EDUCATIVE COMMENTARY ON JEE 2011 MATHEMATICS PAPERS

Contents

The pattern of JEE 2011 is the same as that of the previous year. The number of questions in Paper 1 has been reduced from 28 to 23. This is a welcome change. But in Paper 2 the number of questions has been increased from 19 to 20. What is worse, the last two questions have four parts each which are totally independent of each other. So, in effect, Paper 2 has 26 questions anyway.

The provisions for negative credit for wrong or ambiguous answers are the same as the last year. But the instructions have been modified so as to leave no ambiguity.

In the commentaries on the JEE mathematics papers for the past several years, frequent references were made to the book *Educative JEE Mathematics* by the author. The second edition of this book has now appeared. However, for the convenience of those who have either edition, references to the book are given in terms of the numbers of the chapter, the comment or the exercise involved, rather than by page numbers.

PAPER 1

Contents

SECTION I

Single Correct Choice Type

This section contains **seven** multiple choice questions. Each question has 4 choices out of which **ONLY ONE** is correct.

Q.1 Let α and β be the roots of $x^2 - 6x - 2 = 0$ with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\dfrac{a_{10} - 2a_8}{2a_9}$ is

(A) 1 (B) 2 (C) 3 (D) 4

Answer and Comments: (C). The successive powers of a root of a quadratic satisfy a recurrence relation of order 2. Problems in Comment No. 10 and 11 of Chapter 4 are based on this fact. See also the remark at the end of Comment No. 14 of Chapter 22. In the present problem we have

$$\alpha^2 = 6\alpha + 2 \tag{1}$$

Multiplying throughout by α^{n-2}, we get

$$\alpha^n = 6\alpha^{n-1} + 2\alpha^{n-2} \tag{2}$$

for every $n \geq 2$ (with the understanding that $\alpha^0 = 1$). Similarly,

$$\beta^n = 6\beta^{n-1} + 2\beta^{n-2} \tag{3}$$

for every $n \geq 2$. Subtracting,

$$a_n = 6a_{n-1} + 2a_{n-2} \tag{4}$$

Putting $n = 10$, we get $a_{10} = 6a_9 + 2a_8$ from which the ratio $\dfrac{a_{10} - 2a_8}{2a_9}$ comes out to be 3.

Actually, the same reasoning shows that for every $n \geq 2$, the ratio $\frac{a_n - 2a_{n-2}}{2a_{n-1}}$ equals 3. Note that nowhere we had to evaluate α and β. If instead of 10, the value of the ratio had been asked for some small value of n, say $n = 3$ or 4, it would have been tempting to find it by first finding α and β. By giving a high value like 10, the paper-setters have implicitly given a hint that the brute force method is not the right one. So this is a good problem. The same method would have worked if instead of a quadratic, we had a cubic or a polynomial equation of a higher degree because the powers of the roots of any polynomial satisfy a recurrence relation which can be written down by inspection from the polynomial. Note, however, that this method is applicable only for ratios which can be expressed in terms of the recurrence relation. In the present problem, if we want the value of the ratio $\frac{a_{10} - a_8}{a_9}$ (say), then there is no easy way.

Those who cannot think of a recurrence relation satisfied by the a_n's, are likely to be tempted to attack the problem by taking out the factor $\alpha - \beta$ from every term of the numerator and the denominator and canceling it. But this approach will fail since the other factors are fairly complicated. However, a solution which requires the values of $\alpha + \beta$ and $\alpha\beta$ (which come out to be 6 and -2 respectively) can be given as follows.

$$\begin{aligned} a_{10} &= \alpha^{10} - \beta^{10} = (\alpha + \beta)(\alpha^9 - \beta^9) - \alpha\beta(\alpha^8 - \beta^8) \\ &= 6a_9 + 2a_8 \end{aligned} \tag{5}$$

which gives the given ratio as 3. In this method too, we do not actually find α and β individually. But once again, its limitation is that it works only for some special types of ratios.

Incidentally, the piece of data $\alpha > \beta$ is nowhere needed. Apparently, it is given just to define each a_n unambiguously. But no matter which of the two roots is taken as α, the value of the ratio remains the same.

Q.2 A straight line L through the point $(3, -2)$ is inclined at an angle $60°$ to the line $\sqrt{3} + y = 1$. If L also intersects the x-axis, then the equation of L is

(A) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$ (B) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$
(C) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$ (D) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$.

Answer and Comments: (B). The answer would come as soon as we find the slope, say m, of L. The slope of the other line is $-\sqrt{3}$. There are two lines which are inclined to it at $60°$. So there are two possible values of m. But the last piece of data means that L is not parallel to the x-axis, i.e. its slope is not zero. So we expect that one of the two possible values of m is 0. (As otherwise there would be no point in specifying this last condition.)

Since $\tan 60^\circ = \sqrt{3}$, from the formula for the angle between two lines, we get an equation for m, viz.

$$\frac{m+\sqrt{3}}{1-m\sqrt{3}} = \pm\sqrt{3} \tag{1}$$

The possibility $m = 0$ corresponds to the choice of the positive sign on the R.H.S. Discarding this we get an equation for m, viz.

$$\frac{m+\sqrt{3}}{1-m\sqrt{3}} = -\sqrt{3} \tag{2}$$

solving which we get $m = \sqrt{3}$. Hence the equation of L is

$$y + 2 = \sqrt{3}(x-3) \tag{3}$$

which simplifies to (B).

The other line in the problem is given to have slope $-\sqrt{3}$. The fact that this is the tangent of a familiar angle, viz. 120° can be used to simplify the solution a little. The other line makes an angle of 120° with the x-axis. Since L is inclined to this line at 60°, the angle L makes with the x-axis is 120 ± 60 degrees, i.e. either 180 or 60 degrees. The first possibility will make L parallel to the x-axis and hence has to be discarded. The second possibility gives the answer.

However, even without this simplification, the problem is quite simple and straightforward.

Q.3 Let (x_0, y_0) be the solution of the following equations:

$$\begin{aligned} (2x)^{\ln 2} &= (3y)^{\ln 3} \\ 3^{\ln x} &= 2^{\ln y} \end{aligned}$$

Then x_0 is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 6

Answer and Comments: (C). We can treat the data as a system of equations in the unknowns $\ln x$ and $\ln y$ instead of x and y. This is permissible because a (positive) real number is uniquely determined by its logarithm. So, taking logarithms, we get

$$(\ln 2)^2 + (\ln 2)(\ln x) = (\ln 3)^2 + (\ln 3)(\ln y) \tag{1}$$

$$(\ln 3)(\ln x) = (\ln 2)(\ln y) \tag{2}$$

which can be treated as a system of linear equations in the unknowns $\ln x$ and $\ln y$. Eliminating $\ln y$ from the two equations gives

$$(\ln 2)^3 + (\ln 2)^2(\ln x) - (\ln 3)^2(\ln 2) = (\ln 3)^2(\ln x) \tag{3}$$

which simplifies to

$$[(\ln 3)^2 - (\ln 2)^2](\ln x) = -(\ln 2)[(\ln 3)^2 - (\ln 2)^2] \tag{4}$$

As $(\ln 3) \neq \pm(\ln 2)$, we cancel the bracketed factor and get

$$\ln x = -\ln 2 \tag{5}$$

which gives $x = 1/2$.

Once the key idea strikes, viz. to take $\ln x$ and $\ln y$ as the variables, the problem is straightforward. The properties of logarithms needed are elementary and standard. Nowhere we have to convert logarithms w.r.t. one base to those w.r.t. another.

Q.4 Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then

(A) $P \subset Q$ and $Q - P \neq \emptyset$ (B) $Q \not\subset P$ (C) $P \not\subset Q$ (D) $P = Q$

Answer and Comments: (D). All the four alternatives involve the containment relations between the two sets, viz. P and Q. Ideally, in such cases one should identify the elements of each set and see which of them belong to the other. This is especially necessary when the conditions that define the sets are of very different types, e.g. one of the sets is the set of roots of some polynomial and the other is the set of all solutions of some trigonometric equation. However, in the present problem, the defining conditions for both the sets are very similar. Therefore, even without identifying their solution sets, if we can decide which, if any, of these conditions implies the other, then we would be in a position to tell which one of P and Q is a subset of the other. And, if the conditions are equivalent, then the two sets would be the same.

Elements of P are defined by the equation

$$\sin\theta - \cos\theta = \sqrt{2}\cos\theta \tag{1}$$

which can be rewritten as

$$(\sqrt{2} + 1)\cos\theta = \sin\theta \tag{2}$$

The defining condition of Q can be rewritten as

$$\cos\theta = (\sqrt{2} - 1)\sin\theta \tag{3}$$

We can rewrite (2) after dividing by $\sqrt{2} + 1$ and rationalising as

$$\cos\theta = \frac{1}{\sqrt{2} + 1}\sin\theta = (\sqrt{2} - 1)\sin\theta \tag{4}$$

which is exactly same as (3). So the defining conditions of the two sets are the same. Hence $P = Q$.

This is a tricky problem. Superficially it looks like a problem in trigonometric equations. And indeed it would be so if the elements of the two sets P and Q were to be identified explicitly. But as it stands the nature of these two conditions is such that the two conditions are the same merely by a simple rationalisation.

Q.5 Let $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}, \vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{c} = \hat{\imath} - \hat{\jmath} - \hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by

(A) $\hat{\imath} - 3\hat{\jmath} + 3\hat{k}$ (B) $-3\hat{\imath} - 3\hat{\jmath} - \hat{k}$
(C) $3\hat{\imath} - \hat{\jmath} + 3\hat{k}$ (D) $\hat{\imath} + 3\hat{\jmath} - 3\hat{k}$

Answer and Comments: (C). Note that the data is insufficient to determine $\vec{v}$ uniquely even upto a $\pm$ sign. There are infinitely many vectors in a plane which have a given projection in a given direction. This is all right because the problem merely asks to find only one such vector, and that too from among the four given vectors. So one way to attempt the problem is to try these four vectors one by one to determine (i) if they are coplanar with $\vec{a}$ and $\vec{b}$ (which can be done by checking if the determinants of the components vanish) and (ii) if their projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$ (which can be done by taking dot product with $\vec{c}$). Although this is hardly the recommended approach, in some problems it works quickly especially if the correct choice is (A) or (B).

Let us, however, do the problem in the clean way. Let $\vec{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. Coplanarity of $\vec{v}$ with $\vec{a}$ and $\vec{b}$ gives an equation in x, y, z, viz. $\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$, which simplifies to

$$x = z \tag{1}$$

As it turns out, among the four choices, only the vector in (C) satisfies this condition. So, this alone is enough to identify the correct answer. Surely, this is a lapse on the part of the paper-setters. They could have overcome it by giving at least one false answer in which (1) holds. Let us ignore this lapse and continue the solution. The projection of $\vec{v}$ on $\vec{c}$ is $\frac{\vec{v} \cdot \vec{c}}{\|\vec{c}\|} = \frac{x - y - z}{\sqrt{3}}$. Equating this with $\frac{1}{\sqrt{3}}$ gives one more equation in the unknowns, viz.

$$x - y - z = 1 \tag{2}$$

The equations (1) and (2) have infinitely many solutions, the general solution being of the form $(x, -1, x)$ for $x \in I\!R$. The value $x = 3$ gives the vector in (C).

Instead of starting with $\vec{v}$ as $x\hat{i} + y\hat{j} + z\hat{k}$ where there are three unknowns, we could have taken it as $\lambda\vec{a} + \mu\vec{b}$ with λ and μ as unknowns. This way, the coplanarity of $\vec{a}, \vec{b}$ and $\vec{v}$ has been incorporated to reduce the number of variables by one. The second requirement about $\vec{v}$ now translates into $(\lambda\vec{a} + \mu\vec{b}) \cdot \vec{c} = 1$, i.e. as

$$\lambda(\vec{a} \cdot \vec{c}) + \mu(\vec{b} \cdot \vec{c}) = 1 \tag{3}$$

The values of $\vec{a} \cdot \vec{c}$ and $\vec{b} \cdot \vec{c}$ are -1 and 1 respectively. So we get

$$-\lambda + \mu = 1 \tag{4}$$

In terms of components, $\vec{v}$ can be written as

$$\vec{c} = (\lambda + \mu)\hat{i} + (\lambda - \mu)\hat{j} + (\lambda + \mu)\hat{k} \tag{5}$$

which, in view of (4) becomes

$$\vec{c} = (2\lambda + 1)\hat{i} - \hat{j} + (2\lambda + 1)\hat{k} \tag{6}$$

where λ is a parameter which can take any real value. The value $\lambda = 1$ gives the vector in (C).

