EDUCATIVE COMMENTARY ON JEE 2010 MATHEMATICS PAPERS

Contents

The pattern of JEE 2010 resembles closely that of the previous year. The proportion of questions with single digit answers has gone up. These questions are intended to make up for the inherent limitations of multiple choice questions. But in a way these are multiple choice questions too except that for each question there are 10 possible choices. Also the requirement that the answer has to be a single digit sometimes entails some clumsiness, such as having to multiply some function in the question by a weird factor.

Negative credit for wrong answers was dropped this year. Another novel feature was that in those questions where more than one answers is correct, a candidate who chooses some of these (but no wrong ones) would get partial credit proportional to the number of correct choices he has made. But this instruction was not properly worded and, in absence of any negative credit, could mean that a candidate could score full marks merely by darkening all bubbles!

There was some confusion about the numbering of the questions in the question papers and that in the Objective Response Sheet. In this commentary, questions in Paper I will be numbered serially from 1 to 28 and those in Paper II from 1 to 19.

In the commentaries on the JEE mathematics papers for the past several years, frequent references were made to the book *Educative JEE Mathematics* by the author. At the time of writing the present commentary, this book was out of print. The second edition has appeared subsequently but has a slightly different pagination. As a result, in this year's commentary, references to the book are given in terms of the numbers of the chapter, the comment or the exercise involved, rather than by page numbers for the convenience of those having either of the two editions.

In the earlier vesrion of this Commentary, in the solution to Q.28, the sum S_1 was wrongly taken as 1 instead of 0 (and hence the answer, too, was given as 4 instead of 3). The error was pointed out by Amit Singhal.

PAPER 1

Contents

SECTION I

Single Correct Choice Type

This section contains 8 multiple choice questions. Each question has 4 choices out of which **ONLY ONE** is correct.

Q.1 Let p and q be real numbers such that $p \neq 0$ and $p^3 \neq \pm q$. Let α and β be non-zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$. Then a quadratic equation having α/β and β/α as its roots is

(A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
(B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
(C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
(D) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

Answer and Comments: (B). A straightforward question about the roots of a quadratic equation. The product of the two roots of the desired quadratic is 1 while their sum is $\dfrac{\alpha^2 + \beta^2}{\alpha\beta}$. So the values of $\alpha^2 + \beta^2$ and $\alpha\beta$ have to be determined from those of $\alpha + \beta$ and $\alpha^3 + \beta^3$ which are given to be $-p$ and q respectively. Since $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 + \alpha\beta + \beta^2) = (\alpha + \beta)((\alpha + \beta)^2 - 3\alpha\beta)$, we get $q = -p(p^2 - 3\alpha\beta)$ from which it follows that $\alpha\beta = \dfrac{p^3 + q}{3p}$. From this we also get $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 - 2(\dfrac{p^3 + q}{3p}) = \dfrac{p^3 - 2q}{3p}$. Hence the desired quadratic is

$$x^2 - \frac{p^3 - 2q}{p^3 + q}x + 1 = 0$$

which gives (B). Although the calculations are easy, it is doubtful if one can do them in 2 minutes which is the time allowed.

Q.2 Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is

(A)	$x + 2y - 2z = 0$	(B)	$3x + 2y - 2z = 0$
(C)	$x - 2y + z = 0$	(B)	$5x + 2y - 4z = 0$

Answer and Comments: (C). An extremely straightforward question. As the desired plane passes through the origin (a point on the first line) its equation is of the form $ax + by + cz = 0$. To determine the coefficients a, b, c upto a constant multiple, we need two equations. The first one comes from the fact that this plane contains the first line with direction numbers $2, 3, 4$. This gives

$$2a + 3b + 4c = 0 \tag{1}$$

The plane containing the other two lines will have its normal along the vector $(3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) \times (4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ which comes out to be $8\mathbf{i} - \mathbf{j} - 10\mathbf{k}$. The normal to the desired plane lies along the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Perpendicularity of the two planes gives the perpendicularity of their normals and hence

$$8a - b - 10c = 0 \tag{2}$$

(1) and (2) together give $a : b : c = -26 : 52 : -26$. Canceling the factor -26, we can take $a = 1, b = -2$ and $c = 1$. Hence the desired plane is $x - 2y + z = 0$. Note that we only need the direction of the normal to the second plane. So, it is a waste of time to actually identify the equation of the second plane. This is perhaps the only alertness needed. Otherwise the question is extremely routine. The portion of solid geometry in the JEE syllabus is so limited and elementary that it is difficult for the examiners to come up with a really novel question.

Q.3 Let ω be a complex root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2, r_3 are the numbers obtained on the die, the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is

(A) 1/18 (B) 1/9 (C) 2/9 (D) 1/36

Answer and Comments: (C) The problem is a combination of complex numbers and elementary probability. The possible values of each r_i are from 1 to 6. Hence there are in all 6^3 possibilities, each corresponding to an ordered triple (r_1, r_2, r_3) of integers from 1 to 6. We have to first see for which of these the equation $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ holds. The basic quadratic satisfied by ω is $\omega^2 + \omega + 1 = 0$. We can multiply this throughout by any powers of ω. Moreover since $\omega^3 = 1$, we can multiply any of the terms by ω^3 or any power of it to get another sum of three powers of ω that add to 0. Thus within the permissible values from 1 to 6, we see that one r_i has to be 1 or 4, another 2 or 5 and the third one either 3 or 6. Thus each r_i has two possible values and since they are all distinct, they can be permuted among themselves in 6 ways. Thus the number of favourable cases is $2^3 \times 3!$. Therefore the desired probability is $\frac{8 \times 6}{6^3} = \frac{8}{36} = \frac{2}{9}$.

The problem is a good combination of two essentially unrelated branches of mathematics. Earlier in 1997, there was a problem which asked for the probability that the roots of a certain quadratic equation be real (see Exercise (22.16) (b)).

Q.4 If the angles A, B, C of a triangle are in an arithmetic progression and if a, b, c denote the lengths of the sides opposite to A, B, C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is

(A) $1/2$ (B) $\sqrt{3}/2$ (C) 1 (D) $\sqrt{3}$

Answer and Comments: (D). By the sine rule, we can replace the ratios $\frac{a}{c}$ and $\frac{c}{a}$ by the ratios of the sines of the opposite angles. Then the given expression becomes simply $2(\sin A \cos C + \cos A \sin C) = 2\sin(A + C) = 2\sin B$. But as the angles are in an A.P., $\angle B = \frac{\pi}{3}$. So the given expression equals $2\sin(\frac{\pi}{3}) = \sqrt{3}$.

Questions about triangles with angles in an A.P. have been asked several times. This particular one is very easy the moment the key idea, viz. the sine rule, strikes.

Q.5 Let P, Q, R, S be the points on the plane with position vectors $-2\mathbf{i} - \mathbf{j}, 4\mathbf{i}, 3\mathbf{i} + 3\mathbf{j}$ and $-3\mathbf{i} + 2\mathbf{j}$. Then the quadrilateral $PQRS$ must be a

(A) parallelogram, which is neither a rhombus nor a rectangle

(B) square

(C) rectangle, but not a square

(D) rhombus, but not a square

Answer and Comments: (A). The points P, Q, R, S could have as well been specified by coordinates. The very fact that their position vectors are given suggests that vector methods may be handy. By direct calculations, the sides PQ and SR are represented by the vectors $4\mathbf{i} - (-2\mathbf{i} - \mathbf{j}) = 6\mathbf{i} + \mathbf{j}$ and $3\mathbf{i} + 3\mathbf{j} - (-3\mathbf{i} + 2\mathbf{j}) = 6\mathbf{i} + \mathbf{j}$. Hence the opposite sides are parallel and equal. So $PQRS$ is a parallelogram. To see if it is a rhombus or a rectangle we need to calculate the other sides (actually either one would do). So, QR is represented by the vector $3\mathbf{i} + 3\mathbf{j} - 4\mathbf{i} = -\mathbf{i} + 3\mathbf{j}$. As this vector has a different length than and is not perpendicular to the vector represented by the other pair of sides (as can be seen from their dot product), we see that the quadrilateral is neither a rhombus nor a a rectangle.

Had the problem been stated in terms of the coordinates of the points P, Q, R, S, the work would have been essentially the same. But in that case an alternate approach would have been more tempting. That is, first see if the diagonals have the same mid-point. In the present case, the point $(\frac{1}{2}, 1)$ is indeed the midpoint of both the diagonals. So the quadrilateral is a parallelogram. Further, the lengths of the diagonals are different and so it cannot be a rectangle. Finally, the diagonals are not perpendicular to each other and so $PQRS$ is not a rhombus either.

A very simple problem, no matter which approach is taken.

Q.6 The value of $\lim\limits_{x \to 0} \dfrac{1}{x^3} \displaystyle\int_0^x \dfrac{t \ln t}{t^4 + 1} dt$ is

(A) 0 (B) 1/12 (C) 1/24 (D) 1/64

Answer and Comments: (B). Problems of this type have been asked several times. The key idea is *not to* evaluate the integral. Our interest is only in the limit and that too the limit of a ratio, for which other methods are often available. In fact, in the present problem, both the numerator (viz. the integral $\displaystyle\int_0^x \dfrac{t \ln(1+t)}{t^4 + 4} dt$) and the denominator (viz. x^3) tend to 0 as $x \to 0$. So this is a limit of the $\dfrac{0}{0}$ form and therefore L'hôpital's rule is a popular tool. To take the derivative of the numerator, we need the second form of the Fundamental Theorem of Calculus, which gives

$$\frac{d}{dx} \int_0^x \frac{t \ln(1+t)}{t^4 + 4} dt = \frac{x \ln(1+x)}{x^4 + 4} \tag{1}$$

Hence the given limit equals $\lim\limits_{x \to 0} \dfrac{x \ln(1+x)}{3x^2(x^4+4)} = \dfrac{1}{12} \lim\limits_{x \to 0} \dfrac{\ln(1+x)}{x}$. The fans of L'ôpital's rule can apply it again to evaluate the last limit. But that is hardly necessary because if we write it as $\lim\limits_{x \to 0} \dfrac{\ln(1+x) - \ln(1+0)}{x - 0}$ we see that it is simply the derivative of $\ln(1+x)$ at $x = 0$ and hence equals $\dfrac{1}{1+0} = 1$. Whichever way we go, the given limit is $\dfrac{1}{12}$.

The problem resembles Exercise (17.18), especially Part (ii) of it.

Q.7 Let $f(x), g(x), h(x)$ be real-valued functions defined on the interval $[0, 1]$ by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2 e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maxima of f, g and h on $[0, 1]$, then

(A) $a = b$ and $c \neq b$ (B) $a = c$ and $a \neq b$
(C) $a \neq b$ and $c \neq b$ (D) $a = b = c$

Answer and Comments: (D). All the three functions are differentiable everywhere. Therefore the easiest way to find their extrema on $[0,1]$ is by studying the signs of their derivatives. A direct calculation gives

$$\begin{aligned}
f'(x) &= 2xe^{x^2} - 2xe^{-x^2} = \frac{2x(e^{2x^2}-1)}{e^{x^2}} && (1)\\
g'(x) &= 2x^2e^{x^2} + e^{x^2} - 2xe^{-x^2} = \frac{(2x^2+1)e^{2x^2}-2x}{e^{x^2}} && (2)\\
\text{and}\quad h'(x) &= 2xe^{x^2} + 2x^3e^{x^2} - 2xe^{-x^2} = \frac{(2x+2x^3)e^{2x^2}-2x}{e^{x^2}} && (3)
\end{aligned}$$

The denominators of the last expressions in each formula are all positive for all x and so their signs are determined from those of their numerators. In each case, we have $e^{2x^2} > 1$ for all $x > 0$. This immediately implies that $f'(x) > 0$ for all $x > 0$. Hence by Lagrange's Mean Value Theorem, $f(x)$ is strictly increasing on $[0,1]$. So, its maximum on $[0,1]$ occurs at the end point 1. In (2), for $x > 0$, the numerator is at least $2x^2 + 1 - 2x = x^2 + (x-1)^2$ which is always positive. So, $g(x)$ is also strictly increasing on $[0,1]$. In (3) too, the numerator is at least $(2x + 2x^3) - 2x = 2x^3$ and hence is positive for $x > 0$. Hence all the three functions attain their maxima on $[0,1]$ at 1. Since $f(1) = g(1) = h(1)$ we see that $a = b = c$.

The problem is a combination of maxima/minima and inequalities. Although all the three functions are strictly increasing, the arguments needed to show this are slightly different. Since no reasoning is to be given, an unscrupulous student who simply assumes this saves a lot of time as compared to a sincere student who carefully analyses the expressions as above and shows that they are all positive. To reward the sincere student it would have been better if one of the three functions was so designed that its maximum did not occur at the end.

Q.8 The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions is

(A) 0 (B) $2^9 - 1$ (C) 168 (D) 2

Answer and Comments: (A). This is one of those questions where some irrelevant pieces of data are deliberately inserted. The given system of equations is a non-homogeneous system of three linear equations in three unknowns. Such a system has either a unique solution, or no solution or infinitely many solutions, depending on the rank of the matrix A and the rank of the augmented 3×4 matrix obtained by adding one more column to A. It can never have exactly two solutions. The fact that the entries of the matrix A are either 0 or 1 has no bearing on the question. It is just meant to fool the undiscerning candidates into counting the number of such matrices (which comes out to be 2^9).

Those not familiar with properties of matrices can do the problem geometrically. Each equation represents a plane in the three dimensional space. The solutions of the system correspond to points of intersection of these three planes. But no three planes can meet in only two points.

SECTION II

Multiple Correct Choice Type

This section contains five multiple choice questions. Each question has four choices out of which **ONE OR MORE** may be correct.

Q.9 Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as a diameter, then the slope of the line joining A and B can be

(A) $-1/r$ (B) $1/r$ (C) $2/r$ (D) $-2/r$

Answer and Comments: (C), (D). In such problems it is convenient to take the parametric representation of the parabola, viz. $x = t^2, y = 2t$. Let the points A and B be $(t_1^2, 2t_1)$ and $(t_2^2, 2t_2)$. Then the slope, say m, of AB is given by

$$m = \frac{2(t_1 - t_2)}{(t_1^2 - t_2^2)} = \frac{2}{t_1 + t_2} \tag{1}$$

We are given that the circle with AB as a diameter touches the axis of the parabola, viz. the x-axis. This means that the midpoint of AB is at a distance r from the x-axis. Hence its y-coordinate is $\pm r$. This gives

$$(t_1 + t_2) = \pm r \tag{2}$$

(1) and (2) together imply that $m = \pm\frac{2}{r}$. So (C) as well as (D) can be correct.

A very simple problem once the key idea, viz. using parametric representation strikes. Even if A, B are taken as (x_1, y_1) and (x_2, y_2), the calculations are not so prohibitive since in that case we would get

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_1^2 - y_2^2}{(x_1 - x_2)(y_1 + y_2)} = \frac{4}{y_1 + y_2} \tag{3}$$

Moreover the y coordinate of the midpoint of AB is $\frac{y_1 + y_2}{2}$. And equating this with $\pm r$ and using (3) will give the same answer.

Q.10 The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$ is (are)

(A) $\dfrac{22}{7} - \pi$ (B) $2/105$ (C) 0 (D) $\dfrac{71}{15} - \dfrac{3\pi}{2}$

Answer and Comments: (A). The substitution $x = \tan\theta$ is tempting because the denominator of the integrand is $1 + x^2$. But that leads to a complicated integrand involving high powers of $\cos\theta$ in the denominator. The best method to evaluate the integral is by applying long division to the integrand. A direct expansion using the binomial theorem gives

$$\begin{aligned}\frac{x^4(1-x)^4}{1+x^2} &= \frac{x^8 - 4x^7 + 6x^6 - 4x^5 + x^4}{1+x^2} \\ &= \frac{(x^8+x^6) - 4(x^7+x^5) + 5(x^6+x^4) - 4(x^4+x^2) + 4(x^2+1) - 4}{1+x^2} \\ &= x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} \qquad (1)\end{aligned}$$

Term by term integration gives

$$\begin{aligned}\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx &= \frac{1}{7} - \frac{4}{6} + 1 - \frac{4}{3} + 4 - 4\tan^{-1}1 \\ &= \frac{1}{7} + 3 - 4 \times \frac{\pi}{4} \\ &= \frac{22}{7} - \pi \qquad (2)\end{aligned}$$

So, out of the given options, only (A) is correct. If we accept $\dfrac{22}{7}$ as an approximate value of π then (C) is also (approximately) correct. Apparently, the intention behind giving the choice (C) is to test if the candidate knows that π does not equal $\dfrac{22}{7}$. (In fact, π is not a rational number.) The trouble is that a candidate has no way to know this. Outside mathematics, π is often taken to equal $\dfrac{22}{7}$. Even in mathematics, in numerical problems such as those involving mensuration it is not uncommon to replace π by $\dfrac{22}{7}$ to get the answer in a numerical form. (This is also the reason why in numerical problems the radii are often multiples of 7.) Since the possibility of more than one answer being correct is not ruled out, even a good student is likely to mark both (A) and (C) correct, because he may think that the paper-setters want to test if he knows that π is approximately $\dfrac{22}{7}$. In that case, he will be unduly penalised. It would have been better to put this question in Section I where there is only one correct answer. That would not confuse anybody.

