EDUCATIVE COMMENTARY ON JEE 2009 MATHEMATICS PAPERS

Contents

The pattern of JEE 2009 closely resembles that of 2007 and 2008. So, this year too there were two papers, each covering all the three subjects, viz. Mathematics, Physics and Chemistry. Most of the questions are of the multiple choice type except Section IV of Paper 2 where each question has an integer as an answer and the candidate has to indicate it by bubbles. However, even on these questions the candidate does not have to show his work. The number of questions has gone down a little so that now Paper 1 has 20 questions for each of Chemistry, Mathematics and Physics while Paper 2 has 19. But, once again, because Paper 1 contains two matrix matching questions each with four mutually unrelated subquestions, the effective number of questions in Paper 1 is 26 in each of the three subjects.

One welcome departure is that questions of the assertion and reasoning type have been dropped. Apparently, this is in an answer to the criticism that such questions can be controversial as shown in the commentary last year. Another welcome change is that in the matrix matching question, each part given credit independently of the others. As a result, a candidate does not lose the fruit of his good work because of some bad, unrelated work.

Last year's commentary was written after the Model Answers were displayed on the official websites of the IITs. A mistake in them was pointed out. But by that time the JEE results were already declared and even admissions had been made. So it was too late to take any corrective action. A suggestion was made that to avoid recurrences of such incidents in future, the IITs should display the model answers to the public before they are frozen. There were newspaper reports that the suggestion was accepted and would be implemented from 2009. But this has not been done for 2009. One can only hope that the model answers this year would be free of mistakes.

The questions in Paper 1 are numbered from 21 to 40 while those in Paper 2 are numbered 20 to 38. As in the case of the educative commentaries on the JEE papers of the last few years, the references given here refer to the author's *Educative JEE Mathematics*, unless otherwise stated.

QUESTIONWISE PAGINATION

Paper 1

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PAPER 1

Contents

SECTION I

Single Correct Choice Type

This section contains 8 multiple choice questions. Each question has 4 choices out of which **ONLY ONE** is correct. There are 3 marks for a correct answer, 0 mark if the question is not answered and -1 mark in all other cases.

Q.21 Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum\limits_{m=1}^{15} \mathrm{Im}(z^{2m-1})$ at $\theta = 2^\circ$ is

(A) $\dfrac{1}{\sin 2^\circ}$ (B) $\dfrac{1}{3\sin 2^\circ}$ (C) $\dfrac{1}{2\sin 2^\circ}$ (D) $\dfrac{1}{4\sin 2^\circ}$

Answer and Comments: (D). Call the given sum as T. Since the imaginary part of a sum of complex numbers equals the sum of the imaginary parts of the summands, T equals $\mathrm{Im}(\sum\limits_{m=1}^{15} z^{2m-1})$. This sum is the sum of a G.P. with the first term z and common ratio z^2. So, if we call the sum as S, we have

$$S = \frac{z(1-(z^2)^{15})}{1-z^2} = \frac{z - z^{31}}{1-z^2} \qquad (1)$$

Our interest is in finding the imaginary part of this complex number. So we multiply and divide it by the complex conjugate of the denominator and then simplify using DeMoivre's rule, keeping in mind that since $|z| = 1$, $\overline{z}$ also equals z^{-1}. We then get

$$\begin{aligned} S &= \frac{(z - z^{31})(1 - z^{-2})}{1 - z^2 - \overline{z}^2 + |z^2|^2} \\ &= \frac{z - z^{31} - z^{-1} + z^{29}}{2 - 2\mathrm{Re}(z^2)} \end{aligned}$$

$$= \frac{\cos 29\theta - \cos 31\theta + i(2\sin\theta + \sin 29\theta - \sin 31\theta)}{2 - 2\cos 2\theta} \quad (2)$$

Taking only the imaginary part,

$$\begin{aligned} T &= \frac{\sin\theta - \cos 30\theta \sin\theta}{2\sin^2\theta} \\ &= \frac{1 - \cos 30\theta}{2\sin\theta} \quad (3) \end{aligned}$$

When $\theta = 2^\circ$, the value of T is $\dfrac{1-\frac{1}{2}}{2\sin 2^\circ}$ i.e. $\dfrac{1}{4\sin 2^\circ}$.

In this approach, we expressed the desired sum T as the imaginary part of another sum, S, of complex numbers in a G.P. There is also a way to evaluate T directly if one wants. By DeMoivre's rule z^{2m-1} equals $\cos((2m-1)\theta) + i\sin((2m-1)\theta)$ whose imaginary part is $\sin((2m-1)\theta)$. So, we have

$$T = \sum_{m=1}^{15} \sin((2m-1)\theta) \quad (4)$$

which can be evaluated by using a formula for the sum of the sines of angles in an A.P., viz.

$$\sin\alpha + \sin(\alpha+\beta) + \ldots + \sin(\alpha + (k-1)\beta) = \frac{\sin(\alpha + \frac{k-1}{2}\beta)\sin\frac{k\beta}{2}}{\sin\frac{\beta}{2}} \quad (5)$$

which is valid whenever β is not an even multiple of π. Putting $\alpha = \theta$, $\beta = 2\theta$ and $k = 15$ in this formula, we get

$$T = \frac{\sin(\theta + 14\theta)\sin(15\theta)}{\sin\theta} = \frac{\sin^2(15\theta)}{\sin\theta} \quad (6)$$

whose value for $\theta = 2^\circ$ is simply $\dfrac{1}{4\sin 2^\circ}$. This alternate derivation is certainly simpler for those who remember the formula (5), which can be proved by induction (or even otherwise). But the most elegant way to prove (5) is by combining it with its companion formula for the sum of the cosines of angles in an A.P., viz.

$$\cos\alpha + \cos(\alpha+\beta) + \ldots + \cos(\alpha + (k-1)\beta) = \frac{\cos(\alpha + \frac{k-1}{2}\beta)\sin\frac{k\beta}{2}}{\sin\frac{\beta}{2}} \quad (7)$$

As shown in Comment No. 6 of Chapter 7, instead of proving (5) and (7) individually, it is better to prove them together by observing that the two sums are nothing but the imaginary and the real parts of the complex sum $\sum_{r=1}^{k} e^{i(\alpha+(r-1)\beta)}$ which is the sum of a G.P. with first term $e^{i\alpha}$ and common ratio $e^{i\beta}$. So, the two solutions above are not radically different. In effect, the first solution derives the formula on which the second solution is based. That is why the first solution is a little longer than the second. So, this is a problem where memorising a particular formula (which is not used so commonly as many other trigonometric identities) pays in terms of time.

Q.22 The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only, is

(A) 55 (B) 66 (C) 77 (D) 88

Answer and Comments: (C). On the face of it, this appears to be a problem in number theory. But that is an illusion. The actual seven digit numbers that are constructed in the problem have no role here. It is only the number of such integers that matters. So, the problem is purely combinatorial. In essence, we have seven boxes, numbered from 1 to 7 and we are to put 10 identical balls into them so that each box contains only 1, 2 or 3 balls. The problem asks for the number of such placements.

Thus the problem amounts to counting the number of ways to place 10 identical objects into 7 distinct boxes so that no box is empty, and further, no box can contain more than 3 balls. Let us begin every such placement by first putting one ball in each of the seven boxes. We are now left with 3 balls and we are to put them into these seven boxes so that no box contains more than 2 of these (remaining 3) balls. There are only two ways this can happen. Either all three balls go to distinct boxes or two of them go to one box and the third one to some other. In the first case, these three boxes can be chosen in $\binom{7}{3} = 35$ ways. In the second case, there are $7 \times 6 = 42$ ways. So, together, the placements can be done in $35 + 42 = 77$ ways.

The problem can also be looked at as a problem of counting permutations with repetitions. There are in all seven objects, which fall into three kinds, the 1'2, the 2's and the 3's. The sum of the values of the objects is to be 10. So, clearly there can be at most one 3. Therefore there are tow possibilities:

(i) there is one 3, one 2 and remaining five are 1's

(ii) there is no 3, three 2's and four 1's.

In (i), we have a permutation of 7 objects of which five are repeated. There are $\dfrac{7!}{5!} = 42$ such permutations. In (ii) we have a permutation of length 7 with two types of objects, one repeated 3 times and the other 4 times. The number of such permutations is $\dfrac{7!}{3!4!} = 35$. Adding, we get 77 as the answer. Note that the two figures 42 and 35 also appeared in the first solution. This is not a coincidence. In essence the two solutions are the same.

As remarked earlier, this problem is purely combinatorial and not number theoretic. Some number theory could have been introduced into it if it asked not only to find how many numbers of the given description are possible, but also to find the sum of all these 77 numbers. The brute force method is to actually write down all these 77 seven digit numbers and add. But surely, there must be a better way out. We describe one such.

Call the sum of all these 77 numbers as S. In the solution above we classified these numbers into two types (i) and (ii). There are 42 numbers of the first type. Let us denote their sum by A. Similarly, let B denote the sum of the 35 numbers of type (ii). Clearly, then, $S = A + B$. So our problem is reduced to finding A and B. We do this separately.

Note that a typical number of type (i) is of the form $x_1x_2x_3x_4x_5x_6x_7$ where one of the x_i's is 3, one is 2 and the remaining five are 1 each. The value of this number is

$$x_1 \times 10^6 + x_2 \times 10^5 + x_3 \times 10^4 + \ldots + x_6 \times 10 + x_7 \times 1 \qquad (8)$$

We can find the sum A of all 42 numbers of this form, by adding the contributions made by the digits in each place. Consider, for example, the first digit. Out of the 42 numbers of type (i), in 6 of them x_1 equals 3, in 6 of them it equals 2 and in the remaining 30 numbers it equals 1. So, the sum of the contributions of the first digit is

$$6 \times 3 \times 10^6 + 6 \times 2 \times 10^6 + 30 \times 1 \times 10^6 = 60 \times 10^6 \qquad (9)$$

The sum of the contributions made by the second digit x_2 can be calculated in the same manner. The only change is that its place value is 10^5 instead of 10^6. So the sum of the contributions from the second digit will be 60×10^5. The same reasoning applies to the remaining digits. Adding, we get

$$\begin{aligned} A &= 60 \times (10^6 + 10^5 + 10^4 + 10^3 + 10^2 + 10^1 + 10^0) \\ &= 60 \times 1111111 = 66666660 \qquad (10) \end{aligned}$$

The calculation of B is similar. We now have 35 numbers of type (ii). In each of these, 2 appears 3 times and 1 appears 4 times. So, the first digit x_1 will equal 2 in 15 of these numbers while it will equal 1 in the remaining 20 numbers. Therefore the sum of the contributions from x_1 is

$$15 \times 2 \times 10^6 + 20 \times 1 \times 10^6 = 50 \times 10^6 \qquad (11)$$

The same logic applies to the other digits. So, analogously to (10), we get

$$\begin{aligned} B &= 50 \times (10^6 + 10^5 + 10^4 + 10^3 + 10^2 + 10^1 + 1) \\ &= 50 \times 1111111 = 55555550 \qquad (12) \end{aligned}$$

Adding (10) and (12) we get $S = 110 \times 1111111 = 122222210$. This problem is far more interesting than the one asked. But, of course, it can hardly be asked for 3 marks. In the past, questions of comparable difficulty and work could be asked as full length questions. Nowadays, the severe restrictions on the number of marks makes it impossible to ask such really testing problems. Still, the present problem along with this extra stuff could make a nice package for some paragraph.

Q.23 Let $P(3,2,6)$ be a point in space and Q be a point on the line $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(-3\hat{i} + \hat{j} + 5\hat{k})$. Then the value of μ for which $\overrightarrow{PQ}$ is parallel to the plane $x - 4y + 3z = 1$ is

(A) $\frac{1}{4}$ (B) $-\frac{1}{4}$ (C) $\frac{1}{8}$ (D) $-\frac{1}{8}$

Answer and Comments: (A). A straightforward problem about lines and planes. First we calculate the vector $\overrightarrow{PQ}$.

$$\begin{aligned} \overrightarrow{PQ} &= (1-3\mu)\hat{i} + (-1+\mu)\hat{j} + (2+5\mu)\hat{k} - (3\hat{i} + 2\hat{j} + 6\hat{k}) \\ &= (-3\mu - 2)\hat{i} + (\mu - 3)\hat{j} + (5\mu - 4)\hat{k} \qquad (1) \end{aligned}$$

This vector will be parallel to the plane $x - 4y + 3z = 1$ if and only if it is perpendicular to the normal to the plane, i.e. to the vector $\hat{i} - 4\hat{j} + 3\hat{k}$. So, taking dot product, we get $(-3\mu - 2) - 4(\mu - 3) + 3(5\mu - 4) = 0$, i.e. $8\mu - 2 = 0$ or $\mu = \frac{1}{4}$.

Q.24 Let f be a non-negative function defined on the interval $[0,1]$. If $f(0) = 0$ and

$$\int_0^x \sqrt{1 - (f'(t))^2}\, dt = \int_0^x f(t)\, dt$$

for $0 \leq x \leq 1$, then

(A) $f(\frac{1}{2}) < \frac{1}{2}$ and $f(\frac{1}{3}) > \frac{1}{3}$ (B) $f(\frac{1}{2}) > \frac{1}{2}$ and $f(\frac{1}{3}) > \frac{1}{3}$

(C) $f(\frac{1}{2}) < \frac{1}{2}$ and $f(\frac{1}{3}) < \frac{1}{3}$ (D) $f(\frac{1}{2}) > \frac{1}{2}$ and $f(\frac{1}{3}) < \frac{1}{3}$

Answer and Comments: (C). The fact that the given equality involves both f and f' indicates that this is a problem about differential equations. But the differential equation is given in a twisted manner. Instead of saying that two functions are equal, we are given that their integrals are equal (or, more precisely, the functions defined by their integrals are equal.) So, the first thing is to apply the Fundamental Theorem of Calculus (second form) to differentiate both the sides. This gives

$$\sqrt{1 - (f'(x))^2} = f(x) \qquad (1)$$

for $0 \leq x \leq 1$. (Incidentally, this means that the information given in the data, viz. that f is non-negative, was redundant.) Squaring, and writing y for $f(x)$, this becomes a differential equation in y, viz.

$$(y')^2 = 1 - y^2 \tag{2}$$

This is a quadratic in y'. So first we have to solve for y'. In the present case, as there is no linear term in y', we merely take square roots to get

$$y' = \pm\sqrt{1 - y^2} \tag{3}$$

If we write $\dfrac{dy}{dx}$ for y', this can be cast in the separate variables form as

$$\frac{dy}{\sqrt{1 - y^2}} = \pm dx \tag{4}$$

Integrating both the sides,

$$\sin^{-1} y = \pm x + c \tag{5}$$

The initial condition $f(0) = 0$ means $y = 0$ when $x = 0$ and this determines c as 0 regardless of whether we take the $+$ or the $-$ sign. (If the initial condition was the value of f at a point other than 0, the values of c would change depending upon the choice of the sign.) So we have $y = \pm \sin x$ as possible solutions. But as f is non-negative, we must choose the $+$ sign for $x \in [0, 1]$. We now use a well-known inequality

$$\sin x < x \tag{6}$$

for all $x > 0$. It then follows immediately that $f(\frac{1}{2}) < \frac{1}{2}$ and also that $f(\frac{1}{3}) < \frac{1}{3})$.

The problem is a hotch-potch of several unrelated concepts. Basically, the problem is to solve the differential equation (2) subject to an initial condition. This itself is very easy. So, as if to make up for it, some gadgets are added. First, instead of giving the d.e. in a straightforward manner, it is given in a twisted form. And, even after obtaining the solution $y = \sin x$, the problem goes on to test the knowledge of some properties of the sine function which are totally unrelated to the d.e. The solution would be really interesting if the conclusions about the values of $f(x)$ could be drawn from the d.e. *without* actually solving it. In the present problem, this is indeed possible. From (3) we see that regardless of which sign holds there,

$$y' = f'(x) \leq 1 \tag{7}$$

for all $x \in [0, 1]$. Let $g(x) = x$. Then $g'(x) = 1$ for all x. The initial condition gives $f(0) = g(0)$. Now, the inequality (7) means that f does not grow faster than g. So, for all $x \in [0, 1]$, we must have

$$f(x) \leq x \tag{8}$$

This is enough to weed out the possibilities $f(\frac{1}{2}) > \frac{1}{2}$ as well as $f(\frac{1}{3}) > \frac{1}{3}$ and thereby to eliminate (A), (B) and (D) as possible answers. For a rigorous formulation, suppose, say, $f(\frac{1}{2}) > \frac{1}{2}$. Since $f(0) = 0$ we have

$$\frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0} > 1 \tag{9}$$

But by Lagrange's MVT, there exists some $c \in (0, \frac{1}{2})$ such that the L. H. S. of (9) equals $f'(c)$ which would contradict (8).

Actually, with a little more care, one can show that strict inequality holds in (8) for all $x \in (0, 1]$. For, we can write

$$f(x) - x = f(x) - x - (f(0) - 0) = \int_0^x f'(t) - 1 \, dt \tag{10}$$

By (8) the integrand is non-positive for all $x \in [0, 1]$ and hence so is the integral. But as the integrand is continuous, if it is negative at even one point, then it would be so in a neighbourhood of that point and then the integral in (10) would also be negative. So, if $f(x) = x$ for some $x \in (0, 1]$, that would force $f'(t) = 1$ for *all* $t \in [0, x]$. But, in that case, integrating and using $f(0) = 0$ we would get $f(t) = t$ for all $t \in [0, x]$. But that would not satisfy the differential equation (1).

As the candidates do not have to show their reasoning, it is hard to guess whether those who answered this question correctly did so with or without actually solving the d.e. (1). In fact, it is not clear whether the paper-setters had this alternate clever argument in their minds when they designed the question. If they did and their intention was to reward only those candidates who could think of the sophisticated solution, they probably would have given a d.e. which is sufficiently complicated to solve. For example, instead of (1), had they given

$$\sqrt{1 - (f'(x))^4} = f(x) \tag{11}$$

then instead of (2), the d.e. would have become

$$(y')^4 = 1 - y^2 \tag{12}$$

This can still be solved quite easily for y'. But integrating it afterwards is not easy. But even without solving it, (7) would still hold and hence the same conclusion can be reached. In this formulation, the question would have really set the thinkers apart from the mediocre.