Another very straightforward problem. Although several approaches are available, in a trivial problem like this, it is pointless to compare their merits and demerits.

Q.6 The value of $\displaystyle\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)}\, dx$ is

(A) $\frac{1}{4} \ln \frac{3}{2}$ (B) $\frac{1}{2} \ln \frac{3}{2}$ (C) $\ln \frac{3}{2}$ (D) $\frac{1}{6} \ln \frac{3}{2}$

Answer and Comments: (A). The substitution $x^2 = u$ suggests itself and transforms the integral, say I, into

$$I = \frac{1}{2} \int_{\ln 2}^{\ln 3} \frac{\sin u}{\sin u + \sin(\ln 6 - u)}\, du \tag{1}$$

The substitution $t = \tan(u/2)$ will convert this into an integral of a rational function in t. But usually finding the antiderivatives of rational functions is a cumbersome task. Anyway our interest is not so much in the indefinite integral but only in a particular definite integral and sometimes this can be done without finding an antiderivative of the integrand. Note that $\ln 6 = \ln 2 + \ln 3$ which is precisely the sum of the lower and the upper limit of the integral. This suggests that the following formula may work.

$$\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx \tag{2}$$

Indeed, with this formula we have

$$I = \int_{\ln 2}^{\ln 3} \frac{\sin(\ln 6 - u)}{\sin(\ln 6 - u) + \sin u}\, du \tag{3}$$

which is not any simpler than (1). But the denominators of the integrands in both (1) and (2) are the same. So they can be added easily to get

$$2I = \int_{\ln 2}^{\ln 3} 1\, du = \ln 3 - \ln 2 = \ln \frac{3}{2} \tag{4}$$

which implies $I = \frac{1}{2} \ln \frac{3}{2}$.

This would have been a good problem if nothing like it was asked in any previous JEE. But that is far from the case. Usually every year there is some integral based on the same trick (see the problems in Comment No. 14 of Chapter 18 and the corresponding exer cises). Any student who has seriously prepared for JEE is sure to see the trick. So problems like this have lost their ability to test originality of thought.

Q.7 Let the line $x = b$ divide the area enclosed by $y = (1-x)^2, y = 0$ and $x = 0$ into two parts $R_1(0 \leq x \leq b)$ and $R_2(b \leq x \leq 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

Answer and Comments: (B). The given curve is a vertically upward parabola which touches the x-axis at $(1, 0)$ and cuts the y-axis at $(0, 1)$. Obviously $0 \leq b \leq 1$. By integration, we have

$$R_1 = \int_0^b (x-1)^2\, dx = \left.\frac{(x-1)^3}{3}\right|_0^b = \frac{(b-1)^3 + 1}{3} \tag{1}$$

and

$$R_2 = \int_b^1 (x-1)^2\, dx = \left.\frac{(x-1)^3}{3}\right|_b^1 = \frac{-(b-1)^3}{3} \tag{2}$$

The condition $R_1 - R_2 = \frac{1}{4}$ gives an equation in b, viz.

$$\frac{2(b-1)^3 + 1}{3} = \frac{1}{4} \tag{3}$$

or equivalently,

$$\frac{2(b-1)^3}{3} = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12} \tag{4}$$

which yields $(b-1)^3 = -\dfrac{1}{8}$. Solving, $b - 1 = -\dfrac{1}{2}$ which determines b as $\dfrac{1}{2}$.

Yet another very straightforward problem.

SECTION II

Multiple Correct Choice Type

This section contains **four** multiple choice questions. Each question has four choices out of which **ONE OR MORE** may be correct.

Q.8 Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to

(A) M^2 (B) N^2 (C) M^2 (D) MN

Answer and Comments: There is a glaring mistake in the very statement of the problem. A skew-symmetric matrix of an odd order is always singular. So the data is vacuous. Actually, the order of the matrices is unimportant in the present problem. All that needs to be given is that M, N are some square matrices (of equal orders) which are non-singular and skew-symmetric. On this assumption, a solution can be given as follows.

Call the given expression as E. Since $MN = NM$, we have $M^2N^2 = M(MN)N = M(NM)N = (MN)^2$. Also $M^T = -M$. As a result, we obtain

$$E = -(MN)^2(MN)^{-1}(MN^{-1})^T = -MN(MN^{-1})^T \quad (1)$$

From the properties of transposes,

$$(MN^{-1})^T = (N^{-1})^T M^T = (N^T)^{-1}M^T = (-N)^{-1}(-M) = N^{-1}M \quad (2)$$

From (1) and (2),

$$E = -MNN^{-1}M = -M^2 \quad (3)$$

So the correct answer would be (C). However, as the question stands, its hypothesis is never satisfied. So technically, any conclusion drawn from it is true. Such statements are said to be **vacuously true**. It is like saying that if $x \in \emptyset$, then any statement you make about x is true because there is no x to render it false!

Undoubtedly some scrupulous students would be unsettled by the mistake in the statement of the problem. Those who realise that the order of the matrices is nowhere needed will get through. But then that would also include those who are not sharp enough to see the mistake. A gross injustice both to the candidates and to the selection process, wrought out by the multiple choice format of the test. In a conventional examination a candidate can write down that there is a mistake and can be given some credit for it.

Q.9 The vector(s) that is/are coplanar with the vectors $\hat{\imath}+\hat{\jmath}+2\hat{k}$ and $\hat{\imath}+2\hat{\jmath}+\hat{k}$ and perpendicular to the vector $\hat{\imath}+\hat{\jmath}+\hat{k}$ is/are

(A) $\hat{\jmath}-\hat{k}$ (B) $-\hat{\imath}+\hat{k}$ (C) $\hat{\imath}-\hat{\jmath}$ (D) $-\hat{\jmath}+\hat{k}$

Answer and Comments: (A), (D). This problem is strikingly similar to Q.5. Again, we are given some unknown vector, say $\vec{v}$ which is coplanar with two given vectors. In Q.5 the projection of this vector on a third given vector $\vec{c}$ was given as $\dfrac{1}{\sqrt{3}}$. In the present problem this projection is 0 because of perpendicularity. So in a way this problem is even simpler and can be tried with either of the methods used in Q.5. So, let $\vec{v}x\hat{\imath}+y\hat{\jmath}+z\hat{k}$. Then its coplanarity with the first two vec tors gives $\begin{vmatrix} x & y & z \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0$, i.e.

$$-3x+y+z=0 \tag{1}$$

Perpendicularity of $\vec{v}$ with the vector $\hat{\imath}+\hat{\jmath}+\hat{k}$ gives one more equation, viz.

$$x+y+z=0 \tag{2}$$

Together (1) and (2) give $x=0$ and $y=-z$. Out of the given vectors, those in (A) and (D) satisfy these conditions.

It is difficult to see what is gained by asking two essentially similar and absolutely straightforward questions in the same paper.

Q.10 Let $f : I\!R \longrightarrow I\!R$ be a function such that $f(x+y)=f(x)+f(y)$ for all $x, y \in I\!R$. If $f(x)$ is differentiable at $x=0$, then

(A) $f(x)$ is differential only in a finite interval containing 0
(B) $f(x)$ is continuous for every $x \in I\!R$
(C) $f'(x)$ is constant for every $x \in I\!R$
(D) $f(x)$ is differentiable except at finitely many points.

Answer and Comments: (B), (C), (D). The condition satisfied by f is an example of what is called a functional equation. These are discussed in Chapter 20. The present problem is exactly the problem from Comment

No. 3 of this chapter where it is shown that $f(x)$ is of the form cx for some constant c. In fact, Comment No. 4 deals with the more challenging problem of drawing the same conclusion under the weaker hypothesis that $f(x)$ is continuous at 0.

We duplicate here the proof of the easier assertion (which is the present problem). Putting $x = y = 0$ in the given functional equation, we get $f(0) = f(0+0) = f(0) + f(0) = 2f(0)$ which means $f(0) = 0$. We are given that $f'(0)$ exists. let $c = f'(0)$. We claim that $f'(x) = c$ for all $x \in I\!R$. To see this,

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{f(x)+f(h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{f(h)}{h} \\ &= \lim_{h\to 0}\frac{f(h)-f(0)}{h} \\ &= f'(0) = c \qquad (1)\end{aligned}$$

Thus we have shown that f is differentiable everywhere and further that $f'(x)$ is a constant (viz. c). As a result the statements (B) and (C) are true while (A) is false. A controversy can arise about (D). Many people will interpret it to mean that there are finitely many points, say $x_1, x_2, \ldots, x_n$ at which f is not differentiable and at all other points it is differentiable. With this interpretation (D) is false. But technically, it is a true statement. What it says is that there exists a finite subset, say F of $I\!R$ such that f is differentiable at all points of $I\!R - F$ but not differentiable at any points of F. Now there is nothing to stop us from taking F to be the empty set $\emptyset$ because the empty set is, after all, a finite set. So (D) is also correct. A similar interpretation is followed many times. For example, if a theorem is proved about functions with only finitely many discontinuities it is applicable to continuous functions too (i.e. functions with no discontinuities).

The problem is unsuitable to be asked as a multiple choice question. The real challenge in it is to *prove* that $f(x)$ is of the form cx for some constant c. The mediocre students will simply *assume* this and will still get rewarded. The scrupulous students stand to lose. Also it is better to avoid controversial alternatives such as (D) because then the answer depends on the interpretation adopted and so no matter which interpretation is taken as correct, some section of the candidates stands to suffer, even though they have the right thought (which they are not in a position to explain).

Q.11 Let the eccentricity of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ be the reciprocal of that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

(A) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(B) a focus of the hyperbola is $(2, 0)$

(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(D) the equation of the hyperbola is $x^2 - 3y^3 = 3$.

Answer and Comments: (B), (D). Writing the ellipse in the standard form as

$$\frac{x^2}{4} + \frac{y^2}{1} = 1 \tag{1}$$

we see that its eccentricity is $\frac{\sqrt{4-1}}{2} = \frac{\sqrt{3}}{2}$ and its foci are at $(\pm 2 \times \frac{\sqrt{3}}{2}, 0)$ i.e. at $(\pm\sqrt{3}, 0)$.

Therefore the eccentricity, say e of the given hyperbola is $\frac{2}{\sqrt{3}}$. From the formula for the eccentricity, we get

$$b^2 = a^2(e^2 - 1) = a^2(\frac{4}{3} - 1) = \frac{a^2}{3} \tag{2}$$

This gives one equation in a and b. To determine the hyperbola we need another. It comes from the fact that the hyperbola passes through one of the points $(\pm\sqrt{3}, 0)$. (Because of the symmetry of the hyperbola, it does not matter which one we take.) We thus get

$$\frac{3}{a^2} - \frac{0}{b^2} = 1 \tag{3}$$

i.e. $a^2 = 3$. From (2) we now get $b^2 = 1$. Hence the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{1} = 1$, or $x^2 - 3y^3 = 3$ which makes (D) true and (A) false. Its eccentricity is already known to be $\frac{2}{\sqrt{3}}$. So its foci are at $(\pm\sqrt{3}\frac{2}{\sqrt{3}}, 0)$ i.e. at $(\pm 2, 0)$. So (C) is false while (B) is true.

A straightforward problem once you remember the standard formulas for the eccentricity and the foci of conics. Really challenging problems about conics are difficult to ask in a multiple choice format and that too in a question that has to be answered in a couple of minutes. That explains the occurrence of a large number of such questions in the recent JEE papers. In the good old days this question could have been asked as "If the eccentricities of two conics C_1 and C_2 with axes along the same pair of lines are reciprocals of each other and C_1 passes through the foci of C_2, prove that C_2 passes through the foci of C_1."

SECTION III

Paragraph Type

This section contains two paragraphs. There are two multiple choice questions based on the first paragraph and three on the second. All questions have ONLY ONE correct answer.

Paragraph for Q. 12 and 13

Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls and U_2 contains only 1 white ball. A fair coin is tossed. If a head appears then one ball is drawn at random and put into U_2. However, if a tail appears then two balls are drawn at random from U_1 and put into U_2. Now one ball is drawn at random from U_2.

