, in Q.11 in Paper 2, it is given that $\beta^2 \neq -1$ which is nowhere needed in the solution. Similarly, in Q.14 of Paper 1, the data that $g(0) \neq 0$ and $g''(0) \neq 0$ is quite redundant. Unless the very purpose of a question is to test a candidate's ability to pick out the relevant facts from a huge collection of data, such redundancies can confuse a good student by making him believe that if he has not used some part of the hypothesis, then perhaps he has missed some vital hitch and that may prompt him to waste his precious time checking his work again and again.

As already commented, Statement-1 in Q.14 of Paper 1 ought to have been given after interchanging the two sides of the equation. Although mathematically the statement remains the same, such an interchange might have helped a good stuent to arrive at the correct answer. In fact, in that case it would have been a good question. The way it stands at present, it is doubtful how many got the intended interpretation and answered it correctly.

Far more serious are the errors in Q.23 in Paper 1 and Q. 21(A) in Paper 2. From the model answers displayed on the JEE website, the former appears to have been considered while finalising the model answers. But the latter mistake remains uncatered to. As remarked in the comments to that question, the mistake arises because the word 'minimum' is used instead of the phrase 'local minimum'. Since most good books emphasise the difference between these two concepts, a good candidate relying on such books will be confused on seeing that none of the answers is correct. So, he may simply leave a blank in the answer. There may be some unscrupulous books or teachers who do not distinguish between a 'minimum' and a 'local minimum'. If the idea was that candidates in remote areas who have to depend exclusively on such shoddy sources should not be penalised, then perhaps the answer given in the model answers can be justified as a charity on social but non-mathematical grounds. But why should such charity be at the cost of a good candidate? Just as the finalised model answers allowed credit to several options in the case of Q.23 of Paper 1, the same could have been done in the case of Q.21(A) of Paper 2. That is, if a candidate does not match (A) in Column I with anything in Column II, that also ought to have been treated as correct.

The real harm caused by the failure of the model answers to do so is not confined only to the good candidates who answered Q.21(A) correctly. A far greater casualty is mathematics itself, because the model answers amount to giving a sancion to wrong mathematics. Next year the IITs will have no moral right to penalise a candidate who confuses ‘minimum’ with ‘local minimum’.

An even more serious mistake in model answers is in Q.7 of Paper 1. As discussed in the comments in the solution to that question, (D) is the only correct answer. But the model answers gives both (B) and (D) as correct. Since no reasoning is displayed in support of the model answers, one can only guess what might have led to the mistake. One possibility is that the problem was not designed by the paper-setters, but was borrowed from some source and its subtlety eluded everybody, including the paper-setters and those who scrutinised the model answers. Another possibility is that just as the papersetters confused ‘minimum’ with ‘local minimum’ in Q.21(A) of Paper 2, they confused ‘$>$’ with ‘$\geq$’. If so, it will send wrong signals to the candidates of the coming JEEs. If they make a similar mistake, the IITs will have no moral right to penalise them.

The question papers of any examination are a jealously guarded secret and so the only persons who can rectify mistakes in a question paper are the paper-setters themselves. But as far as model answers are concerned, there is hardly any reason to hide them from the eyes of those who may be victimised because of mistakes in them. It will be highly desirable if the model answers prepared by the paper-setters and scrutinised by JEE experts are made open to public scrutiny before freezing. Several examinations of the JEE type, conducted by the various states in India are already following this transparent policy, which is to the benefit of everybody. There are reports that from 2009, the JEE will also do so. If so, that will be a reform long overdue and in that case one can say that the mistakes in the model answers in 2008 have served some purpose for the future. It is also important that the IITs admit this year’s mistakes in the model answers lest they are guilty of perpetuating wrong mathematics.