Q.11 Let ABC be a triangle such that $\angle ACB = \pi/6$ and let a, b, c denote the lengths of the sides opposite to A, B, C respectively. The value(s) of x for which $a = x^2 + x + 1, b = x^2 - 1$ and $c = 2x + 1$ is (are)

(A) $-(2+\sqrt{3})$ (B) $1+\sqrt{3}$ (C) $2+\sqrt{3}$ (D) $4\sqrt{3}$

Answer and Comments: (B). Using the cosine formula we get

$$\begin{aligned}\cos C &= \frac{a^2+b^2-c^2}{2ab} \\ &= \frac{(x^2+x+1)^2+(x^2-1)^2-(2x+1)^2}{2(x^2+x+1)(x^2-1)} \qquad (1)\end{aligned}$$

Since $\angle C$ is given to be $\pi/6$ and $\cos(\pi/6) = \sqrt{3}/2$, (1) gives us an equation for x, viz.

$$(x^2+x+1)^2+(x^2-1)^2-(2x+1)^2 = \sqrt{3}(x^2+x+1)(x^2-1) \qquad (2)$$

This is an equation of degree 4 and in general not easy to solve. But if we combine the first and the third term on the L.H.S. it factors as $(x^2+3x+2)(x^2-x)$ and further as $(x+1)(x+2)x(x-1)$. Thus each side of (2) has x^2-1 as a factor. Moreover this factor is non-zero since $x^2 = 1$ would mean that the side b is 0. So, canceling this factor, (2) reduces to a quadratic

$$x^2+2x+x^2-1 = \sqrt{3}(x^2+x+1) \qquad (3)$$

i.e.

$$(2-\sqrt{3})x^2+(2-\sqrt{3})-(1+\sqrt{3}) = 0 \qquad (4)$$

Before solving this, we divide it throughout by $2-\sqrt{3}$ and note that the reciprocal of $2-\sqrt{3}$ is simply $2+\sqrt{3}$. So, (5) becomes

$$x^2+x-(5+3\sqrt{3}) = 0 \qquad (5)$$

whose roots are $\dfrac{-1 \pm \sqrt{21+12\sqrt{3}}}{2}$. The negative sign is ruled out as that would make the side c of the triangle negative. The problem is thus reduced to checking which of the four given numbers equals $\dfrac{-1+\sqrt{21+12\sqrt{3}}}{2}$. This can be done by checking these numbers one-by-one. But if we observe that $21+12\sqrt{3}$ is simply $(3+2\sqrt{3})^2$, we see that $\dfrac{-1+\sqrt{21+12\sqrt{3}}}{2}$ equals $1+\sqrt{3}$. Hence (B) is the only correct answer.

Although superficially a problem in trigonometry, the trigonometrical part of the problem is very elementary. The real problem lies in simplification of polynomials and surds. It is not immediately obvious that x^2-1 is a factor of both the sides of (2). Nowadays questions on surds are also not very common and therefore it is not easy to strike that $21+12\sqrt{3}$ is the square of some simple surd. Both these factors make the problem very tricky. Even if a candidate gets these tricks quickly, the rest of the work is also considerable and can hardly be expected in the proportionate time for the question. It may be argued that the other trigonometric question about triangles, viz. Q.4, was exceptionally simple and hence the two questions balance each other.

Another possible solution is based on the identity

$$(x^2-1)^2+(2x+1)^2-(x^2+x+1)^2 = -(x^2-1)(2x+1) \tag{6}$$

which immediately implies that $\cos A = -\dfrac{1}{2}$ and hence that $\angle A = 2\pi/3$. As $\angle C$ is given to equal $\pi/6$, we see that the remaining angle $\angle B$ also equals $\pi/6$. Therefore the triangle ABC is isosceles with $b = c$. This gives us a much simpler quadratic for x, viz. $x^2 - 2x - 2 = 0$, solving which we get that $x = 1+\sqrt{3}$. It is not difficult to prove (6), the key idea once gain being that $x^2 - 1$ is a factor of both the sides. But this is certainly not one of the standard identities and it is doubtful if anybody would think of it as a first step to the solution. There are some identities such as

$$(x^2-1)^2+(2x)^2 = (x^2+1)^2 \tag{7}$$

which implies that a triangle with sides $x^2-1, 2x$ and x^2+1 is a right angled one. This is quite well-known and is indeed used in determining all Pythagorean triples of integers. But (6) is far from well-known. In a conventional examination, the question could have asked a candidate to first show that under the hypothesis, the triangle is isosceles and then to determine the value of x. The candidate would then begin by showing that the remaining angles are either $\pi/6$ and $2\pi/3$ or else $5\pi/12$ each. The first possibility is simpler and that would lead him to come up with (6). But as the question stands, the approach we have given is the only natural one and even then the problem is very tricky.

Q.12 Let z_1 and z_2 be two distinct complex numbers and let $z = (1-t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg (w) denotes the principal argument of a non-zero complex number w, then

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ (B) $\text{Arg}(z - z_1) = \text{Arg}(\text{z} - \text{z}_2)$

(C) $\begin{vmatrix} z - z_1 & \overline{z} - \overline{z_1} \\ z_2 - z_1 & \overline{z_2} - \overline{z_1} \end{vmatrix} = 0$ (D) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

Answer and Comments: (A), (C), (D). A complex number of the form $(1-t)z_1 + tz_2$ (with t real) represents a point on the line passing through z_1 and z_2. The points z_1 and z_2 correspond to $t = 0$ and $t = 1$, while for $0 < t < 1$, the corresponding point lies on the segment from z_1 to z_2. (See Comment No. 16 of Chapter 8 and also Comment No. 15 for what happens if t is complex.) If this is understood then (A) is obvious. The direction of the ray from z to z_1 is opposite to that of the ray from

z to z_2 but is the same as the ray from z_2 to z_1. Even without explicit calculations, this shows that (B) is false while (D) is true. This also implies that $z - z_1 = \lambda(z_2 - z_1)$ for some real λ. But then we also have $\overline{z} - \overline{z_1} = \overline{\lambda}(\overline{z_2} - \overline{z_1}) = \lambda(\overline{z_2} - \overline{z_1})$. Hence the first row of the determinant in (C) is λ times the second. So the determinant vanishes. (Note that for (C) we only need collinearity and not that $0 < t < 1$.)

This is a good problem because once the essential idea, viz. collinearity of z_1, z_2 and z (with z lying in between z_1 and z_2), is understood, no computations are needed. The parameter t has the significance that the point z divides the segment in the ratio $t : 1-t$. This is known as the Section Formula. But that is nowhere needed in the present problem.

Q.13 Let f be a real valued function defined on the interval $(0, \infty)$ by

$$f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\; dt$$

Then which of the following statements is (are) true?

(A) $f''(x)$ exists for all $x \in (0, \infty)$

(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(D) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \leq \beta$ for all $x \in (0, \infty)$.

Answer and Comments: (B), (C). By the second fundamental theorem of calculus, we have

$$f'(x) = \frac{1}{x} + \sqrt{1 + \sin x} \tag{1}$$

for all $x \in (0, \infty)$. Note that the square root function is differentiable when the argument is positive but not (right) differentiable at 0, even though it is continuous there. As a result the second term on the R.H.S. of (1) is not differentiable at points x for which $\sin x = -1$. But it is continuous everywhere as is also the first term. So, (A) is false but (B) holds. For (C) note that for $x \geq 1$, $|f'(x)| = f'(x) \leq 1 + \sqrt{2}$ which is a fixed number. But $f(x)$ is strictly increasing and tends to ∞ as $x \to \infty$. So, there will be some α such that for every $x > \alpha$, $f(x) > 1 + \sqrt{2}$. For any such α, (C) holds. We also have that $f(x)$ is unbounded and so (D) is false as otherwise both f and f' would be bounded.

SECTION III

Paragraph Type

This section contains two paragraphs. There are two multiple choice questions based on the first paragraph and three on the second. All questions have ONLY ONE correct answer.

Paragraph for Q. 14 and 15

The circle $x^2 + y^2 - 8x = 0$ and the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ intersect at the points A and B.

Q.14 Equation of a common tangent with positive slope to the circle as well as to the hyperbola is

(A) $2x - \sqrt{5}y - 20 = 0$ (B) $2x - \sqrt{5}y + 4 = 0$
(C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

Answer and Comments: (B). The easiest way is to begin by assuming that the common tangent has an equation of the form

$$y = mx + c \tag{1}$$

The condition for tangency to each curve will give an equation in m and c. Eliminating c from these two equations, we shall get an equation for m which we then have to solve.

To begin, (1) is a tangent to the hyperbola if and only if

$$c^2 = 9m^2 - 4 \tag{2}$$

(This is a standard result. It is obtained by putting $y = mx + c$ into the equation of the hyperbola and then writing down the condition that the resulting quadratic in x has coinciding roots.)

We can similarly derive a condition that $y = mx + c$ touch the circle $x^2 + y^2 - 8x = 0$. But it is easier to do this geometrically. Completing squares, the centre of the circle is at $(4, 0)$ while its radius is 4. So equating the perpendicular distance of the centre from the line $y = mx + c$ with the radius we get

$$\pm \frac{4m + c}{\sqrt{m^2 + 1}} = 4 \tag{3}$$

which upon squaring gives

$$c^2 + 8mc = 16 \tag{4}$$

To eliminate c between (2) and (4), we first put (2) into (4) to get

$$8mc = 20 - 9m^2 \tag{5}$$

Squaring and again substituting from (2) gives $64m^2(9m^2 - 4) = (20 - 9m^2)^2$ which, upon simplification becomes

$$495m^4 + 104m^2 - 400 = 0 \tag{6}$$

Although superficially this is a fourth degree equation in m, there are no odd degree terms. So we put $u = m^2$ to get

$$495u^2 + 104u - 400 = 0 \tag{7}$$

which is a quadratic in u. We want only the positive root of this which comes out as $\dfrac{-52 + \sqrt{(52)^2 + 495 \times 400}}{495}$, i.e. as $\dfrac{-52 + 4\sqrt{169 + 495 \times 25}}{495}$. The arithmetic involved is rather prohibitive. But if carried out, we get $u = \dfrac{4}{5}$. Recalling that $u = m^2$ and we want only the positive value of m, we get that the slope of the common tangent is $\dfrac{2}{\sqrt{5}}$. As we already know $8mc = 20 - 9m^2$ we get the value of c as $\dfrac{4}{\sqrt{5}}$. Therefore the common tangent $y = mx + c$ becomes $y = \dfrac{2x + 4}{\sqrt{5}}$ or $2x - \sqrt{5}y + 4 = 0$.

The complicated arithmetic involved in solving (7) can be bypassed if we factor the L.H.S. of (7) as $(5u - 4)(99u + 100)$. But this is more like an after thought. If one tries to solve (7) honestly, the time taken is enormous. Here an alert student can take advantage of the fact that the question is designed as a multiple choice question. The slopes of the lines in the options (A) to (D) are, respectively $\dfrac{2}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}, \dfrac{3}{4}$ and $\dfrac{4}{3}$. The corresponding values of u are $\dfrac{4}{5}, \dfrac{4}{5}, \dfrac{9}{16}$ and $\dfrac{16}{9}$. By a direct substitution, the last two are ruled out as roots of (7).

Probably the paper-setters intended to allow this sneak path. Of course a student who wants to rely solely on sneak paths can as well do the sneaking even further and directly verify that the condition for tangency to the hyperbola, viz. $c^2 = 9m^2 - 4$ is satisfied only in (B). So, without even checking if this line touches the circle, he can safely tick (B) as the answer since exactly one of the answers is given to be correct.

Q.15 Equation of the circle with AB as its diameter is

(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$
(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

Answer and Comments: (A). The straightforward way is to begin by actually identifying the points A and B by solving the equations

$$\begin{aligned} x^2 + y^2 - 8x &= 0 \quad (8) \\ \text{and} \quad \frac{x^2}{9} - \frac{y^2}{4} &= 1 \quad (9) \end{aligned}$$

simultaneously. Eliminating y and simplifying we get a quadratic in x, viz.

$$13x^2 - 72x - 36 = 0 \quad (10)$$

which has 6 and $-6/13$ as its roots. When we put $x = 6$ in (9) we get $y^2 = 12$. So $(6, \sqrt{12})$ and $6, -\sqrt{12})$ are two points of intersection. The other value of x, viz. $-6/13$ would make y^2 negative and hence has to be discarded. So the points A and B are $(6, \pm\sqrt{12})$. The equation of the circle with diameter AB can now be written as

$$(x-6)(x-6) + (y - \sqrt{12})(y + \sqrt{12}) = 0 \quad (11)$$

which, upon simplification coincides with (A). Actually, an alert candidate can bypass a lot of work involved. In general two conics intersect in four points. But in the present problem they are *given* to intersect only in two points as otherwise the question would not have a unique meaning. So he need not bother to check what happens when the other root of (10) is put into (9). Actually, the sneaking can go even further. The moment we know that the x-coordinates of the points of intersection of both A and B are 6, we know that in the equation of the circle with AB as a diameter we will have $x^2 - 12x$ and some other terms. In the given options, (A) is the only one where this holds. So, without any further work it is the right answer. Had the paper-setters been a little careful, among the fake options they would have included one in which the x terms are x^-12x.

As it often happens with the so called paragraph type questions, the two questions are totally unrelated. Even though they deal with the same pair of curves, the work done in any one of them does not help in the solution of the other.

Paragraph for Q.16 to Q.18

Let p be an odd prime number and T_p be the following set of 2×2 matrices

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, \ldots, p-1\} \right\}$$

Q.16 The number of A in T_p such that A is either symmetric, or skew symmetric or both and det (A) is divisible by p is

(A) $(p-1)^2$ (B) $2(p-1)$ (C) $(p-1)^2+1$ (D) $2p-1$

Answer and Comments: (D). Let us first assume that A is symmetric. Then $c = b$ and the determinant of A is a^2-b^2 which factors as $(a+b)(a-b)$. Since p is a prime, if it divides $a^2 - b^2$, it has to divide either $a + b$ or $a - b$. Since a and b lie between 0 and $p-1$, the second possibility means $a = b$ while the first one can hold only if $a + b = 0$ or p. $a = b$ has p solutions. As for $a + b = 0$, the only solution is $a = 0, b = 0$ which is already counted. For $a + b = p$, a can take any value from 1 to $p-1$ and then b is uniquely determined. Also note that $a + b = p$ and $a = b$ cannot hold simultaneously since p is odd. Hence in all there are $p + (p-1)$ i.e. $2p-1$ possibilities, each of which gives exactly one symmetric matrix $A \in T_p$ with determinant divisible by p.

Now assume A is skew symmetric. Then $a = 0$ and $c = -b$. Then $\det(A) = b^2$ which is divisible by p only when $b = 0$. But in that case A is the zero matrix which is already counted as a symmetric matrix. Finally, the only matrix which is both symmetric and skew symmetric is the zero matrix which is already counted. So the net count remains at $2p-1$.

The essential idea in the problem is that when a prime divides a product of integers, it must divide at least one of the factors. The material from the matrices needed is little beyond the definitions of symmetry, skew symmetry and the determinant. The counting involved is also elementary. So this problem is a good combination of elementary ideas from three fields. The only disturbing part is that since only the symmetric matrices contribute to the counting, a candidate who eliminates the case of skew symmetric ones by reasoning cannot be distinguished from someone who blissfully ignores them.

Q.17 The number of A in T_p such that the trace of A is not divisible by p but $\det(A)$ is divisible by p is

(A) $(p-1)(p^2-p+1)$ (B) $p^3-(p-1)^2$
(C) $(p-1)^2$ (D) $(p-1)(p^2-2)$

[**Note:** The trace of a matrix is the sum of its diagonal entries.]

Answer and Comments: (C). The trace is $2a$ which is divisible by p if and only if $a = 0$. So we assume $a \neq 0$. For each such fixed a we now need to determine all ordered pairs (b, c) for which $a^2 - bc$ is divisible by p. Note that even though $1 \leq a \leq p-1$, a^2 may be bigger than p. Let r be the remainder when a^2 is divided by p. Then $r \neq 0$ (as otherwise p would divide a) and so $1 \leq r \leq p-1$. The condition that $a^2 - bc$ is divisible by p is equivalent to saying that the integer bc also leaves the remainder r when divided by p. This rules out the possibility that $b = 0$. For every b with $1 \leq b \leq p-1$, we claim that there is precisely one c with $1 \leq c \leq p-1$ such that bc leaves the remainder r when divided by p. To see this consider the remainders when the multiples $b, 2b, 3b, \ldots, (p-1)b$ are divided by p. None of these remainders is 0 as otherwise p would divide b. We claim these remainders are all distinct. For, suppose ib and jb leave the same remainder for some i, j with $1 \leq i < j \leq p-1$. Then p divides $(j-i)b$ which would mean that either p divides $j-i$ or it divides b, both of which are impossible since $j-i$ and b both lie between 1 and $p-1$. Since the remainders left by $b, 2b, \ldots, (p-1)b$ are all distinct, and there are $p-1$ possible values for the remainder, we conclude that the remainder r occurs precisely once in this list. Put differently, for each $b \in \{1, 2, \ldots, p-1\}$, there is precisely one c such that bc leaves the same remainder as a^2 does, which is equivalent to saying that $a^2 - bc$ is divisible by p. So, for each fixed $a \in \{1, 2, \ldots, p-1\}$ there are $p-1$ matrices of the desired type. Since a itself takes $p-1$ distinct values, the total number of desired matrices is $(p-1)^2$.

For those familiar with the language of congruence modulo an integer, (introduced in Comment No. 15 of Chapter 4) the argument above can be paraphrased slightly. Instead of saying that r is the remainder when a^2 is divided by p we say that let $a^2 \equiv r\ (mod\ p)$. The crux of the argument above is that given any $a \not\equiv 0\ (mod\ p)$ and any $b \not\equiv 0\ (mod\ p)$, there is precisely one c such at $a^2 \equiv bc\ (mod\ p)$. The proof, however, remains essentially the same. The proof is interesting because it does not actually exhibit such c, but proves its existence by showing that the congruence classes of the $p-1$ integers $b, 2b, \ldots, (p-1)b$ are all distinct and since there are only $p-1$ (non-zero) congruence classes in all, exactly one of them is the congruency class of a^2. This proof, therefore, is an application of the well known **pigeon hole principle** introduced in Comment No.s 16 and 17 of Chapter 6. In terms of multiplication modulo p, the same argument can be used to show that if b is not divisible by p, then b has an inverse modulo p, i.e. an integer d such that $bd \equiv 1$ (modulo p). In fact, if we use this result, then the integer c above can be quickly identified as the unique integer between 1 to $p-1$ whose congruence class modulo p is the same as that of a^2d, because in the modulo p arithmetic, $a^2 = a^2bd = bc$.