Q.12 The probability of the drawn ball from U_2 being white is

(A) $\frac{13}{30}$ (B) $\frac{23}{30}$ (C) $\frac{19}{30}$ (D) $\frac{11}{30}$

Answer and Comments: (B). There are two mutually exclusive and exhaustive events, each of probability 1/2 depending upon whether a head or a tail shows. In the first event, the probability that U_2 will contain two white balls is 3/5 while the probability that it will contain one white and one red ball is 2/5. In the first case the ball drawn from U_2 will be white with probability 1, in the second it will be so with probability 1/2.

If a tail shows then the probability that the two balls drawn from U_1 will be both white or both red or one white and one red with probabilities $\frac{\binom{3}{2}}{\binom{5}{2}}, \frac{\binom{2}{2}}{\binom{5}{2}}$ and $\frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}}$ i.e. $\frac{3}{10}, \frac{1}{10}$ and $\frac{6}{10}$ respectively. In the first case, U_2 will have 3 white balls and so the ball drawn from U_2 will be white with probability 1. In the second case, U_2 will have 1 white and 2 red balls and so the ball drawn from U_2 will be white with probability 1/3. In the last case U_2 will have 2 white and 1 red balls and so the ball drawn from U_2 will be white with probability 2/3.

Putting all these pieces together, the probability that the ball drawn from U_2 is white is

$$\frac{1}{2} \times (\frac{3}{5} \times 1 + \frac{2}{5} \times \frac{1}{2}) + \frac{1}{2}(\frac{3}{10} \times 1 + \frac{1}{10} \times \frac{1}{3} + \frac{6}{10} \times \frac{2}{3}) \qquad (1)$$

which comes out to be $\frac{2}{5} + \frac{11}{30} = \frac{23}{30}$.

A straightforward problem on probability. The reasoning in arriving at (1) is too long to write down. But it does not take so long to do it as a rough work, especially with the aid of a tree diagram, as shown in Comment No. 11 of Chapter 22. Moreover, in such problems it is very unlikely that the answer would come out to be correct by a fluke or by cancellation of two mistakes. So, if a candidate has given the correct answer, his thinking is undoubtedly correct. Such problems are therefore well suited for multiple choice tests. Not surprisingly, similar problems have been asked many times in the past and so such problems do not test originality of thinking.

Q.13 Given that the ball drawn from U_2 is white, the probability that a head appeared on the coin is

$$\text{(A)} \ \frac{17}{23} \quad \text{(B)} \ \frac{11}{23} \quad \text{(C)} \ \frac{15}{23} \quad \text{(C)} \ \frac{12}{23}$$

Answer and Comments: (D). This is a problem on conditional probability. Let A be the event that a head occurs. Let B be the event that the ball drawn from U_2 is white. We are asked to find $P(A|B)$, i.e. the probability of A given B. This equals $\frac{P(A \cap B)}{P(B)}$. In the last question we already calculated $P(B)$ as $\frac{23}{30}$. As for the numerator it is the probability that a head occurs and the ball drawn from U_2 is white. This is precisely the first summand in the expression (1) above. So,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/5}{23/30} = \frac{12}{23} \tag{2}$$

The calculations needed in this problem were already done in the last question. So it is inherently unfair to give the same weightage to the two questions. In a conventional examination only the present question can be asked and the last question would be an essential step in its solution, which can be given substantial partial credit.

Once again, problems of finding conditional probabilities are common and standard. Given the constraints of the paper-setting, it is difficult to come up with a radically new problem.

One wishes that at least some of the fake answers to Q.13 had a denominator different from 23. With all the denominators being 23, an alert student who knows that a conditional probability is a ratio will get a hint that the correct answer to Q.12 would have 23 in the numerator. He is further helped by the fact that all the alternatives in Q. 12 have the same denominator, viz. 30. So his non-mathematical skills will give him the answer to Q.12 without doing any work!

Paragraph for Q.14 to Q.16

Let a, b, c be three real numbers satisfying

$$[a\ b\ c] \begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ 0\ 0] \text{............(E)}$$

Q.14 If the point (a, b, c), with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is

(A) 0 (B) 12 (C) 7 (D) 6

Answer and Comments: (D). The matrix equation (E) is a compact way of expressing a homogeneous system of three linear equations in the three unknowns a, b, c, viz.

$$a + 8b + 7c = 0 \qquad (1)$$
$$9a + 2b + 3c = 0 \qquad (2)$$
$$\text{and} \qquad 7a + 7b + 7c = 0 \qquad (3)$$

The very fact that the present question stipulates some other data to determine the value of $7a+b+c$ indicates that (E) alone does not determine the values of a, b, c uniquely. This is verified by showing that the matrix in (E) is singular. Before proceeding further, we need the general solution of the system of equations. From (1) and (3) we have $a + 8b = 7a + 7b$ i.e. $b = 6a$ and hence from (3) we get $c = -a - b = -7a$. So the general solution of (E) is

$$a = t, b = 6t, c = -7t \qquad (4)$$

where t is a real parameter. If the point (a, b, c) is to lie on the plane $2x + y + z = 1$, then we have $2a + b + c = 1$ which gives $t = 1$. So $a = 1, b = 6$ and $c = -7$. Hence $7a + b + c = 6$. (Although this is simple enough by directly substituting the values of a, b, c, a few seconds can be saved if we notice that one of the equations is $a + b + c = 0$ and so $7a + b + c = 6a$.)

Q.15 Let ω be a solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If $a = 2$ with b and c satisfy (E), then the value of $\dfrac{3}{\omega^a} + \dfrac{1}{\omega^b} + \dfrac{3}{\omega^c}$ is equal to

(A) -2 (B) 2 (C) 3 (D) -3

Answer and Comments: (A). Since $a = 2$, from (1), $b = 12$ and $c = -14$. The powers of ω repeat in a cycle of length 3 with $\omega^3 = 1$. Hence $\omega^{-2} = \omega, \omega^{12} = 1$ and $\omega^{14} = \omega^2$. Therefore the given sum equals $3\omega+1+3\omega^2$. Further $\omega^2+\omega+1 = 0$ (as can be seen by factorising ω^3-1 and noting that $\omega \neq 1$). So the expression equals $1 + 3(\omega^2 + \omega) = 1 - 3 = -2$.

An easy problem based on very standard properties of ω

Q.16 Let $b = 6$ with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n$ equals

(A) 6 (B) 7 (C) $\dfrac{6}{7}$ (D) ∞

Answer and Comments: (B). As $b = 6$, in (1) $t = 1$ and so $a = 1$ and $c = -7$. Hence the quadratic is $x^2 + 6x - 7$. Therefore $\alpha + \beta = -6$ and $\alpha\beta = -7$. So, the expression $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ equals $\dfrac{\alpha + \beta}{\alpha\beta} = \dfrac{-6}{-7} = \dfrac{6}{7}$. The given infinite series is a geometric series with common ratio less than 1 in magnitude. Therefore it is convergent with sum $\dfrac{1}{1 - \frac{6}{7}} = 7$.

All the three problems in this paragraph are very simple. But, unlike in the last paragraph, the questions in this paragraph are completely unrelated to each other. Their only common feature is that they all have three parameters a, b, c, which are related to each other by (E). The values of these parameters are different in the the three problems and have to be obtained by solving (E). A numerical slip in solving (E) will be penalised thrice, which is inherently unfair. The purpose of the common factor 7 in the third column of the given matrix is not clear. Perhaps just to improve the chances of numerical mistakes!

SECTION IV

Integer Type

This section contains seven questions. The answer to each question is a single digit integer, ranging from 0 to 9.

Q.17 The minimum value of the sum of the real numbers $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8$ and a^{10} with $a > 0$ is ....

Answer and Comments: 8. The direct way to approach the problem is to consider the function

$$f(x) = \frac{1}{x^5} + \frac{1}{x^4} + \frac{3}{x^3} + 1 + x^8 + x^{10} \qquad (1)$$

defined for all $x > 0$ and to find the minimum of f on the set $(0, \infty)$. If we have to do this using calculus, we shall have to consider

$$f'(x) = -\frac{5}{x^6} - \frac{4}{x^5} - \frac{9}{x^4} + 8x^7 + 10x^9 \qquad (2)$$

The critical points of f are the positive roots of the polynomial

$$p(x) = 10x^{15} + 8x^{13} - 9x^2 - 4x - 5 \qquad (3)$$

By inspection, $x = 1$ is a critical point. But there could be many others and they are not easy to identify. So although $x = 1$ is a possible candidate for the minimum of f, we cannot conclusively say that the minimum occurs at it. The second derivative test will at the most give us that there is a local minimum at 1. But that is not good enough.

So we look for some other method. One of the terms in the sum is the constant 1. Such a term plays no role in finding the minimum (or the maximum). The very fact that it is given indicates that it has some other role to play. One such role is that when 1 occurs as a factor (rather than as a summand), it serves as a device to increase the number of factors without affecting the product. There are some places where it is the number of factors changes an expression even though the product remains the same. For example, the geometric mean of x and y is $(xy)^{1/2}$ but that of $1, x$ and y is $(xy)^{1/3}$.

This suggests that the problem may be amenable to the A.M.-G.M. inequality. If we apply it directly, we get

$$a^{-5} + a^{-4} + 3a^{-1} + 1 + a^8 + a^{10} \geq 6(3a^8)^{1/6} \geq 6 \times 3^{1/6}a^{4/3} \qquad (4)$$

for all $a > 0$ with equality holding only when all terms on the L.H.S. are equal. But there is no value of a fro which all the six terms are equal and so this inequality does not give the minimum. Moreover the R.H.S. is not a fixed real number.

So, this approach is also not workable. However, as pointed out at the end of Comment No. 6 of Chapter 6, in applying the A.M.-G.M. inequality, sometimes it pays to split some terms. This idea provides the key to the solution of the present problem. If we rewrite $3a^{-3}$ as $a^{-3} + a^{-3} + a^{-3}$ then, the L.H.S. of (4) becomes a sum of 8 terms, whose product is $a^{-5}a^{-4}a^{-9}a^8a^{10}$ which is simply 1. Therefore the G.M. is 1 and so we get

$$a^{-5} + a^{-4} + 3a^{-3} + 1 + a^8 + a^{10} \geq 8 \times 1^{1/8} = 8 \qquad (5)$$

It is tempting to think now that 8 is the minimum of the given sum. But this reasoning is incomplete. In the case of the A.M.-G.M. inequality, equality is possible only when all the terms are equal. So, unless there is some value of a for which all the terms $a^{-5}, a^{-4}, a^{-3}, 1, a^8$ and a^{10} are equal, we cannot conclude that the minimum value of the sum is 8. If there is no such value of a then all we can say is that the minimum is *at least* 8. It could be higher and may require a more sophisticated argument.

Fortunately, that does not happen here. At $a = 1$, all the terms are equal and so 8 is indeed the minimum value of the given sum.

Problems where the A.M.-G.M. inequality is applied in a tricky manner have appeared before (see e.g. JEE 2004). When the minimum value of a sum is to be obtained by applying the A.M.-G.M. inequality, it is vital to show that equality holds. In a conventional examination, it is possible to see if a candidate has done that. In the multiple choice format, that is all masked. Yet another instance how the multiple choice format rewards the unscrupulous.

Q.18 The positive integer $n > 3$ satisfying the equation
$\frac{1}{\sin(\frac{\pi}{n})} = \frac{1}{\sin(\frac{2\pi}{n})} + \frac{1}{\sin(\frac{3\pi}{n})}$ is ....

Answer and Comments: 7. let $\theta = \frac{\pi}{n}$. Then the given equation can be written as

$$\frac{1}{\sin\theta} = \frac{1}{\sin 2\theta} + \frac{1}{\sin 3\theta} \tag{1}$$

Note that $\sin 2\theta$ and $\sin 3\theta$ both have $\sin\theta$ as a factor. Canceling this factor which is non-zero since $0 < \theta < \pi$) (1) becomes

$$1 = \frac{1}{2\cos\theta} + \frac{1}{3 - 4\sin^2\theta} \tag{2}$$

which can be written as a cubic equation in $\cos theta$ viz.

$$8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1 = 0 \tag{3}$$

Unfortunately, no roots of this cubic can be found by inspection. So we abandon this approach.

instead, we solve (1) as a trigonometric equation. Clearing out the denominators, (1) becomes

$$\sin 2\theta \sin 3\theta = \sin\theta \sin 2\theta + \sin\theta \sin 3\theta \tag{4}$$

Multiplying throughout by 2 and using the identity

$$2\sin A \sin B = \cos(A - B) - \cos(A + B) \tag{5}$$

we get

$$\cos\theta - \cos 5\theta = \cos\theta - \cos 3\theta + \cos 2\theta - \cos 4\theta \tag{6}$$

which simplifies to

$$\cos 3\theta - \cos 5\theta = \cos 2\theta - \cos 4\theta \tag{7}$$

Using the identity (5) backwards, this becomes

$$2\sin 4\theta \sin\theta = 2\sin 3\theta \sin\theta \tag{8}$$

We can now cancel $\sin\theta$ to get

$$\sin 4\theta = \sin 3\theta \tag{9}$$

Since $0\theta < \pi/2$, the only way this can happen is if $3\theta = \pi - 4\theta$, i.e. $7\theta = \pi$ which means $7\frac{\pi}{n} = \pi$, i.e. $n = 7$.