Q.25 Let $z = x + iy$ be a complex number where x, y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\overline{z}^3 + \overline{z}z^3 = 350$ is

(A) 48 (B) 32 (C) 40 (D) 80

Answer and Comments: (A). The equation about the complex number z can be written as

$$|z|^2(\overline{z}^2 + z^2) = 350 \qquad (1)$$

This can be converted to an equation in x and y by writing $|z|^2$ as $x^2 + y^2$ and $\overline{z}^2 + z^2$ as twice the real part of z^2, i.e. as $2(x^2 - y^2)$. So, (1) becomes

$$(x^2 + y^2)(x^2 - y^2) = 175 \qquad (2)$$

This equation has infinitely many solutions if we allow x, y to be real numbers. But in the present problem, they are integers and that reduces the number of solutions drastically. (The very setting of the question gives an implicit hint that (2) would have 4 integral solutions, corresponding to the four vertices of a rectangle.) It is not a good idea to write the L.H.S. as $x^4 - y^2$. Our goal is to get hold of two equations in x and y and the only way this can be done is by equating factors of the L.H.S. with those of the R.H.S. So we leave the L.H.S. in the factorised form as it is. Note that $x^2 - y^2$ is the smaller factor. So we factorise the R.H.S. into two factors and equate the smaller one with $x^2 - y^2$ and the larger with $x^2 + y^2$. Now, the prime factorisation of 175 is $7 \times 5 \times 5 \times 1$. So the only way it can be factored into two factors is either as 175×1 or as 35×5 or as 25×7. The first factorisation would force us to equate $x^2 - y^2$ with 1 and $x^2 + y^2$ with 175. But that would mean $x^2 = 88$ which is not a perfect square. Similarly, the second factorisation would give $x^2 = 20$ which is not a perfect square either. So we try the third possibility, viz.

$$x^2 - y^2 = 7 \quad \text{and} \quad x^2 + y^2 = 25 \qquad (3)$$

Solving, $x^2 = 16$ and $y^2 = 9$. Thus the four integral solutions of (2) are $x = \pm 4, y = \pm 3$. The corresponding vertices of the rectangle are $(\pm 4, \pm 3)$. Clearly, the length of the rectangle is 8 while its breadth is 6 which gives the area as 48.

The problem is a good combination of complex numbers and a little elementary number theory. The key idea is that given in Comment No. 18 of Chapter 4. Normally, you need as many equations as the number of variables to determine them. But when the variables can take only integers as possible values, a system of equations where there are fewer equations than the number of variables can also suffice to identify all possible (not necessarily unique, but nevertheless only finitely many) solutions. Because of the time constraint on a question, it is not easy to ask really interesting problems in number theory. All one can do is to touch it as is done nicely in the present problem.

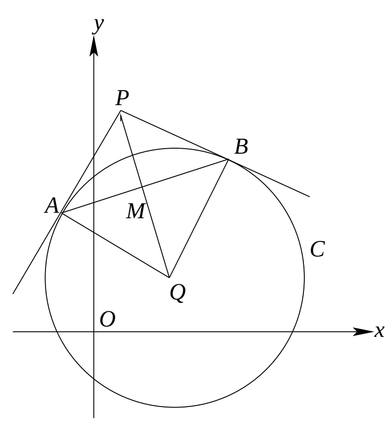
Q.26 Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is

(A) $x^2 + y^2 + 4x - 6y + 19 = 0$ (B) $x^2 + y^2 - 4x - 10y + 19 = 0$

(C) $x^2 + y^2 - 2x + 6y - 29 = 0$ (D) $x^2 + y^2 - 6x - 4y + 19 = 0$

Answer and Comments: (B). The most straightforward method would be to first identify the points A and B and then determine the equation of the circumcircle of the triangle PAB. But this would require too much computation. This itself gives a hint that there must be some short cut. Let us see how we can go about finding it. The particular numerical data such as the coordinates of P provide little clue. Some simplification would have been possible if the circle were centred at the origin. But that is not the case either. But the very fact that it is a circle provides an important clue. The tangents to a circle from any point outside it are equal. (A similar assertion does not hold for tangents to other curves such as ellipses.) So, in the present case, the triangle PAB is isosceles with $PA = PB$. Therefore, if we let M be the midpoint of the segment AB, then the circumcentre (and indeed the other centres such as the orthocentre, the incentre etc.) will lie on the line AM. So, the line AM will be a diameter of the circumcircle. One of the end-points of this diameter is P, which is already given to us. If we could find the other end of the diameter, say Q, then it would be easy to write down the equation of the circumcircle as shown in Comment No. 13 of Chapter 9 or Exercise (9.49)(a) (in the second edition).

So, our problem is reduced to that of identifying Q, the point on the circumcircle diametrically opposite to P. For this we note that the angles $\angle PAQ$ and $\angle PBQ$ must be both right angles. But since PA touches the given circle, say C, at P, the radius of C passing through A is already perpendicular to OA. So, Q must lie on the line containing the radius of C through A. The same holds for the radius of C through B. Therefore, the point Q must be none other than the centre of the circle C. As the equation of C is given to us, we get $Q = (3, 2)$.



Having identified the points P and Q as $(1, 8)$ and $(3, 2)$ respectively, the equation of the circle having PQ as a diameter is easy to write down, based on the fact that the angle in a semicircle is a right angle. It comes out as

$$\frac{y-8}{x-1} \times \frac{y-2}{x-3} = -1 \tag{4}$$

which, after simplification, becomes $(x-1)(x-3)+(y-8)(y-2) = 0$, i.e.

$x^2 + y^2 - 4x - 10y + 19 = 0$. So, this is the equation of the circumcircle of the triangle OAB. For those who are not familiar with this rather tricky way of writing the equation of a circle with a given diameter, a more straightforward approach is to note that the centre of the desired circle is the midpoint of the diameter PQ, viz. $(2, 5)$ while its radius is $\frac{1}{2}PQ$, i.e. $\frac{1}{2}\sqrt{4 + 36} = \sqrt{10}$. Then the equation of the desired circle comes out as $(x - 2)^2 + (y - 5)^2 = 10$ which is the same as before.

Finally, for those who could not even begin with this elegant approach, we give the straightforward solution which begins by identifying the points A and B. Let (x_0, y_0) be a point of contact of a tangent from $P(1, 8)$ drawn to the given circle C whose centre is $(3, 2)$. As the tangent is perpendicular to the radius, we get

$$\frac{y_0 - 8}{x_0 - 1} \times \frac{y_0 - 2}{x_0 - 3} = -1 \tag{5}$$

which is highly analogous to (4) and therefore gives, after simplification,

$$x_0^2 + y_0^2 - 4x_0 - 10y_0 + 19 = 0 \tag{6}$$

Since (x_0, y_0) also lies on the given circle C, we have

$$x_0^2 + y_0^2 - 6x_0 - 4y_0 - 11 = 0 \tag{7}$$

To identify the coordinates of A and B, we must solve (6) and (7) simultaneously. Subtracting (6) from (7) gives $6y_0 = 2x_0 + 30$, or

$$y_0 = \frac{1}{3}x_0 + 5 \tag{8}$$

If we substitute this into (6) we get a quadratic in x_0, viz.

$$x_0^2 + (\frac{1}{3}x_0 + 5)^2 - 4x_0 - 10(\frac{1}{3}x_0 + 5) + 19 = 0 \tag{9}$$

which, upon simplification, becomes

$$\frac{10}{9}x_0^2 - 4x_0 - 6 = 0 \tag{10}$$

The roots of this quadratic will give the x-coordinates of the points A and B. The y-coordinates can then be found from (8). But the roots are irrational and so the expressions for these coordinates would be complicated. Even if one is prepared to persevere, after getting the coordinates of A and B, one still has to find the centre of the circumcircle, say (a, b) by equating its distances from the points P, A and B. This part can be bypassed by writing down the equation of a circle through three points in a determinant form. But after that the 4×4 determinant will have to be expanded. A good student will recognise that either way it is a horrendous task.

But, there is an unexpected shot of luck even in this approach. After getting that the coordinates of the points A and B satisfy (6), if one pauses a little, he will notice that (6) is also satisfied if we put $x_0 = 1$ and $y_0 = 8$. But these are precisely the coordinates of the point P. Put together, this means that the coordinates of all the three points P, A and B satisfy the equation

$$x^2 + y^2 - 4x - 10y + 19 = 0 \tag{11}$$

But this equation represents some circle as there is no xy term and the coefficients of x^2 and y^2 are equal. Since there is only one circle through three (non-collinear) points, it follows that this circle must be the circumcircle of the triangle PAB ! So, sometimes even if you miss the bus, if you are a fast runner you can run after it and catch it.

Of course, a question that arises is why would anyone think of checking whether $x_0 = 1, y_0 = 8$ is a solution of (6)? To this, there is no convincing answer because this is like inquiring about the genesis of a stroke of genius. But in the present problem, there is a non-mathematical answer, viz. its close resemblance with one of the four alternatives given, specifically (B). Whatever be the correct answer, the point $(1, 8)$ has to lie on it. The tragedy is that none of the other three alternatives has this property. So, on this preliminary ground alone, they are eliminated. Surely, the paper-setters did not intend to reward a candidate who gets the correct answer by such an unwarranted short-cut. They should have taken care to see that at least some of the fake alternatives are satisfied by the point $(1, 8)$.

If done honestly, the present problem is an excellent one, because although there is a trick to its (intended) solution, the trick is not something that strikes by an accident. Rather it can be arrived at by a systematic reasoning as shown above. The key idea comes from pure geometry, rather than from coordinate geometry. The replacement of pure geometry by coordinate geometry in the JEE syllabus killed many interesting problems. With the further adoption of short questions, the coordinate geometry problems became even more dull. (Q. 23 above is a sample.) But, sometimes there is still room to ask imaginative problems and the paper-setters deserve to be commended for this problem. But as it happens many times in a multiple choice test, the excellence of the problem is marred by an unwarranted short cut.

Q.27 The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

Answer and Comments: (D). Over the past few years, the JEE papersetters are getting addicted to ask some problems on conics which are meant more to test vocabulary than mathematics, since they freely use archaic expressions such as 'abscissa' or 'latus rectum'. The present problem also belongs to this category.

Let us begin by writing the equation of the ellipse in the standard form as

$$\frac{x^2}{9} + \frac{y^2}{1} = 1 \tag{1}$$

This immediately identifies A as $(3, 0)$ and B as $(0, 1)$. ($(-3, 0)$ and $(-1, 0)$ are also possibilities. But because of the symmetry of an ellipse about both its axes, it does not matter which possibility we take. So, as usual, we select the one which avoids the error-prone negative signs.) The equation of the auxiliary circle, say C, is

$$x^2 + y^2 = 9 \tag{2}$$

The point M is the (other) point of intersection of C with the line AB (the point A already being one of the points of intersection). To find it we first write the equation of AB using the two intercepts form as

$$\frac{x}{3} + \frac{y}{1} = 1 \tag{3}$$

or, $x = 3 - 3y$. Substituting this into (2), we get a quadratic in y, viz.

$$10y^2 - 18y = 0 \tag{4}$$

which has 0 as one root, corresponding to the point A. The other root, $\frac{9}{5}$, corresponds to the point M. From (3), we get M as $(-\frac{12}{5}, \frac{9}{5})$. Now that we know the three points A, M and O, the origin, the area of the triangle can be found by the determinant formula as $\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ -\frac{12}{5} & \frac{9}{5} & 1 \end{vmatrix}$ which comes out as $\frac{27}{10}$. (In fact, a clever candidate who writes this determinant first will realise that its value is independent of the entry in its third row and first column and can save some time by not calculating the x-coordinate of M using (3).)

Q.28 If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are unit vectors such that $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 1$ and $\vec{a} \cdot \vec{c} = \frac{1}{2}$, then

(A) $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar
(B) $\vec{b}, \vec{c}, \vec{d}$ are non-coplanar
(C) $\vec{b}, \vec{d}$ are non-parallel
(D) $\vec{a}, \vec{d}$ are parallel and $\vec{b}, \vec{c}$ are parallel

Answer and Comments: (C). This is one of those problems where a single piece of data implies a lot because it deals with an extreme case. For example, if x, y are real then an equation like $x^2 + y^2 = c$ where c is a positive real number does not tell us the values of x and y. But if $c = 0$, then the fact that the minimum value of a sum of squares of real numbers is 0 and occurs when all those numbers are 0, forces us to conclude that $x = y = 0$. Similarly, as pointed out at the beginning of Comment No. 8 of Chapter 14, merely knowing the value of $\cos A + \cos B + \cos C$ does not determine the angles A, B, C of a triangle uniquely. But if this value is given to be $\frac{3}{2}$, then they must all be $60°$. This happens because $\frac{3}{2}$ is the maximum possible value of the expression $\cos A + \cos B + \cos C$ and it is attained only for an equilateral triangle.

So, let us consider the given relation about the dot product, viz.

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 1 \tag{1}$$

There is an inequality about the dot product (which is, in fact, a generalisation of the Cauchy-Schwarz inequality as shown in Comment No. 4 of Chapter 21), viz.

$$\vec{u} \cdot \vec{v} \ \leq \ ||\vec{u}|| \, ||\vec{v}|| \tag{2}$$

with equality holding if and only if the vectors $\vec{u}, \vec{v}$ are parallel to each other. Similarly, there is an inequality about the length of a cross product in terms of the lengths of the two vectors, viz.

$$||\vec{u} \times \vec{v}|| \ \leq \ ||\vec{u}|| \, ||\vec{v}|| \tag{3}$$

with equality holding if and only if the vectors $\vec{u}, \vec{v}$ are mutually perpendicular.

In the present problem, all the four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are given to be unit vectors. So, by (3)

$$||\vec{a} \times \vec{b}|| \ \leq \ 1 \tag{4}$$

$$\text{and} \quad ||\vec{c} \times \vec{d}|| \ \leq \ 1 \tag{5}$$

Putting these into (2),

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) \ \leq \ ||\vec{a} \times \vec{b}|| \, ||\vec{c} \times \vec{d}|| \ \leq \ 1 \tag{6}$$

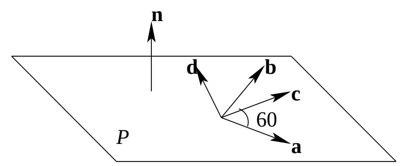
But we are given that the first and the last terms are equal. This means equality must hold in (4) and (5) as well. So, we conclude that: (i) $\vec{a}, \vec{b}$ are mutually perpendicular, (ii) $\vec{c}, \vec{d}$ are mutually perpendicular, and finally, (iii) $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$ are parallel to each other. As both of these are also unit vectors, they must be equal. Let us call this common vector $\vec{n}$. Let P be a plane perpendicular to $\vec{n}$. Then we get that $\vec{a}, \vec{b}$ is a pair of mutually perpendicular vectors in the plane P and also that $\vec{c}, \vec{d}$ is another pair of mutually perpendicular vectors in P. Thus we see that all the four vectors are coplanar. This rules out (A) and (B) as possible choices.

So far we have not used the second piece of data, viz.

$$\vec{a} \cdot \vec{c} = \frac{1}{2} \tag{7}$$

As both $\vec{a}, \vec{c}$ are unit vectors, this means that the angle between them is $60°$. Since $\vec{b}, \vec{d}$ are perpendicular to $\vec{a}$ and $\vec{c}$ respectively and all are in the same plane, the angle between $\vec{b}$ and $\vec{d}$ is also $60°$. So, they are not parallel to each other. Hence (C) is the correct alternative.

As only one of the alternatives is given to be correct, we need not now show that (D) is false. Still, from an educative point of view, we note that as the vector $\vec{a}$ makes an angle of $60°$ with the vector $\vec{c}$, it must make an angle of 60 ± 90 degrees with the vector $\vec{d}$ which is perpendicular to $\vec{c}$. So, $\vec{a}, \vec{d}$ can never be parallel. Similarly the angle between $\vec{b}$ and $\vec{c}$ is either 30 or 150 degrees and hence they cannot be parallel to each other either.



This is a good problem because it involves thinking and very little computation. But it is rather tricky because the key idea can strike you only if you realise that the first piece of data can hold only in an extreme case. Instead of giving the alternative (C) as it is now, the paper-setters could have given it as ‘ the angle between $\vec{b}$ and $\vec{d}$ is $60°$’. As the angle between $\vec{a}$ and $\vec{c}$ is also (indirectly) given to be $60°$, this new formulation of (C) would have given a good candidate a hint that he must somehow relate the angle between $\vec{b}$ and $\vec{d}$ to the angle between $\vec{a}$ and $\vec{c}$, and one of the ways to do that is to show that $\vec{b}, \vec{d}$ are perpendicular (or parallel) to the vectors $\vec{a}, \vec{c}$ respectively. Such a mild hint provides help without diluting the problem. (Of course, some of the fake alternatives should also involve the angle between some vectors to preclude an unwarranted short cut.)

SECTION II

Multiple Correct Choice Type

This section contains 4 multiple correct answer(s) type questions. Each question has 4 choices (A), (B), (C) and (D) out of which **ONE OR MORE** is/are correct. There are 4 marks for a completely correct answer (i.e. all the correct and only the correct options are marked), zero mark if no answer and -1 mark in all other cases.

Q.29 Let

$$L = \lim_{x\to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}, \quad a > 0.$$

If L is finite, then

(A) $a = 2$ (B) $a = 1$ (C) $L = \frac{1}{64}$ (D) $L = \frac{1}{32}$

Answer and Comments: (A, C). The given function, say $f(x)$ is a ratio of two functions, $\frac{g(x)}{h(x)}$ where

$$\begin{aligned} g(x) &= a - \sqrt{a^2 - x^2} - \frac{x^2}{4} \quad (1) \\ \text{and} \quad h(x) &= x^4 \quad (2) \end{aligned}$$

Here both the numerator and the denominator tend to 0 as x tends to 0 and so the limit of the ratio depends upon which of them tends to 0 more rapidly. As explained in Comment No. 6 of Chapter 15, the limit will be zero if the numerator tends to 0 more rapidly than the denominator and some finite non-zero number if the two tend to 0 equally rapidly. Put differently, the question asks for which value of a the order of magnitude of $g(x)$ is x^4 or higher. So qualitatively the question is of the same spirit as the JEE 2002 problem considered there which asks to determine the value of n for which $\lim_{x\to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is finite and non-zero.

If $g(x)$ were a polynomial in x then, of course, its order of magnitude is the exponent of its lowest degree term. The present function $g(x)$ is not a polynomial because of the radical term $\sqrt{a^2 - x^2}$ term in it. But we can expand it in powers of x. One way to to this is to use the binomial theorem with a fractional exponent, viz. $\frac{1}{2}$. Thus, we have

$$\begin{aligned} \sqrt{a^2 - x^2} &= (a^2 - x^2)^{1/2} \\ &= a(1 - (\frac{x}{a})^2)^{1/2} \\ &= a\left[1 - \frac{1}{2}\frac{x^2}{a^2} + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!}\frac{x^4}{a^4} - \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{3!}\frac{x^6}{a^6} + \ldots\right] \\ &= a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{3x^6}{24a^5} + \ldots) \quad (3) \end{aligned}$$

If we put this into (1) we get

$$g(x) = \frac{x^2}{2a} - \frac{x^2}{4} + \frac{x^4}{8a^3} + \frac{x^6}{8a^5} + \ldots \quad (4)$$

Here the coefficient of x^2 is $\frac{1}{2a} - \frac{1}{4}$. Unless it is 0, $g(x)$ will be of order 2 and then the limit L would not be finite. So we must have $2a = 4$, i.e. $a = 2$ if L is to be finite. Also, for this value of a, using (4), we have

$$f(x) = \frac{g(x)}{h(x)} = \frac{1}{64} + \frac{1}{256}x^6 + \dots \tag{5}$$

from which we see that $L = \lim_{x \to 0} f(x) = \frac{1}{64}$.