As in the last question, the knowledge of matrices needed is minimal. The question is more on number theory. It is not easy to come up with good short questions in number theory. The paper-setters deserve to be commended for achieving this in the present problem.

Q.18 The number A in T_p such that $\det(A)$ is not divisible by p is

(A) p^2 (B) $p^3 - 5p$ (C) $(p-1)^2$ (D) $p^3 - p^2$

Answer and Comments: (D). The very format of the question suggests that complementary counting is the right tool. Since every matrix in T_p is determined by three mutually independent parameters a, b, c each taking p possible values, in all there are p^3 matrices in the set T_p. We are interested in counting how many of these have a determinant not divisible by p. So let us count how many matrices in T_p have determinants divisible by p. In the last question we did this count when the trace was not divisible by p and that count was $(p-1)^2$. Let us now add to this the number of matrices with both the trace and the determinant being divisible by p. The trace $2a$ is divisible precisely when a is divisible by p, i.e. when $a = 0$. In this case, the determinant is simply $-bc$ which is divisible by p if and only if b or c (or both) is 0. There are $2p-1$ ways this can happen. Thus the number of matrices with determinant divisible by p is $(p-1)^2 + 2p - 1$ which is simply p^2. So, the number of desired matrices is $p^3 - p^2$.

As the answer to the second question of the paragraph is crucially needed to answer the third question, this is a well designed paragraph of questions unlike the last paragraph where the two questions had nothing to do with each other.

SECTION IV

Integer Type

This section contains ten questions. The answer to each question is a single digit integer, ranging from 0 to 9.

Q.19 let f be a real-valued differentiable function defined on $I\!R$ (the set of all real numbers) such that $f(1) = 1$. If the y-intercept of the tangent at any point $P(x, y)$ on the curve $y = f(x)$ is equal to the cube of the abscisa of P, then the value of $f(-3)$ is equal to .... .

Answer and Comments: 9. We first have to formulate the given geometric condition in terms of some equation about the function f and then solve this equation to determine this function. Since the condition involves tangents and hence derivatives, the resulting equation will be a differential equation. This is therefore a problem on geometric applications of differential equations.

Let $P = (x_0, y_0)$ be a typical point on the graph of the function. (In the statement of the question, this point is taken as (x, y). But in that case the same variables cannot be used to write down the equation of the tangent etc. Such an equation is then written with some other variables such as X and Y instead of x and y. We prefer to reserve x and y for the coordinates of any point in the plane and so denote the point P by (x_0, y_0).)

The equation of the tangent at P is

$$y - y_0 = f'(x_0)(x - x_0) \tag{1}$$

The y intercept of this line is $y_0 - x_0 f'(x_0)$ and the given condition says

$$y_0 - x_0 f'(x_0) = x_0^3 \tag{2}$$

Replacing x_0, y_0 by x, y and f' by $\frac{dy}{dx}$ this becomes a differential equation, viz.

$$\frac{dy}{dx} - \frac{y}{x} = -x^2 \tag{3}$$

which is a linear differential equation with integrating factor $e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$. Hence the general solution of (3) is

$$y = f(x) = x[(\int -xdx)] = -\frac{x^3}{2} + cx \tag{4}$$

where c is a constant. The condition $f(1) = 1$ gives $c = \frac{3}{2}$. Hence we have

$$f(x) = -\frac{x^3}{2} + \frac{3}{2}x \tag{5}$$

from which $f(-3) = \frac{27}{2} - \frac{9}{2} = \frac{18}{2} = 9$.

Applications of differential equations of this sort are very common and so the problem is straightforward. But the work involved is time consuming and prone to errors. The only hint is that if the answer is not an integer from 0 to 9, then something has gone wrong.

Q.20 The number of values of θ in the interval $(-\pi/2, \pi/2)$ such that $\theta \neq n\pi/2$, $n = 0, \pm 1, \pm 2$, $\tan\theta = \cot 5\theta$ and $\sin 2\theta = \cos 4\theta$ is .... .

Answer and Comments: 3. Here we have to solve two trigonometric equations in θ simultaneously. The excluded values are those where $\sin 5\theta = 0$ and hence $\cot 5\theta$ is undefined.

We solve the equations separately by converting each to a trigonometric equation where only one trigonometric function of θ is involved. The first equation can be rewritten as $\tan\theta \tan 5\theta = 1$ which gives $\cos 5\theta \cos\theta - \sin 5\theta \sin\theta = 0$, i.e. as

$$\cos 6\theta = 0 \tag{1}$$

whose general solution is $6\theta = (2n+1)\frac{\pi}{2}$ or

$$\theta = (2n+1)\frac{\pi}{12} \tag{2}$$

where n is an integer. The interval $(-\pi/2, \pi/2)$ contains six such values of θ, viz.

$$\theta = \pm\frac{\pi}{12}, \pm\frac{3\pi}{12}, \pm\frac{5\pi}{12} \tag{3}$$

The second equation, viz. $\sin 2\theta = \cos 4\theta$ can be rewritten as

$$\sin(\frac{\pi}{2} - 4\theta) = \sin(2\theta) \tag{4}$$

whose general solution is

$$\frac{\pi}{2} - 4\theta = (-1)^n 2\theta + n\pi \tag{5}$$

where n is an integer. Depending upon whether $n = 2k$ or $2k + 1$ this splits into two sets of solutions, viz.

$$\begin{aligned} 6\theta &= \frac{\pi}{2} - 2k\pi & (6) \\ \text{and} \quad 2\theta &= \frac{\pi}{2} - (2k+1)\pi & (7) \end{aligned}$$

where k is an integer. As we want to compare these values with those in (3) we rewrite them in terms of multiples of $\frac{\pi}{12}$, viz.

$$\begin{aligned} \theta &= \frac{\pi}{12}(1 - 4k) & (8) \\ \text{and} \quad \theta &= \frac{\pi}{12}(3 - 6(2k+1)) & (9) \end{aligned}$$

where again k is an integer. Both of these give only odd multiples of $\frac{\pi}{12}$ and we have to see which of these occur in (3). For (8) this happens when $k = 0, \pm 1$ which give $\theta = \frac{\pi}{12}, -\frac{3\pi}{12}$ and $\theta = \frac{5\pi}{12}$ while for (9) this happens only for $k = 0$ which gives $\theta = -\frac{3\pi}{12}$ which is already covered by (8). Thus we see that the two given equations have 3 solutions in common, viz. $\frac{\pi}{12}, -\frac{3\pi}{12}$ and $\frac{5\pi}{12}$.

This problem, too, is straightforward. But the numerical work needed is lengthy and prone to errors.

Q.21 The maximum value of the expression

$$\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$$

is .... .

Answer and Comments: 2. Denote the denominator by $f(\theta)$. If we complete the squares the denominator can be written as

$$f(\theta) = (\sin\theta + \frac{3}{2}\cos\theta)^2 + \frac{11}{4}\cos^2\theta \tag{1}$$

which shows that the denominator is always positive. Hence the given expression will be maximum when $f(\theta)$ is minimum. This can be found by differentiating $f(\theta)$. A simple calculation gives

$$\begin{aligned} f'(\theta) &= 2\sin\theta\cos\theta + 3\cos^2\theta - 3\sin^2\theta - 10\sin\theta\cos\theta \\ &= 3\cos 2\theta - 4\sin 2\theta \qquad (2) \end{aligned}$$

so that $f'(\theta) = 0$ gives $\tan 2\theta = \frac{3}{4}$. This has two possible solutions viz. $\sin\theta = \pm\frac{3}{5}$, $\cos 2\theta = \pm\frac{4}{5}$. Clearly one possibility gives the maximum while the other gives the minimum of $f(\theta)$. It is not necessary to find the values of θ for which these equations hold. We are interested only in the minimum value of $f(\theta)$ and not the points where it occurs. And this becomes possible by expressing $f(\theta)$ as a function of 2θ. Indeed, we have

$$f(\theta) = 1 + \frac{3}{2}\sin 2\theta + 2 + 2\cos 2\theta \qquad (3)$$

When $\sin 2\theta = \frac{3}{5}, \cos 2\theta = \frac{4}{5}$, we get the maximum while $\sin\theta = -\frac{3}{5}, \cos 2\theta = -\frac{4}{5}$ gives the mimimum of $f(\theta)$. The latter equals $3 - \frac{9}{10} - \frac{8}{5} = \frac{1}{2}$. As observed before, the maximum values of the given expression is the reciprocal of this, viz. 2.

We could have as well found the minimum of $f(\theta)$ directly from (3), without differentiation. For this we use the result that for any positive real numbers the maximum and the minimum of $a\cos 2\theta + b\sin 2\theta$ are $\pm\frac{1}{\sqrt{a^2+b^2}}$. The easiest way to show this is to write the expression as $\sqrt{a^2+b^2}(\sin\alpha\cos 2\theta + \cos\alpha\sin 2\theta) = \sqrt{a^2+b^2}\sin(\alpha + 2\theta)$ where $\alpha = \sin^{-1}(\frac{a}{\sqrt{a^2+b^2}})$. So, in the present case, the minimum value of $f(\theta)$ is $3 - \sqrt{\frac{9}{4} + 4} = 3 - \frac{5}{2} = \frac{1}{2}$, the same value as before.

There are two key ideas in the solution. First, the maximum of the reciprocal is the reciprocal of the minimum. Secondly, for trigonometric functions involving only sines and cosines, the extrema can often be found without differentiation using the fact that the maximum and the minimum of the sine and the cosine functions are 1 and -1. For the first assertion to be valid, the expression must be shown to be positive throughout. Once again, a sincere candidate who spends time proving this is at a disadvantage as compared to a candidate who simply ignores this point.

Q.22 Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

is

Answer and Comments: 1. We already encountered one problem (Q.3) on ω, the complex cube root of unity. The solution to it was based on the equation $\omega^2 + \omega + 1 = 0$. In fact this is the case with most problems involving ω at the JEE level and the present problem is no exception. The value of the given determinant is a function of z and so let us denote it by $D(z)$. It is obvious that $D(z)$ is a cubic polynomial in z and so in general it will have three roots. But they may not be all distinct. To see if this is so, we need to evaluate $D(z)$. This can be done by direct expansion. But the relation $\omega^2 + \omega + 1 = 0$ suggests that a row reduction is a better idea. Indeed if we add the second and the third row to the first, then the first row has a common factor, viz. $z + 1 + \omega + \omega^2$ which is simply z. So we get

$$D(z) = z \begin{vmatrix} 1 & 1 & 1 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} \tag{1}$$

which is manageable enough for a direct expansion. Those who shun direct expansions can subtract the first column from each of the remaining ones and expand w.r.t. the first row. Whichever method is applied, we get (keeping in mind that $\omega^3 = 1$)

$$D(z) = z[z^2 + \omega z + \omega^2 z - \omega z - \omega^2 z] \tag{2}$$

But this is simply z^3. So, the given equation has a triple root 0. Therefore there is only one complex number satisfying it.

This is a good problem but somewhat marred by the duplication of the essential idea which was also used in Q.3.

The given determinant has a special significance. We see that $D(-z)$ is simply the characteristic polynomial of the matrix $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = A$ (say). The roots of this polynomial are called its characteristic roots or eigenvalues and are very important in applications. The present problem says that 0 is an eigenvalue of multiplicity 3 of the matrix A. That 0 is an eigenvalue of A can also be seen from the fact that the matrix A is singular (which follows by showing that its determinant is 0, since the sum of the rows is identically 0. Those familiar with the concept of the rank of a matrix and the nullity of a matrix will recognise that the rank of A is 2 because its second and third columns are multiples of the first. Therefore its nullity is 2, which means that the multiplicity of 0 as an eigenvalue is 2. However, these observations do not prove that it is 3 as is needed in the present problem.

Some structural features of the matrix A are noteworthy. Of course it is a symmetric matrix. But it is much more than that. Each column in it except the first one is obtained from its previous column by cyclically shifting its entries one row upwards. For this reason, a matrix of this form is called a **circulant matrix**.

Q.23 If the distance between the planes $Ax-2y+z=d$ and the plane containing the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$

is $\sqrt{6}$, then $|d|$ is .... .

Answer and Comments: 6. There is an implied hint in the problem that the two planes are parallel to each other. Hence when their equations are written the coefficients of the linear terms will be proportional. This will enable us to identify the unknown A in the equation of the first plane, provided we first identify the second plane. We already know that $(1,2,3)$ is a point on the second plane. Also a normal, say $\mathbf{n}$ to it is given by the cross product of the vectors parallel to the given lines. Thus

$$\mathbf{n}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\mathbf{i}+2\mathbf{j}-\mathbf{k} \tag{1}$$

Hence the equation of the second plane is $-(x-1)+2(y-2)-(z-3)=0$ i.e.

$$-x+2y-z=0 \tag{2}$$

This plane will be parallel to the first plane if and only if $A=1$. Once we know this, the perpendicular distance between the two planes is $\dfrac{|d-0|}{\sqrt{1+4+1}} = \dfrac{|d|}{\sqrt{6}}$. For this to equal $\sqrt{6}$, $|d|$ must equal 6.

A very straightforward problem. The idea of specifying a plane by specifying two lines in it was also used in Q.2. This duplication could have been avoided, for example, by specifying three points in the plane.

Q.24 The line $2x+y=1$ is tangent to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is ... .

Answer and Comments: 2. Another instance of duplication of ideas, this time with Q.14). Writing the given line as $y=-2x+1$, the condition for tangency to the hyperbola gives

$$4a^2-b^2=1 \tag{1}$$

If e denotes the eccentricity of the hyperbola then $b^2 = a^2(e^2 - 1)$ and putting this into (1) we get

$$5a^2 - e^2a^2 = 1 \tag{2}$$

To determine e we need one more equation in a and e. This is provided by the data that the line passes through the point of intersection of the nearest directrix with the x-axis. There are only two directrices of the parabola, viz. the lines $x = \pm\frac{a}{e}$. The statement of the question says that we have to take the nearer of the two. But is is not clear nearer to what. Presumably, it is the point of contact of the line $2x + y = 1$ with the hyperbola. To find it we substitute $y = -2x + 1$ into the equation of the hyperbola and get

$$\frac{x^2}{a^2} - \frac{(-2x+1)^2}{b^2} = 1 \tag{3}$$

which, in view of (1) becomes

$$(4a^2 - 1)x^2 - a^2(-2x+1)^2 - a^2(4a^2 - 1) = 0 \tag{4}$$

Further simplification gives

$$-x^2 + 4a^2x - 4a^4 = 0 \tag{5}$$

which has $x = 2a^2$ as a double root. This is hardly surprising because if there were no double roots then the line will not be a tangent. The point to note is that the x-coordinate of the point of contact of the given line is positive. Therefore, even without finding its y-coordinate, we see that the directrix $x = \frac{a}{e}$ is nearer to it than the directrix $x = -\frac{a}{e}$.

Thus the data now means that the line $2x + y = 1$ passes through the point $(\frac{a}{e}, 0)$. This gives

$$\frac{2a}{e} = 1 \tag{6}$$

or $a = e/2$. Putting this into (2) gives a quartic

$$e^4 - 5e^2 + 1 = 0 \tag{7}$$

which gives $e^2 = 1$ or $e^2 = 4$. The first possibility is ruled out because the eccentricity of a hyperbola is always bigger than 1. The second possibility gives $e = 2$.

Yet another problem which, although straightforward, demands a lot of computation. Especially disturbing is the fact that the data is ambiguous. As shown above, if the nearest directrix is with reference to the point of contact, then some work is needed to make the choice. A candidate who simply assumes the correct choice stands to gain undeservedly in terms of the precious time saved.

Q.25 For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by

$$f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd} \\ 1 + [x] - x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\dfrac{\pi^2}{10} \displaystyle\int_{-10}^{10} f(x) \cos \pi x \, dx$ is .... .

Answer and Comments: 4. The function $[x]$ is popularly called the greatest integer function or the integral part function and it is an unwritten rule that every JEE paper must contain at least one question based on it ! The paper-setters have obeyed the rule. The reason for the popularity of this function is that it is a standard example of a step function. So this function and some other functions derived from it such as the **fractional part function** $\{x\}$ defined as $x - [x]$ figure in many problems about continuity and differentiability. In the present problem, however, we need that this function is periodic with period 1. Further, as the definitions are slightly different depending upon whether $[x]$ is even or odd, we see that the given function $f(x)$ has a period 2 and not 1. Note further that the function $\cos \pi x$ is also periodic with period 2. As the two periods match, we see that the integrand $f(x) \cos \pi x$ is a periodic function of x with period 2π.