This is a good problem on trigonometric equations. It is important to realise that the approach in which (3) has to be solved is not going to work. The problem is the essence of a geometric problem asked in 1994 JEE (see the Main Problem of Chapter 10). However, that problem could also be tackled by pure geometry (using Ptolemy's theorem) or using complex numbers (see Comments 2 and 3 of Chapter 10). If one wants, one can attack the present problem too with complex numbers. Put $z = e^{i\theta} = \cos\theta + i\sin\theta$. Then $\sin\theta = \dfrac{1}{2i}(z - \overline{z}) = \dfrac{1}{2i}(z - \dfrac{1}{z}) = \dfrac{z^2-1}{2iz}$. Similarly, $\sin 2\theta = \dfrac{z^4-1}{2iz^2}$ and $\sin 3\theta = \dfrac{z^6-1}{2iz^3}$. Substituting these into (1) we get

$$\frac{z}{z^2-1} = \frac{z^2}{z^4-1} + \frac{z^3}{z^6-1} \tag{10}$$

Canceling z from the numerators and $z^2 - 1$ from the denominators, and simplifying, we get

$$z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = 0 \tag{11}$$

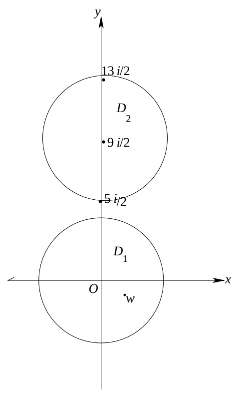
Multiplying by $z + 1$, we get

$$z^7 + 1 = 0 \tag{12}$$

which means $\cos 7\theta + i\sin 7\theta = 1$. Hence $\sin 7\theta = 0$. From this we determine n as before.

Q.19 If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is ..... .

Answer and Comments: Note that $|2z - 6 + 5i| = 2|z - 3 + \frac{5}{2}i|$ and so it suffices to minimise the second absolute value. Put $w = z - 3 - 2i$. Then $z - 3 + \frac{5}{2}i = w + \frac{9}{2}i$. We are given that w varies over the disc, say D_1, of radius 2 centred at the origin. And we have to minimise $|w + \frac{9}{2}i|$ as w varies over this disc. Now, as w varies over D_1, $w + \frac{9}{2}i$ varies over the disc, say D_2, of radius 2 and centred at $\frac{9}{2}i$. This disc lies entirely above the x-axis and is centred at a point on the y-axis. So the point of D_2 that is closest to the origin is the lower end of its vertical diameter, viz. $\frac{9}{2}i - 2i = \frac{5}{2}i$. Hence the minimum of $|z - 3 + \frac{5}{2}i|$ is $\dfrac{5}{2}$. Therefore the minimum we want is twice this, viz. 5.



A simple but good problem testing geometric visualisation of complex numbers and their absolute values.

Q.20 Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is .... .

Answer and Comments: 1. By the chain rule, the desired derivative is $\frac{f'(\theta)}{\frac{d}{d\theta}\tan\theta} = \frac{f'(\theta)}{\sec^2\theta}$. However, finding $f'(\theta)$ directly from its formula is cumbersome. So, let us see if we can first simplify $f(\theta)$. (Trigonometric functions of the angles that are expressed as the inverse trigonometric functions of some quantities can often be simplified.)

Let $\alpha = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)$. Then $\tan\alpha = \frac{u}{v}$ where $u = \sin\theta$ and $v = \sqrt{\cos 2\theta}$. So we have

$$\begin{aligned} \sin\alpha &= \frac{u}{\sqrt{u^2+v^2}} \\ &= \frac{\sin\theta}{\sqrt{\sin^2\theta + \cos 2\theta}} \\ &= \frac{\sin\theta}{\sqrt{\cos^2\theta}} && (1) \\ &= \frac{\sin\theta}{\cos\theta} && (2) \\ &= \tan\theta && (3) \end{aligned}$$

where, in going from (1) to (2) we have used that $\cos\theta > 0$ for the given range of θ.

Therefore $f(\theta) = \sin(\alpha)$ is simply $\tan\theta$. So, its derivative w.r.t. itself is simply 1.

This is a problem on simplification of inverse trigonometric functions. The calculus part of it is negligible.

Q.21 Let $a_1, a_2, \ldots, a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i$, $1 \leq p \leq 100$. For any integer n with $1 \leq n \leq 20$, let $m = 5n$. If $\frac{S_n}{S_m}$ does not depend on n, then a_2 is ..... .

Answer and Comments: 3 or 9. This is a problem on the sum of the terms of an A.P. Although normally we assume that the common difference, say d of an A.P. is non-zero, there is nothing in the definition to preclude it. For such a degenerate progression, all the terms are the same and so the ratio of the sum of *any* $5n$ of them will always be 5 times the sum of any n terms. So in this case the hypothesis of the problem holds trivially. Therefore $a_2 = a_1 = 3$ is a valid answer.

We assume that the paper-setters did not want to design such a trivial problem (although, going by some other problems, this assumption is shaky!). So, let us see if some other value of d will satisfy the hypothesis. From the standard formulas for the sums of the terms in an A.P., we have

$$S_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} 3 + (i-1)d = 3n + \frac{1}{2}n(n-1)d \tag{1}$$

with a similar formula for S_m. Since $n = 5m$, we have

$$\begin{aligned} \frac{S_m}{S_n} &= \frac{6m + m(m-1)d}{6n + n(n-1)d} \\ &= \frac{30n + 5n(5n-1)d}{6n + n(n-1)d} \\ &= 5\frac{6 + 5nd - d}{6 + nd - d} \end{aligned} \tag{2}$$

The ratio will be independent of n in two cases : (i) $d = 0$ when the ratio will be 5 and (ii) $6 - d = 0$, i.e. $d = 6$ (in which case the ratio will be 25). We already considered the first case and got $a_2 = a_1 + d = 3$. In the second case we get $a_2 = a_1 + 6 = 9$.

This is a good question, because although problems involving sums of arithmetic progressions are common, the condition of some expression being independent of some parameters is fairly uncommon and requires a candidate to think. Unfortunately, the trivial possibility eluded the paper-setters (as, otherwise, they could have designed the problem so that a_2 would not lie between 0 and 9, e.g. by giving $a_1 = -1$ instead of $a_1 = 3$). This would not have mattered in a conventional examination where the candidates would be asked to show their work. But as the problem is to have only one correct answer, this lapse is likely to unsettle some candidates.

Q.22 Let $f : [1, \infty) \longrightarrow [2, \infty)$ be a differentiable function such that $f(1) = 2$. If $6\int_1^x f(t)dt = 3xf(x) - x^3$ for all $x \geq 1$, then the value of $f(2)$ is .... .

Answer and Comments: 6. Differentiating the given equation w.r.t. x and using the Fundamental Theorem of Calculus (second from) we get

$$6f(x) = 3xf'(x) + 3f(x) - 3x^2 \tag{1}$$

If we call $f(x)$ as y, then this is a differential equation in y, viz.

$$\frac{dy}{dx} - \frac{y}{x} = x \tag{2}$$

This is a linear differential equation of order 1 with integrating factor $e^{-\ln x} = \frac{1}{x}$. Multiplying by it the equation becomes

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 1 \tag{3}$$

i.e.

$$\frac{d}{dx}(\frac{y}{x}) = 1 \tag{4}$$

whose general solution is

$$\frac{y}{x} = x + c \tag{5}$$

where c is a constant. The initial condition $f(1) = 2$ determines c as 1. So, we have

$$f(x) = x^2 + x \tag{6}$$

for $x \geq 1$, whence $f(2) = 4 + 2 = 6$.

There is a minor inconsistency in the data. From the given equation we get $0 = 6\int 1_1 f(t)\, dt = 3 \times 2 - 1 = 5$, a contradiction. To avoid this inconsistency, the given equation should have been $6\int_1^x f(t)dt = 3xf(x) - x^3 - 5$. The mistake does not affect either the answer or the method. In fact, many candidates might not even notice it. Still, a prestigious examination should not contain such lapses.

This is a simple problem and problems like this have appeared several times after differential equations were introduced into the JEE syllabus. But as with many other problems of this kind, the work involved is in excess of what is proportional to the credit given. Instead of specifying the differential equation directly, it is given in a twisted form.

Q.23 Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P(\frac{1}{2}2)$, and let Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is ..... .

Answer and Comments: 2. In the various formulas pertaining to a parabola, the parabola is generally taken to be in the standard from, viz. $y^2 = 4ax$. In the present problem $a = 2$. But it is *not* a good idea to use this value in the working because it may lead to numerical mistakes, without achieving any simplification. Therefore it is a good idea to work out the problem for the parabola $y^2 = 4ax$ and put $a = 2$ only at the end.

In many problems about parabolas, it is convenient to take its parametric equations, viz.

$$x = at^2, y = 2at \tag{1}$$

where $-\infty < t < \infty$. The end points of the latus rectum correspond to the values $t = \pm 1$ while in the present problem, since $a = 2$, the point $P(\frac{1}{2}, 2)$ corresponds to $t = 1/2$. But once again, since the formulas for the equations of the tangents are for any arbitrary value of the parameter, little is to be gained by taking specific values of t. Instead let us suppose that $P_i = (at_i^2, 2at_i)$ $i = 1, 2, 3$ are any three points on the parabola and do the problem for it. At the end we shall put $t_1 = 1, t_2 = -1$ and $t_3 = \frac{1}{2}$.

The tangent at $(at_i^2, 2at_i)$ is $t_i y = x + at_i^2$. Taking intersections two at a time we take $Q_1 = (at_2t_3, a(t_2 + t_3))$ $Q_2 = (at_3t_1, a(t_3 + t_1))$ and $Q_3 = (at_1t_2, a(t_1 + t_2))$. The ratio of the areas of the triangle $P_1P_2P_3$ and $Q_1Q_2Q_3$ equals $|\frac{2D_1}{D_2}|$, where $D_1 = \begin{vmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{vmatrix}$ and $D_2 = \begin{vmatrix} t_2t_3 & t_2 + t_3 & 1 \\ t_3t_1 & t_3 + t_1 & 1 \\ t_1t_2 & t_1 + t_2 & 1 \end{vmatrix}$. The row operations $R_2 \longrightarrow R_2 - R_1$ and $R_3 \longrightarrow R_3 - R_1$ show that each of D_1 and D_2 equals $\pm(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)$.) Therefore the ratio of the areas is 2.

So, the answer is independent not only of the value of a but also of which three points on the parabola are chosen to begin with. This property of the parabolas is fairly well known but not as well-known as some other properties such as the focusing property of the parabola. One of the questions in JEE 1996 asked for a proof of this property. But that time it was a full length question and so those who knew this property already had only a marginal advantage over the others, because they still had to show all the work. In a multiple choice format, those who know this property get the answer instantaneously. Those who don't, have to do a lot of work (equivalent to a full length question in the earlier JEE's requiring about 8 to 10 minutes of work). Yet another unfair consequence of the multiple choice format.

PAPER 2

Contents

SECTION I

Single Correct Choice Type

This section contains **eight** multiple choice questions. Each question has 4 choices out of which **ONLY ONE** is correct.

Q.24 A value of b for which the equations

$$x^2 + bx - 1 = 0 \quad \text{and} \quad x^2 + x + b = 0$$

have one root in common is

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$ (C) $i\sqrt{5}$ (D) $\sqrt{2}$

Answer and Comments: (B). There is a general necessary and sufficient condition for two quadratics (see Exercise (3.10)(d)) $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ to have a common root, viz.

$$(q - s)^2 - pr(q + s) + r^2 q + p^2 s = 0 \tag{1}$$

In the present problem, this reduces to

$$b^3 + 3b = 0 \tag{2}$$

whose solutions are $b = 0$ and $b = \pm i\sqrt{3}$. Out of these only $-i\sqrt{3}$ is given as an alternative.