This solution is sophisticated because it is based on the expansion of $\sqrt{a^2 - x^2}$ as an infinite power series in x. Strictly speaking, this is beyond the JEE syllabus. A more legitimate approach would be to apply Taylor's theorem to expand $g(x)$ upto order 4 and then write an error term which will involve the fifth power of x. For this we need to find the first five derivatives of $g(x)$ and evaluate the first four of them at 0. A direct calculation gives

$$g'(x) = -x(a^2 - x^2)^{-1/2} - 2x \tag{6}$$

$$g''(x) = -(a^2 - x^2)^{-1/2} - x^2(a^2 - x^2)^{-3/2} - 2 \tag{7}$$

The calculations of the subsequent derivatives are cumbersome and so this method is not recommended. Essentially, the same difficulty arises if we try to solve the problem by a mechanical application of the L'Hôpital's rule. The given limit is of the form $\frac{0}{0}$. We shall have to go on differentiating both $g(x)$ and $h(x)$ until we come to a stage where the derivative of the denominator is non-zero. As the denominator is x^4, we shall have to differentiate four times, and, as just observed, this is very laborious.

But a simple trick would simplify the work. Note that the problem involves only even powers of x^2. So, if we put x^2 as u, then $f(x)$ simply becomes $\frac{a - \sqrt{a^2 - u} - \frac{u}{4}}{u^2}$. As $x \to 0, u \to 0^+$. So, we have

$$L = \lim_{u \to 0^+} \frac{a - \sqrt{a^2 - u} - \frac{1}{4}u}{u^2} \tag{8}$$

If we now apply L'Hôpital's rule, we need to go only upto the second derivatives of the denominator and the numerator. To begin with,

$$L = \lim_{u \to 0^+} \frac{\frac{1}{2(a^2 - u)^{1/2}} - \frac{1}{4}}{2u} \tag{9}$$

For this limit to exist, the numerator must tend to 0 as $u \to 0^+$. But the limit of the numerator is simply $\frac{1}{2a} - \frac{1}{4}$. So, we must have $a = 2$. Having found this, we continue the calculation of L from (9) onwards and get

$$L = \lim_{u \to 0^+} \frac{\frac{1}{2}(4 - u)^{-1/2} - \frac{1}{4}}{2u}$$

$$
\begin{aligned}
&= \lim_{u\to 0^+} \frac{\frac{1}{4}(4-u)^{-3/2}}{2} \\
&= \frac{1}{8\times 4^{3/2}} = \frac{1}{64} \qquad (10)
\end{aligned}
$$

Instead of using these advanced methods, there is an elementary way to solve the problem based on a purely algebraic simplification, analogous to rationalisation. We begin by rewriting $f(x)$.

$$
\begin{aligned}
f(x) &= \frac{(a-\sqrt{a^2-x^2})-\frac{1}{4}x^2}{x^4} \\
&= \frac{\dfrac{x^2}{a+\sqrt{a^2-x^2}}-\dfrac{1}{4}x^2}{x^4} \\
&= \frac{\dfrac{1}{a+\sqrt{a^2-x^2}}-\frac{1}{4}}{x^2} \qquad (11)
\end{aligned}
$$

If $f(x)$ is to tend to a finite limit as $x\to 0$, then the numerator must tend to 0. This gives $\dfrac{1}{2a}=\dfrac{1}{4}$ and hence $a=2$. Putting this into (11), and with one more rationalisation, we get

$$
\begin{aligned}
f(x) &= \frac{\dfrac{1}{2+\sqrt{4-x^2}}-\dfrac{1}{4}}{x^2} \\
&= \frac{2-\sqrt{4-x^2}}{4x^2(2+\sqrt{4-x^2})} \\
&= \frac{x^2}{4x^2(2+\sqrt{4-x^2})^2} \\
&= \frac{1}{4(2+\sqrt{4-x^2})^2} \qquad (12)
\end{aligned}
$$

It is now very clear that $L=\lim_{x\to 0} f(x) = \dfrac{1}{4(2+\sqrt{4})^2} = \dfrac{1}{64}$.

Finally, we can also use a trigonometric substitution to get rid of the radical and thereby simplify the work. Put $x=a\sin\theta$. Then $\theta\to 0$ as $x\to 0$. So,

$$
\begin{aligned}
L &= \lim_{\theta\to 0}\frac{a(1-\cos\theta)-\frac{1}{4}a^2\sin^2\theta}{a^4\sin^4\theta} \\
&= \lim_{\theta\to 0}\frac{(1-\cos\theta)-\frac{a}{4}\sin^2\theta}{a^3\sin^4\theta} \qquad (13) \\
&= \lim_{\theta\to 0}\frac{\dfrac{\sin^2\theta}{1+\cos\theta}-\dfrac{a\sin^2\theta}{4}}{a^3\sin^4\theta}
\end{aligned}
$$

$$= \lim_{\theta\to 0} \frac{\dfrac{1}{1+\cos\theta} - \dfrac{a}{4}}{a^3 \sin^2\theta} \tag{14}$$

Once again, for this limit to exist, the numerator must tend to 0 as θ tends to 0 and this gives us $a = 2$. Putting this value into (13) we continue the calculation to get

$$\begin{aligned} L &= \lim_{\theta\to 0} \frac{\dfrac{1}{1+\cos\theta} - \dfrac{1}{2}}{8\sin^2\theta} \\ &= \lim_{\theta\to 0} \frac{1-\cos\theta}{16(1+\cos\theta)\sin^2\theta} \\ &= \lim_{\theta\to 0} \frac{\sin^2\theta}{16(1+\cos\theta)^2\sin^2\theta} \\ &= \lim_{\theta\to 0} \frac{1}{16(1+\cos\theta)^2} \\ &= \frac{1}{64} \end{aligned} \tag{15}$$

This trigonometric version is not radically different from the algebraic approach based on rationalisation. But it is a little more comfortable because of the absence of the radical signs. It is a little more natural to think of rewriting $1-\cos\theta$ as $\dfrac{\sin^2\theta}{1+\cos\theta}$ than to rewrite $a - \sqrt{a^2-x^2}$ as $\dfrac{x^2}{a+\sqrt{a^2-x^2}}$.

The trigonometric conversion of the problem also simplifies the sophisticated solution, because the power series for the trigonometric functions are far more well known that the binomial theorem with a fractional exponent. After obtaining (13) as above, we first replace the factor $\sin^4 theta$ in the denominator by θ^4. We can do so because $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$ and so we can multiply (13) by $\dfrac{\sin^4\theta}{\theta^4}$ without affecting the limit. As for the numerator in (13), we expand $1-\cos\theta$ using the well known power series of $\cos\theta$, viz.

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \tag{16}$$

Similarly we rewrite the last term in the numerator in terms of $\cos 2\theta$ and expand it using (16) (with θ replaced by 2θ.

With these manipulations to (13), we get

$$L = \lim_{\theta\to 0} \frac{(1-\cos\theta) - \dfrac{a}{8}(1-\cos 2\theta)}{a^3\theta^4}$$

$$= \lim_{\theta \to 0} \frac{\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots - \frac{a}{8}(2\theta^2 - \frac{16}{4!}\theta^4 + \frac{64}{6!}\theta^6 + \ldots)}{a^3\theta^4} \quad (17)$$

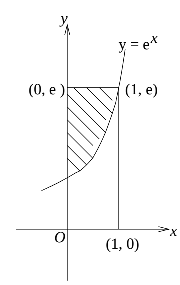
Once again, for this limit to be finite, the coefficient of θ^2 in the numerator must vanish, which gives $a = 2$. After getting the value of a, the limit will be the ratio of the coefficients of θ^4 in the numerator and the denominator, i.e. $L = \frac{\frac{1}{6} - \frac{1}{24}}{8}$ which comes out to be $\frac{1}{64}$.

Problems asking for evaluation of limits are very common in JEE. The present problem is a little different because it involves a parameter (viz. a) and the first task is to determine the value of the parameter for which the limit exists. Once this value is found, the calculation of the limit itself is straightforward but laborious if attempted directly. With suitable substitutions the task is simplified considerably. Even then the total work involved is more than can be expected in the average time allotted to a four point question. It would have been a better idea to make the calculations of a and L as two separate questions in a paragraph.

Q.30 Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is

(A) $e - 1$ (B) $\int_1^e \ln(e + 1 - y)\, dy$ (C) $e - \int_0^1 e^x\, dx$ (D) $\int_1^e \ln y\, dy$

Answer and Comments: (B, C, D). This is a very simple problem once you identify the given region, say R, correctly and this is best done by drawing the graph of the function $y = e^x$, which is a very standard graph. Note that the line $y = e$ cuts this graph at the point $(1, e)$. So the area, say A, of the region R is obtained by subtracting the area under the graph of $y = e^x$ from the area of the rectangle with vertices O, $(1, 0), (1, e)$ and $(0, e)$. This gives



$$A = e - \int_0^1 e^x\, dx = e - (e - 1) = 1 \quad (1)$$

Having found A, we now have to check which of the given alternatives are equal to it. Clearly, (A) is ruled out. The other three alternatives are not given as numbers, but as certain integrals. So we have to evaluate each of these integrals and see which of them equal 1. We do this one-by-one. (C) is the easiest to check and so we begin with it.

$$e - \int_0^1 e^x\, dx = 1 - (e - 1) = 1 \quad (2)$$

which shows that (C) is a correct answer. Note that the actual evaluation of this integral is not needed. If we rewrite it as $\int_0^1 e - e^x \, dx$ we see that it is precisely the area of the region between the line $y = e$ and the graph $y = e^x$, $0 \leq x \leq 1$. But this is exactly the given region R and so the integral equals its area A.

Next in simplicity is (D), which involves integration by parts.

$$\begin{aligned} \int_1^e \ln y \, dy &= (y \ln y)\Big|_1^e - \int_1^e 1 \, dy \\ &= e - 0 - (e - 1) = 1 \end{aligned} \qquad (3)$$

So, (D) is also a correct alternative. Here too, we could have obtained the answer without actually evaluating the integral in (D). If we find the area of the region R by horizontal, rather than vertical slicing, it comes out to be precisely the integral in (D), because, in the region R, y varies from 1 to e and for each fixed $y \in [e, 1]$, x varies from 0 to $\ln y$.

Finally, we tackle (B). This integral can also be evaluated using integration by parts. But that is hardly necessary. By a well-known property of definite integrals (see Equation (38) of Chapter 18), the integrals in (B) and (D) equal each other. So, (D) is also a correct answer.

Apparently, in this question the paper-setters wanted to test the knowledge of certain facts about areas and integrals. The correctness of the answers (C) and (D) comes from the fact that they represent alternate ways to evaluate the area of the same region, while (B) is correct because even without actual evaluation, the integral in (B) equals that in (D) by a certain standard property of the definite integrals. But if this was really the intention of the paper-setters, they have chosen a wrong example. In the present problem all the integrals are so easy to evaluate that their equality can be tested by direct evaluation. Nor is this very time consuming. The problem would have been interesting if some of the integrals were difficult to evaluate, because in that case their equality will have to be established by some reasoning rather than by computation. For example, instead of giving the curve as $y = e^x$, if it were given as $y = e^{x^2}$ (there being no other change), then the desired area would have been $e - \int_0^1 e^{x^2} dx$. This integral cannot be evaluated in a closed form. Still, if (C) and (D) were given as $\int_0^1 e - e^{x^2} \, dx$ and $\int_1^e \sqrt{\ln y} \, dy$ respectively, they would be correct. Alternative (B) could also be changed accordingly to $\int_1^e \sqrt{\ln(e + 1 - y)} \, dy$. As for the fake alternative (A), any number which is obviously not equal to the integral (for example, a number larger than e or a negative number) would have done the job.

Q.31 If $\dfrac{\sin^4 x}{2} + \dfrac{\cos^4 x}{3} = \dfrac{1}{5}$, then

(A) $\tan^2 x = \dfrac{2}{3}$ (B) $\dfrac{\sin^8 x}{8} + \dfrac{\cos^8 x}{27} = \dfrac{1}{125}$

(C) $\tan^2 x = \dfrac{1}{3}$ (D) $\dfrac{\sin^8 x}{8} + \dfrac{\cos^8 x}{27} = \dfrac{1}{125}$

Answer and Comments: (A, B). The expressions in the data as well as all the alternatives can be expressed in terms of $\sin^2 x$ and $\cos^2 x$. But $\cos^2 x$ itself can be expressed in terms of $\sin^2 x$. It is therefore a good idea to make a substitution $u = \sin^2 x$ and work out the problem in terms of u. The given equation then becomes

$$\frac{u^2}{2} + \frac{(1-u)^2}{3} = \frac{1}{5} \qquad (1)$$

which reduces to a quadratic in u, viz.

$$25u^2 - 20u + 4 = 0 \qquad (2)$$

which has $\dfrac{2}{5}$ as a double root. So, $\sin^2 x = \dfrac{2}{5}$ and $\cos^2 x = \dfrac{3}{5}$. Correctness of (A) and incorrectness of (C) follow directly. As for the other two alternatives, we note that

$$\begin{aligned} \frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} &= \frac{1}{8} \times \left(\frac{2}{5}\right)^4 + \frac{1}{27}\left(\frac{3}{5}\right)^4 \\ &= \frac{2}{625} + \frac{3}{625} = \frac{5}{625} = \frac{1}{125} \qquad (3) \end{aligned}$$

Thus (B) is correct and (D) is incorrect.

This is a very straightforward problem about trigonometric equations. The work involved is light. If instead of the quadratic (3), we would have gotten some quadratic which has two distinct real roots then some thought may be necessary to discard one of them (e.g. a negative root or a root bigger than 1). In that case, the fake alternatives could have been designed to correspond to this discarded root and that would have trapped some careless candidates.

Q.32 In a triangle ABC with fixed base BC, the vertex A moves such that

$$\cos B + \cos C = 4 \sin^2 \frac{A}{2}.$$

If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C respectively, then

(A) $b + c = 4a$

(B) $b + c = 2a$

(C) locus of point A is an ellipse

(D) locus of point A is a pair of straight lines

Answer and Comments: (B, C). Yet another routine trigonometric problem, this time about the solution of a triangle. We are given two fixed points B and C at a fixed distance a from each other. The point A is given to move subject to the constraint

$$\cos B + \cos C = 4\sin^2 \frac{A}{2} \tag{1}$$

and from this we are to identify the locus of A. A well-known property of an ellipse is that the sum of the distances of any point on it from its two foci is constant (equal to the length of the major axis of the ellipse). In fact, sometimes this is taken as the definition of an ellipse. It is, therefore, clear that if either (A) or (B) holds then (C) would also be true. Let us therefore first check these two.

One way to do this would be to express $\cos B, \cos C$ and $4\sin^2 \frac{A}{2}$ (which equals $2(1 - \cos A)$ in terms of the sides a, b, c using the cosine formula and substitute into (1). This would give

$$\frac{c^2 + a^2 - b^2}{2ca} + \frac{a^2 + b^2 - c^2}{2ab} = \frac{2bc - b^2 - c^2 + a^2}{bc} \tag{2}$$

Clearing the denominators,

$$bc^2 + ba^2 - b^3 + ca^2 + cb^2 - c^3 = 4abc - 2ab^2 - 2ac^2 + 2a^3 \tag{3}$$

We would like to see if from this we can conclude if either $b + c = 2a$ or $b+c = 4a$. For this we would have to take all terms in (3) on one side and factorise the expression so that one of the factors is $(b+c-2a)$ or $(b+c-4a)$ and then show that the other factor is non-zero. But as the expression is not symmetric in a, b, c its factorisation will not be easy. So, we abandon this approach. Instead we look at the equalities in (A) and (B) a little differently. By the sine rule, the sides of a triangle are proportional to the sines of the opposite angles. As a result, an equality of two expressions involving sides of a triangle can be translated into equalities about the sines of the angles of a triangle. (The solution to Exercise (11.6) is based on this observation. See also the remarks in Comment No. 4 of Chapter 24.)

So, let us see if from (1) we can conclude that $\sin B+\sin C$ equals either $2\sin A$ or $4\sin A$. Of course, we shall also use the fact that $A + B + C =$

$180°$, which, in particular means $\frac{B+C}{2}$ and $\frac{A}{2}$ are complementary angles. As a result, we have

$$\begin{aligned} \cos B + \cos C &= 2\cos(\frac{B+C}{2})\cos(\frac{B-C}{2}) \\ &= 2\sin\frac{A}{2}\cos(\frac{B-C}{2}) \end{aligned} \qquad (4)$$

Combining this with (1) and canceling the factor $2\sin\frac{A}{2}$ we get

$$\cos(\frac{B-C}{2}) = 2\sin\frac{A}{2} \qquad (5)$$

We now work with $\sin B + \sin C$ in a similar manner and use (5) to get

$$\begin{aligned} \sin B + \sin C &= 2\sin(\frac{B+C}{2})\cos(\frac{B-C}{2}) \\ &= 4\cos\frac{A}{2}\sin\frac{A}{2} \\ &= 2\sin A \end{aligned} \qquad (6)$$

As noted before, this is equivalent to saying $b + c = 2a$. As a result, (B) is true and because of the property of an ellipse mentioned above, the locus of A is an ellipse with foci B and C and major axis $2a$.

This problem is a good combination of standard trigonometric identities and a well-known property of ellipses. Once the key idea (viz. the sine rule) strikes, the computation involved is negligible. So this question is quite suitable as a multiple choice question. There is just a minor grammatical error in its statement. The constraint given in the question is about the manner in which the point A moves. So, to describe it we need an adverbial phrase and not an adjective phrase. So the correct wording would have been 'so that' and not 'such that'. If the phrase 'such that' was to be used it should have been an adjective phrase. In that case a wording like, ' The motion of the point A is such that ...' would have been fine.

SECTION III

Comprehension Type

This section contains 2 groups of questions. Each group has 3 multiple choice questions based on a paragraph. Each question has 4 choices out of which **ONLY ONE** is correct. There are 4 marks for a correct answer, 0 marks if the question is not answered and -1 mark in all other cases.

Paragraph for Question No. 33 to 35

Let $\mathcal{A}$ be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.

Q.33 The number of matrices in $\mathcal{A}$ is

(A) 12 (B) 6 (C) 9 (D) 3

Answer and Comments: (A). Let us classify the 9 entries of a 3×3 matrix into three sets, the set, say A, of the diagonal entries, the set, say B, of the entries below the diagonal and the set C of the entries above the diagonal. Each set has 3 entries. For a symmetric matrix, the entries in the sets B and C are replicas of each other. In particular this means that every element occurs an even number of times in $B \cup C$. We are given that 1 occurs 5 times in the matrix. So, it must occur an odd number of times in A. This means that either 1 or all 3 diagonal entries are 1 and the remaining are 0. In the first case, the lone 1 can appear in 3 positions. In the second case, there is only one way to place three 1's on the diagonal.