We now use a crucial property of the integral of a periodic function, viz. that the integral over any interval whose length equals the period is the same, i.e. independent of the starting point of the interval. So, in the present case, we chop the interval $[-10, 10]$ into 10 subintervals of length 2 each and get

$$\frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx = \pi^2 \int_0^2 f(x) \cos \pi x \, dx \tag{1}$$

Since the definition of $f(x)$ differs on $[0, 1]$ and on $[1, 2]$, we split the integral on the R.H.S. to get

$$\begin{aligned} \int_0^2 f(x) \cos \pi x \, dx &= \int_0^1 f(x) \cos \pi x \, dx + \int_1^2 f(x) \cos \pi x \, dx \\ &= \int_0^1 (1 - x) \cos \pi x \, dx + \int_1^2 (x - 1) \cos \pi x \, dx \\ &= \int_0^1 (1 - x) \cos \pi x \, dx - \int_0^1 x \cos \pi x \, dx \\ &= \int_0^1 (1 - 2x) \cos \pi x \, dx \end{aligned} \tag{2}$$

The last integral can be evaluated by parts as

$$\int_0^1 (1 - 2x) \cos \pi x \, dx = = \frac{1}{\pi}(1 - 2x) \sin \pi x \Big|_0^1 + \frac{2}{\pi} \int_0^1 \sin \pi x \, dx$$

$$\begin{aligned} &= 0 - \frac{2}{\pi^2} \cos \pi x \Big|_0^1 \\ &= \frac{4}{\pi^2} \qquad (3) \end{aligned}$$

Substituting this into (1), we get the given integral as 4.

The problem is a good combination of the properties of the integrals of periodic functions and those of the integral part function. But once again, the computations can hardly be completed within the allocated time.

Q.26 If $\mathbf{a}$ and $\mathbf{b}$ are vectors in space given by $\mathbf{a} = \dfrac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}}$ and $\mathbf{b} = \dfrac{2\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{\sqrt{14}}$, then the value of $(2\mathbf{a} + \mathbf{b}) \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} - 2\mathbf{b})]$ is .... .

Answer and Comments: 5. Note that both $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, although this has little bearing on the problem. The question asks for the value of a scalar triple product of three vectors each of which is expressed in terms of the vectors $\mathbf{a}$ and $\mathbf{b}$. As both these vectors are given explicitly, the most straightforward approach is to express all the three vectors in terms of their components and evaluate the determinant of the coefficient matrix. But as it often happens, sometimes certain vector identities lead to a more elegant solution. In the present case we use the following identity about the box product.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) \qquad (1)$$

(which is a reflection of a certain property of determinants). Applying this to the given dot product we see that

$$(2\mathbf{a} + \mathbf{b}) \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} - 2\mathbf{b})] = (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{a} - 2\mathbf{b}) \times (2\mathbf{a} + \mathbf{b})] \qquad (2)$$

The advantage gained is that the cross product inside the brackets on the R.H.S. can be expanded to get

$$(\mathbf{a} - 2\mathbf{b}) \times (2\mathbf{a} + \mathbf{b}) = \mathbf{a} \times \mathbf{b} - 4(\mathbf{b} \times \mathbf{a}) = 5\mathbf{a} \times \mathbf{b} \qquad (3)$$

If we put this into the R.H.S. of (2) it simply becomes $5|\mathbf{a} \times \mathbf{b}|^2$ which is best evaluated by directly calculating $\mathbf{a} \times \mathbf{b}$ as

$$\mathbf{a} \times \mathbf{b} = \frac{1}{\sqrt{70}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ 2 & 1 & 3 \end{vmatrix} = \frac{1}{\sqrt{70}}(-6\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) \qquad (4)$$

which gives $|\mathbf{a} \times \mathbf{b}| = \dfrac{1}{70}(36 + 9 + 25) = 1$. Hence the R.H.S. of (2) is simply 5.

Instead of resorting to (2), we could have simplified the given expression using an identity about the vector triple product, viz.

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \qquad (5)$$

Using this property and the anti-commutativity of the cross product, we get

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}-2\mathbf{b}) &= -(boa-2\mathbf{b})\times(\mathbf{a}\times\mathbf{b})\\ &= (2\mathbf{b}-\mathbf{a})\times(\mathbf{a}\times\mathbf{b})\\ &= [(2\mathbf{b}-\mathbf{a})\cdot\mathbf{b}]\mathbf{a}-[(2\mathbf{b}-\mathbf{a})\cdot\mathbf{a}]\mathbf{b}\\ &= (2\mathbf{b}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{b})\mathbf{a}-(2\mathbf{b}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{a})\mathbf{b} \qquad (6)\end{aligned}$$

The vectors $\mathbf{a}, \mathbf{b}$ are given explicitly in terms of the components. So we can easily calculate the various dot products and express the R.H.S. of (6) as a linear combination of $\mathbf{a}$ and $\mathbf{b}$. Substituting this into the desired scalar triple product and again putting in the values of the various dot product, we get the numerical value of the scalar triple product. The work involved here is more than in the earlier approach. But that is largely because in the earlier approach, the bracketed expression in the R.H.S. of (2) luckily came out to be a scalar multiple of the vector $\mathbf{a}\times\mathbf{b}$. If instead of $\mathbf{a}\times\mathbf{b}$ we had some other vector the computation would not have been so easy. There is therefore really no way to tell beforehand which approach is better in a particular numerical problem.

Q.27 The number of all possible values of θ where $0<\theta<\pi$, for which the system of equations

$$\begin{aligned}(y+z)\cos 3\theta &= xyz\sin 3\theta \qquad (1)\\ x\sin 3\theta &= \frac{2\cos 3\theta}{y}+\frac{2\sin 3\theta}{z} \qquad (2)\\ (xyz)\sin 3\theta &= (y+2z)\cos 3\theta + y\sin 3\theta \qquad (3)\end{aligned}$$

has a solution (x_0, y_0, z_0) with $y_0z_0 \neq 0$, is .... .

Answer and Comments: 3. This is a combination of systems of equations and trigonometric equations. Here if we take the unknowns as x, y, z (a natural choice) the given system is *not* a system of linear equations. But if we multiply (2) throughout it becomes

$$xyz\sin 3\theta = 2y\sin 3\theta + 2z\cos 3\theta \qquad (4)$$

So, if we take the unknowns as xyz, y and z, then (1), (3) and (4) forms a homogeneous linear system which in the matrix notation can be written as

$$\begin{bmatrix}\sin 3\theta & -\cos 3\theta & -\cos 3\theta\\ \sin 3\theta & -2\sin 3\theta & -2\cos 3\theta\\ \sin 3\theta & -\sin 3\theta-\cos 3\theta & -2\cos 3\theta\end{bmatrix}\begin{bmatrix}xyz\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad (5)$$

This system will have a non-trivial solution if and only if the determinant, say D, of the coefficient matrix vanishes. If we take out common factors

$\sin 3\theta, -1$ and $-\cos 3\theta$ from the first, the second and the last column respectively, we get

$$D = \sin 3\theta \cos 3\theta \begin{vmatrix} 1 & \cos 3\theta & 1 \\ 1 & 2\sin 3\theta & 2 \\ 1 & \sin 3\theta + \cos 3\theta & 2 \end{vmatrix} \tag{6}$$

Subtracting the first row from the other two we get

$$\begin{aligned} D &= \sin 3\theta \cos 3\theta \begin{vmatrix} 1 & \cos 3\theta & 1 \\ 0 & 2\sin 3\theta - \cos 3\theta & 1 \\ 0 & \sin 3\theta & 1 \end{vmatrix} \\ &= \sin 3\theta \cos 3\theta (\sin 3\theta - \cos 3\theta) \end{aligned} \tag{7}$$

Thus D will vanish when any one of the factors vanishes. This gives us three trigonometric equations in θ and the union of their solution sets is the set of values of θ for which the system (5) has a non-trivial solution. But our requirement is more specific. A non-trivial solution simply means a triple (x_0, y_0, z_0) for which at least one of $x_0 y_0 z_0, y_0$ and z_0 is non-zero. But we want only those solutions in which *both* y_0, z_0 are non-zero. So the method above is *not* applicable.

We therefore tackle the problem directly. We see that the term $xyz \sin 3\theta$ appears in all the three equations (1), (3) and (4). If we subtract (4) from (3), both x and z get eliminated and we get

$$y(\cos 3\theta - \sin 3\theta) = 0 \tag{8}$$

For this to have a non-zero solution, we must have $\sin 3\theta = \cos 3\theta$, or equivalently, $\tan 3\theta = 0$. In the interval $(0, \pi)$ this happens only for $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$ and $\frac{9\pi}{12}$. (1) and (3) together imply that

$$z \cos 3\theta + y \sin 3\theta = 0 \tag{9}$$

which, in presence of $\sin 3\theta = \cos 3\theta$, becomes $(y+z)\cos 3\theta = 0$ and hence $z = -y$ since $\cos 3\theta$ cannot vanish when it also equals $\sin 3\theta$. Thus we have $z = -y$. Therefore whenever y is non-zero, so is z.

We are not yet done. We have shown that when $\tan 3\theta = 1$, (8) and (9) have a solution in which both y, z are non-zero and $y + z = 0$. We must now check whether the *original* system (1), (2), (3) has a solution for these values of θ. The very first equation gives $x = 0$. This is also consistent with (2) and (3). Thus in all there are three values of θ (already listed above) for which the original system has a solution in which y, z are both non-zero. Indeed we know that the complete solution set is the set of all triples of the form $(0, y_0, -y_0)$ where $y_0 \neq 0$.

This is a very good problem where a wrong technique (viz. the criterion for a homogeneous system of linear equations to have a non-trivial solution) is very tempting. But once again, a candidate who arrives at the correct answer after going through the analysis above cannot be distinguished from one who ignores the factors $\sin 3\theta$ and $\cos 3\theta$ of D and simply solves $\sin 3\theta = \cos 3\theta$.

Q.28 Let $S_k, k = 1, 2, \ldots, 100$ denote the sum of the infinite geometric series whose first term is $\dfrac{k-1}{k!}$ and the common ratio is $\dfrac{1}{k}$. Then the value of $\dfrac{100^2}{100!} + \displaystyle\sum_{k=1}^{100} \left|(k^2 - 3k + 1)S_k\right|$ is .... .

Answer and Comments: 3. This is a problem about getting a closed form expression for a finite sum whose k-th term is specified as a function of k, viz. $|(k^2 - 3k + 1)S_k|$ where S_k is itself specified indirectly, viz. the sum of an infinite geometric series. The formula for this sum is standard, viz. $\dfrac{a}{1-r}$ where a is the first term and r is the common ratio. But this formula is valid only when $|r| < 1$. In the present case, the common ratio being $1/k$, the sum S_1 has to be calculated directly without this formula. When $k = 1$, the first and hence all the terms of the G.P. are zero and so we have $S_1 = 0$. (It is a little misleading to use the term 'common ratio' for such a geometric progression, because when we take the ratio of two numbers, there is a tacit assumption that the denominator is non-zero. But if we interpret the term 'geometric progression' to mean simply a sequence whose terms are of the form $a, ar, ar^2, \ldots, ar^n, \ldots$, where a and r are some numbers, then S_1 can pass as a geometric progression. Note however, that it does not have a uniquely defined common ratio. Such degeneracies occur elsewhere too. For example, the argument of the complex number 0 is not uniquely defined. A zero vector has no unique direction. And so on.)

Since $S_1 = 0$, we might as well replace the given sum by $\displaystyle\sum_{k=2}^{100} \left|(k^2 - 3k + 1)S_k\right|$.

For $k > 1$, the formula is applicable and we have

$$S_k = \frac{(k-1)/k!}{1-(1/k)} = \frac{k(k-1)}{k!(k-1)} = \frac{1}{(k-1)!} \qquad (1)$$

for all $k > 1$. Let us now analyse the k-th term of the given summation, viz. $|(k^2 - 3k + 1)S_k|$. From (1) we have

$$|(k^2 - 3k + 1)S_k| = \left|\frac{k^2 - 3k + 1}{(k-1)!}\right| \qquad (2)$$

There are well-known formulas for sums of the form $\displaystyle\sum_{k=1}^{n} k$ and $\displaystyle\sum_{k=1}^{n} k^2$. But because of the term $(k-1)!$ in the denominator, they are of little use here. The only way to sum a series whose general term is of the type in (2) is to rewrite it as a telescopic series, i.e. to express the k-th term as the difference of two expressions, say as $a_k - b_k$ in such a way that b_k cancels with a_{k+1}. In the present case, we write the numerator $k^2 - 3k + 1$ as $(k-1)^2 - k$. Then we have

$$\begin{aligned} \frac{k^2 - 3k + 1}{(k-1)!} &= \frac{(k-1)^2 - k}{(k-1)!} \\ &= \frac{k-1}{(k-2)!} - \frac{k}{(k-1)!} \end{aligned} \tag{3}$$

the first fraction is greater than the second whenever $(k-1)^2 > k$ which holds for all $k \geq 3$. So,

$$\left| \frac{k^2 - 3k + 1}{(k-1)!} \right| = \begin{cases} \frac{k-1}{(k-2)!} - \frac{k}{(k-1)!} & \text{for } k \geq 3 \\ \frac{k}{(k-1)!} - \frac{k-1}{(k-2)!} & \text{for } k = 2 \end{cases} \tag{4}$$

As a result, the first term of the (reduced) summation is exceptional. Isolating it, we write

$$\begin{aligned} \sum_{k=2}^{100} \left| \frac{k^2 - 3k + 1}{(k-1)!} \right| &= \sum_{k=2}^{k=2} \left| \frac{k^2 - 3k + 1}{(k-1)!} \right| + \sum_{k=3}^{100} \left| \frac{k^2 - 3k + 1}{(k-1)!} \right| \\ &= \frac{1}{1!} + \sum_{k=3}^{100} \left(\frac{k-1}{(k-2)!} - \frac{k}{(k-1)!} \right) \end{aligned} \tag{5}$$

The last series is a telescopic series which adds up to $\dfrac{3-1}{(3-2)!} - \dfrac{100}{(99)!}$ i.e. to $2 - \dfrac{100^2}{(100)!}$. So, the reduced sum and hence also the given sum equals $\dfrac{1}{1!} + 2 = 3$.

The essential idea in the problem is that of a telescopic series. But to make the answer come out as an integer from 0 to 9, the term $\dfrac{100^2}{100!}$ is added. That makes the problem clumsy. The exceptionality of the first two terms of the summation adds to its clumsiness. And as if all this was not enough, instead of giving S_k directly, it is given as the sum of an infinite geometric series (where again, S_1 has to be obtained differently). Such a thing would have been appropriate in the past for a full length question carrying sufficient weightage allowing the candidates to spend about 6 to 7 minutes for the solution. It is true that in the present format, the candidate does not have to show all his work. But he still has to do it as rough work and that takes time. It is high time that the paper-setters themselves do a realistic assessment of the time it takes to put all the pieces together. What is worse still is that a candidate who does the harder part of the problem (viz. writing the series as a telescopic series) but makes a mistake in some relatively minor part (such as realising that $S_1 = 0$) gets the wrong answer and hence the time he has spent is absolutely wasted.

The term S_1 deserves some comment. Note that the first two ratios in (1) make no sense for $k = 1$, because they are of the indeterminate form $\frac{0}{0}$. But the last term, viz. $\frac{1}{(k-1)!}$, is perfectly well defined even for $k = 1$ and has value 1. So, it is very tempting to take S_1 as 1 rather than as 0, in which case the answer to the question would be 4 and not 3. Since the sum of an infinite series is *defined* as the limit of its partial sums, for the series S_1, all the partial sums are 0 and so the correct value of S_1 is 0. But there is some plausible justification for taking S_1 as 1. First of all, S_1 is the sum of an infinite number of terms, each of which is 0. Hence it is of the form $0 \times \infty$, which is another indeterminate form (closely related to the $\frac{0}{0}$ form). The values of such indeterminate forms can be non-zero if they are expressed as limits of some other type. For example, the expressions $x^2 \cot^4 x, x^4 \cot^4 x$ and $x^6 \cot^4 x$ are all of the form $0 \times \infty$ at $x = 0$. But the three functions approach the limits $\infty, 1$ and 0 respectively as $x \to 0$.

In (1), the variable k is a discrete variable. Can we make it into a continuous variable and then regrad S_1 as $\lim\limits_{k \to 1^+} S_k$? The immediate difficulty in doing so is that the factorials are defined only for integral values of k. But this can be rectified by considering a new function, called the **gamma function** defined by

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx \tag{6}$$

It is easy to show that this integral is finite for all $a > 0$ and further, integration by parts gives the functional equation

$$\Gamma(a+1) = a\Gamma(a) \tag{7}$$

for all $a > 0$. A direct calculation gives $\Gamma(1) = 1$ and then repeated applications of (7) give $\Gamma(k+1) = k!$ for every positive integer k. So, in (1), replacing the factorials by the gamma function and using (6), we can write

$$S_k = \frac{(k-1)/\Gamma(k+1)}{1-(1/k)} = \frac{k(k-1)}{\Gamma(k+1)(k-1)} = \frac{1}{\Gamma(k)} \tag{8}$$

which is valid for all real $k > 1$. So, as $k \to 1^+$, S_k tends to $\frac{1}{\Gamma(1)}$ which equals 1 because the gamma function can be shown to be continuous at all $a > 0$. Therefore it is not entirely unreasonable to take S_1 as 1.

Similar ambiguities arise with other indeterminate forms. For example, what meaning do we give to the power 0^0? On one hand, for every non-zero x, the power x^0 equals 1 and so by taking the limit as $x \to 0$, we should set $0^0 = 1$. But we might as well keep the base fixed and let the exponent vary. In that case for every $y \neq 0$, 0^y equals 0 and so 0^0 should be given the value 0. The first interpretation is generally taken to prevail.

The term S_1 has nothing to do with the main theme of the problem, viz. the telescopic series. By including it, the paper-setters probably wanted to test if a candidate is careful enough to realise that it has to be treated differently from S_k for $k > 1$. But then they ought to have exercised equal care by avoiding the phrase 'common ratio' which is not applicable to S_1. Their failure to do so gives some credence to treat S_1 on par with the remaining S_k's and makes the problem controversial. It would have been better to drop this term which anyway has nothing to do with the main theme of the problem, viz. the telescopic series.

PAPER 2

Contents

SECTION I

Single Correct Choice Type

This section contains 6 multiple choice questions. Each question has 4 choices out of which **ONLY ONE** is correct.