However, (1) is complicated to derive and to remember. So we give a direct derivation of (2) which is simpler because in the present problem p and r coincide. Let α be a common root of the two quadratic. Then we get the required condition by eliminating α from the two equations:

$$\alpha^2 + b\alpha - 1 = 0 \tag{3}$$
$$\text{and} \quad \alpha^2 + \alpha + b = 0 \tag{4}$$

Subtracting,

$$\alpha = \frac{b+1}{b-1} \tag{5}$$

Putting this value in (4) we get

$$\left(\frac{b+1}{b-1}\right)^2 + \frac{b+1}{b-1} + b = 0 \tag{6}$$

which, upon simplification, gives (2).

This is good problem on the roots of a quadratic equation.

Q.25 Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$ where each of a, b, c is either ω or ω^2. Then the number of distinct matrices in the set S is

(A) 2 (B) 6 (C) 4 (D) 8

Answer and Comments: (A). The framing of the question is unnecessarily clumsy. Instead of letting S be the set of certain matrices and asking how many distinct matrices it has, the paper-setters could have directly asked the number of non-singular matrices of the given form where a, b, c can be either ω or ω^2.

Since a, b, c each assume two possible values independently, there are, in all 8 possible matrices. One could list down all of them and see which ones among them are non-singular. But that would be too time consuming. Instead, let us go eliminating the possibilities by taking the values of a first. Since the powers of ω recur in cycles of length 3, the second and the third entries of the second column can be written as ω^3 and ω^4 which are, respectively, ω^2 times the second and the third entries of the first column. So, if $a = \omega^2$, then the second column will be a multiple of the first and the matrix would be singular. So, we must have $a = \omega$. By a similar reasoning, if $c = \omega^2$, then the third row will be ω times the second and the matrix would be singular. So we must have $c = \omega$ too. That leaves only b. It can take two possible values. So the correct answer to the problem would be 0, 1 or 2. Since only 2 is included among the answers, it must be the right answer. However, instead of resorting to this sneaky method, after determining that a and c must both be ω each, we can expand the determinant $\begin{vmatrix} 1 & \omega & b \\ \omega & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$. It comes out as $1 - \omega^2 + \omega - \omega^2 = 1 + \omega - \omega^2$ which is non-zero and independent of b. So b can assume any of the two given values.

In the solution above, the elimination of the possibilities $a = \omega^2$ and $c = \omega^2$ was rather tricky. A more systematic solution would begin by expanding the determinant, say Δ, of the given matrix as a function of a, b and c. It comes out as

$$\begin{aligned} \Delta = \Delta(a,b,c) &= 1 - \omega c + a(\omega^2 c - \omega) + b(\omega^2 - \omega^2) \\ &= 1 + ac\omega^2 - \omega(a+c) \\ &= (1 - a\omega)(1 - c\omega) \end{aligned} \quad (1)$$

which is independent of b and vanishes only when $a\omega = 1$ or $c\omega = 1$. The first possibility occurs when $a = \omega^2$ and the second when $c = \omega^2$. So the only way the matrix is non-singular is when $a = c = \omega$. Since b can have either of the two values, there are two non-singular matrices of the given type.

The problem is a good combination of the properties of ω and of determinants. However, to preclude the sneaky approach, at least one of 0 and 1 should have been given as possible answers.

Q.26 Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in I\!R$. Then the set of all x satisfying $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$, where $(f \circ g)(x) = f(g(x))$, is

(A) $\pm\sqrt{n\pi},\ n \in \{0, 1, 2, \ldots\}$
(B) $\pm\sqrt{n\pi},\ n \in \{1, 2, \ldots\}$
(C) $\frac{\pi}{2} + 2n\pi,\ n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
(D) $2n\pi,\ n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Answer and Comments: (A). The L.H.S. is the composite of four functions. When written out directly, it equals $\sin^2(\sin x^2)$. The R.H.S. is the composite of three functions and equals $\sin(\sin x^2)$. If we call $\sin(\sin x^2)$ as u, then the given equation becomes

$$u^2 = u \quad (1)$$

which is an algebraic equation with thw possible solutions, viz. 1 and 0.

The first possibility gives the trigonometric equation

$$\sin(\sin x^2) = 1 \quad (2)$$

which is possible only when $\sin x^2 = 2k\pi + \frac{\pi}{2}$ for some integer k. The numbers of this set are $\ldots, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$. Since $\pi/2 > 1$, none of these numbers is in $[-1, 1]$, which is the range of the sine function. It follows that the equation (2) has no real solutions.

Thus we are only left with the possibility $u = 0$, which leads to the equation

$$\sin(\sin x^2) = 0 \quad (3)$$

which is possible only when $\sin x^2 = n\pi$ for some integer n. The numbers of this set are $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$. Out of these, only 0 lies in the range of the sine function. So the problem no reduces to solving

$$\sin(x^2) = 0 \tag{4}$$

whose solutions are given by $x^2 = n\pi$ where n is an integer. Obviously n cannot be negative. So the solutions are as in (A).

This is a simple but good problem which tests several abilities, first of all, unravelling a composite of three or four functions, then reducing an equation step by step to an equation whose solutions are known or can be easily found and finally, the knowledge of the solutions of some (albeit very elementary) trigonometric equations. The value of π is rarely needed in mathematics questions. But in the present problem it is needed to eliminate all possibilities in (2) and all but one possibilities in (3).

Q.27 Let (x, y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from $(0, 0)$ to (x, y) in the ratio $1 : 3$. Then the locus of P is

(A) $x^2 = y$ (B) $y^2 = 2x$ (C) $y^2 = x$ (D) $x^2 = 2y$

Answer and Comments: (C). The standard method to find the locus of a point is to let (h, k) be the current coordinates of the point whose locus is to be found. We then use the conditions which this point is given to obey to get an equation in h and k. In the present problem, let $P = (h, k)$. Since P divides the segment from $(0, 0)$ to (x, y) in the ratio $1 : 3$, we have

$$h = \frac{1}{4}x, \quad k = \frac{1}{4}y \tag{1}$$

This gives $x = 4h$ and $y = 4k$. But the point (x, y) is given to lie on the parabola $y^2 = 4x$. So we have

$$16k^2 = 4(4h) \tag{2}$$

Hence $h = k^2$ is the equation satisfied by the coordinates of the moving point P. The locus is obtained by replacing h by x and k by y.

An extremely simple problem on locus. In fact one wonders if such a simple problem fits the standards of JEE. But perhaps that is all that can be expected when the question is to be answered in less than two minutes.

Q.28 Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at $(9, 0)$, then the eccentricity of the hyperbola is

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

Answer and Comments: (B). Another straightforward problem in coordinate geometry. The eccentricity, say e, of a hyperbola in the standard form is given by

$$b^2 = a^2(e^2 - 1) \tag{1}$$

So the problem is to determine a and b. For this we need two equations. One of them is given by the condition that $P(6,3)$ lies on the hyperbola, which gives

$$\frac{36}{a^2} - \frac{9}{b^2} = 1 \tag{2}$$

We need one more equation in a and b. (Actually, as seen from (1), the eccentricity depends only on the ratio $\frac{a}{b}$. But, unfortunately, (2) does not give this ratio. So we need another equation.) It can be obtained from the data that the normal at the point $(6,3)$ passes through $(9,0)$. Hence its slope is $\frac{3-0}{6-9} = -1$. But the slope of the normal at a point (x_0, y_0) on the hyperbola is $-\frac{y_0 a^2}{x_0 b^2}$. Hence the slope of the normal at $(6,3)$ is $-\frac{3a^2}{6b^2} = -\frac{a^2}{2b^2}$. Equating the two expressions for the slope we get

$$a^2 = 2b^2 \tag{3}$$

(Ironically, this makes (2) redundant. But it is natural to begin with (2) since that is the most immediate inference from the data.) From (1) and (3), we get the eccentricity as $\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}}$.

The solution requires little more than knowledge of standard formulas.

Q.29 The circle passing through the point $(-1,0)$ and touching the y-axis at $(0,2)$ also passes through the point

(A) $(-\frac{3}{2}, 0)$ (B) $(-\frac{5}{2}, 2)$ (C) $(-\frac{3}{2}, \frac{5}{2})$ (D) $(-4, 0)$

Answer and Comments: (D). The equation of a circle has three independent parameters and can, therefore, be determined by three pieces of data. Sometimes the pieces of data can be incorporated in the (unknown) equation of the circle so as to get rid of some of the parameters. The present problem is of this type. Call the given circle as C. Since it is given to touch the y-axis at the point $(0,2)$, its centre must lie on the line $y = 2$ and further its distance from $(0,2)$ must equal its radius. So, the centre of C is at a point of the form $(h, 2)$ for some real number h and its radius must be $|h|$. Thus the problem is now reduced to finding only one parameter, viz. h. For this we need one more condition. Since C is given to pass through the point $(-1,0)$, its radius equals the distance between the centre $(h,2)$ and $(-1,0)$. This gives

$$(h+1)^2 + (2-0)^2 = h^2 \tag{1}$$

which gives $2h+5=0$, i.e. $h = -\frac{5}{2}$. So the centre of C is at $(-\frac{5}{2}, 2)$ and its radius is $\frac{5}{2}$. We now have to see which of the four given alternatives is a point whose distance from $(-\frac{5}{2}, 2)$ is $\frac{5}{2}$. By trial, this point is $(-4,0)$.

A simple problem, which tests the ability to utilise the data in a most convenient form. It is a little unfortunate that the final answer has to be obtained by trial. Instead, the desired point could have been specified, say, as the point where the circle touches some line or some other given circle. In that case, it could be arrived at with some work.

Q.30 If $\lim_{x\to 0^+} \left[1+\ln(1+b^2)\right]^{\frac{1}{x}} = 2b\sin^2\theta$, $b>0$ and $\theta \in (-\pi,\pi)$, then the value of θ is

(A) $\pm\frac{\pi}{4}$ (B) (B) $\pm\frac{\pi}{3}$ (C) $\pm\frac{\pi}{6}$ (D) $\pm\frac{\pi}{2}$

Answer and Comments: (D). Call the given limit as L. To find it we use logarithms. Call the given function as $f(x)$. Clearly, $f(x)>0$ for all $x>0$. So we can take its logarithm. So, let

$$g(x) = \frac{\ln[1+x\ln(1+b^2)]}{x} \tag{1}$$

Call the numerator as $h(x)$. Note that $h(0)=0$. Therefore

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{h(x)-h(0)}{x-0} \tag{2}$$

which is nothing but $h'_+(0)$, the right-handed derivative of h at 0. Now, since $h(x) = \ln[1+x\ln(1+b^2)]$, we have

$$h'(x) = \frac{\ln(1+b^2)}{1+x\ln(1+b^2)} \tag{3}$$

In particular, $h'(0) = \ln(1+b^2)$. So, we have shown $\lim_{x\to 0^+} = \ln(1+b^2)$. (This could also have been obtained by applying l'Hôpital's rule to (1).But the work involved is the same.)

Now we use the fact that $f(x) = e^{g(x)}$ and hence

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= e^{\lim_{x\to 0^+} g(x)} \\ &= e^{\ln(1+b^2)} = 1+b^2 \end{aligned} \tag{4}$$

(Although not expected at the JEE level, the justification needed here is the continuity of the exponential function.)

Having evaluated the limit, we turn to the remainder of the problem. We are given

$$1 + b^2 = 2b \sin^2 \theta \tag{5}$$

where $b > 0$. Applying the A.M.-G.M. inequality to the L.H.S., we get

$$1 + b^2 \geq 2b \geq 2b \sin^2 \theta \tag{6}$$

Equality of the first and the last term implies equality everywhere. That is, $1 + b^2 = 2b$ and $\sin^2 \theta = 1$. The first equality implies $b = 1$ but is not of interest to us. The second one gives $\sin \theta = \pm 1$. Since θ is given to lie in $(-\pi, \pi)$, this means $\theta = \dfrac{\pi}{2}$.