Let us now see the corresponding placements in the set B. In the first case, B has two 1's and one 0 and these can be placed in 3 different ways. Therefore the number of matrices in which there is only one 1 on the diagonal are $3 \times 3 = 9$. As for matrices with three 1's on the diagonal, the set B has one 1 and two 0's and these can be placed in 3 ways. So, in all the number of matrices is $9 + 3 = 12$.

Q.34 The number of matrices A in $\mathcal{A}$ for which the system of linear equations

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

has a unique solution, is

(A) less than 4 (B) at least 4 but less than 7
(C) at least 7 but less than 10 (D) at least 10

Answer and Comments: (B). The given system of equations is a non-homogeneous system of 3 linear equations in 3 unknowns. The condition for a unique solution is that the coefficient matrix is non-singular, i.e. has a non-zero determinant. So, we have to count how many of the 12 matrices in the set $\mathcal{A}$ are non-singular. We already classified these 12 matrices into two categories, those in which there are three 1's on the diagonal and those in which there is only one 1 on the diagonal. The second category can be further divided into three subcategories depending upon where 1 occurs on the diagonal.

Thus, in all the 12 matrices in $\mathcal{A}$ fall into four types, each class containing 3 matrices. It is cumbersome to write a full 3×3 matrix everytime. As the classification is based on the diagonal entries, for the sake of brevity of notation, we denote the diagonal by an ordered triple. For example a matrix of type $(1, 0, 0)$ means a matrix whose diagonal entries are 1, 0 and 0. So we now have four types of matrices, viz. $(1, 1, 1), (1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ with three matrices of each type. Let us identify the non-singular ones in each type.

A typical matrix A of type $(1, 1, 1)$ looks like $\begin{bmatrix} 1 & \times & \times \\ \times & 1 & \times \\ \times & \times & 1 \end{bmatrix}$. In the blanks below the diagonal, we have to put only one 1 and by symmetry this also decides the position of the 1 above the diagonal. All the remaining four entries are 0. It is clear that no matter where we put 1 below the diagonal, some two columns of A (and hence also some two rows of A) will be identical and so A will be singular. So there are no non-singular matrices in $\mathcal{A}$ of the type $(1, 1, 1)$.

A typical matrix A in $\mathcal{A}$ of type $(1, 0, 0)$ looks like $\begin{bmatrix} 1 & \times & \times \\ \times & 0 & \times \\ \times & \times & 0 \end{bmatrix}$. This time two of the blanks below the diagonal are to be filled with 1's and the third one with a 0. This gives us three possibilities, viz. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. It is clear that out of these three, the first one is singular while the other two are non-singular. Thus $\mathcal{A}$ contains two non-singular matrices of type $(1, 0, 0)$.

There is no need to freshly calculate how many non-singular matrices of type $(0, 1, 0)$ there are in $\mathcal{A}$. Although we normally expand a determinant by its first row, it can as well be done w.r.t. any other row as the roles of all rows are exactly the same. So, without any further work, we find that $\mathcal{A}$ contains two non-singular matrices each of type $(0, 1, 0)$ and $(0, 0, 1)$. Thus in all there are 6 matrices for which the given system of equations has a unique solution. Hence (B) is the correct answer.

Q.35 The number of matrices A in $\mathcal{A}$ for which the system of linear equations

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is inconsistent, is

(A) 0 (B) more than 2 (C) 2 (D) 1

Answer and Comments: (B). The system of equations is the same as in the last question. In the last question we found that 6 of the 12 matrices in $\mathcal{A}$ are non-singular and for these the system has a unique solution. (In fact, this is the case regardless of the entries on the right hand side of the equations.) When A is singular, two extreme things can occur. Either there is no solution (i.e. the system is inconsistent) or there are infinitely many solutions. Which possibility holds depends not only on the matrix A but also on the column vector on the right. The present question asks us to decide for how many of the 6 singular matrices in A the first possibility holds when the column vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Unfortunately, a complete criterion to decide which of the two possibilities holds is fairly complicated (see Exercise (3.20)). As commented in the answer to that exercise, the criterion can be expressed succinctly using the concept of the rank of a matrix. But that is beyond the JEE syllabus. Nevertheless, there is one condition which is *necessary* for the existence of a solution (and hence of infinitely many solutions). Let us call the determinant of A as Δ. Let $\Delta_1, \Delta_2, \Delta_3$ denote, respectively, the determinants of the matrices obtained when the first, the second and the third column vector of A is replaced by the column vector on the right, viz. by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. In other words, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{and} \quad \Delta_1 = \begin{vmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} \quad \text{etc.} \tag{1}$$

Now, when $\Delta = 0$, a *necessary* condition for the system to have infinitely many solutions is that Δ_1, Δ_2 and Δ_3 should all vanish. Taking contrapositive, when at least one of these three is non-zero, the system is inconsistent. As we saw in the solution to the last problem, there are 6 matrices in $\mathcal{A}$ for which Δ vanishes, viz. all three matrices of the type $(1,1,1)$ and one matrix of each of the types $(1,0,0), (0,1,0)$ and $(0,0,1)$. let us see one-by-one for how many of these six matrices, at least one of the three determinants Δ_1, Δ_2 and Δ_3 is non-zero. As there are only six matrices it is best to do this by trial and error.

Let us first consider the three singular matrices of type $(1,1,1)$, say,

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{and} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{2}$$

By direct substitution we see that if the first column of A_1 is replaced by the column vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, then the resulting matrix has a non-zero determinant, viz. 1. So, for the matrix A_1, $\Delta_1 \neq 0$. Therefore the given system of equations is inconsistent if the matrix A in it is A_1. Similarly, in the case of A_2 also, the replacement of the first column by the given column vector gives a non-singular matrix. So, with $A = A_2$ also the given system is inconsistent. For A_3, however, the first column coincides with the given column vector and so no matter which column is replaced, we shall get a singular matrix. This, by itself, *does not* mean that the system $A_3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is consistent, because the criterion we are adopting is only a necessary one. To decide if the system indeed has a solution will require further work. Instead of doing it right away, let us first see if we can identify any one of the remaining three singular matrices in $\mathcal{A}$ for which the system is inconsistent on this preliminary ground. For, if we are successful in identifying even one such matrix, then alternative (B) would be correct and obviate the need for any further work.

The remaining three singular matrices in A are of the type $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively. Specifically, they are

$$A_4 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \quad A_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{3}$$

With A_4 we have no luck, because since the given column vector coincides with two columns of A_4, replacement of any column by it will give a singular matrix. So, let us try A_5. Here we are very lucky, because replacing the very first column by the given column vector gives a non-singular matrix. Now that we have identified at least three matrices in $\mathcal{A}$ (viz. A_1, A_2 and A_5) for which the given system of equations is inconsistent, we know that the alternative (B) is correct (and also that all the other three are incorrect). So, it is not necessary to do any further testing.

As the number of variables in the system of equations is small, we can also check its consistency by directly attempting to solve it. If we denote the (i, j)-th entry of the matrix A by a_{ij} then the given system, written in the conventional form reads

$$a_{11}x + a_{12}y + a_{13}z = 1 \tag{4}$$
$$a_{21}x + a_{22}y + a_{23}z = 0 \tag{5}$$
$$a_{31}x + a_{32}y + a_{33}z = 0 \tag{6}$$

The last two equations are homogeneous and always have infinitely many solutions which are easy to identify. In fact, when the coefficients are only 0 and 1, this can be done almost by inspection. The system will be

> consistent if and only if at least one of these solutions satisfies (4) too. The advantage of this method is not only that it is more direct but also that it gives a necessary as well as sufficient condition, whereas the determinant criterion given above gives only a necessary condition for consistency of the system. As a result, the calculations become simpler and moreover, we can dispose off cases which were left unfinished in the earlier approach. Take, for example, the case where $A = A_5$. We proved the inconsistency above by the determinant criterion. But if we rewrite the system in the conventional form, the equations are simply, (i) $y = 1$, (ii) $x+y+z = 0$ and (iii) $y = 0$. It is instantaneous that (i) and (iii) can never hold together. So the system is inconsistent. On the other hand, take the case $A = A_3$ where the determinant criterion did not give a conclusive answer. Here the first equation is $x = 1$ while the other two both say $y + z = 0$. So the system has infinitely many solutions of the form $x = 1, y = t, z = -t$ where t is a real parameter. In particular, it is consistent.

The paragraph is a combination of combinatorics, matrices and the theory of linear equations. The first question is purely combinatorial. It is not clear what is gained by giving the answers to the second question in a vague form, when it was easy to tell exactly how many matrices in $\mathcal{A}$ were non-singular. In the last question, however, the vagueness of the correct alternative (B) is understandable if the problem is attempted by the determinant criterion as we did. So apparently, the intention of the paper-setters was to test if a candidate knows this criterion. But if we identify all the 6 singular matrices in $\mathcal{A}$ as we have done, then the direct method is much easier and shows that the given system is inconsistent when A equals A_1, A_2, A_5 or A_6. So the exact answer to Q.35 is 4.

But no matter which method is followed, there is considerable duplication of work in both Q.34 and Q.35. For both of them the 12 matrices in $\mathcal{A}$ have to be classified into four types, viz. $(1,1,1), (1,0,0), (0,1,0)$ and $(0,0,1)$ there being 3 matrices of each type. In Q.34, we have to find out, for each of these types, the number of non-singular matrices of that type. We did this for the first two types and then observed that there was no need to duplicate the work for the last two types since they followed the pattern of the second type because of the symmetry of the determinant w.r.t. its rows. But in Q.35, no such short cut was possible because the given column vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is *not* symmetric in all its rows. If it were something like $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then some duplication could be avoided. Even then, replacing the three columns of a given matrix by the given column vector one-by-one and calculating the determinant in each case is itself a very repetitious job. To have to do it at least three times is hardly justifiable.

It is tempting to try to avoid the duplication of work by writing a typical matrix of each type by filling in the blanks (given in the solution to Q.34) with some variables. As the matrix is symmetric, we need only three variables. For example, a typical matrix of the type $(1, 1, 1)$ is of the form

$$A = \begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix} \tag{7}$$

where two of the unknowns a, b, c are 0 and the third one is 1. By a direct calculation, we get

$$\Delta = \det(A) = 1 - a^2 - b^2 - c^2 + 2abc \tag{8}$$

Because of the restrictions on a, b, c, it is clear that $\Delta = 0$ in every case. Hence all the three matrices of type $(1, 1, 1)$ are singular. But this is something that could be done just as easily by writing all these 3 matrices out. Moreover, in Q.35, this approach does not save much work since the given column vector is not symmetric in its rows. So the various types have to be considered separately.

Paragraph for Question No.s 36 to 38

A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required.

Q.36 The probability that $X = 3$ equals

(A) $\frac{25}{216}$ (B) $\frac{25}{36}$ (C) $\frac{5}{36}$ (D) $\frac{125}{216}$

Answer and Comments: (A). There could hardly be a more straightforward problem on probability. To say that $X = 3$ means that the die is tossed thrice and the first two tosses are different from six while the last one is a six. As the die is fair, all six scores are equally likely. So the probability that a six shows on a given toss is $\frac{1}{6}$ while the probability that some other figure shows is the complementary probability $1 - \frac{1}{6} = \frac{5}{6}$. In absence of any information to the contrary, we have to assume that the tosses are independent. Therefore the desired probability is simply the product $\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216}$.

Q.37 The probability that $X \geq 3$ is

(A) $\frac{125}{216}$ (B) $\frac{25}{36}$ (C) $\frac{5}{36}$ (D) $\frac{25}{216}$

Answer and Comments: (B). Now the event is that a six occurs on the n-th toss for the first time, for some $n \geq 3$. This is really a combination of infinitely many, mutually exhaustive events. For every positive integer

n, let E_n be the event that the first six occurs on the n-th toss. The event in the last question was nothing but E_3. Duplicating the reasoning used there, we have

$$P(E_n) = \left(\frac{5}{6}\right)^{n-1} \times \frac{1}{6} \tag{1}$$

As the event X is the disjunction of the events $E_n, n = 3, 4, 5, \ldots$, we have

$$P(X \geq 3) = \sum_{n=3}^{\infty} P(E_n) = \sum_{n=3}^{\infty} \left(\frac{5}{6}\right)^{n-1} \times \frac{1}{6} = \sum_{i=0}^{\infty} \frac{25}{216} \left(\frac{5}{6}\right)^{i} \tag{2}$$

This is an infinite geometric series with first term $\frac{25}{216}$ (which was the answer to the last question), and common ratio $\frac{5}{6}$. As the common ratio is less than 1 in magnitude the series is convergent. Using the well-known formula for the sum of a geometric series (see Equation (5) in Chapter 23), we get

$$P(X \geq 3) \quad = \quad \frac{\frac{25}{216}}{1 - \frac{5}{6}} = \frac{25}{216} \times 6 = \frac{25}{36} \tag{3}$$

In this approach we split the even $X \geq 3$ into an infinite number of events E_n. There is a much easier way to find $P(X)$ by finding its complementary probability, i.e. the probability of the complementary event. It is clear that the complementary event of $X \geq 3$ is $X \leq 2$, which itself is the disjunction of the two events $X = 1$ and $X = 2$. Using the same reasoning once again, the probabilities of these events are $\frac{1}{6}$ and $\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$ respectively. Therefore, we get

$$P(X \geq 3) = 1 - \left(\frac{1}{6} + \frac{5}{36}\right) = 1 - \frac{11}{36} = \frac{25}{36} \tag{4}$$

which is the same answer as before.

An interesting observation is that $\frac{25}{36}$ is also the probability that a six does not appear in the first two rounds. Let us call this event as F_2. Thus $P(F_2)$ is the same as $P(X \geq 3)$. This is *not* a coincidence. Let S be the event that when a fair die is cast in succession an infinite number of times, a six shows at least once. $P(S)$ can be calculated as a geometric series in an analogous manner to get

$$P(S) = \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{n-1} \times \frac{1}{6} = \frac{1}{6} \sum_{i=0}^{\infty} \left(\frac{5}{6}\right)^{i} = \frac{1}{6} \frac{1}{1 - \frac{5}{6}} = 1 \tag{5}$$

In other words, the event S is a sure success (that is why we have denoted it by an S), even though its complementary event, viz. getting a figure different from 6 on *every* round is not impossible. Verbally, if you keep on tossing a fair die then you are sure to get a six sooner or later. This happens because in infinitistic probability, an event which is not impossible may have probability 0. (See Comment No. 2 of Chapter 3 for an explanation of this apparently paradoxical situation.)

If we keep this in mind, we get a very quick solution to the present problem. The event $P(X \geq 3)$ consists of the first tosses not showing a six and thereafter getting a six sooner or later. So, $P(X \geq 3)$ is the same as $P(F_2)P(L)$, which is simply $\frac{25}{36}$. The same reasoning shows that for every positive integer k

$$P(X \geq k) = P(F_{k-1})P(L) = \left(\frac{5}{6}\right)^{k-1} \tag{6}$$

where F_n is the event that six never shows on any of the first n tosses.

Q. 38 The conditional probability that $X \geq 6$ given that $X > 3$ equals

(A) $\frac{125}{216}$ (B) $\frac{25}{216}$ (C) $\frac{5}{36}$ (D) $\frac{25}{36}$

Answer and Comments: (D). This question is also straightforward. For notational brevity, call the events $X \geq 6$ and $X > 3$ as A and B respectively. Then clearly, A is a sub-event of B and so, even without using the law of conditional probability, the desired probability is simply the ratio $\frac{P(A)}{P(B)}$. We already know from (6) that $P(A) = \left(\frac{5}{6}\right)^5$. As for $P(B)$ note that B is the same as the event $X \geq 4$. So again by (6), $P(B) = \left(\frac{5}{6}\right)^3$. Taking ratio, the desired conditional probability is simply $\left(\frac{5}{6}\right)^2 = \frac{25}{36}$. We could, of course, also calculate $P(A)$ and $P(B)$ using infinite series or complementary probabilities. But that would take longer than a solution based on (6) which is instantaneous.

All three questions in this paragraph are simple. The last one is a bit tricky if attempted using (6). But even otherwise, the work is not unreasonable. The last paragraph entailed a lot of repetitious work, demanding more time than justified by the credit allotted. The present paragraph restores the balance.

SECTION IV

Matrix- Match Type

This section contains 2 questions. Each question contains statements given in two columns. Match those in **Column I** with those in **Column II**. The same statement may have more than one correct matches. There are two marks for each statement which is matched correctly. There is no negative marking.

Q.39 Match the statements/expressions in **Column I** with the open intervals in **Column II**.

Column I		**Column II**	
(A)	Interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2y' + y = 0$	(p)	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(B)	Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)\,dx$	(q)	$\left(0, \frac{\pi}{2}\right)$
(C)	Interval in which at least one of the points of local maximum of $\cos^2 x + \sin x$ lies	(r)	$\left(\frac{\pi}{8}, \frac{5\pi}{8}\right)$
(D)	Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(s)	$\left(0, \frac{\pi}{8}\right)$
		(t)	$(-\pi, \pi)$

Answers and Comments: (A) $\to$ (p, q, s), (B) $\to$ (p, t), (C) $\to$ (p, q, r, t), (D) $\to$ (s).

The entries in **Column I** are totally unrelated to each other. So, in effect, what we have here is a bunch of four separate multiple choice questions, each having 5 alternatives, out of which one or more may be correct. The only thing they have in common is the 5 alternatives. But that does not help in their solution.

So, we solve the four problems given in **Column I** one by one. The differential equation in (A) can be written in the separate variables form as

$$\frac{dy}{y} = -\frac{dx}{(x-3)^2} \tag{1}$$

Integrating both the sides we get the general solution as $\ln y = \frac{1}{x-3} + c$, or equivalently

$$y = ke^{1/(x-3)} \tag{2}$$

where k is some constant. No initial condition is given. So we cannot determine k. Instead, we have to consider the entire family of solutions for different values of k. They have a common domain, viz. the set $\mathbb{R}-\{3\}$ which does not depend on the value of k. (For $k = 0$ we get the identically 0 solution which is defined everywhere, but which is to be excluded.) So the problem asks to identify those intervals in **Column II** which *do not* contain the point 3. Taking the approximate value of π as 3.1416, we easily see that $\frac{\pi}{2} < \ 3 \ < \pi$. So, (p), (q) and (s) are the correct answers. on k.