Q.1 Let $S = \{1, 2, 3, 4\}$. The total number of unordered pairs of disjoint subsets of S is equal to

(A) 25 (B) 34 (C) 42 (D) 41

Answer and Comments: (D). Let us first consider all *ordered* pairs $(A, B))$ of mutually disjoint subsets of S. The number of elements in the subset A can be any integer, say r from 0 to 4. For each such r, A can be chosen in $\binom{4}{r}$ ways. Once the subset A is fixed is fixed, the subset B can be any subset of $S - A$, the complement of A in S. Since $S - A$ has $4 - r$ elements, the number of its subsets (including the empty set) is 2^{4-r}. Hence the total number, say T, of ordered pairs of mutually disjoint subsets of S is

$$T = \sum_{r=0}^{4} \binom{4}{r} 2^{4-r} \tag{1}$$

As there are only five terms in the sum, it is possible to evaluate it simply by computing each term and adding. The count comes out as

$$T = 2^4 + 4 \times 2^3 + 6 \times 2^2 + 4 \times 2^1 + 2^0 = 16 + 32 + 24 + 8 + 1 = 81 \tag{2}$$

An alert reader will hardly fail to notice that the coefficients are precisely those that would occur in a binomial expansion and so there has to be an easier way to evaluate this sum. This guess is correct. If we multiply the r-th term in (1) by the power 1^r which equals 1 for every r, we see that

$$T = \sum_{r=0}^{4} \binom{4}{r} 2^{4-r} 1^r = (2 + 1)^4 = 3^4 = 81 \tag{3}$$

In fact, this shows that if the set S had n elements to begin with then the number, say T_n, of all ordered pairs (A, B) of mutually disjoint subsets of S would be 3^{n+1}.

Now, coming to the problem, it asks for *unordered* pairs $\{A, B\}$ instead of ordered pairs (A, B) of mutually disjoint subsets. In other words we now have to identify the pair (A, B) with (B, A). These two pairs are distinct since two mutually disjoint subsets can never be the same, except when both of them are the empty sets. hence out of the 81 ordered pairs, 80 pairs pair off into 40 pairs, each giving rise to an unordered pair, while the pair $(\emptyset, \emptyset)$ remains the same after interchanging its entries. Thus in all there are 41 unordered pairs.

More generally, if S had n elements the answer would be $\dfrac{3^n - 1}{2} + 1$.

There is a far more elegant way to count the number T_n of all mutually disjoint pairs of subsets of a set S with n elements. (In our problem, $n = 4$.) The idea is to get a one-to-one correspondence between the set of all ordered pairs (A, B) of mutually disjoint subsets of S and the set of all functions from the set S to a set with three elements. Given any such ordered pair, for every $x \in S$, exactly one possibility holds, either x is in A, or it is in B or it is in neither. We let $f(x) = a, b, c$ depending upon which possibility holds. This gives us a function from S to the three element set $\{a, b, c\}$. Conversely every such function determines an ordered pair (A, B) of mutually disjoint subsets of S, if we let $A = f^{-1}(a)$ and $B = f^{-1}(b)$. There are in all 3^n functions from S to $\{a, b, c\}$. Hence we have $T_n = 3^n$. If we want unordered pairs, the answer is $\dfrac{3^n - 1}{2}$.

The problem is a good one, based on elementary counting. The cardinality of the set S is kept small enough to enable those who can't think of either the solution using the binomial theorem or the elegant solution at the end. Problems based on this idea have been asked before in the JEE. (See the 1990 problem in Comment No. 17 of Chapter 5, and also its alternate solution in Chapter 24.)

Q.2 For $r = 0, 1, \ldots, 10$, let A_r, B_2, C_r denote respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}, (1 + x)^{20}$ and $(1 + x)^{30}$. Then

$$\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$$

equals

(A) $B_{10} - C_{10}$ (B) $A_{10}(B_{10}^2 - C_{10}A_{10})$ (C) 0 (D) $C_{10} - B_{10}$

Answer and Comments: (D). This is a binomial sum. The terms A_{10}, B_{10} and C_{10} are independent of r. Hence the given sum, say S, equals

$$S = B_{10} \sum_{r=1}^{10} A_r B_r - C_{10} \sum_{r=1}^{10} A_r A_r \tag{1}$$

There is in general no formula for the sums of products of the binomial coefficients. But in the present case because of the symmetry relations for the binomial coefficients, we have

$$\begin{aligned} B_r &= B_{20-r} && (2) \\ \text{and} \quad A_r &= A_{10-r} && (3) \end{aligned}$$

for every $r = 1, 2, \ldots, 10$. So, we can rewrite S as

$$S = B_{10} \sum_{r=1}^{10} A_r B_{20-r} - C_{10} \sum_{r=1}^{10} A_r A_{10-r} \tag{4}$$

The reason for these replacements is that there is a way to sum the products of binomial coefficients when their suffixes add to a fixed number. Specifically, we have the following identity for any non-negative integers p, q, m.

$$\sum_{r=0}^{m} \binom{p}{r} \binom{q}{m-r} = \binom{p+q}{m} \tag{5}$$

which is proved by writing $(1+x)^p(1+x)^q$ as $(1+x)^{p+q}$ and collecting the coefficient of x^m from both the sides. (If the value of some r exceeds the exponent p or q, then the corresponding binomial coefficient is to be taken as 0. That way we do not have to worry about how big m is as compared to p and q. Indeed, it could even be bigger than both p and q.)

We apply (5) with $m = 10$ to each of the sums in (4). Note that in both these sums the index variable only varies from 1 to 10 and not from 0 to 10. So allowing for the missing terms, we have

$$\begin{aligned} S &= B_{10}(C_{10} - A_0 B_{20} - C_{10}(B_{10} - A_0 A_{10} \\ &= B_{10}(C_{10} - 1) - C_{10}(B_{10} - 1) \\ &= C_{10} - B_{10} && (6) \end{aligned}$$

This is a problem based on two identities about the binomial coefficients. Both are standard and problems based on them have appeared many times (see Chapter 5 for examples). In the multiple choice format, it is very difficult to ask any radically new problem.

Q.3 Let f be a real valued function defined on the interval $(-1, 1)$ such that

$$e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4 + 1}\, dt$$

for all $x \in (-1, 1)$ and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to

(A) 1 (B) 1/3 (C) 1/2 (D) $1/e$

Answer and Comments: (B). Note that the question does not ask us to show that the inverse function exists. To see this, rewrite the given equation as

$$f(x) = 2e^x + e^x \int_0^x \sqrt{t^4+1}\, dt \tag{1}$$

The function e^x is strictly increasing everywhere. As the integrand $\sqrt{t^4+1}$ is positive everywhere, the function defined by its integral is also strictly increasing as x increases. Hence $f(x)$ is strictly increasing on $(-1, 1)$ (in fact on the entire real line) and therefore the inverse function definitely exists (as a consequence of the Intermediate Value property of continuous functions), its domain being the range of the function $f(x)$. For notational simplicity, denote the inverse function f^{-1} by g. We are asked to find $g'(2)$.

The theorem about the derivative of an inverse function says that if $f(x_0) = y_0$, then

$$g'(y_0) = \frac{1}{f'(x_0)} \tag{2}$$

In the present problem y_0 is given as 2. So first we have to find x_0 such that $f(x_0) = y_0$. Had we known a formula for g we would simply let $x_0 = g(x_0)$. But in the present problem, it is not easy to express $g(x)$ by an explicit formula. Nevertheless, by inspection we see from (1) that $f(0) = 2$. (Probably the reason to specify the data in a different form than (1), which is the most natural way to express a function, was to make it a little more difficult to see that $f^{-1}(2) = 0$.)

Our problem is now reduced to finding $f'(0)$. For this we differentiate (1) using the second form of the fundamental theorem of calculus to get

$$f'(x) = 2e^x + e^x \int_0^x \sqrt{t^4+1}\, dt + e^x\sqrt{x^4+1} \tag{3}$$

which directly gives $f'(0) = 2 + 1 = 3$. Hence $(f^{-1})'(2) = 1/3$.

The crux of the problem is that even though the inverse function f^{-1} cannot be described by an explicit formula, its derivative at a particular point y_0 can be obtained if we are able to find (or rather guess) the inverse image of this particular point y_0. The problem per se is very simple.

Q.4 If the distance of the point $P(1, -2, 1)$ from the plane $x + 2y - 2z = \alpha$ where $\alpha > 0$, is 5, then the foot of the perpendicular to the plane is

(A) (8/3, 4/3, −7/3) (B) (4/3, 4−/3, 1/3)
(C) (1/3, 2/3, 10/3) (D) (2/3, −1/3, 5/2)

Answer and Comments: (A). A straightforward problem in solid coordinate geometry. The formula for the perpendicular distance of a point from a plane gives

$$5 = \frac{|1 - 4 - 2 - \alpha|}{\sqrt{1 + 4 + 4}} = \frac{|5 + \alpha|}{3} \tag{1}$$

which implies that $\alpha + 5 = \pm 15$ and hence $\alpha = 10$ or $\alpha = -20$. As α is given to be positive, we have $\alpha = 10$. So the equation of the plane now is

$$x + 2y - 2z = 10 \tag{2}$$

Let $Q = (x_0, y_0, z_0)$ be the foot of the perpendicular from P to this plane. We express the coordinates of Q in terms of a parameter. For this we note that the line joining P to Q is perpendicular to the plane and hence parallel to the normal to the plane. So we must have

$$\frac{x_0 - 1}{1} = \frac{y_0 + 2}{2} = \frac{z_0 - 1}{-2} \tag{3}$$

Let each ratio equal t. Then we have $Q = (t + 1, 2t - 2, -2t + 1)$. As this point lies on the plane (2), we have $t + 1 + 4t - 4 + 4t - 2 = 10$ which yields $t = 5/3$. Hence from (3), $x_0 = 5/3 + 1 = 8/3$. We can similarly determine y_0 and z_0. But an alert student will not waste his time in doing so, since among the given options (A) is the only one where the first coordinate is 8/3. At least for such an extremely straightforward problem like this the paper-setters should have designed the fake answers so as to preclude such short cuts.

Q.5 A signal which can be green or red with probability 4/5 or 1/5 respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is 3/4. If the signal received at station B is green, then the probability that the original signal was green is

(A) 3/5 (B) 6/7 (C) 20/23 (D) 9/20

Answer and Comments: An interesting problem on probability. Also the setting is different from the usual tossing of coins or dice or drawing balls from an urn or cards from a pack. The estimation of accuracy in signal transmission is an important problem in the theory of communication. So apart from its mathematical content, the paper-setters deserve some appreciation for giving a very inviting (and colourful!) garb to the problem.

Now, coming to the problem itself, we are given that the signal received by B is green. This can happen in any of the following mutually exclusive ways:

E_1: the original signal was green and both A and B received their respective signals correctly

E_2: the original signal was green and both the stations received their respective signals wrongly

E_3: the original signal was red and A received it correctly but B received the retransmitted signal wrongly, and

E_4: the original signal was red, A received it wrongly but B received the transmitted signal correctly.

Denote the disjunction of these four events by E and the disjunction of the first two events by F. The problem amounts to finding the conditional probability of F given E. The easiest way to do this is to calculate the probability $P(E_i)$ for $i = 1, 2, 3, 4$ and express $P(E)$ and $P(F)$ in terms of these.

Let us begin by $P(E_1)$. This is a conjunction of three mutually independent events, viz. the original signal is green, the reception at station A is correct and thirdly, that at B is also correct. The probabilities of these three are given to be $\frac{4}{5}$, $\frac{3}{4}$ and $\frac{3}{4}$ respectively. So, we have

$$P(E_1) = \frac{4}{5} \times \frac{3}{4} \times \frac{3}{4} = \frac{36}{80} \tag{1}$$

By a similar reasoning, we have

$$P(E_2) = \frac{4}{5} \times \frac{1}{4} \times \frac{1}{4} = \frac{4}{80} \tag{2}$$

$$P(E_3) = \frac{1}{5} \times \frac{3}{4} \times \frac{1}{4} = \frac{3}{80} \tag{3}$$

$$P(E_4) = \frac{1}{5} \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{80} \tag{4}$$

Adding all four,

$$P(E) = P(E_1) + P(E_2) + P(E_3) + P(E_4) = \frac{46}{80} \tag{5}$$

while, adding only the first two,

$$P(F) = P(E_1) + P(E_2) = \frac{40}{80} \tag{6}$$

The desired probability is the ratio $P(F)/P(E)$ which comes out as 40/46 i.e. 20/23.

It is easy to generalise this to the case where instead of two stations we have a series of stations, say, $A_1, A_2, \ldots, A_n$, each (except the last) transmitting the received signal to the next one. Assume that the probability of correct reception at each station A_i is p. (In a more realistic situation, these probabilities could change from station to station. But we do only the simple case.) Let α be the probability that the original signal is green. Now suppose that the signal received by the last station is green. Then there are two possibilities: (i) the original signal was green and the reception at an even number of the stations was wrong or (ii) the original signal was red and the reception at an odd number of stations was wrong. Call these events as A and B respectively. Then we have

$$\begin{aligned} P(A) &= \alpha \sum_{r \text{ even}}^{n} \binom{n}{r} p^{n-r} q^r \\ \text{and} \quad P(B) &= (1-\alpha) \sum_{r \text{ odd}}^{n} \binom{n}{r} p^{n-r} q^r \end{aligned} \tag{7}$$

where, as usual, $q = 1 - p$ is the probability of a wrong reception. The desired probability now is $\dfrac{P(A)}{P(A)+P(B)}$. In some cases (e.g. when $p = q = \frac{1}{2}$) it is possible to express these sums in a closed form.

overall, this is a good problem on conditional probability.

Q.6 Two adjacent sides of a parallelogram $ABCD$ are given by $\overrightarrow{AB} = 2\vec{i} + 10\vec{j} + 11\vec{k}$ and $\overrightarrow{AD} = -\vec{i} + 2\vec{j} + 2\vec{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then cosine of the angle α is given by

(A) $8/9$ (B) $\sqrt{17}/9$ (C) $1/9$ (D) $4\sqrt{5}/9$

Answer and Comments: (B). The problem is apparently designed to test a candidate's ability to weed out the irrelevant portions of the data and focus only on the essential part. It is not clear what role is played by the parallelogram. In essence, we are given three vectors $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AD'}$ all in the same plane along with the information that $\overrightarrow{AD'}$ is obtained from $\overrightarrow{AD}$ by a rotation through an acute angle α and further that it is perpendicular to the vector $\overrightarrow{AB}$ and we are asked to find $\cos\alpha$. Under these circumstances it is very obvious that the angle between $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is $\frac{\pi}{2} - \alpha$ and therefore

$$\cos\alpha = \sin\theta \tag{1}$$

where θ is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$. We are given both these vectors explicitly. So, we have

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AD}}{|\overrightarrow{AB}|\,|\overrightarrow{AD}|}$$

$$= \frac{-2+20+22}{\sqrt{4+100+121}\sqrt{1+4+4}}$$
$$= \frac{40}{15\times 3} = \frac{8}{9} \qquad (2)$$

It is now immediate that $\cos\alpha = \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(8/9)^2} = \sqrt{17}/9$.

The essence of the problem is merely that if three lines in the same plane are concurrent and two of them are mutually perpendicular, then the acute angles made by the third line with these two lines are complementary.

SECTION II

Integer Type

This section contains five questions. The answer to each question is a single digit integer, ranging from 0 to 9.

Q.7 Consider a triangle ABC and let a, b, c denote the lengths of the sides opposite to the vertices A, B, C respectively. Suppose $a = 6, b = 10$ and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and r denotes the radius of the incircle of the triangle, then r^2 equals .... .

Answer and Comments: 3. Yet another problem on solving a triangle. There were already two questions in Paper I (Q.4 and Q.11) where it was given that a, b, c denote the lengths of the sides opposite to the vertices A, B, C respectively. This is such a standard practice that many times in the past, JEE questions have been asked without this elaboration. One wonders why this has been done now and that too several times. Perhaps the idea is to help those who have studied in other media. The notation r for the inradius is also standard.

Coming back to the problems, there is a huge number of identities involving the sides, the angles, the inradius, the circumradius, the area, the perimeter and so on of a triangle. It is therefore often possible to arrive at the answer in a variety of ways (see Chapter 11 for many illustrations of this.) Usually, it is a good idea to spend some time to see which approach will be better for a given problem. In the present problem, we are given the area, say Δ of the triangle and asked to find the inradius r. There is a simple formula which connects these two., viz.

$$r = \frac{\Delta}{s} \qquad (1)$$

where $s = \frac{1}{2}(a+b+c)$ is the semi-perimeter of the triangle. We are already given a and b. So we would be done if we can find the third side c. This can be done by yet another formula which expresses the area directly in terms of the sides, viz.

$$16\Delta^2 = (a+b+c)(a+b-c)(a-b+c)(b+c-a) \tag{2}$$

Here everything except c is known and so substituting their values, we shall get an equation in c. superficially this will be a fourth degree equation in c. But this need not deter us. If we carefully group the factors, we see that there are no odd degree terms in c. (This is similar to the equation in m we encountered in Q.14 of Paper I.) So, if we put $\Delta^2 = (15\sqrt{3})^2 = 675, a = 6$ and $b = 10$ in (2) we get

$$10800 = (16+c)(16-c)(c-4)(c+4) = (256-c^2)(c^2-16) \tag{3}$$

If we put $u = c^2$, this is a quadratic in u. Unfortunately, the arithmetic involved would be rather complicated. (This was also the case in Q.14 of Paper I. But there we could sneak by looking at the given answers. In the present problem this is not easy since the question does not ask for c but something else.) So, let us try some other formula for the area. There is a formula which gives the area in terms of any two sides and the included angle, viz.

$$\Delta = \frac{1}{2}ab\sin C \tag{4}$$

which gives $15\sqrt{3} = 30\sin C$ and hence $\sin C = \frac{\sqrt{3}}{2}$. Hence $\cos C = \pm\frac{1}{2}$. But as $\angle C$ is given to be obtuse, we have $\cos C = -\frac{1}{2}$. The cosine formula

$$\cos C = \frac{a^2+b^2-c^2}{2ab} \tag{5}$$

now gives $a^2+b^2-c^2 = -ab$ i.e. $c^2 = a^2+b^2+ab = 36+100+60 = 196$. Hence $c = 14$. (We can now easily verify that this value of c does satisfy (3). But that is an afterthought.) Therefore the semi-perimeter s equals 15 and hence using (1) we get the inradius r equals $\sqrt{3}$. Therefore $r^2 = 3$.