The problem is an artificial combination of three very different topics, the limits of functions, inequalities and trigonometric equations. Such combinations are useful when the paper-setters want to cover all parts of the syllabus with a few questions. In the present set up, there is hardly any need to do so. Already there is a full separate question (Q. 26 above) for trigonometric equations. Also Q. 17 and 18 cater, respectively, to the A.M.-G.M. inequality and to the trigonometric equations. It is difficult to see what is gained by adding these appendages to a problem in calculus. If a candidate does the calculus part (which is obviously the heart of the question) correctly but fumbles later on, he is unfairly penalised.

Q.31 Let $f : [-1, 2] \longrightarrow [0, \infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1, 2]$. Let $R_1 = \displaystyle\int_{-1}^{2} x f(x)\, dx$ and R_2 be the area of the region bounded by $y = f(x), x = -1, x = 2$ and the x-axis. Then

(A) $R_1 = 2R_2$ (B) $R_1 = 3R_2$ (C) $2R_1 = R_2$ (D) $3R_1 = R_2$

Answer and Comments: (C). R_2 is given as an area. In terms of integrals, it is simply

$$R_2 = \int_{-1}^{2} f(x)\, dx \tag{1}$$

R_1 is given in a straight manner, viz.

$$R_1 = \int_{-1}^{2} x f(x)\, dx \tag{2}$$

Since the function $f(x)$ is not given, neither integral can be evaluated explicitly. Nor is it necessary, because the question merely asks the relationship between the two integrals and this can be derived from the functional equation

$$f(x) = f(1-x) \tag{3}$$

which the function is given to satisfy. Using the formula

$$\int_a^b g(x)\,dx = \int_a^b g(a+b-x)\,dx \tag{4}$$

and the functional equation above we get

$$\begin{aligned} R_1 &= \int_{-1}^{2} (1-x)f(1-x)\,dx \\ &= \int_{-1}^{2} (1-x)f(x)\,dx \\ &= \int_{-1}^{2} f(x)\,dx - \int_{-1}^{2} xf(x)\,dx \\ &= R_2 - R_1 \end{aligned} \tag{5}$$

from which we immediately get $2R_2 = R_1$.

The trick used in this problem is the same as that in the answer to Q.6 in Paper 1, and, as commented there, many questions in the past JEE papers are based on it. It is difficult to see what is gained by the duplication.

SECTION II

Multiple Correct Answer(s) Type

This section contains **four** multiple choice questions. Each question has four possible answers out of which **ONE OR MORE** may be correct.

Q.32 Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If $P(T)$ denote the probability of the occurrence of the event T, then

(A) $P(E) = \frac{4}{5}, P(F) = \frac{3}{5}$ (B) $P(E) = \frac{1}{5}, P(F) = \frac{2}{5}$

(C) $P(E) = \frac{2}{5}, P(F) = \frac{1}{5}$ (D) $P(E) = \frac{3}{5}, P(F) = \frac{4}{5}$

Answer and Comments: (A) and (D). Denote $P(E)$ and $P(F)$ by x and y respectively. We need two equations in x and y to determine them. Let E', F' denote the complementary events of E and F respectively. Then $P(E') = 1 - x$ and $P(F') = 1 - y$. To say that exactly one of E and F occurs means that either E and F' occur, or else E' and F occur.

Moreover these two possibilities are disjoint. So, the first piece of data gives

$$P(E \cap F') + P(E' \cap F) = \frac{11}{25} \tag{1}$$

Since E, F are independent, so are E, F' and $E'F$. So $P(E \cap F') = P(E)P(F') = x(1-y)$. Similarly, $P(E' \cap F) = (1-x)y$. So (1) becomes

$$x(1-y) + (1-x)y = x + y - 2xy = \frac{11}{25} \tag{2}$$

The second piece of data says that $P(E' \cap F') = \dfrac{2}{25}$ which translates as

$$(1-x)(1-y) = \frac{2}{25} \tag{3}$$

i.e.

$$x + y - xy = 1 - \frac{2}{25} = \frac{23}{25} \tag{4}$$

From (2) and (4) we get

$$xy = \frac{12}{25}, \quad x + y = \frac{35}{25} \tag{5}$$

We can eliminate either one of the variables and solve the resulting quadratic equation for the other. Or even by inspection, we see that one of x and y is $\dfrac{4}{5}$ and the other is $\dfrac{3}{5}$. Because of the symmetry of the data, it is impossible to tell which is which. Hence either (A) is true or (D) is true. Obviously both cannot be true. Normally, the alternatives given in a multiple correct question are not mutually exclusive. That is, they can hold true simultaneously. For example, a triangle may be isosceles as well as right angled. In this respect, the present problem is misleading.

Q.33 Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point $(9, 6)$, then L is given by

(A)	$y - x + 3 = 0$	(B)	$y + 3x - 33 = 0$
(C)	$y + x - 15 = 0$	(D)	$y - 2x + 12 = 0$

Answer and Comments: (A), (B), (D). A typical point on the parabola $y^2 = 4x$ is of the form $(t^2, 2t)$ for a unique $t \in I\!R$. The slope of the normal at this point is $-t$. Hence the equation of the normal at this point is

$$y - 2t + t(x - t^2) = 0 \tag{1}$$

This normal will pass through the point $(9, 6)$ if and only if
$6 - 2t + t(9 - t^2) = 0$ or equivalently,

$$t^3 - 7t - 6 = 0 \tag{2}$$

This is a cubic with an obvious root $t = -1$. Writing $t^3 - 7t + 6$ as $(t+1)(t^2 - t - 6)$ and further as $(t+1)(t-3)(t+2)$ the roots of (2) are $-1, -2$ and 3. The corresponding normals have slopes $1, 2$ and -3 and their equations come out to be those in (A), (D) and (B) respectively as can be verified from (1).

The problem is based on the concept of conormal points, i.e. points the normals at which are concurrent.

Q.34 Let $f : (0, 1) \longrightarrow I\!R$ be defined by $f(x) = \dfrac{b - x}{1 - bx}$, where b is a constant such that $0 < b < 1$. Then

(A) f is not invertible on $(0, 1)$

(B) $f \neq f^{-1}$ on $(0, 1)$ and $f'(b) = \dfrac{1}{f'(0)}$

(C) $f = f^{-1}$ on $(0, 1)$ and $f'(b) = \dfrac{1}{f'(0)}$

(D) f^{-1} is differentiable on $(0, 1)$

Answer and Comments: (A). In order for f^{-1} to exist f must be one-to-one and onto. Differentiating $f(x)$, we have

$$f'(x) = \frac{b^2 - 1}{(1 - bx)^2} \tag{1}$$

for all $x \in (0, 1)$. Since $0 < b < 1$, $f'(x) < 0$ for all $x \in (0, 1)$. So f is strictly decreasing on $(0, 1)$. This proves it is injective. Further its range is obtained by considering the limits of $f(x)$ as x approaches the end-points of the interval $(0, 1)$. Specifically the range of f is the interval (L_1, L_2) where

$$L_1 = \lim_{x \to 1^-} \frac{b - x}{1 - bx} \tag{2}$$

$$\text{and } L_2 = \lim_{x \to 0^+} \frac{b - x}{1 - bx} \tag{3}$$

By direct calculation, $L_1 = -1$ and $L_2 = b$. So the range of f is only $(-1, b)$. Therefore f is not onto and hence f^{-1} does not exist.

As mentioned in Comment No. 3 of Chapter 6, there are other ways to find the range of a function which is the ratio of two polynomials as in the present problem. The range, say R, is the set of those points y for which the equation

$$\frac{b - x}{1 - bx} = u \tag{4}$$

has at least one solution in the domain interval $(0, 1)$. This can be solved explicitly for x as

$$x = \frac{b-u}{1-bu} \tag{5}$$

This will lie in the interval $(0, 1)$ if and only if $b - u$ and $1 - bu$ have the same sign and $|b - u| < |1 - bu|$. This runs into two cases depending on the sign of the numerator and a discussion of all cases is time consuming, So this approach is not workable here.

The main theme of the problem is the concept of an inverse function. By definition, a function $f : X \longrightarrow Y$ is called invertible if there exists a function $g : B \longrightarrow A$ such that (i) $g(f(x)) = x$ for all $x \in X$ and (ii) $f(g(y)) = y$ for all $y \in Y$. When such a function g exists it is unique and generally denoted by f^{-1}. It is easy to show that a function is invertible if and only if it is both one-to-one and onto. The first condition depends only on the domain set X. But the second condition involves both the domain set X and the codomain set Y.

In the present problem, f is not onto because the codomain of f is given to be $I\!R$ while its range comes out to be only $(-1, b)$ which is a proper subset of $I\!R$. Therefore f is not invertible, i.e. f^{-1} does not exist and that rules out the options (B), (C) and (D) since they all refer to f^{-1}, something which does not exist.

However, many times if a function $f : X \longrightarrow Y$ has range R, then it is considered as a function from X to R. If f is not onto, then $R \neq Y$. So, technically it is a different function. But it acts on elements of X exactly the same way as the original function f. If f is given by some formula then the same formula also gives this new function from X to R. So, this new function is often denoted by f itself. If further f is one-to-one but not onto, then regarded as a function from X to R it is both one-to-one and onto and hence invertible. So, we talk about the inverse function f^{-1} with the understanding that it is defined only on R and not on the entire set Y. Classic examples of this situation are the inverse trigonometric functions $\sin^{-1}$ and $\cos^{-1}$. The sine function from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to $I\!R$ is one-to-one but its range is only $[-1, 1]$. Still we define $\sin^{-1}$ with the understanding that it is defined only on $[-1, 1]$. Similarly, even though the exponential function is not invertible as a function from $I\!R$ to $I\!R$, we still say that the logarithm function is the inverse of the exponential function, the understanding being that it is defined only on the set of positive real numbers (which is the range of the exponential function).

In the present problem, the function f is one-to-one but not onto. However, if we treat it invertible in the sense just described, then the formula for the inverse function is (5) which is the same as that for f. So, if we equate a function with its formula, then $f^{-1} = f$. Moreover, if we extend the formula for $f(x)$ to the end point 0 of its domain, we see that $f(0) = b$ and so the derivative of the inverse function at b is the reciprocal of the derivative of f at 0. So, with these allowances, the statement (C) is also true. So, unless the very idea of the question is to test the ability to distinguish between a function and its formula, (C) may also be given as a correct answer. At the JEE level a distinction of this kind is rather fussy.

Q.35 Let $f(x) = \begin{cases} -x - \dfrac{\pi}{2}, & x \leq -\dfrac{\pi}{2} \\ -\cos x, & -\dfrac{\pi}{2} < x \leq 0 \\ x - 1, & 0 < x \leq 1 \\ \ln x, & x > 1 \end{cases}$. Then,

(A) $f(x)$ is continuous at $x = -\dfrac{\pi}{2}$
(B) $f(x)$ is not differentiable at $x = 0$
(C) $f(x)$ is differentiable at $x = 1$
(D) $f(x)$ is differentiable at $x = -\dfrac{3}{2}$

Answer and Comments: (A), (B), (C), (D). The four parts are unrelated to each other since the function is defined by different formulas in their neighbourhoods. Each part has to be tackled separately.

At $-\dfrac{\pi}{2}$, both the left and the right-handed limits are 0. Also $f(0) = \dfrac{\pi}{2} - \dfrac{\pi}{2} = 0$. So f is continuous at $-\dfrac{\pi}{2}$.

The left hand derivative of f at 0 is the same as that of $-\cos x$ which equals $\sin 0$, i.e. 0. The right hand derivative of f is the same as that of $x - 1$ which is 1. So, f is not differentiable at 0.

By a similar reasoning, the left hand derivative of f at 1 is 1 while its right hand derivative is $\left(\dfrac{d}{dx}(\ln x)\right)_{x=1}$ which is also 1. So, f is differentiable at 1,

Finally, $-\dfrac{3}{2} \in (-\dfrac{\pi}{2}, 0)$ on which $-\cos x$ is differentiable. So f is differentiable at $-\dfrac{3}{2}$.

Thus all the four statements are true.

A simple problem. Writing down the justifications in precise forms takes time. But it takes very little time to think of them. So a question like this is ideal for a multiple choice test.

SECTION III

Integer Type

This section contains **six** questions. The answer to each question is a single digit integer, ranging from 0 to 9.

Q.36 The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is ....