Call the integral in (B) as I. It can be evaluated by actually expanding the integrand as a polynomial of degree 5. But that would be too laborious. Instead we notice that the integrand is symmetric about the point 3 which is also the mid-point of the interval of integration, viz. $[1, 5]$. This suggests some short cuts to evaluate I. We can make a substitution $u = x-3$ which will convert I as

$$\begin{aligned} I &= \int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)\,dx \\ &= \int_{-2}^2 (u-2)(u-1)u(u+1)(u+2)\,du \qquad (3) \end{aligned}$$

Now the integrand is an odd function of u and the interval of integration is symmetric about 0. So the integral I is 0 without actual calculation. Alternately, we can apply the property of definite integrals used in the solution to Q.30. Letting $f(x)$ denote the integrand of I, we get

$$\begin{aligned} I &= \int_1^5 f(x)\,dx \\ &= \int_1^6 f(6-x)\,dx \\ &= \int_1^5 (5-x)(4-x)(3-x)(2-x)(1-x)\,dx \\ &= -\int_1^5 (x-1)(x-2)(x_3)(x-4)(x-5)\,dx = -I \qquad (4) \end{aligned}$$

Thus $I = -I$ and so $I = 0$. We now have to identify which of the intervals in **Column II** contain the point 0. Clearly (p) and (t) are the correct answers. Note that (q) and (s) are incorrect because we are dealing with *open* intervals and so they do not contain their end-points.

Coming to (C) in **Column I**, call $\cos^2 x + \sin x$ as $f(x)$. Then $f(x)$ has derivatives of all orders everywhere. Therefore the points of local maximum can be found using derivatives. But a little more intelligent way is to write $f(x)$ as a quadratic in $\sin x$, viz. as $-\sin^2 x + \sin x + 1$ or, completing the square, as $-(\sin x - \frac{1}{2})^2 + \frac{5}{4}$. Then its local maximum occurs when $\sin x = \frac{1}{2}$ i.e. when $x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$ where k is an integer. All the intervals in **Column II** contain the point $\frac{\pi}{6}$ and so (p), (q), (r), (t) are correct answers while (s) is not because it does not contain any of these points.

Finally, for (D), let us call the function $\tan^{-1}(\sin x + \cos x)$ as $f(x)$. Note that $f(x)$ is the composite of the function $g(x)$ defined by $g(x) = \sin x + \cos x$ and the arctan function. The latter is strictly increasing on the entire real line and so $f(x)$ is increasing over an interval if and only if $g(x)$ is increasing over it. So the problem is reduced to determining the intervals over which $g(x)$ is increasing. So we consider $g'(x) = \cos x - \sin x$ and find out where it is non-negative. (Those who miss this point and fail to reduce the problem to $g'(x)$ do not lose much because $f'(x) = \dfrac{\cos x - \sin x}{(1 + (\sin x + \cos)^2}$ and as the denominator is always positive, the determination of the sign of $f'(x)$ reduces to that of the numerator anyway.)

To find points where $\cos x \geq \sin x$ we first find points where they are equal. These are points where $\tan x = 1$, i.e. all points of the form $n\pi + \frac{\pi}{4}$ where n is an integer. We also see that on intervals of the form $[2k\pi + \frac{\pi}{4}, (2k+1)\pi + \frac{3\pi}{4}]$ (where k is an integer), $\sin x \geq \cos x$ while on the complementary intervals of the form $[(2k-1)\pi + \frac{\pi}{4}, 2k\pi + \frac{\pi}{4}]$, $\cos x \geq \sin x$. We are interested in intervals of the latter type. Among the intervals in **Column II**, the only interval which is contained in an interval of the form $[(2k-1)\pi + \frac{\pi}{4}, 2k\pi + \frac{\pi}{4}]$ is $(0, \frac{\pi}{8})$, (with $k = 0$). All other intervals contain the point $\frac{\pi}{4}$ at which a change of behaviour occurs. So, (s) is the only correct answer.

Q.40 Match the conics in **Column I** with the statements/expressions in **Column II**.

	Column I		**Column II**
(A)	Circle	(p)	The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(B)	Parabola	(q)	Points z in the complex plane satisfying $\lvert z + 2\rvert - \lvert z - 2\rvert = \pm 3$
(C)	Ellipse	(r)	Points of the conic have parametric representation $x = \sqrt{3}\dfrac{1 - t^2}{1 + t^2}, y = \dfrac{2t}{1 + t^2}$
(D)	Hyperbola	(s)	The eccentricity of the conic lies in the interval $1 \leq x < \infty$
		(t)	Points z in the complex plane satisfying $\mathrm{Re}(z + 1)^2 = \lvert z\rvert^2 + 1$

Answer and Comments: (A) → (p), (B) → (s, t), (C) → (r), (D) → (q, s). It would be more logical to interchange the two columns because the natural approach is that you start with a set with a given description and then inquire what type of a conic it represents. Then there is a unique answer.

So, let us take the entries in **Column II** one by one. In (p), those who remember the condition for tangency of a given line to a given conic will have an easier time. The rest of the mortals can equate the distance of the centre of the circle from the given line with the radius of the given circle to get,

$$\left|\frac{1}{\sqrt{h^2+k^2}}\right| = 2 \tag{1}$$

which, upon squaring, becomes $h^2 + k^2 = \frac{1}{4}$. This is a circle. So, (A, p) is a correct match. For (B), the point moves so that the difference of its distances from two fixed points in the plane is a constant. It is well-known that the locus is a hyperbola with foci at these two points. (It is necessary that this constant be less than the distance between those two points, as otherwise, by the triangle inequality, the locus will degenerate to a single point or even be an empty set. In the present problem this condition holds true and in the conventional type of examination, a candidate could be penalised if he missed this subtle point. But in a multiple choice test, there is no way to test this. In fact, scruples cost precious time and hence do not pay.) So (D, q) is a correct match. (The analogous property of ellipses was used in the solution to Q.32. So, there is some duplication of ideas in the paper.) For (r), the identity $(1-t^2)^2 + (2t)^2 = (1+t^2)^2$ allows us to eliminate the parameter t between the two equations to get

$$\frac{x^3}{3} + y^2 = 1 \tag{2}$$

which is an ellipse. So (C, r) is a correct match. For (s), no work is needed except to recall that a conic is an ellipse, parabola or a hyperbola according as its eccentricity is less than 1, equal to 1 or greater than 1. So, both (B) and (D) match with (s).

Finally, for (t), let $z = x + iy$. Then $(z+1)^2 = (x+1+iy)^2$ whose real part is $(x+1)^2 - y^2$. So, in terms of the real coordinates the given equation reduces to

$$x^2 + 2x + 1 - y^2 = x^2 + y^2 + 1 \tag{3}$$

which, upon simplification, becomes simply, $y^2 = x$, which is a parabola. So, (B, t) is a correct match. (Here too, we note that the idea of converting a complex equation into a real one was also central in Q. 25.)

All the parts in both the questions of matrix matching are so easy that there is not much to comment on any of them. In terms of proportional time each question has 6 minutes. It would be far better to ask a single good, thought provoking question for 8 marks than ask these tidbits which really do not test anything, except possibly speed.

PAPER 2

Contents

SECTION I

Single Correct Choice Type

This section contains 4 multiple choice questions. Each question has 4 choices out of which **ONLY ONE** is correct. There are 3 marks for a correct answer, 0 mark if the question is not answered and -1 mark in all other cases.

Q.20 A line with positive direction cosines passes through the point $P(2, -1, 2)$ and makes equal angles with coordinate axes. The line meets the plane

$$2x + y + z = 9$$

at point Q. The length of the line segment equals

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2

Answer and Comments: (C). Another very routine problem in solid coordinate geometry. Call the line as L. Let the direction cosines of L be (l, m, n). Then $l^2 + m^2 + n^2 = 1$. Moreover, as L makes equal angles with the axes, we also have $l = m = n$. Together this gives

$$l = m = n = \frac{1}{\sqrt{3}} \tag{1}$$

The point $P(2, -1, 2)$ is given to lie on L. So the parametric equations of L are

$$x = 2 + \frac{t}{\sqrt{3}},\ y = -1 + \frac{t}{\sqrt{3}}, z = 2 + \frac{t}{\sqrt{3}} \tag{2}$$

When this line meets the plane $2x + y + z = 9$ we have

$$5 + \frac{4t}{\sqrt{3}} = 9 \tag{3}$$

which gives $t = \sqrt{3}$. So, Q, the point of intersection of L with the given plane is $(3, 0, 3)$. Hence, $PQ = \sqrt{1+1+1} = \sqrt{3}$. (Actually, there is some redundancy in finding Q and then calculating its distance from P. Since $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ are not just the direction ratios but the direction cosines of L, the parameter t in (2) (or rather $|t|$) *is* the distance between P and the point given by (2). So, no further work is necessary after we find t. Of course, in a trivial problem like this, such alertness has little value except to save a few seconds.)

Q.21 If the sum of the first n terms of an A. P. is cn^2, then the sum of squares of these n terms is

(A) $\dfrac{n(4n^2-1)c^2}{6}$ (B) $\dfrac{n(4n^2+1)c^2}{3}$

(C) $\dfrac{n(4n^2-1)c^2}{3}$ (D) $\dfrac{n(4n^2+1)c^2}{6}$

Answer and Comments: (C). Following the usual notation, let T_k be the k-th term of the A.P. and T_n be the sum of the first n terms. Then,

$$T_k = a + (k-1)d \tag{1}$$

where a and d are respectively, the first term and the common difference of the progression. The sum of the first n terms is given by

$$\begin{aligned} S_n &= \sum_{k=1}^{n} S_k = \sum_{k=1}^{n}(a + (k-1)d) \\ &= na + d\sum_{k=1}^{n}(k-1) = na + \frac{n(n-1)d}{2} \\ &= \frac{d}{2}n^2 + (a - \frac{d}{2})n \end{aligned} \tag{2}$$

We are given that this equals cn^2 for all n. So the coefficient of n^2 must be c while that of the linear term n must vanish. This gives us two equations in the two unknowns a and d, viz. $\dfrac{d}{2} = c$ and $a = \dfrac{d}{2}$. So, $d = 2c$ and $a = c$. We have now identified the k-th term of the A.P. as

$$T_k = c + 2c(k-1) = 2ck - c \tag{3}$$

There is also a short cut to get this. We note that S_k is obtained by adding T_k to S_{k-1}. As we are given that $S_k = ck^2$ for every k, we get

$$\begin{aligned} T_k &= S_k - S_{k-1} \\ &= ck^2 - c(k-1)^2 = ck^2 - c(k^2 - 2k + 1) = 2ck - c \end{aligned} \tag{4}$$

Now that we have got hold of the general term T_k, the sum of the squares of the first n terms is given by

$$\begin{aligned}\sum_{k=1}^{n} T_k^2 &= \sum_{k=1}^{n} c^2(2k-1)^2 \\ &= 4c^2\sum_{k=1}^{n} k^2 - 4c^2\sum_{k=1}^{n} k + nc^2 \\ &= c^2\left(\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n\right) \\ &= nc^2\left(\frac{8n^2+12n+4-12n-12+6}{6}\right) \\ &= nc^2\frac{(4n^2-1)}{3} \qquad (5)\end{aligned}$$

Note that the number c has no role in the problem. It is a (non-zero) constant. If we divide every term of an A.P. by c, it still remains an A.P. and the sum of the squares of its terms gets divided by c^2. So, we may have assumed, without loss of generality, that $c = 1$. However, this does not simplify the solution substantially in the present problem. But it can inspire a sneaky solution. Suppose we take $c = 1$. Then the data of the problem reads that the sum of the first n terms of the A.P. is n^2. In particular, the first term is 1. So, the sum of the squares of the first n terms is also 1 for $n = 1$. But out of the given four answers, (C) is the only one which reduces to 1 (with $c = 1$). As only one alternative is correct, it has to be (C). (Of course, this sneaky approach is also possible even if we do not assume $c = 1$. It is just that it is a little easier to think of if c is absent.)

We gave two derivations of (3), one a straightforward and the other a 'smart' one. The paper-setters must have expected a good candidate to come up with the smart derivation. What they did not realise is that some candidate could prove too smart for them. To restrain him, at least one of the fake alternatives should have been designed so that it holds true for some lower values of n. Of course, a super-smart candidate may observe that all the alternatives are cubic polynomials in n. A cubic polynomial is uniquely determined by its values at any four (distinct) points. So, if one of the given alternatives holds true for some four values of n, then it must be true for all n. So, without working out the problem he can correctly identify it by just checking its truth for $n = 1, 2, 3, 4$. But this does involve some work (in fact, more work than in a straightforward solution). And, in any case, a candidate who is smart enough to see this deserves to be rewarded. But this hardly applies for a candidate who gets the correct answer merely by checking all alternatives for $n = 1$.

Q.22 The locus of the orthocentre of the triangle formed by the lines

$$(1+p)x - py + p(1+p) = 0$$

$$(1+q)x - qy + q(q+1) = 0$$

and $y = 0$, where $p \neq q$ is

(A) a hyperbola
(B) a parabola
(C) an ellipse
(D) a straight line

Answer and Comments: (D). The fact that we are not asked the actual locus but only the type of the curve it represents suggests that there may be some way to answer the question without actually finding the equation of the locus. (We came across such problems in the last question of Paper 1.) But there does not seem any easy way of identifying the type of the locus in the present problem. So, we take the straightforward approach of first identifying the orthocentre, say P, of the triangle formed by the given lines. There is a formula for this, but it is quite complicated. So we look for some other method. Let L_1 and L_2 be the lines with equations

$$(1+p)x - py + p(1+p) \quad = \quad 0 \qquad (1)$$

$$(1+q)x - qy + q(q+1) \quad = \quad 0 \qquad (2)$$

respectively. These lines cut the x-axis at $A(-p, 0)$ and $B(-q, 0)$ as is easily checked. So, these are two of the vertices of our triangle. If we want, we can solve (1) and (2) together to get the third vertex, say C, of the triangle and then apply the formula for the orthocentre of a triangle whose three vertices are known (see Comment No. 3 of Chapter 8). But this formula, too, is very complicated. Nor is it necessary. To find the orthocentre it is enough to know any two altitudes and we already have the necessary data to write down the equations of the altitudes through A and B. Moreover, there is a bonus. As the data is symmetric between p and q, as soon as we find the equation of any one of these two altitudes, we get the other merely by interchanging p and q.

Let us find the equation of the altitude through $A(-p, 0)$. The side opposite to A lies along L_2 and so has slope $\frac{1+q}{q}$. Hence the slope of the altitude through A is $-\frac{q}{1+q}$ and hence its equation is

$$y = \frac{-q}{q+1}(x+p) \qquad (3)$$

As noted before, the equation of the altitude through B is obtained by interchanging p and q. It comes out as

$$y = \frac{-p}{p+1}(x+q) \qquad (4)$$

The orthocentre P lies on both (3) and (4). Equating the expressions on the R.H.S. and multiplying by $(p+1)(q+1)$ gives

$$x(p-q) = pq(p-q) \tag{5}$$

from which we conclude that $x = pq$ since $p \neq q$. The other coordinate y then comes out as $-pq$. So, regardless of what p and q are, the orthocentre P always lies on the line $x + y = 0$. Therefore its locus is a straight line.

It is, however, a little misleading to call this line the 'locus' of the orthocentre P. As explained in Comment No. 8 of Chapter 9, a locus problem typically involves a 1-parameter family of curves, such as the family of all straight lines passing through a given point or the family of all circles touching a given line at a given point. If $P(h,k)$ is the point whose locus is to be found, then the standard procedure is to express both h and k in terms of a parameter which parametrises the given family of curves (e.g. the family of all lines through a given point can be parametrised by their slopes). We then eliminate this parameter to get an equation in h and k. Replacing h and k by x and y respectively gives the equation of the locus. Of course, sometimes a one parameter family may be represented superficially by two parameters. But these are usually related by some equation and so in effect we have only one parameter. (See the comments on the alternate solution to finding the locus of a point on a moving ladder.)

The present problem is qualitatively different. The point, say $P(h,k)$, whose locus is to be found is the orthocentre of a certain triangle, formed by three lines given to us. One of the lines is fixed, viz. the x-axis. But the other two lines are given in terms of the parameters p and q respectively and there is nothing to indicate that these two parameters are related to each other. It is just by chance that from the two equations $h = pq$ and $k = -pq$, the two parameters p and q could be eliminated. Normally, this does not happen because you need three equations to eliminate two parameters. Instead of the orthocentre, suppose we let $Q(h,k)$ be the centroid of the triangle formed by the three given lines. We already identified two vertices $A(-p,0)$ and $B(-q,0)$ of this triangle. The third vertex C which we did not need comes out as $(pq, pq+p+q+1)$ by solving (1) and (2) together. So, we would get

$$h = \frac{pq-p-q}{3} \quad k = \frac{pq+p+q+1}{3} \tag{6}$$

It is impossible to eliminate p and q from these two equations. In fact, these two equations can be solved to express p and q in terms of h and k. Indeed these two equations give

$$p+q = \frac{3k-3h-1}{2}, \quad \text{and} \quad pq = \frac{3h+3k-1}{2} \tag{7}$$

from which we see that p and q are the roots of the quadratic

$$2x^2 + (3h-3k+1)x + (3h+3k-1) = 0 \tag{8}$$

This quadratic will have real and distinct roots whenever $(3h-3k+1)^2 > 8(3h+3k-1)$, which reduces to

$$3h^2 + 3k^2 - 6hk - 6h - 10k + 3 > 0 \tag{9}$$

Let R be the region in the plane defined by

$$R = \{(x,y) : 3x^2 + 3y^2 - 6xy - 6x - 10y + 3 > 0\} \tag{10}$$

Then R is the region lying on one side of a parabola. For suitable values of p and q the centroid $Q(h,k)$ can be made to lie anywhere in this region. So the locus of Q is a plane region and a region is usually not called a 'locus', except in a very generalised sense.

Q.23 The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the mid-point of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points

(A) $\left(\pm\dfrac{3\sqrt{5}}{2}, \pm\dfrac{2}{7}\right)$ (B) $\left(\pm\dfrac{3\sqrt{5}}{2}, \pm\dfrac{\sqrt{19}}{4}\right)$

(C) $\left(\pm 2\sqrt{3}, \pm\dfrac{1}{7}\right)$ (D) $\left(\pm 2\sqrt{3}, \pm\dfrac{4\sqrt{3}}{7}\right)$

Answer and Comments: (C). Another straightforward but laborious problem about conics. First we have to find the locus of the point M. As usual, let $M = (h,k)$. We begin by writing the ellipse in the standard form as

$$\frac{x^2}{16} + \frac{y^2}{4} = 1 \tag{1}$$

It is convenient to represent the point P in a parametric form as $(4\cos\theta, 2\sin\theta)$ for some $\theta \in [0, 2\pi]$. Then the slope of the normal at P is $2\tan\theta$ and so the equation of the normal at P is

$$y - 2\sin\theta = 2\tan\theta(x - 4\cos\theta) \tag{2}$$

This meets the x-axis at $Q(3\cos\theta, 0)$. Since M is the midpoint of PQ, we have

$$2h = 7\cos\theta \tag{3}$$

$$\text{and} \quad 2k = 2\sin\theta \tag{4}$$

Eliminating θ we get $\dfrac{4h^2}{49} + k^2 = 1$. So the locus of M is

$$\frac{4x^2}{49} + y^2 = 1 \tag{5}$$

which is also an ellipse. We now have to find the points of intersection of this ellipse with the latus recta of the original ellipse. There are two latus recta and each one of them meets (5) in two points. The foci of the original ellipse are $(\pm 4e, 0)$, where e, the eccentricity, comes out as $\sqrt{1 - \frac{4}{16}} = \frac{\sqrt{3}}{2}$. Hence the equations of the latus recta are $x = \pm 2\sqrt{3}$. Putting these values in (5) and solving for y we get $y = \pm\frac{1}{7}$. So the four points of intersection are $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$.