The problem is easy provided you identify the easy approach correctly.

Q.8 Let $a_1, a_2, \ldots, a_{11}$ be real numbers satisfying $a_1 = 15, 27 - a_2 > 0$ and

$$a_k = 2a_{k-1} - a_{k-2}$$

for $k = 3, 4, \ldots, 11$. If $\frac{a_1^2 + a_2^2 + \ldots + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + \ldots + a_{11}}{11}$ is .... .

Answer and Comments: 0. If we rewrite the formula for a_k as

$$a_k - a_{k-1} = a_{k-1} - a_{k-2} \tag{1}$$

for $k \geq 3$, then it is clear that the numbers are in an A.P. Once this idea strikes the rest of the solution is routine. We are given that the first term of this A.P. is 15 but we are not given the common difference. Instead, we are given the sum of the squares of the terms. From this we first have to find the common difference, say d. We have the formula

$$\begin{aligned}\sum_{k=1}^{11} a_k^2 &= \sum_{k=0}^{10}(15+kd)^2 \\ &= 11 \times 225 + 30d\sum_{k=0}^{10} k + d^2\sum_{k=0}^{10} k^2 \\ &= 11 \times 225 + 30d \times 55 + \frac{10 \times 11 \times 21}{6} d^2 \qquad (2)\end{aligned}$$

We are given that this sum equals 990. Canceling the factors 11 and 5 from all the terms this gives us a quadratic in d, viz.

$$7d^2 + 30d + 27 = 0 \qquad (3)$$

whose roots are $\dfrac{-15 \pm \sqrt{225-189}}{7} = \dfrac{-15 \pm 6}{7}$, i.e. -3 and $-\dfrac{9}{7}$. With these values a_2 becomes $15 - 3$ and $15 - (9/7)$. The latter possibility contradicts $a_2 < 13.5$. Hence we must have $d = -3$. To get the answer we can evaluate $\dfrac{a_1 + a_2 + \ldots + a_{11}}{11}$ by a summation formula. But that is not necessary. The desired number is the arithmetic mean of an odd number of terms in an A.P. So it simply equals the middle term, i.e. the 6-th term in the present case. Since $a_6 = a_1 + 5d = 15 - 15 = 0$, the given expression is 0.

This is a straightforward problem on arithmetic progressions. But some twists are given. For example, it is not given directly that the numbers are in an A.P. Secondly, the common difference d has to be obtained after solving a quadratic in d and discarding one of the roots using the condition on the second term. And finally, the candidate also has to realise that the arithmetic mean of the terms of an A.P. is the middle term (or the average of the middle two terms in case their number is even). The last twist does test some alertness on the part of the candidate. The other twists only serve to complicate the problem.

Q.9 For a square matrix M, let adj$M$ denote its adjoint and for a real number k, let $[k]$ denote the largest integer less than or equal to k. Now for a positive real number k, let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If $\det(\text{adj}A) + \det(\text{adj}B) = 10^6$, then $[k]$ equals .... .

Answer and Comments: 4. Yet another problem where the main idea has been given several twists. The main idea of this problem is the determinant of the adjoint of a square matrix. If M is an $n \times n$ matrix then we have the following result about its relationship with its adjoint.

$$M\text{adj}(M) = DI_n \tag{1}$$

where D is the determinant of M and I_n is the identity matrix of order n. Note that the matrix on the R.H.S. is a diagonal matrix of order n whose diagonal entries all equal D. Obviously, its determinant is D^n. On the other hand the L.H.S. is a product of two matrices and so its determinant is the product of the determinants of the two matrices. Keeping in mind that $\det(M) = D$, (1) implies that

$$D\det(\text{adj}M) = D^n \tag{2}$$

If M is non-singular, we can divide by D to get

$$\det(\text{adj}M) = D^{n-1} \tag{3}$$

which remains valid even when M is singular, for in that case $D = 0$ and the R.H.S. of (1) is the zero matrix, whence $\text{adj}(M)$ is also singular and hence has a vanishing determinant. This is the basic formula we need in the present problem. We have included its derivation because at the JEE level this formula is not so well-known. We apply this formula with $n = 3$ to the 3×3 matrices A and B. A direct calculation gives

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix} \\
&= (2k-1)(4k^2-1) + 2\sqrt{k}(4k\sqrt{k} + 2\sqrt{k}) + 2\sqrt{k}(4k\sqrt{k} + 2\sqrt{k}) \\
&= 8k^3 - 4k^2 - 2k + 1 + 8k^2 + 4k + 8k^2 + 4k \\
&= 8k^3 + 12k^2 + 6k + 1 \\
&= (2k+1)^3 \qquad (4)
\end{aligned}$$

We can similarly calculate $\det(B)$. But there is a better way. Note that B is skew-symmetric. So, by definition, $B^t = -B$ where the exponent t stands for the transpose and not for the power. Since a matrix has the same determinant as its transpose, we have $\det(B) = \det(B^t) = \det(-B) = (-1)^3\det(B) = -\det(B)$ which shows that $\det(B) = 0$. (More generally, the same argument shows that the determinant of every skew-symmetric matrix of odd order is 0.)

Now that we know the determinants of the matrices A and B, (3) gives us the determinants of their adjoints as $((2k+1)^3)^2 = (2k+1)^6$ and 0 respectively. We are given that these add up to 10^6. Thus we have $2k+1 = \pm 10$. The negative sign is excluded since k is given to be positive.

So we have $2k + 1 = 10$ and hence $k = 9/2$. Therefore the integral part $[k]$ equals 4.

The problem is a good application of the formula for the adjoint of a determinant. But this formula is not so well known at the JEE level. Those who do not know it will foolishly try to calculate $\det(\text{adj}A)$ by actually writing down all the nine entries of the adjoint matrix. Even those who know the formula are likely to commit a numerical slip in the calculation of $\det(A)$. The addition of the matrix B to the problem only serves to complicate it. The work needed to show that its determinant is 0 is quite independent of that needed to calculate the determinant of A, which itself is quite substantial. So, those who calculate $\det(A)$ correctly but miss on $\det(B)$ get heavily penalised as there is no provision for partial credit. Finally, it is not clear what is achieved by designing the problem so as to make k come out to be fractional. The knowledge of the definition of the integral part of a number has been tested several times, and in any case, it is very elementary as compared to the knowledge of a formula like (3) or the knowledge of the fact that every skew-symmetric matrix of an odd order is singular. Why unnecessarily add to the work of a candidate when it is totally irrelevant to the main theme of the problem? Perhaps the idea was to make the answer come out as an integer. But that could have as well been served by asking the value of $2k$ rather than k. Or, still better, if the sum of the determinants of the adjoints of the two matrices were given as 11^6 instead of 10^6, then k itself would come out as an integer, viz. 5 and this ugly addition of the integral part could have been avoided.

Problems testing several (possibly independent) ideas can be asked as full length questions where a candidate has to show his work and therefore gets some partial credit. In a multiple choice test with a large number of questions, where a candidate can barely spend a minute or two on each question, designing such questions is not only unfair to the candidates, but it also distorts the selection. A good candidate may get rejected despite despite doing nearly all the hard work correctly but then committing some trivial slip at the end (e.g. writing $[9/2]$ as 5 instead of 4 in the present problem). This is rather like rejecting an otherwise excellent essay on the biography of Mahatma Gandhi simply because it gives his birth date wrong. (Another example occurred in Q.28 of Paper I.) The present problem would have been an excellent question in a conventional examination. But the present multiple choice format of JEE has killed it.

Q.10 Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3} + 1$ apart. If the chords subtend at the centre angles of π/k and $2\pi/k$, where $k > 0$, then the value of $[k]$ is .... .

Answer and Comments: 3. Yet another problem where the integral part had to be introduced to make the answer come out as an integer. The problem itself is a simple trigonometric problem. As the distance between the two chords exceeds 2, they must lie on opposite sides of the centre. Let d_1, d_2 be their perpendicular distances from the centre. Then we have

$$d_1 + d_2 = \sqrt{3} + 1 \tag{1}$$

If θ is the angle which a chord of a circle of radius r subtends at the centre, then its distance from the centre is $r\cos(\theta/2)$. In our problem, $r = 2$ and the angles subtended by the chords are given to be π/k and $2\pi/k$. Substituting into (1) we get

$$2\cos(\pi/2k) + 2\cos(\pi/k) = \sqrt{3} + 1 \tag{2}$$

Let us call $\pi/2k$ as θ. Then (2) becomes a trigonometric equation in θ, viz.

$$\cos\theta + \cos 2\theta = \frac{\sqrt{3}}{2} + \frac{1}{2} \tag{3}$$

To solve this we reduce it to a quadratic in $\cos\theta$, viz.

$$2\cos^2\theta + \cos\theta - \frac{3}{2} - \frac{\sqrt{3}}{2} = 0 \tag{4}$$

whose roots are $\cos\theta = \dfrac{-1 \pm \sqrt{13 + 4\sqrt{3}}}{4}$. The negative square root is to be discarded since it would make $\cos\theta < -1$. Further simplification hinges on recognising $13 + 4\sqrt{3}$ as $(1 + 2\sqrt{3})^2$. We now have

$$\cos\theta = \frac{-1 + 1 + 2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \tag{5}$$

which gives $\theta = \pi/6$. So, we have $\pi/2k = \pi/6$ which gives $k = 3$ and hence $[k] = 3$ too. As k itself comes out to be an integer, asking to find out its integral part is only a dirty trick played with the candidates by inducing them to think that k would come out to be fractional.

We have solved the trigonometric equation (3) systematically and as in Q.11 of Paper I, the success depended on recognising a simple surd as the square of another simple surd. (It is significant that that question was also on trigonometry.) But when you merely have to guess a solution of (3), we rewrite the R.H.S. as $\cos\dfrac{\pi}{6} + \cos\dfrac{\pi}{3}$. In other words it is the sum of the cosines of two angles one of which is the double of the other. But so is the L.H.S. So equating the smaller angles of both the sides we get $\theta = \dfrac{\pi}{6}$. This approach is time saving but not mathematically sound. It will be so if we show that if α, β are two angles in $(0, \pi/2)$, then $\cos\alpha + \cos 2\alpha = \cos\beta + \cos 2\beta$ can hold only when $\alpha = \beta$. This can be done algebraically. But it follows more slickly if we observe that if $\alpha < \beta$ (say), then we would also have $2\alpha < 2\beta$ and hence $\cos\alpha > \cos\beta$ and also $\cos 2\alpha > \cos 2\beta$ since the cosine function is strictly decreasing on the interval $[0, \pi]$ which contains $\alpha, \beta, 2\alpha$ and 2β. But then we would have $\cos\alpha + \cos 2\alpha > \cos\beta + \cos 2\beta$, a contradiction.

This problem is of the same spirit as the Main Problem of Chapter 10. In both the problems, a geometric data is reduced to a trigonometric equation and the solution to this equation leads to a solution of the original problem. In other words, it is a problem where trigonometric equations are applied to solve a problem in geometry. Such problems are not very common and the paper-setters have come up with a good one. It is tempting to bypass trigonometric equations and try to solve the problem by getting a system of algebraic equations for d_1 and d_2. We already have one such equation, viz. (1). The second one can be obtained by using the fact that the angle subtended at the centre by one of the chords is double that subtended by the other. Let these angles be α and 2α. Without loss of generality, we assume that $d_1 \geq d_2$. Then we have

$$\sin(\alpha/2) = \frac{\sqrt{4-d_1^2}}{2} \tag{6}$$

$$\cos(\alpha/2) = \frac{d_1}{2} \tag{7}$$

$$\text{and} \quad \sin\alpha = \frac{\sqrt{4-d_2^2}}{2} \tag{8}$$

Eliminating α from these three equations and squaring gives

$$4 - d_2^2 = d_1^2(4 - d_1^2) \tag{9}$$

We now have to solve (1) and (9) simultaneously. Once again, we can *guess* a solution in which $d_1 = \sqrt{3}$ and $d_2 = 1$. But *arriving at* it is not easy. We can easily eliminate one of d_1 and d_2. But no matter which one we eliminate, we shall be left with a fourth degree equation in the other variable which will *not* be easy to solve, because it will involve terms of degree one too in that variable. So, in this case trigonometric equations is the only way. A similar situation occurred in the solution to the Main Problem of Chapter 10. As elaborated in Comment No. 1 there, if we try to solve that problem by reducing the data to a cubic equation, we are in trouble because the cubic we get has no obvious root and hence is not easy to solve. In fact, as shown there, sometimes a cubic can be solved by converting it to a suitable trigonometric equation.

The Main Problem of Chapter 10 was a full length question in 1994 JEE. It would have been better if the present problem were also a full length question because in that case it would be possible to see if the candidate actually solves (3) or guesses a solution and substantiates it with reasoning (given above) or merely guesses a solution and runs away. In the new JEE format, the runner is the winner as his scrupulous competitors foolishly waste their time in giving justification to their own conscience. A telling example of the gentleman finishing the last.

Q.11 Let f be a function defined on $I\!R$ (the set of real numbers) such that

$$f'(x) = 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4$$

for all $x \in I\!R$. If g is a function defined on $I\!R$ with values in $(0, \infty)$ such that $f(x) = \ln g(x)$ for all $x \in\in I\!R$, then the number of points in $I\!R$ at which g has a local maximum is .... .

Answer and Comments: 1. The number 2010 and its neighbours have absolutely no role in the problem. The number 2010 is indicative of the fact that the question appears in an examination held in the year 2010. This practice is consistently followed in the International Mathematics Olympiads. In JEE 2005, Q.13 of the screening paper had the figure 2005 thrown for no mathematical reason. (See the Commentary for that year.)

Now, coming to the problem itself, the function $g(x)$ is specified by $f(x) = \ln g(x)$. This is equivalent to defining $g(x)$ as $e^{f(x)}$. In fact, with this definition there would be no need to specify that $g(x)$ takes values in the interval $(0, \infty)$. One really wonders why $g(x)$ has been specified so indirectly. Just to give an unwarranted twist to the problem perhaps. We are asked how many local maxima g has. For this we can differentiate and get

$$g'(x) = f'(x)e^{f(x)} \tag{1}$$

Since the exponential function is always positive, the critical points of g are the same as those of f and the problem of finding the maxima of g is the same as that of finding those of f. We could have also seen this without differentiation. Since the exponential function is strictly increasing, the function $e^{f(x)}$ increases or decreases according as $f(x)$. Hence the two have the same points of local maxima/minima.

So, we now focus entirely on $f(x)$. We are not given $f(x)$. But we are given $f'(x)$ and that is what we really need. So even though we can find $f(x)$ by integrating $f'(x)$, it is a waste of time to do that. (See Exercise (17.16)(a) for a similar situation.) We are given

$$f'(x) = 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4 \tag{2}$$

The constant 2010 is positive and hence plays no role in the sign determination of $f'(x)$. We simply ignore it and concentrate on the remaining factors of $f'(x)$. f' has four zeros, 2009, 2010, 2011 and 2012 of multiplicities 1,2,3 and 4 respectively. A change of sign occurs only at zeros of odd multiplicities, i.e. at 2009 and 2011. For $x < 2009$, all the ten non-constant factors of $f'(x)$ are negative and so $f'(x)$ is positive. If x lies between 2009 to 2010 $f'(x)$ is negative since nine of its factors are negative. So f' changes sign from positive to negative at 2009. Hence the behaviour of f changes from increasing to decreasing at 2009. Therefore there is a local maximum of f at 2009. This is also a local maximum of $g(x)$ as noted above.

The only other zero of odd multiplicity of $f'(x)$ is 2011. For $x \in (2010, 2011)$, $f'(x) < 0$ since three factors of f' are positive and the remaining 7 are negative. But for $x \in (2011, 2012)$, 6 factors are positive and 4 factors are negative.So f' changes sign from negative to positive at 2011 and correspondingly the behaviour of $f(x)$ changes from decreasing to increasing at 2012. So there is a local minimum of $f(x)$ and hence of $g(x)$ at 2011. All told, $f(x)$ and hence $g(x)$ has precisely one local maximum.

This is a simple problem where the extrema of a function have to be found knowing only its derivative. Probably to make up for its simplicity, the function $g(x)$ has been added. In the printed version of the original question paper, the dash in $f'(x)$ is alarmingly close to the left parenthesis. It appears as $f'\!(x)$ and actually almost as $f(x)$. So, unless one reads it very carefully, it is likely to be read as $f(x)$. This completely changes the problem and makes it considerably more difficult. Possible sources of confusion like this need to be weeded out at the time of proof reading. A mis-spelt word generally does not cause so much confusion since it can often be corrected from the context. But confusing $f'(x)$ with $f(x)$ can be killing.

SECTION III

Paragraph Type

This section contains two paragraphs. Based on each of the paragraphs, there are three multiple choice questions. Each of these questions has four choices out of which ONLY ONE is correct.

Paragraph for Questions 12 to 14

Consider the polynomial

$$f(x) = 1 + 2x + 3x^3 + 4x^3.$$

Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$.