Answer and Comments: 2. Let $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$. It is not easy to identify the zeros of f. Nor is it necessary, because the question merely asks for the number of distinct zeros and not their values. Clearly, f is continuous on $\mathbb{R}$ and $f(0) = -1 < 0$. But $f(x) \to \infty$ as $x \to \infty$ and also as $x \to -\infty$. So by the Intermediate Value Property, f has at least one zero in each of $(-\infty, 0)$ and $(0, \infty)$. Hence it has at least two distinct zeros. To see if it could have more, we use Rolle's theorem which asserts that between any two distinct zeros of f, there is at least one zero of f'. So, if f had 3 (or more) zeros, then f' would have at least two distinct zeros and so by Rolle's theorem again, f'' would have at least one root. But in the present case

$$f'(x) = 4x^3 - 12x^2 + 24x + 1 \quad (1)$$
$$f''(x) = 12(x^2 - 2x + 2) = 12[(x-1)^2 + 1] \quad (2)$$

which shows that $f''(x)$ is always positive. So, $f''(x)$ has no root and therefore f can have at most two roots.

A very standard application of Rolle's theorem.

Q.37 Let $y'(x) + y(x)g'(x) = g(x)g'(x), y(0) = 0, x \in \mathbb{R}$, where $f'(x)$ denotes $\dfrac{d\,f(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on $\mathbb{R}$ with $g(0) = g(2) = 0$. Then the value of $y(2)$ is ....

Answer and Comments: 0. Here we are given a linear differential equation which, in a more familiar form, looks

$$\frac{dy}{dx} + g'(x)y = g(x)g'(x) \quad (1)$$

Multiplying by the integrating factor $e^{\int g'(x)\,dx} = e^{g(x)}$ and integrating,

$$e^{g(x)}y = \int e^{g(x)}g(x)g'(x)\,dx \quad (2)$$

To find the integral on the R.H.S., put $u = e^{g(x)}$. Then $du = e^{g(x)}g'(x)dx$ and so the integral equals $\displaystyle\int \ln u\,du$. On integrating by parts this comes out as $u \ln u - u$, i.e. $e^{g(x)}(g(x) - 1)$. So, from (2) we get

$$e^{g(x)}y = e^{g(x)}(g(x) - 1) + c \quad (3)$$

where c is an arbitrary constant to be determined from the initial condition $y(0) = 0$. As we are given $g(0) = 0$, putting $x = 0$ in (3) gives

$$1 \times 0 = 1 \times (0 - 1) + c \tag{4}$$

from which we get $c = 1$. Next we put $x = 2$. We are given $g(2) = 0$ and we already know that $c = 1$. So, we have

$$y(2) = -1 + 1 = 0 \tag{5}$$

Note that the hypothesis that $g(x)$ is not identically constant was never used in the solution. That provides a sneaky way to get the answer. Nothing else is given about $g(x)$ other than $g(0) = g(2) = 0$. So, if the answer is to depend only on these two values, it can as well be obtained by taking $g(x)$ to be identically zero, In that case the differential equation reduces to

$$\frac{dy}{dx} = 0 \tag{6}$$

which is extremely easy to solve. The solution is $y = c$, a constant. Since $y(0) = 0$, we have $c = 0$ and so $y(2) = 0$ too, In a conventional examination this solution will be marked as outrightly wrong because it violates the hypothesis. But in an examination where as long as your answer is correct, nobody questions how you got it, such a question rewards the unscrupulous.

Q.38 Let $\vec{a} = -\hat{i} - \hat{k}, \vec{b} = -\hat{i} + \hat{j}$ and $\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} \cdot \vec{a} = 0$, then the value of $\vec{r} \cdot \vec{b}$ is ..... .

Answer and Comments: 9. We can begin by taking $\vec{r}$ as $x\hat{i} + y\hat{j} + z\hat{k}$ and use the data to get a system of equations in the unknowns x, y, z. Solving this system, we shall get $\vec{r}$ and then we can compute $\vec{r} \cdot \vec{b}$. But we can do better by using the condition $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ cleverly to reduce the number of unknowns from 3 to just 1. By properties of the cross product, this equation gives

$$(\vec{r} - \vec{c}) \times \vec{b} = \vec{0} \tag{1}$$

which, in turn, means that the vector $\vec{r} - \vec{c}$ is a scalar multiple of the vector $\vec{b}$, say $\lambda\vec{b}$ for some $\lambda \in I\!R$. This gives

$$\begin{aligned} \vec{r} &= \vec{c} + \lambda\vec{b} \\ &= (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + \hat{j}) \\ &= (1 - \lambda)\hat{i} + (2 + \lambda)\hat{j} + 3\hat{k} \end{aligned} \tag{2}$$

Thus to find $\vec{r}$ we only need to determine the value of λ. This can be done using the second condition, $\vec{r}\cdot\vec{a} = 0$, which means

$$-(1-\lambda) - 3 = 0 \tag{3}$$

which gives $\lambda = 4$. Hence $\vec{r} = -3\hat{\imath} + 6\hat{\jmath} + 3\hat{k}$. Therefore $\vec{r}\cdot\vec{b} = 3 + 6 = 9$.

This is a simple but good problem as it tests the ability to utilise some pieces of data so as to reduce the number of unknowns right at the start, instead of starting with a large number of unknowns and eliminating some of them later.

Q.39 The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \leq 6$ into two parts. If $S = \{(2, \frac{3}{4}), (\frac{5}{2}, \frac{3}{4}), (\frac{1}{4}, -\frac{1}{4}), (\frac{1}{8}, \frac{1}{4})\}$, then the number of point(s) in S lying inside the smaller part is .... .

Answer and Comments: 2. Call the given line L. The crucial point to observe is that the two regions lie on the opposite sides of L. Since L does not pass through the centre O, of the region, the origin lies in the larger part of the region. So a point (x_0, y_0) lies in the smaller region if and only if (i) it lies inside the circle $x^2 + y^2 = 6$ and (ii) it lies on the opposite side of L as the origin.

The first condition is easy to tackle. It simply means $x_0^2 + y_0^2 < 6$. By an easy calculation, all points of S except $(\frac{5}{2}, \frac{3}{4})$ are inside the circle. To tackle the second condition we recast the equation of L as

$$2x - 3y - 1 = 0 \tag{1}$$

On one side of the line the expression $2x - 3y - 1$ is positive and on the other side it is negative. At the origin, it is -1 which is negative. Therefore for all points of the larger part it is negative and for all points in the smaller part it is positive. Thus a point (x, y) is on the opposite side of the origin if and only if $2x - 3y - 1 > 0$, i.e. $3y < 2x - 1$. By direct calculation, this condition holds at $(2, \frac{3}{4})$ and $(\frac{1}{4}, -\frac{1}{4})$ but fails at $(\frac{1}{8}, \frac{1}{4})$. So only two of the four given points lie in the smaller part of the circular region.

The central idea of this problem is that every curve divides the plane into two regions of which it is the common boundary. To decide which of these two regions a given point belongs to, one has to cast the equation of the curve in the form $f(x, y) = 0$ and then consider the sign of the expression $f(x_0, y_0)$.

This is a very good problem testing the knowledge of how a curve divides the plane. But there is considerable duplication of work as the calculations have to be carried out at each of the three points. It would have been better to ask this question as a multiple choice question where only one point is given and the candidate is asked to decide which of the given regions it lies in. Instead of the analytic approach given above, the answer can also be obtained by inspection by drawing a diagram which shows the line, the circle and the points. But in a problem like this, in order to ensure the correctness of the answer, such diagrams cannot be merely illustrative but have to be drawn to scale.

Q.40 Let M be a 3×3 matrix satisfying

$$M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, M \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \text{ and } M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}.$$

Then the sum of the diagonal entries of M is ..... .

Answer and Comments: 9. A 3×3 matrix has 9 entries. Each of the given equations is an equality of two 3×1 matrices and therefore equivalent to three linear equations. Thus we can convert the data into a system of 9 equations in 9 unknowns (viz. the entries of M) and solve it. But this is very time consuming. Instead, we use a simple fact which quickly identifies the columns of a matrix by postmultiplying it (i.e. multiplying on the right) by some special matrices. Specifically, let E_1, E_2 and E_3 be the three column vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ respectively. Then for any 3×3 matrix M, the matrix products ME_1, ME_2 and ME_3 are precisely the first, the second and the third columns of M. This can be verified by a direct calculation.

In the present problem, the L.H.S. of the first condition is simply ME_2. So the R.H.S. must be the second column of M. To identify the first column, we need to find ME_1. This is not given to us. However, we have

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{1}$$

and therefore

$$\begin{aligned} ME_1 = M \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= M \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \end{aligned} \tag{2}$$

Thus we have found the first column of M too. We adopt a similar trick to find the third column of M. That is, we write E_3 in terms of what is given or known to us and then premultiply by M. Thus,

$$E_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{3}$$

and so,

$$\begin{aligned} ME_3 = M\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - M\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} \end{aligned} \tag{4}$$

Now that we have identified all the three columns of M, we get

$$M = \begin{bmatrix} 0 & -1 & 1 \\ 3 & 2 & -5 \\ 2 & 3 & 7 \end{bmatrix} \tag{5}$$

Therefore the sum of the diagonal entries of M is $0 + 2 + 7 = 9$.

This is a very good problem on matrices. It is simple once the key idea strikes, viz. identifying the columns of a matrix. Those who cannot conceive this trick can still solve the problem by letting M be the matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. Out of the three given equalities, the first one directly gives the values of a_2, b_2 and c_2 (which amounts to determining the second column of M). The second given equality gives the values of the differences $a_1 - a_2, b_1 - b_2$ and $c_1 - c_2$. As the values of a_2, b_2, c_2 are already known, we now get the values of a_1, b_1 and c_1 (which amounts to finding the first column of M). The last equality will lead to the values of a_3, b_3 and c_3. Thus no matter which approach we follow the work involved is basically the same. But the first approach is more elegant because it makes good use of some elementary properties of matrices. (Specifically, what is needed here is the distributivity of matrix multiplication over matrix addition.)

Q.41 Let $\omega = e^{i\pi/3}$ and a, b, c, x, y, z be non-zero complex numbers such that

$a + b + c = x,$
$a + b\omega + c\omega^2 = y$
$a + b\omega^2 + c\omega = z.$

Then the value of $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is ..... .

Answer and Comments: Incorrect question. It is implicit in the question that the given ratio is a constant (i.e. independent of the values of

a, b, c) and the problem asks you to find this constant. In order to form the ratio in the question, all that is needed is that a, b, c are not all zero, i.e. at least one of them is non-zero. It is then easy to show that if the ratio is constant when a, b, c are all non-zero, then it would also equal this constant when at least one of a, b, c is non-zero. But this does not turn out to be the case. For example, if we take $a = 1, b = c = 0$, then we get $x = y = z = 1$ and the ratio $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ comes out to be 3. But if we let $a = b = c = 1$ then we get $x = 3, y = z = 1 + \omega + \omega^2$. But $\omega = \cos(\pi/3) + i\sin(\pi/3) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\omega^2 = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. So, $y = z = 1 + \sqrt{3}i$. In this case, the given ratio equals $\frac{17}{3}$. Thus the ratio is not a constant.

It is not hard to guess what the paper-setters intended. The symbol ω is used universally to denote a complex cube root of unity. Actually, there are two such roots, viz. $\frac{-1 \pm i\sqrt{3}}{2}$. By convention, ω is taken as to correspond to the + sign. But many times it does not matter which sign is chosen because many properties of ω (e.g. that its powers recur in a cycle of length 3 or the identity $1 + \omega + \omega^2 = 0$) are independent of which sign is chosen.

Already there are two problems (Q.15 and Q.25) involving ω. In the exponential form ω comes out to be $e^{2\pi i/3}$. This is what the paper-setters had in mind. With this choice of ω, which is, in fact, the standard choice, the given ratio indeed comes out to be a constant. But the proof is not so easy if the absolute values are calculated directly in terms of the real and imaginary parts of a, b, c. However, with complex conjugation, the absolute value of a complex number, say α, can be expressed succinctly as $\sqrt{\overline{\alpha}\alpha}$. Moreover, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are any complex numbers then there is an expression for the sum of the squares of their absolute values, viz.

$$\sum_{k=1}^{n} |\alpha_k|^2 = \sum_{k=1}^{n} \overline{\alpha_k}\alpha_k \tag{1}$$

We shall apply this simple observation (with $k = 3$) to evaluate the numerator and the denominator of the ratio, say R, appearing in the statement of the problem.