Incidentally, the phrase 'latus rectums' in the statement of the question is grammatically wrong. The word 'rectum' is a Greek word and its correct plural, even when used in English, is 'recta' much the same way as the plural of 'medium' is media or the plural of 'maximum' is 'maxima'. If the paper-setters are fond of using archaic words, they better use them correctly!

SECTION II

Multiple Correct Choice Type

This section contains 5 multiple correct answer(s) type questions. Each question has 4 choices (A), (B), (C) and (D) out of which **ONE OR MORE** is/are correct. There are 4 marks for a completely correct answer (i.e. all the correct and only the correct options are marked), zero mark if no answer and -1 mark in all other cases.

Q.24 An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is the reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then

(A) Equation of ellipse is $x^2 + 2y^2 = 2$

(B) The foci of the ellipse are $(\pm 1, 0)$

(C) Equation of ellipse is $x^2 + 2y^2 = 4$

(D) The foci of ellipse are $(\pm\sqrt{2}, 0)$

Answer and Comments: (A,B). In the standard form the hyperbola is

$$\frac{x^2}{1/2} - \frac{y^2}{1/2} = 1 \qquad (1)$$

This is a rectangular hyperbola and so its eccentricity is $\sqrt{2}$. (Those not familiar with this may calculate e from the relations $a^2 = b^2 = \frac{1}{2}$ and

$b^2 = a^2(e^2 - 1)$.) So, the eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$. Therefore, if a and b are the lengths of its semi-major and semi-minor axes, then we must have

$$b^2 = a^2(1 - \frac{1}{2}) = \frac{a^2}{2} \tag{2}$$

which means $a^2 = 2b^2$. As the ellipse has its axes along the coordinate axes, its equation is

$$\frac{x^2}{2b^2} + \frac{y^2}{b^2} = 1 \tag{3}$$

Note that so far we have not found the value of b. To find it we need that this ellipse intersects the given hyperbola orthogonally, i.e. that at their points of intersection, the tangents to the ellipse are at right angles to those to the hyperbola. The condition for orthogonality of two circles can be written down by a formula even without finding the points of intersection, because in the case of a circle a tangent at any point is perpendicular to the radius through that point. For ellipses and hyperbolas, there is no such simplifying feature and so we have to begin by first finding the points of intersection of the two curves. Fortunately, as both the curves are symmetric about both the coordinate axes, the four points of intersections will be symmetrically located in the four quadrants and it suffices to consider orthogonality at any one of them. To avoid having to work with negative signs, we let P be the point of intersection of the hyperbola and the ellipse in the first quadrant. It is obtained by solving (1) and (3) simultaneously to get

$$P = (\sqrt{\frac{2}{3}b^2 + \frac{1}{3}}, \sqrt{\frac{2}{3}b^2 - \frac{1}{6}}) \tag{4}$$

Both the coordinates of P are horribly complicated. If we write them as such in every line we shall waste a lot of time and also run the risk of a mistake in carrying them from one line to the next. Let us therefore call P as (u, v) and only at the end put the values of u and v.

We now calculate the slopes of the tangents to (1) and (3) at the point (u, v). There are standard formulas for these. Or we can apply differentiation. So, differentiating (1) we get

$$x - y\frac{dy}{dx} = 0 \tag{5}$$

i.e. $\frac{dy}{dx} = \frac{x}{y}$. Hence the slope of the tangent to the hyperbola at the point $P(u, v)$ is $\frac{u}{v}$. Similarly, differentiating (3) we get $x + 2y\frac{dy}{dx} = 0$, i.e. $\frac{dy}{dx} = -\frac{x}{2y}$. So the slope of the tangent to the ellipse at the point P is

$-\dfrac{u}{2v}$. As these two tangents are perpendicular to each other, the product of these two slopes is -1, or in other words,

$$u^2 = 2v^2 \tag{6}$$

We *now* substitute the values of u and v. Fortunately, the radicals are gone and we get

$$\frac{2}{3}b^2 + \frac{1}{3} = \frac{4}{3}b^2 - \frac{1}{3} \tag{7}$$

which gives $b^2 = 1$. Putting this into (2), the equation of the ellipse comes out as $x^2 + 2y^2 = 2$. Also, now we know $a^2 = 2b^2 = 2$ and so $a = \sqrt{2}$. As we already know the eccentricity of the ellipse as $\dfrac{1}{\sqrt{2}}$, we see that its foci are at $(\pm 1, 0)$. So, (A) and (B) are correct alternatives.

This problem tests the knowledge of quite a few facts about conics. But, once you know these facts the work involved is highly mechanical. Questions like this are therefore, more a test of knowledge of certain facts than the ability to apply them intelligently.

Q.25 For $0 < \theta < \dfrac{\pi}{2}$, the solution(s) of

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right) \operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$

is (are)

(A) $\dfrac{\pi}{4}$ (B) $\dfrac{\pi}{6}$ (C) $\dfrac{\pi}{12}$ (D) $\dfrac{5\pi}{12}$

Answer and Comments: (C, D). This is a problem on solutions of trigonometric equations. But the catch lies in first simplifying the given sum, say $f(\theta)$. The angles involved in this sum are of the form $\theta + \dfrac{m\pi}{4}$ for $m = 0, 2, \ldots, 6$. These angles do form an A.P. But the given sum does not come under any standard type such as the sum of the sines or cosines of the angles in an A.P. The only way to sum such series is to use the 'split and regroup' technique explained in Comment No. 5 of Chapter 24, where it was shown how it can be applied to evaluate the sum of the telescopic series $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$. Here the idea is to split every term into two parts so that the second part of each term cancels with the first part of the next term. (In the given example we write the k-th term as $\dfrac{1}{k} - \dfrac{1}{k+1}$.)

Let us see if a similar approach will work here. We first rewrite the m-th term of the series, say T_m, as

$$T_m = \frac{1}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right) \sin\left(\theta + \frac{m\pi}{4}\right)} \tag{1}$$

It would be great if this equalled

$$\frac{1}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)} - \frac{1}{\sin\left(\theta + \frac{m\pi}{4}\right)} \tag{2}$$

for then we can sum the series as a telescopic series. But, unfortunately, (1) and (2) are not equal. On the contrary, the expression in (2) equals

$$\frac{\sin\left(\theta + \frac{m\pi}{4}\right) - \sin\left(\theta + \frac{(m-1)\pi}{4}\right)}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)} \tag{3}$$

If only the difference of the sines of two angles were equal to the sine of their difference then the numerator would equal to $\sin[\left(\theta + \frac{m\pi}{4}\right) - \left(\theta + \frac{(m-1)\pi}{4}\right)]$ which is a constant, viz. $\dfrac{1}{\sqrt{2}}$ and then (2) could be used to rewrite the given series as a telescopic series.

Unfortunately, $\sin(A-B)$ does not, in general, equal $\sin A - \sin B$. But in the present case, $A - B$ equals $\dfrac{\pi}{4}$ and so $\sin(A-B)$ equals $\dfrac{1}{\sqrt{2}}(\sin A - \sin B)$. Here the coefficient $\dfrac{1}{\sqrt{2}}$ is independent of m. This suggests that even though the given series cannot be written as a telescopic series using (2), if we rewrite its terms using this coefficient $\dfrac{1}{\sqrt{2}}$, then we may be able to do so in some other manner. This is the key idea of the solution. Once it strikes, the rest of the work is routine. We have

$$\begin{aligned}
T_m &= \frac{\sin(\frac{\pi}{4})}{\frac{1}{\sqrt{2}}\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)} \\
&= \frac{\sqrt{2}\sin\left[\left(\theta + \frac{m\pi}{4}\right) - \left(\theta + \frac{(m-1)\pi}{4}\right)\right]}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)} \\
&= \sqrt{2}\frac{\sin\left(\theta + \frac{m\pi}{4}\right)\cos\left(\theta + \frac{(m-1)\pi}{4}\right) - \cos\left(\theta + \frac{m\pi}{4}\right)\sin\left(\theta + \frac{(m-1)\pi}{4}\right)}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)} \\
&= \sqrt{2}\left[\cot\left(\theta + \frac{(m-1)\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right)\right] \qquad (4)
\end{aligned}$$

which is indeed a telescopic series. It is now an easy matter to find the sum function $f(\theta)$.

$$f(\theta) = \sum_{m=1}^{6} T_m$$

$$
\begin{aligned}
&= \sqrt{2}\sum_{m=1}^{6}\left[\cot\left(\theta+\frac{(m-1)\pi}{4}\right)-\cot\left(\theta+\frac{m\pi}{4}\right)\right] \\
&= \sqrt{2}\left[\cot\theta-\cot\left(\theta+\frac{3\pi}{2}\right)\right]=\sqrt{2}\,[\cot\theta+\tan\theta] \\
&= \sqrt{2}\left(\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}\right)=\sqrt{2}\frac{1}{\sin\theta\cos\theta}=\frac{2\sqrt{2}}{\sin 2\theta} \qquad (5)
\end{aligned}
$$

So the given equation becomes

$$2\sqrt{2}=4\sqrt{2}\sin(2\theta) \qquad (6)$$

i.e. $\sin(2\theta)=\frac{1}{2}$. As θ lies in $(0,\frac{\pi}{2})$, the possible values of θ are $\frac{\pi}{12}$ and $\frac{5\pi}{12}$.

This is a very good problem. But the trick involved in converting the given series to a telescopic series is not easy to come without spending considerable thought and time. And even after getting it, there is a fair amount of mechanical work. In the past this could be asked as a full length question allowing a candidate about 8 to 10 minutes. Asking it for four marks (which, on a proportionate basis, allows only 3 minutes) is hardly fair.

Q.26 If

$$I_n=\int_{-\pi}^{\pi}\frac{\sin nx}{(1+\pi^x)\sin x}dx, \quad n=0,1,2,\ldots$$

then

(A) $I_n=I_{n+2}$

(B) $\sum_{m=1}^{10} I_{2m+1}=10\pi$

(C) $\sum_{m=1}^{10} I_{2m}=0$

(D) $I_n=I_{n+1}$

Answer and Comments: (A, B, C). This is a problem about reduction formulas for definite integrals whose integrands involve an integer parameter n. Such integrals are usually not evaluated directly except for some small values of n. Instead, we relate the n-th integral, say I_n, with some 'lower' integrals like I_{n-1} or I_{n-2}. (In effect, this is like a recurrence relation for the sequence $\{I_n\}$. See Chapter 18 for problems involving reduction formulas.)

Let us begin by rewriting I_n using a well-known property of definite integrals (which was also used in the solution to Q.30 in Paper 1).

$$I_n = \int_{-\pi}^{\pi}\frac{\sin n(-x)}{(1+\pi^{-x})\sin(-x)}dx$$

$$= \int_{-\pi}^{\pi} \frac{\pi^x}{1+\pi^x} \frac{\sin nx}{\sin x} dx \quad (1)$$

Adding this to the given expression for I_n, we get

$$\begin{aligned} 2I_n &= \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x} dx + \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1+\pi^x)\sin x} dx \\ &= \int_{-\pi}^{1} \frac{\sin nx}{\sin x} dx \quad (2) \end{aligned}$$

whence we get $I_n = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx$. As the integrand is an even function of x and the interval of integration is symmetric about the origin, this integral equals $2\int_0^{\pi} \frac{\sin nx}{\sin x} dx$. So, we finally get

$$I_n = \int_0^{\pi} \frac{\sin nx}{\sin x} dx \quad (3)$$

which is a much more manageable expression for I_n than the one given in the statement of the problem. The alternatives (A) and (D) give two possible reduction formulas for I_n and we have to see which ones (if any) out of them are true. Of course, if (D) is true then applying it twice we would get $I_n = I_{n+1} = I_{n+2}$ and so (A) would also be true. But it is easily seen that (D) is not true, because from (3) we get $I_0 = 0$ while $I_1 = \pi$ which are not equal to each other. Let us, therefore, see whether (A) holds true independently of (D), i.e. whether $I_{n+2} = I_n$ for every $n = 0, 1, 2, \ldots$. As a starter, $I_2 = \int_0^{\pi} 2\cos x \, dx = 2\sin x\Big|_0^{\pi} = 0$ which does equal I_0. Of course this does not mean (A) is true fro *all* n. But it does mean that it cannot be ruled out as easily as (D).

To check if (A) holds for all n, we must show $I_{n+2} - I_n = 0$. This is best done by a direct calculation.

$$\begin{aligned} I_{n+2} - I_n &= \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x} dx \\ &= \int_0^{\pi} \frac{2\cos(n+1)x \sin x}{\sin x} dx \\ &= \int_0^{\pi} 2\cos(n+1)x dx \\ &= \int_0^{\pi} \frac{2}{n+1} \sin(n+1)x\Big|_0^{\pi} = 0 \quad (4) \end{aligned}$$

Thus we see that (A) is indeed true. We already calculated I_0 and I_1 as 0 and π respectively. So, repeated application of (A) gives us $I_n = 0$ for all even n and $I_n = \pi$ for all odd n. So, in the sum in (B) each of the 10 terms equals π and so the sum is 10π. Hence (B) is true. Similarly, (C) is true because every term is 0.

The problem consists of two essentially unrelated parts, the reduction of the given integral to (3) and then proving a reduction formula for it. The first part itself has two steps, viz. application of a standard property of definite integrals and then an application of a property of integrals of even functions. The property needed in the first part was already useful in the solutions to Q.30 and Q.39 (B) of Paper 1. But there it was not mandatory because those integrals could also be evaluated by other methods. In the present problem, this property is vitally needed. The second part of the problem, viz., verifying that the integral in (3) satisfies a certain recurrence relation is routine. It is analogous to but much simpler than the JEE 1995 problem of obtaining a reduction formula for the integral $\int_0^{\pi} \frac{1-\cos mx}{1-\cos x} dx$, solved in Comment No. 8 of Chapter 18.

Q.27 For the function

$$f(x) = x \cos \frac{1}{x}, \quad x \geq 1,$$

(A) for at least one x in the interval $[1, \infty)$, $f(x+2) - f(x) < 2$

(B) $\lim_{x \to \infty} f'(x) = 1$

(C) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$

(D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

Answer and Comments: (B, C, D). The alternatives (B) and (D) make a direct reference to the derivative $f'(x)$. The other two alternatives make an indirect reference to it because the inequalities in them can be paraphrased in terms of the ratio $\frac{f(x+2) - f(x)}{2}$ and by Lagrange's Mean Value Theorem, this ratio equals $f'(c)$ for some $c \in (x, x+2)$.

So, it is best to begin by calculating $f'(x)$. This is straightforward because since the point 0 is not in the domain of the function, we do not have to worry about the vanishing of the denominator of $\frac{1}{x}$.

$$f'(x) = \cos \frac{1}{x} + \frac{1}{x} \sin \frac{1}{x} \tag{1}$$

As $x \to \infty$, $\frac{1}{x} \to 0$ while $\sin \frac{1}{x}$ remains bounded. So, $\lim_{x \to \infty} f'(x) = \cos 0 + 0$ $= 1$. Thus (B) is correct.

Next, we consider (D). Since $f'(x)$ is differentiable everywhere on $[1, \infty)$, the easiest way to check if it is increasing/decreasing is by considering the sign of its derivative $f''(x)$. Differentiating (1) we get,

$$f''(x) = \frac{1}{x^2}\sin\frac{1}{x} - \frac{1}{x^2}\sin\frac{1}{x} - \frac{1}{x^3}\cos\frac{1}{x} = -\frac{1}{x^3}\cos\frac{1}{x} \qquad (2)$$

For $x \geq 1$, $\frac{1}{x} \in (0, 1] \subset (0, \frac{\pi}{2})$ and so $\cos\frac{1}{x} > 0$. As the second factor, viz. $-\dfrac{1}{x^3}$ is negative for all $x \geq 1$, we get that $f''(x) < 0$ for all $x \in [1, \infty)$ and therefore by Lagrange's MVT, $f'(x)$ is strictly decreasing on $[0, \infty)$. Hence (D) is also true.

It remains to check (A) and (C). Clearly both of them cannot hold together, because except for the strictness of the inequality sign, they are the logical negations of each other. As noted earlier, the key idea is to note that the ratio $\dfrac{f(x+2) - f(x)}{2}$ equals $f'(c)$ for some $c \in (x, x+2)$. So, the answer depends on the sign of $f'(c)$. We do not know the point c. But no matter what x is, c lies in $[1, \infty)$ because $(x, x+2) \subset [1, \infty)$. We already proved that $f'(x)$ is strictly decreasing on $[1, \infty)$ and also that it tends to 0 as $x \to \infty$. Therefore $f'(x) > 0$ for all $x \in [0, \infty)$. In particular, $f'(c) > 0$ and this implies that $f(x+2) - f(x) > 2$. So, (C) is true.

The problem is a simple but good application of the Lagrange's MVT. The main assertion is (C) and (B), (D) are steps in its proof. In a multiple choice test, no proofs can be asked and, as a result, questions based on theoretical calculus take a severe beating. Within this strong constraint, the paper-setters have done the best they could.

Q.28 The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose

(A) vertex is at $\left(\dfrac{2a}{3}, 0\right)$ (B) directrix is $x = 0$

(C) latus rectum $\dfrac{2a}{3}$ (D) focus is $(a, 0)$

Answer and Comments: (A, D). Yet another problem on vocabulary testing, this times that pertaining to a parabola. All the four terms in the alternatives make sense for a parabola, an ellipse and a hyperbola as well. The paper-setters have, however, been merciful enough to tell the candidates that the locus of the centroid is a parabola. If they wanted, they could have withheld this information and in that case they could have given an alternative involving the eccentricity too!