Q.12 The real number s lies in the interval

(A) $(-1/4, 0)$ (B) $(-11, -3/4)$ (C) $(-3/4, -1/2)$ (D) $(0, 1/4)$

Answer and Comments: (C). As $f(x)$ is a cubic polynomial with real coefficients, it has either one or three real roots. If it had three distinct real roots, then by Rolle's theorem its derivative $f'(x) = 12x^2 + 6x + 2$ would have at least two zeros. But this quadratic polynomial has a negative discriminant $36-96=-60$. So, $f(x)$ has only one real root, say α and s equals this lone real root. It is not easy to identify it explicitly. But the Intermediate Value Property (IVP) of continuous functions can be applied to locate α within an interval. Clearly $\alpha<0$ and so (D) is ruled out as an answer. To find out which is of the remaining is the correct answer, we can evaluate f at the endpoints of the given intervals till we come across an interval (a,b) such that $f(a)$ and $f(b)$ are of opposite signs. But this is essentially a trial and error method. To carry out the search for α more systematically, we imitate the standard proof of the IVP given in Comment No. 3 of Chapter 16. We first identify *some* reasonably small interval which contains α. Clearly we have $f(0)=1>0$ while $f(-1)=1-2+3-4=-2<0$. Hence α lies in $(-1,0)$. To narrow the search further, we evaluate f at the midpoint, $-1/2$. Again a direct computation gives $f(-1/2)=1-1+\dfrac{3}{4}-\dfrac{4}{8}=\dfrac{1}{4}>0$. So α lies somewhere in $(-1,-1/2)$. None of the given intervals contains this interval. So we need to carry out the search further. So again we evaluate f at the mid-point $-3/4$ of this interval. This time we have $f(-3/4)=1-\dfrac{3}{2}+\dfrac{27}{16}-\dfrac{27}{16}=-\dfrac{1}{2}<0$. Thus we see that the root α lies in the interval $(-3/4,-1/2)$. Luckily this happens to coincide with the interval given in (C). But it would have sufficed if it were merely contained in it.

The method adopted here is called **successive bisection** or **binary search** because at each stage, we are narrowing down the search to one of the two halves of the previous interval. If we carry out the binary search a sufficient number of times, we can approximate the (unknown) root as closely as we want. Of course, to begin with the initial interval has to be identified by trial. But thereafter the method proceeds like an algorithm. The initial interval should not be extravagantly large. But even if it is, the binary search method is fairly efficient because the powers of 2 increase very fast (an exponential growth in the language of Exercise (6.51).

The problem is a good but standard application of two important results in theoretical calculus, viz. Rolle's theorem and the IVP.

Q.13 The area bounded by the curve $y=f(x)$ and the lines $x=0, y=0$ and $x=t$, lies in the interval

(A) (3/4, 3) (B) (21/64, 11/16) (C) (9, 10) (D) (0, 21/64)

Answer and Comments: (A). Denote the area by A. We know that

$$A=\int_0^t 1+2x+3x^2+4x^3\,dx=t+t^2+t^3+t^4 \qquad (1)$$

Here $t=|s|=-\alpha$ where α is the only real root of $f(x)$. Let $g(x)=x+x^2+x^3+x^4$. Then $A=g(t)$. From the last question we know that

$t \in (1/2, 3/4)$. As the function $g(x)$ is obviously strictly increasing for all $x > 0$ we definitely have the following estimate on $A = g(t)$

$$g(\frac{1}{2}) \ < \ A \ < \ g(\frac{3}{4}) \tag{2}$$

To calculate the functional values of g, we simplify $g(x)$ to $x(1+x+x^2+x^3) = \dfrac{x(x^4-1)}{x-1}$ which is valid for all $x \neq 1$. For $x < 1$ we rewrite this as $\dfrac{x(1-x^4)}{1-x}$. Then $g(1/2) = \dfrac{15}{16}$ and $g(3/4) = 3(1 - \dfrac{81}{256}) = \dfrac{3 \times 175}{256} = \dfrac{525}{256}$. Hence from (2), we get

$$A \in (\frac{15}{16}, \frac{525}{256}) \tag{3}$$

If we can show that this interval is contained in any one of the four given intervals, then that interval would be the answer. The intervals in (C) and (D) are clearly ruled out because they are disjoint from the interval $(\dfrac{15}{16}, \dfrac{525}{256})$. For easy comparison, we bring all fractions involved to a common denominator 256. Then A lies in $(\dfrac{240}{256}, \dfrac{525}{256})$. The intervals in (A) and (B) are, respectively, $(\dfrac{192}{256}, \dfrac{768}{256})$ and $(\dfrac{84}{256} \dfrac{176}{256})$. Out of these two only the first one contains the interval in (3). So the correct choice is (A).

The problem is straightforward. But the arithmetic involved is a little too much. The problem would have been more interesting if none of the given choices contained the interval in (3). That does not necessarily mean that the question is wrong. It only means that the estimate in (3) is too crude and needs to be improved. For this, we need a sharper estimate on the root α of $f(x)$. We already know that α lies in the interval $(-3/4, -1/2)$. To get a better estimate we divide this interval at its midpoint $-5/8$. Then we have $f(-\dfrac{5}{8} = \dfrac{111}{256} > 0$. So the root α lies in the interval $(-\dfrac{3}{4}, -\dfrac{5}{8})$ and, as a result we now have $\dfrac{5}{8} < t < \dfrac{3}{4}$ and further $g(\dfrac{5}{8}) < A < g(\dfrac{3}{4})$. We already know $g(\dfrac{3}{4})$ as $\dfrac{525}{256}$. The calculation of $g(\dfrac{5}{8})$ will introduce fractions with denominators $8^4 = 4096$. We can then see if this new interval is contained in any one of the given ones. If not, we refine the estimate for α still further and so on. Obviously the computations involved would be horrendous if attempted by hand. But the problem would have really tested the thinking ability of the candidate. As it stands, the given problem involves less thinking and more computation. The paper-setters are apparently aware of this and in order to simplify the calculations they have given the function $f(x)$ in such a way that its antiderivative $g(x)$ will come in a very simple form, viz. as a polynomial in which all coefficients are 1.

Q.14 The function $f'(x)$ is

(A) increasing in $(-t, -1/4)$ and decreasing in $(-1/4, t)$

(B) decreasing in $(-t, -1/4)$ and decreasing in $(-1/4, t)$

(C) increasing in $(-t, t)$

(D) decreasing in $(-t, t)$

Answer and Comments: (B). We have $f'(x) = 2 + 6x + 12x^2$ and the question asks for the increasing/decreasing behaviour of $f'(x)$ on certain intervals. As $f'(x)$ is differentiable everywhere, the problem reduces to checking the sign of $f''(x) = 6 + 24x$ on these intervals. As the factor 6 is positive, we might as well check the sign of $4x + 1$. Let us call this function $h(x)$. It has only one zero, viz. $-1/4$. And $h(x)$ is positive for $x > -1/4$ and negative for $x < -1/4$. Now, all the given intervals involve t whose exact value is not known to us. But we already have estimates on its value and, if need arises, we know how to improve these estimates. Let us first see what information we can get from the estimate we already have, viz.

$$\frac{1}{2} < t < \frac{3}{4} \tag{4}$$

which also means

$$-\frac{3}{4} < -t < -\frac{1}{2} \tag{5}$$

(which is the same as the estimate for α we had obtained since $\alpha = -t$.) Put together this means

$$(-\frac{1}{2}, \frac{1}{2}) \subset (-t, t) \subset (-\frac{3}{4}, \frac{3}{4}) \tag{6}$$

In particular, the point $-\frac{1}{4}$ at which h changes sign lies in $(-t, t)$. So there is a change of increasing/decreasing behaviour of $f'(x)$ in these intervals. This rules out (C) and (D) as possible answers. In the remaining two options, the interval $(-t, t)$ has been split into two parts at the point $-1/4$, which is the point at which $h(x)$ changes sign from negative to positive. So $f'(x)$ is decreasing on $(-t, -1/4)$ and increasing on $(-1/4, t)$. Therefore (B) is the correct answer.

This bunch of questions fits for a paragraph because the answer to the first question is crucially needed in the other two. But there is considerable duplication of ideas. Moreover, the essential idea, viz. the increasing/decreasing behaviour of a function has already occurred in Q.7 of Paper I and again in Q.11 of Section II.

Paragraph for Questions 15 to 17

Tangents are drawn from the point $P(3,4)$ to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at the points A and B.

Q.15 The coordinates of A and B are

(A) (3,0) and (0,2) (B) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and ($\left(-\frac{9}{5}, \frac{8}{5}\right)$

(C) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0,2) (D) (3,0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

Answer and Comments: (D). We are asked to find the points of contact of the tangents from the given point P, i.e. points on the ellipse such that the tangents at them pass through P. For this it is convenient to take parametric coordinates of the given ellipse, viz.

$$x = 3\cos\theta, \quad y = 2\sin\theta \tag{1}$$

Now suppose $(3\cos\theta, 2\sin\theta)$ is a point on the ellipse. Then the slope of the tangent at this point is $-\frac{2\cos\theta}{3\sin\theta}$ and so its equation is

$$y - 2\sin\theta = -\frac{2\cos\theta}{3\sin\theta}(x - 3\cos\theta) \tag{2}$$

which simplifies to

$$3\sin\theta\, y + 2\cos\theta\, x = 6 \tag{3}$$

(which can be derived in other ways too). As the tangent passes through $P(3,4)$, we get

$$2\sin\theta + \cos\theta = 1 \tag{4}$$

which is a trigonometric equation in θ. To solve it we introduce an angle α by

$$\cos\alpha = \frac{2}{\sqrt{5}} \quad \text{and} \quad \sin\alpha = \frac{1}{\sqrt{5}} \tag{5}$$

(We could also have introduced α as the acute angle for which $\tan\alpha = \frac{1}{2}$.) Then (4) becomes

$$\sin(\alpha+\theta) = \frac{1}{\sqrt{5}} \tag{6}$$

The R.H.S. equals $\sin\alpha$. Hence the two solutions are given by $\alpha + \theta = \alpha$ and $2\alpha + \theta = \pi$, which gives $\theta = 0$ or $\pi - 2\alpha$. The first value gives

$(3\cos\theta, 2\sin\theta)$ as $(3,0)$. The second value gives it as $(-3\cos 2\alpha, 2\sin 2\alpha)$, which from the values of $\sin\alpha$ and $\cos\alpha$ comes out as $(-\frac{9}{5}, \frac{8}{5})$. Therefore these are the points of contact of the two tangents from P.

Q.16 The orthocentre of the triangle PAB is

(A) (5, 8/7) (B) (7/5, 25/8) (C) (11/5, 8/5) (D) (8/25, 7/5)

Answer and Comments: (C). There is a formula (given in Comment No. 3 of Chapter 8) which gives the orthocentre of a triangle directly in terms of the coordinates of its vertices. But it is far too complicated. Here we have a special triangle, viz. a triangle formed by a point outside a conic and the chord of contact of this point w.r.t. that conic. But even in this special case, there is no handy readymade formula. So we proceed from scratch by finding two of the altitudes of the triangle PAB and identifying their point of intersection. In the present case, from the last question, we can take A as $(3,0)$ and B as $(-9/5, 8/5)$ (without loss of generality). Since $P = (3,4)$. So, obviously the side PA is along the line $x = 3$. Hence the altitude through B is the line $y = \frac{8}{5}$. Among the given four choices (C) is the only one where the y-coordinate is $\frac{8}{5}$. So without any further work, if at all one of the answers has to be correct, it must be (C).

But if we do not want to take an unfair advantage of the carelessness on the part of the paper-setters, then we need one more altitude of the triangle PAB. It is preferable to take the altitude through P because to find it we would first need to find the equation of the line AB and this equation will also be needed in the next question. (This kind of a non-mathematical alertness pays off sometimes.) Since $A = (3,0)$ and $B = (-9/5, 8/5)$, we get the equation of the line AB as $y = \frac{8/5}{-9/5-3}(x-3)$ which, upon simplification becomes

$$y = -\frac{1}{3}x + 1 \tag{7}$$

Hence the slope of the altitude through P is 3 and therefore its equation is

$$y - 4 = 3(x-3) \tag{8}$$

Solving this simultaneously with $y = \frac{8}{5}$ gives $x = 11/3$. Therefore the orthocentre of of the triangle PAB is at $(\frac{11}{5}, \frac{8}{5})$.

Q.17 The equation of the locus of the point whose distances from the point P and the line AB are equal, is

(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

(B) $x^2 + 9y^2 + 6xy - 54x - 62y - 241 = 0$

(C) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

(D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

Answer and Comments: (A). By definition, the locus is a parabola with focus at P and the line AB as its directrix. But this is not much of a help in finding the equation. That is best done by equating the two distances. Let (h, k) be the moving point. Its distance from P is $\sqrt{(h-3)^2 + (k-4)^2}$. We already know the equation of AB from (7). So the distance of (h, k) from it is $\dfrac{|k + \frac{1}{3}h - 1|}{\sqrt{1 + 1/9}}$. Equating the distances and squaring gives

$$(h-3)^2 + (k-4)^2 = \frac{(h + 3k - 3)^2}{10} \tag{9}$$

Replacing h, k by x, y and simplifying, the locus is

$$10(x^2 + y^2 - 6x - 8y + 25) = x^2 + 9y^2 + 6xy - 6x - 18y + 9 \tag{10}$$

i.e. $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$.

The entire paragraph is routine and computational.

SECTION IV

Matrix Type

This section contains 2 questions. Each question has four statements (A, B, C and D) given in **Column I** and five statements (p, q, r, s and t) in **Column II**. Any given statement in Column I can have correct matching with one or more statement(s) given in Column II.

Q.18 Match the statements in **Column I** with those in **Column II**.

[**Note** : Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z respectively.]

Column I	**Column II**
(A) The set of points z satisfying $$\|z - i\|z\|\| = \|z + i\|z\|\|$$ is contained in or equal to	(p) an ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $$\|z + 4\| + \|z - 4\|$$ is contained in or equal to	(q) the set of points z satisfying Im $z = 0$
(C) If $\|w\| = 2$, then the set of points z satisfying $z = w - \frac{1}{w}$ is contained in or equal to	(r) the set of points z satisfying $\|$ Im $z\| \leq 1$
(D) If $\|w\| = 1$, then the set of points z satisfying $z = w + \frac{1}{w}$ is contained in or equal to	(s) the set of points z satisfying $\|$ Re $z\| \leq 2$
	(t) the set of points z satisfying $\|z\| \leq 3$

Answer and Comments: (A) $\rightarrow$ (q, r) (B) $\rightarrow$ (p) (C) $\rightarrow$ (p, s, t) (D) $\rightarrow$ (q, r, s, t).

If this were a one-to-one matching, the answers would be easier to arrive at. The sets given in Column II are either given or are easy to identify geometrically. The entries in Column I are easy to identify qualitatively. And if we are given beforehand that each entry in Column I has only one match then it can usually be found from this qualitative description. But in the present problem several entries may match with the same entry. And that makes it necessary to identify the sets in Column I more precisely. This can be done by translating the descriptions in terms of the real and imaginary parts of z. As usual we denote these by x and y respectively.

Let us first describe the sets in Column II in terms of x and y. For simplicity, denote these subsets by P, Q, R, S and T respectively. The set P is not uniquely defined. But for the remaining sets, we have

$$\begin{aligned} Q &= \{(x, y) : y = 0\} && (1) \\ R &= \{(x, y) : |y| \leq 1\} && (2) \\ S &= \{(x, y) : |x| \leq 2\} && (3) \\ T &= \{(x, y) : x^2 + y^2 \leq 9\} && (4) \end{aligned}$$

It is easy to describe these geometrically. Q is the x-axis, R is the horizontal strip bounded by the lines $y = \pm 1$, S is the vertical strip bounded by the lines $x = \pm 2$ and T is the disc of radius 3 centred at the origin. Clearly $Q \subset R$ and so if (q) is a correct answer to some entry in Column I, then (r) is also correct.

Let us now describe the sets in Column I. For simplicity we denote them by A, B, C and D respectively. In (A), no matter what z is, the points $i|z|$ and $-i|z|$ are on the y-axis, symmetrically located w.r.t. the $x$axis. The condition in (A) implies that z is equidistant from these two points and hence lies on the perpendicular bisector of the segment joining them. So, z must lie on the x-axis. So it lies in Q and hence in R too. Thus we have $A \subset Q$ and $A \subset R$.

Geometrically, the set B is an ellipse with foci at ± 4. Let e be the eccentricity of the ellipse and let $(a, 0), (-a, 0)$ be its vertices. Then we have two $2a = 10$ whence $a = 5$. But then $ae = 4$ implies $e = 4/5$. So (p) holds. Also (q) is ruled out because an ellipse cannot be completely contained in its major axis. To check the remaining options, note first that the points $(\pm 5, 0)$ lie on the ellipse but are obviously outside the vertical strip S as well as the disc T. But we must check if the ellipse lies in the horizontal strip R. For this we need to find the semi-minor axis, say b of the ellipse. This is given by $a\sqrt{1-e^2} = 5 \times \frac{3}{5} = 3$. So the points $(0, \pm 3)$ lie on the ellipse but not in R. Hence only (p) holds for (B).

For C and D, note that w is a complex number which varies over some subset of the complex plane and z is given as a function of w. In (C), we can take $w = 2\cos\theta + 2i\sin\theta$ for some θ. Then

$$\frac{1}{w} = \frac{1}{2(\cos\theta + i\sin\theta)} = \frac{1}{2}(\cos\theta - i\sin\theta) \tag{5}$$

This gives

$$\begin{aligned} w - \frac{1}{w} &= 2(\cos\theta + i\sin\theta) - \frac{1}{2}(\cos\theta - i\sin\theta) \\ &= \frac{3}{2}\cos\theta + i\frac{5}{2}\sin\theta \end{aligned} \tag{6}$$

So, we get

$$C = \{(x, y) : x = \frac{3}{2}\cos\theta, \ \ y = \frac{5}{2}\sin\theta\} \tag{7}$$

But these are precisely the parametric equations of an ellipse with major and minor axis 5 and 3 respectively. (Note that the major axis is along the y-axis.) So the eccentricity is $\frac{\sqrt{25-9}}{5} = \frac{4}{5}$. Hence (p) holds. But unlike A, this time the ellipse C is contained in the strip S as well as the disc T. Hence (s) and (t) also hold true.

Finally, for D the reasoning is analogous to that in (C). This time w is of the form $\cos\theta + i\sin\theta$ and so $w + \frac{1}{w}$ comes out to be simply $2\cos\theta$. So the set D is the segment of the x-axis from $(-2, 0)$ to $(2, 0)$. Clearly (q), (r), (s) and (t) all hold true.

The problem is simple but once the basic ideas are understood, there is considerable duplication of work. Since every option in each column has to be tried for every option in the other column, in effect, 20 questions are asked here. The paper-setters have designed the numerical data of the problem very carefully. Both (B) and (C) are ellipse. In fact the second ellipse is similar to the conjugate ellipse of the first one by a factor of 1/2. As a result, both the ellipses have the same shape and hence equal eccentricities. But the second ellipse is smaller and so fits inside a strip and a disc while the first one does not.

Q.19 Match the statements in Column I with the values in Column II.

Column I	**Column II**
(A) A line from the origin meets the lines $\dfrac{x-2}{1} = \dfrac{y-1}{-2} = \dfrac{z+1}{1}$ and $\dfrac{x-(8/3)}{2} = \dfrac{y+3}{-1} = \dfrac{z-1}{1}$ at points P and Q respectively. If length $PQ = d$, then d^2 is	(p) -4 (q) 0
(B) The values of x satisfying $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}(\frac{3}{5})$ are	(r) 4
(C) Non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ satisfy $\vec{a} \cdot \vec{b} = 0$, $(\vec{b} - \vec{a}) \cdot (\vec{b} + \vec{c}) = 0$ and $2\vert\vec{b} + \vec{c}\vert = \vert\vec{b} - \vec{a}\vert$. If $\vec{a} = \mu\vec{b} + 4\vec{c}$, then the possible values of μ are	(s) 5
(D) Let f be a function on $[-\pi, \pi]$ given by $f(0) = 9$ and $f(x) = \sin(9x/2)/\sin(x/2)$ for $x \neq 0$. The value of $\dfrac{2}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$ is	(t) 6

Answer and Comments: (A) $\to$ (t), (B) $\to$ (p, r), (C) $\to$ (q, s), (D) $\to$ (r).

Unlike in the last question, here the entries in Column I have numerical answers. In (A) and (D), they have to be found by systematic calculations. In n(B) and (C) we have to solve certain equations and so we can, in theory, try all the alternatives in Column II one by one and see which of them are correct. But this is hardly practicable. Nor is it mathematically sound. So we tackle each problem systematically.

In (A), taking the parametric equations of the two lies, we see that the points P and Q are of the form

$$P = (2+s, 1-2s, -1+s) \qquad (1)$$

$$\text{and} \quad Q \quad = \quad (\frac{8}{3} + 2t, -3 - t, 1 + t) \tag{2}$$

for some real numbers s and t. We are further given that O (the origin), P and Q are collinear. Hence the coordinates are proportional. This gives a system of two equations in s and t, viz.

$$\frac{2+s}{(8/3)+2t} = \frac{1-2s}{-3-t} = \frac{-1+s}{1+t} \tag{3}$$

We can equate any two of these ratios to get two equations in s and t. But they will involve terms in st and hence will not be linear. Instead we use a few simple properties of ratios to get simpler equations. Call each of the three ratios above as r. Then if we take any linear combination of any two numerators and divide it by the corresponding linear combination of their respective denominators we still get the same ratio r. We can choose these linear combinations so as to eliminate one of the two variables. For example, if we simply add the numerators of the last two ratios and divide it by the sum of their denominators, we get

$$r = \frac{1-2s-1+s}{-3-t+1+t} = \frac{s}{2} \tag{4}$$

Similarly, we can eliminate t using the first and the third ratio to get

$$r = \frac{2+s-2(-1+s)}{\frac{8}{3}+2t-2(1+t)} = \frac{4-s}{\frac{2}{3}} \tag{5}$$

Equating these two, we get a single equation in s, viz.

$$\frac{s}{2} = \frac{4-s}{\frac{2}{3}} \tag{6}$$

which can be easily solved to get $s = 3$. We can similarly eliminate s to get a linear equation for t. Alternately we can put $s = 3$ into any of the two ratios in(3). Doing so for the last two ratios gives

$$\frac{-5}{-3-t} = \frac{2}{1+t} \tag{7}$$

which gives $t = 1/3$. Putting these values into (1) and (2) gives P as $(5, -5, 2)$ and Q as $(10/3, -10/3, 4/3)$. A direct calculation now gives $d^2 = (5/3)^2 + (-5/3)^2 + (2/3)^2 = 54/9 = 6$.

Although straightforward, the calculations involved here are fairly lengthy and prone to errors. Comparatively, the equation we get for x in (B) is much easier. Recognising $\sin^{-1}(\frac{3}{5})$ as $\tan^{-1}(\frac{3}{4})$ and taking the tangents of both the sides, we get

$$\frac{(x+3)-(x-3)}{1+(x+3)(x-3)} = \frac{3}{4} \tag{8}$$

which simplifies to a quadratic in x, viz. $x^2 - 8 = 8$. This gives ± 4 as possible values of x.

In (C), it is not immediately obvious how to get an equation for μ from the data. Although there are three vectors $\vec{a}, \vec{b}, \vec{c}$ in the problem, the vector $\vec{a}$ is given as a linear combination of $\vec{b}$ and $\vec{c}$, viz.

$$\vec{a} = \mu\vec{b} + 4\vec{c} \tag{9}$$

So, we can write each of the three given equations in terms of the vectors $\vec{b}$ and $\vec{c}$ and the scalar μ. When we expand each of the three given equations it becomes an equation that involves the three dot products $\vec{b}\cdot\vec{b}, \vec{b}\cdot\vec{c}$ and $\vec{c}\cdot\vec{c}$. Specifically, the first equation $\vec{a}\cdot\vec{b} = 0$ becomes $(\mu\vec{b} + 4\vec{c})\cdot\vec{b} = 0$ and gives

$$\mu\vec{b}\cdot\vec{b} + 4\vec{b}\cdot\vec{c} = 0 \tag{10}$$

Similarly, the second equation, $(\vec{b} - \vec{a})\cdot(\vec{b} + \vec{c}) = 0$ becomes $\vec{b}\cdot(\vec{b} + \vec{c}) = (\mu\vec{b} + 4\vec{c})\cdot(\vec{b} + \vec{c})$ or equivalently,

$$(\mu - 1)\vec{b}\cdot\vec{b} + (\mu + 3)\vec{b}\cdot\vec{c} + 4\vec{c}\cdot\vec{c} = 0 \tag{11}$$

The third equation, viz. $2|\vec{b} + \vec{c}| = |\vec{b} - \vec{a}|$ is ostensibly in terms of the lengths of the vectors. But if we square both the sides we can write it as an equation in dot products, viz.

$$4(\vec{b}\cdot\vec{b} + 2\vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{c}) = \vec{b}\cdot\vec{b} - 2\vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{a} \tag{12}$$

Putting $\vec{a} = \mu\vec{b} + 4\vec{c}$ and collecting the like terms, we get

$$(\mu^2 - 2\mu - 3)\vec{b}\cdot\vec{b} + (8\mu - 16)\vec{b}\cdot\vec{c} + 12\vec{c}\cdot\vec{c} = 0 \tag{13}$$

We regard (10), (11) and (13) as a system of three homogeneous linear equations in three unknowns $\vec{b}\cdot\vec{b}$, $\vec{b}\cdot\vec{c}$ and $\vec{c}\cdot\vec{c}$. Since the vectors $\vec{b}, \vec{c}$ are given to be non-zero, so are the dot products $\vec{b}\cdot\vec{b}$ and $\vec{c}\cdot\vec{c}$. Hence this system has at least one non-trivial solution. Therefore setting its determinant equal to 0 we get the desired and much awaited equation for μ, viz.

$$\begin{vmatrix} \mu & 4 & 0 \\ \mu - 1 & \mu + 3 & 4 \\ \mu^2 - 2\mu - 3 & 8\mu - 16 & 12 \end{vmatrix} = 0 \tag{14}$$

Dividing the last column by 4 and then expanding directly, this reduces to a quadratic in μ, viz.

$$\mu(-5\mu + 25) + 4(\mu^2 - 5\mu) = 0 \tag{15}$$

i.e. $-\mu^2 + 5\mu = 0$. Hence the two possible values of μ are 0 and 5.

Unlike Part (A) which is most computational, here the thinking ability is also tested. But the computations involved after getting the key idea are also substantial. The work can be shortened considerably by introducing suitable coordinates, taking advantage of the orthogonality of $\vec{a}$ and $\vec{b}$. So, without loss of generality, assume $\vec{a} = a\vec{i}$ and $\vec{b} = b\vec{j}$. Then $\vec{c} = \frac{1}{4}(\vec{a} - \mu\vec{b}) = \frac{1}{4}(a\vec{i} + (4-\mu)b\vec{j})$ and the condition $(\vec{b} - \vec{a}) \cdot (\vec{b} + \vec{c}) = 0$ becomes $a^2 = (4-\mu)b^2$ while the condition $2|\vec{b} - \vec{c}| = |\vec{b} - \vec{a}|$ reduces to $4(a^2 + b^2) = a^2 + (4-\mu)^2 b^2$. Eliminating a in these two equaltions gives

$$4(5-\mu)b^2 = [(4-\mu) + (4-\mu)^2]b^2$$

As $b \neq 0$, canceling it we get a quadratic in μ which is the same as before. (Actually, the coefficient of b^2 on the R.H.S. can be written as $(4-\mu)(5-\mu)$, which makes it possible to solve the quadratic simply by inspection.)

However, even this slicker solution takes time way beyond that justified by the credit allotted to the question. The problem is more suitable for a full length question. But the present JEE does not permit such questions.

Finally, let us denote the integral in (D) by I. Note that the integrand is *not* discontinuous at 0, because by L'ôpital's rule (or simply by multiplying and dividing by $9x/2$), one can easily show that $\lim_{x \to 0^+} \frac{\sin(9x/2)}{\sin(x/2)}$ equals 9. By a change of variable $x = 2\theta$ we get

$$\begin{aligned} I &= 2\int_{-\pi/2}^{\pi/2} \frac{\sin 9\theta}{\sin\theta}\, d\theta && (16) \\ &= 4\int_{0}^{\pi/2} \frac{\sin 9\theta}{\sin\theta}\, d\theta && (17) \end{aligned}$$

where the last equality follows since the integrand is an even function of θ. It is possible to express $\sin 9\theta$ as a polynomial of degree 9 in $\sin\theta$ (analogous to $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$) and then integrate each term. But that would be very complicated. So we try some other method. We note first that there is nothing special about the integer 9. We could just as well replace it by any odd positive integer, say $2n+1$. Let us do that and call the resulting integral as I_n. In other words,

$$I_n = \int_0^{\pi/2} \frac{\sin(2n+1)\theta}{\sin\theta}\, d\theta \qquad (18)$$

With this notation the integral in (9) is simply I_4. Although our interest is only in this particular integral, it is sometimes easier to tackle the general problem of evaluating I_n for every $n \geq 0$, if we are able to find a **reduction formula** for it, i.e. a formula which expresses I_n in terms of I_{n-1} (or other similar integrals with lower values of the suffix.) This technique of evaluating certain special definite integrals without identifying antiderivatives of their integrands is explained in detail in Comments 5 to 9 of Chapter 18.

In the present problem a reduction formula for I_n can be obtained as follows. Note that

$$\begin{aligned} I_n - I_{n-1} &= \int_0^{\pi/2} \frac{\sin(2n+1)\theta - \sin(2n-1)\theta}{\sin\theta}\, d\theta \\ &= \int_0^{\pi/2} \frac{2\cos 2n\theta \sin\theta}{\sin\theta}\, d\theta \\ &= 2\int_0^{\pi/2} \cos 2n\theta\, d\theta \\ &= \frac{1}{n}\sin 2n\theta\Big|_0^{\pi/2} = 0 \qquad (19) \end{aligned}$$

In other words, we have proved that $I_n = I_{n-1}$. Of course, not all reduction formulas are so simple. In fact, many of them are quite complicated and for many integrals involving an integer parameter, no reduction formulas exist. But as far as our integral is concerned, by repeated applications of the reduction formula, we get that $I_n = I_{n-1} = I_{n-2} = \ldots = I_2 = I_1 = I_0$. The last integral I_0 has to be computed directly. But that is easy. As the integrand $\dfrac{\sin\theta}{\sin\theta}$ is the constant 1, we have $I_0 = \dfrac{\pi}{2}$ and hence $I_n = \dfrac{\pi}{2}$ for every n. In particular $I_4 = \dfrac{\pi}{2}$. Going back to our original integral I, by (17) it equals $4I_4 = 2\pi$. Hence the expression $\dfrac{2}{\pi}\displaystyle\int_{-\pi}^{\pi} \frac{\sin(9x/2)}{\sin(x/2)}\, dx$ equals 4.

The problem is easy once the idea of the reduction formula strikes. The problem is closely related to the problem of showing that $\displaystyle\int_0^{\pi} \frac{1-\cos mx}{1-\cos x}\, dx$ equals $m\pi$ for every positive integer m. In fact, the two problems can be converted to each other. This latter problem was asked as a full length question in JEE 1995. (See Comment No. 8 of Chapter 18 for a solution.)

Except for (B), all entries in Column I require considerable thinking and work. It is very unlikely that even a good candidate will be able to give the time demanded by them.

CONCLUDING REMARKS

The paper-setters have obviously worked very hard. Even where there is considerable numerical work in a problem, the data has been so designed that the final answer would look manageable. In other words, the paper-setters have actually done the computations down to the last end. This has paid off. There are no mathematical errors and hardly any ambiguities or obscurities in both the papers. The two exceptions are Q.24 of Paper 1 in which a reference is made to the 'nearest directrix' which is wrong on two counts. First, a hyperbola has only two directrices. So it would be more appropriate to say the 'nearer directrix'. But more seriously, it is not made clear how nearness is to be measured. One simply has to prseume that it is in terms of the distance from the point of contact. Secondly, in Q.28 of Paper 1, the phrase 'common ratio' is confusing for a progression all whose terms vanish. Another place where confusion may arise is Q.10 in Paper 1 where the correct answer is 0 but since more than one answer are allowed to be correct, a candidate may get confused as to whether $\frac{22}{7} - \pi$ is also to be taken as a correct answer. As already pointed out, in the text of Q.11 of Paper 2, as it appeared in the actual question papers given to the candidates, $f'(x)$ looked more like $f(x)$. But that is a fault of the proof-readers than the paper-setters.

The paper-setters do, however, have a tendency to give unwarranted twists to a problem, which are irrelevant to the main theme of the problem and only serve to take up additional time on the part of the candidate without testing any desirable quality on his part. Examples include Q.28 in Paper 1 (on a telescopic series), Q. 8 in Paper 2 (on arithmetic progressions) , Q. 9 in Paper 2 (on adjoint matrix) and Q.11 of Paper 2 (on local maximum of a function). There are also questions where arriving at the solution honestly is quite time consuming, but guessing it from the alternatives given is easy, e.g. Q. 14 of Paper 1 and Q. 10 of Paper 2. One wonders if the idea was to test some non-mathematical alertness on the part of a candidate. In Q.15 and 16 of Paper 1, a scrupulous answer must eliminate certain possibilities. But these possibilities do not affect the solutions. So a candidate who is blissfully unaware of them gets rewarded in terms of the time he saves. There are also problems where in order that the answer comes out as a single digit integer the problem has been made clumsy.

There is some duplication of ideas. For example, both Q.3 and Q.22 of Paper 1 are based on the same basic property of ω, the complex cube root of unity. One of these problems could have been replaced so as to make room for some untouched area such as even and odd functions or a functional equation. Numerous problems on solid coordinate geometry are asked. At the JEE level there is not much variety in these problems. So, no harm would have been committed if one of them were replaced by differential equations which have been paid only a lip service.

But despite such deficiencies (some of which are essentially matters of taste) the paper-setters have undoubtedly come up with some interesting problems, often covering some rare areas. Examples are arithmetic modulo a prime (Q.17

of Paper 1), the adjoint of a matrix (Q. 9 of Paper 2) and a geometric application of a trigonometric equation (Q.10 of Paper 2). The binary search needed for the location of a zero of a continuous function in Q.12 of Paper 2 is also unprecedented. Although reduction formulas for definite integrals were commonly asked in the past, after the JEE became totally objective type, they did not figure. But the paper-setters have managed to ask a question based on a reduction formula in Part (D) of Q.19 of Paper 2. Part (C) of the same question is also an unusual problem on vectors. Although questions on solving a triangle are frequent, the way the triangle is determined in Q.11 of Paper 1 is quite tricky. But what takes the cake is Q. 27 of Paper 1 where an indiscriminate application of the determinant criterion for the solution of a homogeneous system of linear equations is a very tempting trap.

Unfortunately, the multiple choice format and the huge number of routine questions do not do justice to these good problems. There is simply no time to solve them honestly. So if a candidate has answered them correctly that is not necessarilty an indication of any acumen on his part. He could, in fact, be a very mediocre candidate, as is often revealed in the B. Tech. courses he takes after getting into the IITs.

One can only dream that a handful of such well-chosen problems is all that is asked in a two or three hour examination, in which the candidate has to show all his work. That is how JEE was a long long time ago. Those were the days.