Unfortunately the generosity of the paper-setters is of little help here though, because even if we are given the locus to be a parabola, to answer questions about its focus etc, we have to actually identify the locus and not just its type. So, let $M = (h, k)$ be the centroid of the triangle PTN. We need to get parametric equations for h and k. So a natural start would be to take P in a parametric form as $(at^2, 2at)$. The equation of the tangent at P then is

$$y - 2at = \frac{1}{t}(x - at^2) \tag{1}$$

This meets the x-axis (which is also the axis of the parabola) at $(-at^2, 0)$. Hence

$$T = (-at^2, 0) \tag{2}$$

Similarly the equation of the normal at P is

$$y - 2at = -t(x - at^2) \tag{3}$$

which gives

$$N = (at^2 + 2a, 0) \tag{4}$$

As $M(h, k)$ is the centroid of the triangle PTN, we have

$$3h = at^2 + 2a \tag{5}$$
$$3k = 2at \tag{6}$$

Eliminating t from these two equations, we get $3h = a(\frac{3k}{2a})^2 + 2a = \frac{9k^2}{4a} + 2a$. So, the locus of M is

$$y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right) \tag{7}$$

If we shift the origin to the point $\left(\frac{2a}{3}, 0\right)$ and introduce new coordinates by

$$X = x - \frac{2a}{3},\ Y = y \tag{8}$$

then in the new coordinates, the equation of the parabola becomes

$$Y^2 = 4bX \tag{9}$$

where $b = \frac{a}{3}$. Thus we have the equation of a parabola in the XY-plane in the standard form. Its vertex is at $(0, 0)$, directrix is $X = -b$, latus rectum is $4b$ and focus is at $(b, 0)$, all in terms of new coordinates. To get the answer in old coordinates we use (8). So, the vertex is at $\left(\frac{2a}{3}, 0\right)$, the equation of the directrix is $x - \frac{2a}{3} = -b$, i.e. $x = \frac{2a}{3} - b = \frac{a}{3}$ and the focus is at $\left(b + \frac{2a}{3}, 0\right)$, i.e. at $(a, 0)$. The latus rectum is independent of the coordinate system and hence is $4b = \dfrac{4a}{3}$. Thus we see that (A) and (D) are correct.

Yet another straightforward but laborious problem. Finding the locus of M is extremely routine and after getting it in the form (7), the rest of the work can be done just by inspection. So, a problem like this does not test who can solve it but only who can solve it fast.

SECTION III

Matrix- Match Type

This section contains 2 questions. Each question contains statements given in two columns. Match those in **Column I** with those in **Column II**. The same statement may have more than one correct matches. There are two marks for each statement which is matched correctly. There is no negative marking.

Q.29 Match the statements/expressions in **Column I** with the values given in **Column II**.

	Column I		**Column II**
(A)	Roots of the equation $$2\sin^2\theta + \sin^2 2\theta = 2$$	(p)	$\dfrac{\pi}{6}$
(B)	Points of discontinuity of the function $$f(x) = \left[\dfrac{6x}{\pi}\right]\cos\left[\dfrac{3x}{\pi}\right]$$ where $[y]$ denotes the largest integer less than or equal to y	(q)	$\dfrac{\pi}{4}$
(C)	Volume of the paralleopiped with its edges represented by the vectors $$\hat{i}+\hat{j}, \hat{i}+2\hat{j} \text{ and } \hat{i}+\hat{j}+\pi\hat{k}$$	(r)	$\dfrac{\pi}{3}$
		(s)	$\dfrac{\pi}{2}$
(D)	Angle between vectors $\vec{a}$ and $\vec{b}$ where $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors satisfying $$\vec{a}+\vec{b}+\sqrt{3}\vec{c}=\vec{0}$$	(t)	π

Answer and Comments: (A) $\to$ (q, s), (B) $\to$ (p, r, s, t), (C) $\to$ (t), (D) $\to$ (r).

Once again what we have here is a bunch of four totally unrelated questions, each with 5 answers out of which one or more may be correct. We have to tackle them one by one.

The trigonometric equation in (A) can be written as a quadratic in $\sin^2\theta$, viz.

$$\sin^2\theta + 2\sin^2\theta(1-\sin^2\theta) = 1 \qquad (1)$$

and can be solved as such to get the possible values of $\sin^2\theta$. This approach is strikingly similar to that used in the solution to Q. 31 of Paper 1. But in the present case there is a simpler approach if we transfer $2\sin^2\theta$ to the R.H.S. For, then we get

$$4\sin^2\theta\cos^2\theta = 2 - 2\sin^2\theta = 2\cos^2\theta \qquad (2)$$

So, the possibilities are (i) $\cos^2\theta = 0$ and (ii) $\sin^2\theta = \frac{1}{2}$. (i) gives odd multiples of $\frac{\pi}{2}$ as possible solutions. The only entry in **Column II** which is of this type is (s). The possibility (ii) gives $\sin\theta = \pm\frac{1}{\sqrt{2}}$ whose solutions are all odd multiples of $\frac{\pi}{4}$. Again (q) is the only entry which fits this description.

In (B), the given function $f(x)$ is a product of two functions, say $g(x) = \left[\frac{6x}{\pi}\right]$ and $h(x) = \cos\left[\frac{3x}{\pi}\right]$. As cautioned at the end of Comment No. 15 of Chapter 15, we have to be wary in identifying the points of discontinuity of such a function. The present problem, however, does not ask us to identify all the points of discontinuity of $f(x)$. It only asks us to test its continuity at the five points listed in **Column II**. The integral part function is discontinuous precisely at the integers. So $g(x)$ is discontinuous whenever x is an integral multiple of $\frac{\pi}{6}$. In **Column II** there are four such points, viz. $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$, and π. Out of these, $\frac{\pi}{3}$ and π are also integral multiples of $\frac{\pi}{3}$ and so the the second function $h(x)$ is discontinuous at these values. In fact, $h(x) \to \cos 1$ as $x \to \frac{\pi}{3}^+$ and $h(x) \to \cos 0 = 1$ as $x \to \frac{\pi}{3}^-$. So the left and right handed limits of $f(x)$ at $\frac{\pi}{3}$ are $2\cos 1$ and 1 which are unequal. So, $f(x)$ is discontinuous at $\frac{\pi}{3}$. By a similar reasoning it is discontinuous at π. As for the other values, viz. $\frac{\pi}{6}$ and $\frac{\pi}{2}$, the second factor $h(x)$ is continuous and non-zero. As the first factor $g(x)$ is discontinuous, the product is discontinuous. So, $f(x)$ is discontinuous at all the four points $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ and π. The sad part is that an unscrupulous candidate who concludes this simply on the basis of the discontinuity of $g(x)$ at these points also gets the correct answer with considerable saving in time. It would have been nice if the function $h(x)$ had been given in such a way that it 'cures' the discontinuity of $g(x)$ at some points thereby making $f(x)$ continuous at these points. An extreme example of such a trick is the JEE 1981 question (see Comment No. 15 of Chapter 15) asking for points of discontinuity of the function $\dfrac{\tan(\pi[x-\pi])}{1+[x]^2}$. Here the denominator has many discontinuities. But the numerator vanishes identically and so the function has no discontinuities!

There could hardly be a more straightforward question than (C). The volume is simply the box product of the three vectors and hence equals the absolute value of the determinant $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix}$. By direct evaluation of the determinant, the volume is π cubic units.

Finally, in (D) we are given that $\vec{c} = -\frac{1}{\sqrt{3}}(\vec{a}+\vec{b})$. Therefore $||\vec{c}||^2 = \frac{1}{3}||\vec{a}+\vec{b}||^2 = \frac{1}{3}(||\vec{a}||^2 + ||\vec{b}||^2 + 2\vec{a}\cdot\vec{b}) = \frac{2}{3}(1+\vec{a}\cdot\vec{b})$. But $\vec{c}$ is also a unit vector. So we get $\frac{3}{2} - 1 = \vec{a}\cdot\vec{b}$ which means $\vec{a}\cdot\vec{b} = \dfrac{1}{2}$. Hence the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{6}$.

Parts (A), (C) and (D) are routine. It is really shocking that a question like (C) which involves nothing more than plugging values in a formula be apart of the selection process of a prestigious institute. Only Part (B) required a little thinking if done honestly. But the multiple choice format where no reasoning has to be given, takes the steam out of it too.

Q.30 Match the statements/expressions given in **Column I** with the values given in **Column II**.

	Column I	**Column II**
(A)	The number of solutions of the equation $$xe^{\sin x} - \cos x = 0$$ in the interval $\left(0, \dfrac{\pi}{2}\right)$	(p) 1 (q) 2
(B)	Value(s) of k for which the planes $kx+4y+z=0$, $4x+ky+2z=0$ and $2x+2y+z=0$ intersect in a straight line	(r) 3
(C)	Values of k for which $$\lvert x-1\rvert + \lvert x-2\rvert + \lvert x+1\rvert + \lvert x+2\rvert = 4k$$ has integer solution(s)	(s) 4
(D)	If $y' = y+1$ and $y(0) = 1$, then value(s) of $y(\ln 2)$	(t) 5

Answer and Comments: (A) $\to$ (p), (B) $\to$ (q, s), (C) $\to$ (q, r, s, t), (D) $\to$ (r).

In (A) the given equation involves algebraic, trigonometric as well as exponential expressions. Such equations are usually not easy to solve exactly and fully. Sometimes some solutions can be identified by inspection. But there can be some other solutions which are not so obvious and ignoring them can result in an error. (An example of such an erroneous problem was given in Comment No. 14 of Chapter 10.) Fortunately, in (A) we are asked only to find the number of solutions and not the solutions *per se*. And this can often be done using existence theorems such as the Intermediate Value Property of continuous functions or the Lagrange's Mean Value Theorem. In the present problem, call $xe^{\sin x} - \cos x$ as $f(x)$. On the interval $\left[0, \frac{\pi}{2}\right]$, the functions $x, \sin x$ and $-\cos x$ are all strictly increasing. The exponential function is strictly increasing on the entire real line. So, $f(x)$ is a strictly increasing function of x on the interval $\left[0, \frac{\pi}{2}\right]$. (Those who prefer to do things mechanically rather than with reasoning, can get this by finding $f'(x) = e^{\sin x} + xe^{\sin x}\cos x + \sin x$ and noting that it is positive for all $x \in \left(0, \frac{\pi}{2}\right)$.) A strictly increasing function can have either no roots or only one root in a given interval. To find out which possibility holds, let us apply the Intermediate Value Property (IVP) of continuous functions to $f(x)$ on the interval $[0, \pi/2]$. By a direct calculation $f(0) = -1 < 0$ while $f(\pi/2) = \frac{\pi e}{2} > 0$. So, by the IVP, f has at least one zero in $(0, \pi/2)$. We already showed it cannot have more than one zeros. So it has exactly one zero, i.e. the given equation has exactly one solution in the interval $\left(0, \frac{\pi}{2}\right)$. So (p) matches with (A).

For (B), all the three planes pass through the origin. Whether this is the only point of intersection depends on the determinant $\Delta = \begin{vmatrix} k & 4 & 1 \\ 4 & k & 2 \\ 2 & 2 & 1 \end{vmatrix} = k^2 - 6k + 8$. When $\Delta \neq 0$, there is no solution besides $(0, 0, 0)$ and the three planes meet at the origin. When $\Delta = 0$, they intersect either in a straight line or a plane. $\Delta = 0$ if and only if $k = 2, 4$. For both these values of k, the first and the third plane are different and so the intersection is a straight line and a plane. Hence alternatives (q) and (s) are correct. Again, the question should have been framed so that for some value of k, all the three planes become the same. In that case a candidate who fails to discard it would have been duly penalised.

In (C), let $f(x) = |x-1| + |x-2| + |x+1| + |x+2|$. Depending upon where x lies in $I\!R$, the expression for $f(x)$ will change. For $x \leq -2$, we have $f(x) = 1 - x + 2 - x + (-1) - x + (-2) - x = -4x$. With similar calculations (which are simplified by the observation that $f(x) = f(-x)$, i.e. f is an even function of x), we get

$$\begin{aligned} f(x) &= -4x, \;\; x \leq -2 & (1) \\ f(x) &= 4 - 2x, \;\; -2 \leq x \leq -1 & (2) \\ f(x) &= 6, \;\; -1 \leq x \leq 1 & (3) \\ f(x) &= 2x + 4, \;\; 1 \leq x \leq 2 & (4) \\ f(x) &= 4x, \;\; 2 \leq x & (5) \end{aligned}$$

We now have to consider equations of the form

$$f(x) = 4k \tag{6}$$

where k is an integer and see which of them have integral solutions (in x). Evidently, the integer k has to be positive since $f(x) > 0$ for all x. In the 5 possible expressions we have listed above, (2), (3) and (4) can never be integral multiples of 4 for any integer x. So we only consider (1) and (5). When combined with (6), these mean $x = -k$ and $x = k$ respectively. It is therefore tempting to think that that for *every* integer k, the equation $f(x) = 4k$ has integral solutions. But the catch is that in (1), there is a restriction on x, viz. $x \leq -2$ while in (5), we must have $x \geq 2$. Therefore if there is an integral solution to $f(x) = 4k$ with $f(x) = -4x$, then $4k = f(x) = -4x \geq 8$ which means $k \geq 2$. Similarly, if (6) has an integral solution with $f(x) = 4x$ then $x \geq 2$ and so $4k = 4x \geq 8$. In both the cases we get $k \geq 2$. Therefore only for these values of k, will (6) have an integral solution. In **Column II**, the entries (q), (r), (s) and (t) meet this requirement.

The problem is worded somewhat clumsily. It ought to have been made clear that the solution is in x and not in k. Also, it is not clear what is gained by requiring the solutions to be integral. The essential idea of the problem is that if a, b are real numbers with $a < b$, then the minimum value of the expression $|x-a|+|x-b|$ is $b-a$ and occurs for any $x \in [a, b]$. If we apply this to the pairs $|x+1|+|x-1|$ and $|x+2|+|x-2|$, we see that the minimum value of $f(x)$ is 6. Moreover $f(x) \to \infty$ as $x \to \infty$. So, by the IVP for continuous functions, $f(x)$ assumes every value in $[6, \infty)$.

Finally, we tackle (D). In the separate variable form the differential equation is

$$\frac{dy}{y+1} = dx \tag{7}$$

whose general solution is

$$|y+1| = ke^x \tag{8}$$

The initial condition $y(0) = 1$ determines k as 2. So, we have

$$|y+1| = 2e^x \tag{9}$$

Hence $|y(\ln 2) + 1| = 2e^{\ln 2} = 4$. This gives $y(\ln 2)$ as 3 or -5. But only the first value appears in **Column II**. It is really difficult to see what is gained by asking such a straightforward differential equation, especially so when in Q.24 of Paper 1, a qualitatively similar but more complicated d.e. was asked.

SECTION IV

Integer Answer Type

This section contains 8 questions. The answer to each question is a single digit integer, ranging from 0 to 9. There are 4 marks for each correct answer, no marks if a question is not answered and -1 marks in all other cases.

Q.31 If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is ..... .

Answer and Comments: 2. This is a problem about inverse functions. Let $y = f(x) = x^3 + e^{x/2}$. Then the inverse function g is such that $f(g(y)) = y$ for all y in the range of the function $f(x)$. To find it, we have to solve the equation $f(x) = y$ for x. If we could do this then we could find $g'(y)$ directly for any y. Unfortunately, solving the equation $f(x) = y$ for x is often not easy. In our problem, for example, we would have to solve

$$x^3 + e^{x/2} = y \tag{1}$$

which cannot be done explicitly since the L.H.S. is a mixture of an algebraic and an exponential function. The existence of the solution can be proved using the IVP for continuous functions and the facts that $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$. Moreover the solution is unique since $f(x)$ is strictly increasing on the entire real line. So, the inverse function $g(y)$ certainly exists. But we are not able to express it by a formula in y.

Still, we can find $g'(y)$ by implicitly differentiating both the sides of (1) w.r.t. y, to give

$$(3x^2 + \frac{1}{2}e^{x/2})\frac{dx}{dy} = 1 \tag{2}$$

which gives

$$g'(y) = \frac{dx}{dy} = \frac{1}{3x^2 + \frac{1}{2}e^{x/2}} \tag{3}$$

(Note that in the statement of the question, the notation $g(x)$ is used for the inverse function. This should create no confusion if it is understood that x is only a dummy variable. But beginners often find this confusing. So, we have denoted the inverse function by $g(y)$ and its derivative by $g'(y)$. The confusion arises because of the practice of denoting functions by symbols like $f(x), g(y), u(t)$ etc. instead of merely f, g, u.)

As our interest is only in finding g' at a particular point, viz. 1, we need to solve (1) only for $y = 1$. This can be done by inspection to get $x = 0$. Now by (3) $g'(y)$ is simply the reciprocal of $f'(0)$. By a direct calculation, $f'(0) = \frac{1}{2}$. So, $g'(1) = 2$.

This is a simple problem once the key idea, viz. relationship between the derivatives of a function and of its inverse function strikes. By choosing the function $f(x)$ in such a way that a direct expression for $g(x)$ is impossible, the paper-setters have ensured that only those who get the key idea will get the answer, a care which they failed to exercise in several other problems (e.g. Q.30 of Paper 1).

Q.32 Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1, 2$ and

$$\lim_{x \to 0} \left(1 + \frac{p(x)}{x^2}\right) = 2.$$

Then the value of $p(2)$ is ..... .

Answer and Comments: 0. In the last problem, we could find $g'(1)$ without finding a general expression for $g'(x)$. This happened because we could obtain $g(1)$ (viz. 0) by inspection but were not able to obtain a general expression for the inverse function $g(x)$. In the present problem, however, there is no way to obtain $p(2)$ just by inspection. (The fact that 2 is given to be a point of extremum does imply $p'(2) = 0$, but this says little about $p(2)$.)

So, we proceed to identify the polynomial $p(x)$ which is given to have degree 4. So, we can write $p(x)$ as $ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d, e are constants with $a \neq 0$. To determine their values we need a system of five equations in these variables a, b, c, d, e. The fact that $p(x)$ has an extremeum at $x = 1, 2$ gives one equation each, viz. $p'(1) = 0$ and $p'(2) = 0$. The remaining three equations come from the single piece of data that

$$\lim_{x \to 0} \left(1 + \frac{p(x)}{x^2}\right) = 2 \tag{1}$$

So, this is yet another problem where a single piece of data stores a lot of information. One wonders, however, what is the big idea in giving it in an unnecessarily clumsy form. Clearly, (1) is equivalent to

$$\lim_{x \to 0} \frac{p(x)}{x^2} = 1 \tag{2}$$

which is much more natural. Any candidate who can work out the problem from here onwards would surely be able to reduce (1) to (2). So, no new qualities are tested in giving the data as (1) instead of as (2).

If we recall the terminology used in the solution to Q.29 of Paper 1, (2) says that as $x \to 0$, $p(x)$ is of the same order of magnitude as x^2. So we get instantly that the coefficient of x and the constant term in $p(x)$ must vanish and further that the coefficient of x^2 must be 1. For those who are not quick to see this, here is a methodical approach. For $x \neq 0$ we have

$$\frac{p(x)}{x^2} = ax^2 + bx + c + \frac{d}{x} + \frac{e}{x^2} \tag{3}$$

It is tempting to take the limit of each term on the R.H.S. as $x \to 0$ and conclude that d and e must vanish as otherwise the limit would not be finite. But this logic is faulty. The sum of two functions both tending to $\pm\infty$ can tend to a finite limit as we see from examples like $\sec^2 x - \tan^2 x$ as $x \to \pi/2$. The theorem that the limit of a sum equals the sum of the limits holds only when all limits are known to exist beforehand and does not apply in such cases.

A rigorous proof of the vanishing of the coefficients d and e is easy but not trivial. Multiplying both the sides of (3) by x^2 and transferring all but the last term to the L.H.S., we get

$$e = x \times \left(x\frac{p(x)}{x^2} - ax^3 - bx^2 - cx - d \right) \tag{4}$$

Now, as $x \to 0$, the expression in the parentheses tends to $-d$ because now we know beforehand that every term tends to a finite limit. As the first factor tends to 0 as $x \to 0$, we see that the R.H.S. tends to 0 as $x \to 0$. The L.H.S., being a constant, tends to e. So we get $e = 0$. So, now we have

$$p(x) = ax^4 + bx^3 + cx^2 + dx \tag{5}$$

For $x \neq 0$, we have

$$d = x \times \left(\frac{p(x)}{x^2} - ax^2 - bx - c \right) \tag{6}$$

Once again the expression in the parentheses tends to a finite limit (viz. $1-c$) as $x \to 0$. As the first factor tends to 0, we get $d = 0$. This simplifies our polynomial even further to

$$p(x) = ax^4 + bx^3 + cx^2 \tag{7}$$

which we recast as

$$c = \frac{p(x)}{x^2} - ax^2 - bx \tag{8}$$

for $x \neq 0$. Now taking limits of both the sides as $x \to 0$, we get $c = 1$.

Having found $d = e = 0$ and $c = 1$ (whether instinctively or the hard way), we now have

$$p(x) = ax^4 + bx^3 + x^2 \tag{9}$$

Only two constants a and b remain to be determined. For these we use that $p'(x)$ is 0 at $x = 1, 2$ as these are points of extremum. Since $p'(x) = 4ax^3 + 3bx^2 + 2x$, we get a system of two equations in two unknowns, viz.

$$4a + 3b + 2 = 0 \tag{10}$$

$$\text{and} \quad 32a + 12b + 4 = 0 \tag{11}$$

solving which we get $a = \frac{1}{4}$ and $b = -1$. So, $p(x) = \frac{1}{4}x^4 - x^3 + x^2$ which gives $p(2) = 4 - 8 + 4 = 0$.

This is a good problem but the essential idea, viz. the order of magnitude, is a duplication from Q.29 of Paper 1, where it appeared in a more subtle way. The present problem is very simple if you quickly identify the values of c, d and e from the second part of the data. Of course, those candidates who can do it may not all be able to give a rigorous proof as was given above. It may be argued that such proofs are ritualistic. Still, they are a part of their training and the multiple choice format where no reasoning is to be given makes it impossible to test the ability to give correct proofs.

Q.33 Let (x, y, z) be points with integer coordinates satisfying the system of homogeneous equations:

$$\begin{aligned} 3x - y - z &= 0 \\ -3x + z &= 0 \\ -3x + 2y + z &= 0. \end{aligned}$$

Then the number of such points for which $x^2 + y^2 + z^2 \leq 100$ is ... .

Answer and Comments: 7. Superficially, this is a problem about a system of linear equations. But the system is ridiculously easy to solve. The first two equations imply $y = 0$ and $z = 3x$ and the third one gives nothing new. So the solutions are of the form $(x, 0, 3x)$ where x is any real number. But we want only integral solutions, which means x has to be an integer. Further we want $x^2 + y^2 + z^2 \leq 100$ which means $10x^2 \leq 100$ or $x^2 \leq 10$. As x is an integer, the possible values of x are $0, \pm 1, \pm 2$ and ± 3. So in all x can have 7 values and since each one of these gives rise to a unique integral solution, this is also the number of all desired points (x, y, z)

In essence this problem is a combination of two problems, one about solving a linear system and one about counting. Both are absolutely trivial. Elementary but interesting problems about counting which use some number theory can be designed with a little imagination. (Q.7 of JEE 2006 is a good example.) In the present problem, the set whose cardinality is to be counted is defined by an inequality, which is too straightforward. So this cannot be considered as a problem on inequalities either. Perhaps the constraint that the answer has to be in a single digit inhibits the paper-setters severely. (The 2006 problem referred above had 225 as the right answer.)

Q.34 Let $f I\!\!R \longrightarrow I\!\!R$ be a continuous function which satisfies

$$f(x) = \int_0^x f(t)\, dt.$$

Then the value of $f(\ln y)$ is ... .

Answer and Comments: 0. The function that appears on the R.H.S. is an example of what is called a function defined by an integral. Whenever the integrand is continuous, such a function is differentiable and its derivative can be found by the second form of the Fundamental Theorem of Calculus. So, differentiating both the sides gives

$$f'(x) = f(x) \tag{1}$$

In other words, $f(x)$ is a solution of the differential equation

$$y' = y \tag{2}$$

This d.e. is similar to and even simpler than that in Part (D) of Q.30 above. Its general solution is

$$f(x) = ke^x \tag{3}$$

To determine the constant k we need an initial condition. No such condition is given explicitly. But from the equation given in the statement of the problem, we have

$$f(0) = \int_0^0 f(t)\, dt = 0 \tag{4}$$

So $k = 0$. Hence $f(x)$ vanishes identically. In particular $f(\ln 5) = 0$.

The d.e. part of the problem is trivial and hardly different from the the d.e. part of Q.24 of Paper 1 and Q.30(D) of Paper 2. The key idea is only the fundamental theorem of calculus in its second form.

Q.35 The smallest value of k, for which both the roots of the equation

$$x^2 - 8x + 16(k^2 - k + 1) = 0$$

are real, distinct and have values at least 4, is ... .

Answer and Comments: . A routine problem about quadratic equations. The criterion for the roots to be real and distinct is that the discriminant be positive, which gives

$$16k^2 > 16(k^2 - k + 1) \tag{1}$$

which reduces to $k > 1$. But there is one more constraint, viz. that both the roots be at least 4, which, of course, means that the smaller root be at least 4. As the roots are real, the smaller root, say α corresponds to the negative square root in the quadratic formula. Hence

$$\alpha = 4k - 4\sqrt{k-1} \tag{2}$$

So, the constraint is $4k - 4\sqrt{k-1} \geq 4$, or equivalently,

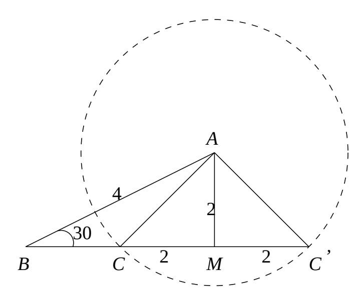
$$k - 1 \geq \sqrt{k-1} \tag{3}$$

We already know that $k - 1$ is positive. A positive real number is bigger than or equal to its square root if and only if it is bigger than or equal to 1. So (3) translates as $k - 1 \geq 1$, i.e. $k \geq 2$. As there are are no further constraints on k, the smallest value of k which satisfies all the constraints given in the question is 2.

Yet another problem which is a combination of problems from different areas, this time quadratic equations and inequalities. Once again, both the parts are absolutely trivial.

Q.36 Let ABC and ABC' be two non-congruent triangles with sides $AB = 4$, $AC = AC' = 2\sqrt{2}$ and angle $B = 30°$. The absolute value of the difference between the areas of these two triangles is ... .

Answer and Comments: 4. The two triangles have the side AB and also the angle B in common. If the side BC were the same as BC' then they would be congruent (in fact, identical), which is not the case. Nevertheless as the side AB and $\angle B$ are common, we get that both C and C' lie on the same line. Let M be the foot of the perpendicular from A to this line. Then the fact that $AC = AC'$ but $C \neq C'$ means that C and C' lie on opposite rays from M and M is their midpoint.



It is now easy to complete the solution. As $\angle B = 30°, AM = \frac{1}{2}AB = 2$. This, coupled with $AC = AC' = 2\sqrt{2}$ gives $MC = MC' = 2$ and hence $CC' = 4$. The difference of the areas of the triangles ABC and ABC' is precisely the area of the triangle ACC' which equals $\frac{1}{2} \times 4 \times 2 = 4$ sq. units.

Yet another simple problem. But this time the candidate has to correctly visualise the data first. After that the calculations are easy and not prone to error. So this is a good problem for the credit allotted.

The trigonometric significance of the data is noteworthy. In schools we have various tests for congruence of two triangles, say ABC and $A'B'C'$. For example, the S - S - S test says that if the corresponding sides are equal (i.e. if $AB = A'B'$, $BC = B'C'$ and $CA = C'A'$) then the two triangles are congruent. Similarly there is the S - A - S test which says that if two pairs of corresponding sides are equal and the included angles are equal (e.g. if $AB = A'B', BC = B'C'$ and $\angle B = \angle B'$) then ABC and $A'B'C'$ are congruent. It is very important here to have the equality of the *included* angles. In the present problem we have two triangles ABC and ABC' with AB common and $AC = AC'$. If the included angles, viz. $\angle BAC$ and BAC' were also equal then S - A - S would apply and the triangles would be congruent to each other. But what is given is only $\angle ABC = ABC'$ and as these angles are not included in the pairs AB, AC and AB, AC' respectively, S - A - S does not apply and the two triangles indeed fail to be congruent.

Trigonometry provides a deeper analysis of the situation. As usaul, denote the sides of ABC by a, b, c. It is customary to say that any three independent attributes of a triangle determine it (upto congruence), that is, any two triangles which match each other in terms of these attributes are congruent to each other. But this is not quite correct. Some attributes do determine the triangle uniquely. For example, the three sides a, b, c. To say that the three sides of a triangle determine it uniquely upto congruence is a sophisticated version of the S - S - S test for congruence. Similarly a, c and $\angle B$ determine ΔABC uniquely upto congruence. This corresponds to the S - A - S test.

In the present problem we are given c, b and $\angle B$. And we have two mutually non-congruent triangles which have these attributes, one of them has $a = 2\sqrt{3} - 2$ and the other has $a = 2\sqrt{3} + 2$. It is interesting to observe that these two possibilities correspond to the two roots of a quadratic. The cosine rule gives

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \tag{1}$$

Here, the values of b, c and $\cos B$ are known while a is an unknown. Let us write (1) as a quadratic in the unknown a.

$$a^2 - (2c \cos B)a + (c^2 - b^2) = 0 \tag{2}$$

The roots of this quadratic will be real and distinct, real and coincident or imaginary depending upon whether $\Delta > 0, \Delta = 0$ or $\Delta < 0$, where

$$\Delta = c^2 \cos^2 B - (c^2 - b^2) = b^2 - c^2 \sin^2 B \tag{3}$$

Thus we see that if $b > c \sin B$, then there are two mutually non-congruent triangles which answer the given description. This is what happens in the present problem, because here $c = 4, \sin B = \frac{1}{2}$ and $b = 2\sqrt{2}$. This is evident geometrically too because a circle of radius $2\sqrt{2}$ centred at A cuts the line BC at two distinct points since the radius exceeds the perpendicular distance of the centre from the line. Note that if b were given to be 2 then it equals $c \sin B$ and then there is only one triangle (viz. the triangle ABM) which answers the description. In this case the line BC touches the circle above. In the last case, viz. $b < 2$, there is no triangle for the given values of b, c and $\angle B$. In this case the circle lies completely on one side of the line.

The term 'determined' is sometimes used in a generalised sense to mean that there are only finitely many possibilities which fit the thing in question. It is only in this generalised sense that we can say that any three independent attributes of a triangle determine it completely.

Q.37 The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x | x^2 + 20 \leq 9x\}$ is .... .

Answer and Comments: 7. A straightforward problem of maximising a given function on a given set. Let us begin by identifying the set A. The condition given in its definition can be rewritten as

$$x^2 - 9x + 20 \leq 0 \tag{1}$$

The L. H. S. is a quadratic in x. As the leading coefficient is positive, the sign of the expression is negative precisely when x lies between the two roots of the quadratic. By inspection, the roots here are 4 and 5. So,

$$A = [4, 5] \tag{2}$$

(Note that we are including the end-points because in (1) possible equality is allowed.) So, the problem now is to maximise the given function $f(x)$ on the interval $[4, 5]$. The standard approach is to identify the intervals where $f(x)$ is increasing/decreasing. This is best done by checking the sign of the derivative.

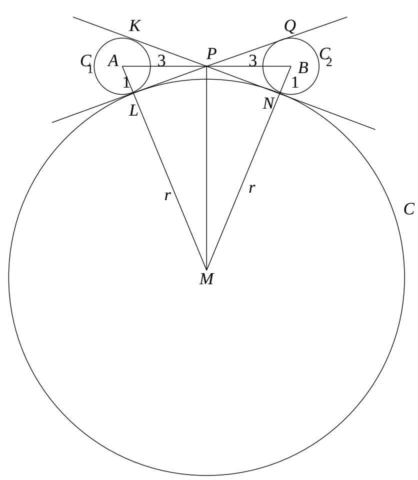
$$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x - 2)(x - 3) \tag{3}$$

So, we see that $f'(x)$ is positive on the intervals $(-\infty, 2)$ and $(3, \infty)$ and negative on the interval $(2, 3)$. Therefore by Lagrange's MVT, $f(x)$ increases strictly on $(-\infty, 2]$, decreases strictly on $[2, 3]$ and then again increases strictly on $[3, \infty)$. Our interest is only in the behaviour of $f(x)$ on the interval $[4, 5]$. On this interval, $f(x)$ is strictly increasing and therefore its maximum value on A is $f(5)$ which comes out to be $250 - 375 + 180 - 48 = 7$.

In this problem too there is a combination of quadratic equations and maxima/minima. Also there is some duplication because checking the sign of a quadratic expression is needed twice, once in (1) and then again in (3). The only catch is that those who are addicted to finding maxima/minima solely through derivatives will get confused that $f(x)$ has no critical points in the set A. The problem would have been more interesting if the set A were given as some interval $[a, b]$ which contains at least one of the two critical points (viz. 2 and 3) in its interior. In that case $f(x)$ will change its increasing/decreasing behaviour at least once in A and so the maximum value will have to be found by comparing the values of f at the critical points inside $[a, b]$ and its values at the end points a and b.

Q.38 The centres of two circle C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the midpoint of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is ....

Answer and Comments: 8. Like many other problems in geometry, in a problem like this, you have to draw a diagram even to understand the problem. let A, B and M be the centres of the circles C_1, C_2 and C respectively. P is the midpoint of AB and so by symmetry, $PM \perp AB$. Let L and N be the points of contacts of C with C_1 and C_2 respectively and let r be the radius of C which we have to find.



There are two tangents from P to C_1, PL and PK (say). By symmetry, both these lines touch C_2 also, at Q (say) and N respectively. Note that

A, L, N as well as B, N, M are collinear. So, PL and PN are tangents to the circle C too. So both the lines KPN and LPQ touch all the three circles C_1, C_2 and C. As a reult, the last piece of data in the statement of the problem is really redundant.

To find r, we first note that as C touches C_1 and C_2 externally,

$$MA = MB = r + 1 \tag{1}$$

From the right-angled triangle ALP, we have

$$PL^2 = PA^2 - AL^2 = 9 - 1 = 8 \tag{2}$$

From the right-angled triangle PLM we have,

$$PM^2 = ML^2 + PL^2 = r^2 + 8 \tag{3}$$

But, from the right angled triangle APM, we have

$$(r+1)^2 = MA^2 = PA^2 + PM^2 = 9 + r^2 + 8 = r^2 + 17 \tag{4}$$

which gives $r^2 + 2r + 1 = r^2 + 17$ i.e. $r = 8$. Essentially, the same argument can be given using coordinates. Take P as the origin and the x-axis along the line AB. Then the points A and B are $(-3, 0)$ and $(3, 0)$ respectively. By symmetry, M must lie on the y-axis and hence must be of the form $(0, h)$ for some h. The equations of the three circles and the two tangents can now be written down. The condition for tangency will give an equation for r. But this approach is not recommended.

This is a good problem to test a candidate's ability to correctly interpret geometric data. Unfortunately, the redundancy in the data can be confusing to a good student by making him think that he is missing something, because normally, every piece of data gives some new information.

CONCLUDING REMARKS

By and large all the problems this year are routine and simple. So, there is not much to comment on them as a whole. The best problems are Q.29 of Paper 1 and Q.25 in Paper 2 where the harder part is to evaluate the given trigonometric sum. This problem deserves more credit because of the ingenuity needed to tackle it. The other multiple correct answer peoblems in Paper 2 also required more work than justified by the credit allotted, except Q.27, which was more reasoning oriented. In contrast, those in Section 4, where the answers were single digit integers are very easy. So perhaps, the time allowed was all right on the average. Still, one wishes the paper-setters had the freedom to allot credit proportional to the level of difficulty and/or the amount of mechanical work needed in the problem.

As pointed out in individual comments, in many questions the paper-setters should have exercised some care to preclude sneaky solutions. There was considerable duplication of ideas. One fails to understand the need to ask separate, long questions on parabola, hyperbola and ellipse. The same property of definite integrals is applicable in as many as three problems, viz. Q. 30 and Q.39(B) of Paper 1 and Q.26 of Paper 2. The differential equations appearing in Q.24 and 39(A) of Paper 1 and in Q.30(D) and Q.34 of Paper 2 are very alike as far the method of solving them is concerned. Surely, at least two of these could have been replaced by some other type. Or, some problems could have been asked to construct a d.e.. Trigonometric equations have figured in as many as four questions (Q.31 and Q.39(C) in Paper 1 and Q.25 and Q.29(A)) and in three of them, the method of solving is similar viz. convert the equation to a quadratic equation by a suitable substitution. One really wonders if this topic deserves so much importance.

As a result of such repetitions, certain interesting topics have got a back seat. For example, number theory and inequalities havefigured only superficially. Matrices appear only in the first paragraph of Paper 1 and those problems involve more combinatorics than matrices. Probability appears only in the second paragraph of Paper 1. But the three problems there are somewhat repetitious and very standard too. (Incidentally, this year too, the word ‘comprehension’ is a misnomer. There is really nothing to ‘comprehend’ in either of the two paragraphs.

Comparatively, the coverage given to geometry problems which can be solved more as pure geometry problems is quite good. The solid coordiante geometry problems are unexciting.

There are no mathematical mistakes in the paper. This is an improvement over the last year. But the wording of Q.30(C) in Paper 2 is somewhat clumsy. Also, as pointed out earlier, in Q.38 of Paper 2, one piece of data is redundant. This can be confusing to a good student and lead him to think that he is missing something because normally every piece of data conveys some new information. There are minor grammatical errors in the wordings of Q.32 of Paper 1 and Q. 23 of Paper 2 as pointed out at the end of the comments on it.