Equation (1) can be recast in terms of matrices. Let M be the $n \times 1$ column vector $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$. Then the Hermitian adjoint, also called the conjugate transpose of M, denoted by M^* is the row vector $[\overline{\alpha_1}\ \overline{\alpha_2}\ \ \ldots \overline{\alpha_n}]$. Since M^* and M are matrices of orders $1 \times n$ and $n \times 1$ respectively, we can multiply them to get a 1×1 matrix, which can be identified as a complex number. In fact, this number is nothing but the R.H.S. of (1). So, we get that for any complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, the sum $\sum_{k=1}^{n} |\alpha_k|^2$ equals M^*M where M is the column vector with entries $\alpha_1, \alpha_2, \ldots, \alpha_n$. In particular, we can now write the numerator and the denominator of the ratio R as

$$|x|^2 + |y|^2 + |z|^2 = [\overline{x}\ \overline{y}\ \overline{z}] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2}$$

$$\text{and} \quad |a|^2 + |b|^2 + |c|^2 = [\overline{a}\ \overline{b}\ \overline{c}] \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{3}$$

So far we have merely expressed the numerator and the denominator of R in terms of certain matrices. To make further progress we relate the matrices in (2) to those in (3). We are given a system of equations which express x, y, z as linear combinations of a, b, c. We can recast this system in terms of a matrix whose entries are the coefficients appearing in these equations. Specifically, let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \tag{4}$$

Then the given system of equations can be written compactly as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{5}$$

Taking Hermitian adjoints of both the sides we get

$$[\overline{x}\ \overline{y}\ \overline{z}] = [\overline{a}\ \overline{b}\ \overline{c}]A^* \tag{6}$$

Substituting (6) and (5) into (3) and using the associativity of matrix multiplication, we get

$$|x|^2 + |y|^2 + |z|^2 = ([\overline{a}\ \overline{b}\ \overline{c}]A^*)(A \begin{bmatrix} a \\ b \\ c \end{bmatrix}) = [\overline{a}\ \overline{b}\ \overline{c}](A^*A) \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{7}$$

We now work with the matrix A^*A. The matrix A is symmetric and so its adjoint A^* is obtained merely by taking the complex conjugates of its entries. So,

$$A^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \overline{\omega} & \overline{\omega^2} \\ 1 & \overline{\omega^2} & \overline{\omega} \end{bmatrix} \tag{8}$$

If we take $\omega = e^{i\pi/3}$, as given in the statement of the problem, then $\overline{\omega} = e^{5\pi i/3}, \omega^2 = e^{2\pi i/3}$ and $\overline{\omega^2} = e^{4\pi i/3}$. With these values the matrix product A^*A does not come out to be anything simple. But if we take $\omega = e^{2\pi i/3}$, which is the standard practice and also probably the intention of the paper-setters, then the situation is very pleasant. For, now ω and ω^2 are complex conjugates of each other. So, we now have

$$A^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} \tag{9}$$

A direct computation, using the properties $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$ now gives

$$A^*A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{10}$$

In other words, A^*A is simply 3 times the identity matrix I_3. Substituting this into (7) we get

$$\begin{aligned} |x|^2 + |y|^2 + |z|^2 &= [\overline{a}\ \overline{b}\ \overline{c}] \begin{bmatrix} 3a \\ 3b \\ 3c \end{bmatrix} \\ &= 3(|a|^2 + |b|^2 + |c|^2) \end{aligned} \tag{11}$$

We now see that the given ratio R is indeed a constant, viz. 3.

The misprint in the definition of ω is indeed an unfortunate one. The symbol ω as a complex number is so standard, that some readers might not even bother to read how it is defined and will simply take it to mean what it is really supposed to mean. (The present author is one such reader.) So, ironically, this question has confused the careful candidates and rewarded the sloppy ones.

Even if the question had no misprint, it is not suitable in a multiple choice test. The real meat of the solution is to *prove* that the given ratio is a constant and this is far from trivial. But we have to merely identify this constant, we can do so by finding the ratio in any *particular* case of our choice. So, an unscrupulous candidates can play a safe gamble and is sure to win.

The problem would have been excellent for a full length question in a conventional examination. Yet another casualty of the particular format of the test. (As brought out in the commentary on the 2006 JEE papers, Q.27 of its first paper also lost its steam for precisely the same reason. Surely, the paper-setters are not learning anything from the mistakes in the past!)

The property $A^*A = 3I$ satisfied by A brings it very close to what is called a unitary matrix. A complex matrix U is called **unitary** if $U^* = U^{-1}$. These matrices are characterised by preservation of the (complex) dot product. That is, an $n \times n$ complex matrix U is unitary if and only if for every complex n-dimensiponal column vectors $\vec{z}, \vec{w}$, the dot product $U\vec{z} \cdot U\vec{w}$ equals $\vec{z} \cdot \vec{w}$. (Note that the complex dot product is defined slightly differently than the real dot product. If $\vec{z} = (z_1, z_2, \ldots, z_n)^t$ and $\vec{w} = (w_1, w_2, \ldots, w_n)^t$ then $\vec{z} \cdot \vec{w}$ is defined not as $z_1w_1 + z_2w_2 + \ldots + z_nw_n$ but as $\overline{z_1}w_1 + \overline{z_2}w_2 + \ldots + \overline{z_n}w_n$.) In particular, taking $\vec{w} = \vec{z}$, we see that unitary matrices preserve lengths of vectors. In the present problem, the expressions $|a|^2 + |b|^2 + |c|^2$ and $|x|^2 + |y|^2 + |z|^2$ are nothing but the squares of the lengths of the column vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ respectively. These two column vectors are related to each other by (5) above. Although the matrix A is not unitary, the matrix $\frac{1}{\sqrt{3}}A$ is so and in essence this is the reason why the ratio in the statement of the problem is constant. (And the constant is 3 because A is $\sqrt{3}$ times a unitary matrix.) Of course, all these things are beyond the JEE level and it is unlikely that any candidate will get an unfair advantage because of this additional knowledge. But it is instructive to know the origin of a problem.

SECTION IV

Matrix Match Type

This section contains 2 questions. Each question has four statements (A, B, C and D) given in **Column I** and five statements (p, q, r, s and t) in **Column II**. Any given statement in Column I can have correct matching with one or more statement(s) given in Column II.

Q.42 Match the statements given in **Column I** with the intervals/union of intervals given in **Column II**.

Column I	Column II
(A) The set $\{\text{Re}\left(\dfrac{2iz}{1-z^2}\right) : z$ is a complex number, $|z| = 1, z \neq \pm 1\}$ is	**(p)** $(-\infty, -1) \cup (1, \infty)$
(B) The domain of the function $f(x) = \sin^{-1}\left(\dfrac{8(3)^{x-2}}{1-(3)^{2(x-1)}}\right)$ is	**(q)** $(-\infty, 0) \cup (0, \infty)$
(C) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\{f(\theta) : 0 \leq \theta < \frac{\pi}{2}\}$ is	**(r)** $[2, \infty)$
(D) If $f(x) = x^{3/2}(3x-10)$, $x \geq 0$, then $f(x)$ is increasing in	**(s)** $(-\infty, -1] \cup [1, \infty)$
	(t) $(-\infty, 0] \cup [2, \infty)$

Answer and Comments: (A, **s**), (B, **t**), (C, **r**) and (D, **r**). The sets given in **Column II** are all distinct and so every entry in **Column I** has at most one match. Let us tackle them one-by-one.

In (A), a direct expression for $\text{Re}\left(\dfrac{2iz}{1-z^2}\right)$ in terms of the real and imaginary parts of z (viz. x and y) would be very complicated. The trick in such problems is to incorporate some parts of the data so that the number of unknowns is reduced. (A similar approach was taken in the solution to Q.38 where the data was used to express the unknown vector $\vec{r}$ in terms of a single unknown λ instead of the usual three unknowns.) This can be done by observing that if $|z| = 1$, then z can be written as $e^{i\theta}$ or as $\cos\theta + i\sin\theta$ for some $\theta \in I\!R$. We take the first form because in this form the expressions for the powers of z are very compact. Now let R be the given ratio $\dfrac{2iz}{1-z^2}$. We first simplify R.

$$\begin{aligned} R &= \frac{2iz}{1-z^2} = \frac{2ie^{i\theta}}{1-e^{2i\theta}} \\ &= \frac{2i}{e^{-i\theta} - e^{i\theta}} \qquad (1) \end{aligned}$$

We now write this in terms of $\cos\theta$ and $\sin\theta$. Note that $e^{-i\theta} = \cos\theta - i\sin\theta$ and therefore the denominator in (1) is simply $-2i\sin\theta$. Hence

$$R = -\frac{2i}{2i\sin\theta} = -\text{cosec}\,\theta \qquad (2)$$

Since R comes out to be real, its real part is R itself. So the problem now boils down to the range of the cosecant function. The range is $(-\infty, 1] \cup [1, \infty)$.

This is a simple but good problem on complex numbers, testing the ability to utilise some parts of the data to make a convenient start.

In (B), $f(x)$ is of the form $\sin^{-1}(g(x))$ and so the problem is equivalent to finding the set, say, S defined by

$$\begin{aligned} S &= \{x \in I\!R : -1 \le \frac{8(3)^{x-2}}{1-3^{2(x-1)}} \le 1\} \\ &= \{x \in I\!R : -1 \le \frac{8(3)^{x-2}}{1-3^{2x-2}} \le 1\} \end{aligned} \tag{3}$$

The variable x appears only in the two exponents. As the base is 3 in both the cases, we can substitute $u = 3^x$ to make the expression more manageable. (This is a valid substitution because as x varies from $-\infty$ to ∞, 3^x varies bijcetively from 0 to ∞. So no values of x are missed out.) Since $3^2 = 9$, multiplying both the numerator and the denominator of the middle term by 9, the inequality (1) reduces to

$$-1 \le \frac{8u}{9-u^2} \le 1 \tag{4}$$

where $u > 0$. Depending upon the sign of the denominator, this runs into two cases.

(i) $0 < u < 3$. Then (4) is equivalent to

$$u^2 - 9 \le 8u \le 9 - u^2 \tag{5}$$

This is a combination of two simultaneous inequalities about the values of a quadratic expression. We tackle them separately. The first inequality holds if and only if $u^2 - 8u - 9 \le 0$, i.e. $(u-9)(u+1) \le 0$ which happens precisely for $u \in [-1, 9]$. The second inequality reduces to $u^2 + 8u - 9 \le 0$, i.e. to $(u+9)(u-1) \le 0$, which holds precisely for $u \in [-9, 1]$. The intersection of these two intervals is the interval $[-1, 1]$. But since u can take only positive values, we conclude that (5) holds only for $u \in (0, 1]$.

(ii) $u > 3$. Here the work involved is similar. (4) now reduces to

$$9 - u^2 \le 8u \le u^2 - 9 \tag{6}$$

The first inequality holds when $u^2 + 8u - 9 \ge 0$ i.e. when $(u-1)(u+9) \ge 0$ which happens precisely for $u \in (-\infty, -9] \cup [1, \infty)$ of which the relevant part is $[1, \infty)$ since $u > 0$. The second inequality reduces to $u^2 - 8u - 9 \ge 0$ which holds for $u \in (-\infty, -1] \cup [9, \infty)$ of which the relevant part is $[9, \infty)$. So (6) holds precisely for $u \in [9, \infty)$.

These two cases together show that (4) holds precisely for $u \in (0,1] \cup [9,\infty)$. Now we cast the answer in terms of the original variable x which equals $\log_3 u$. So, x varies in the set $(-\infty, 0] \cup [2,\infty)$. This is therefore, the domain of the function $f(x)$ given in the statement of the question.

The central idea of this problem is the determination of the sign of a quadratic expression. This is simple enough. But in the present problem it has to be done four times. That makes the work highly repetitious. The paper-setters have combined it with logarithms. and, as if that was not enough, further with the inverse trigonometric functions. This problem fits into the conventional type of examinations where from the work of the candidate we can tell which parts he has handled correctly and give appropriate partial credit. In the multiple choice format, these appendages to the central theme only serve to make the candidate spend more time and more prone to numerical slips. A candidate with a good strategy will simply stay away from a problem like this. And he will be rewarded because in the time saved he can easily bag several times more marks than allotted for this problem. A sad testimony to the charge that the success in the present JEE is more dependent on strategy than expertise.

In part (C), a direct expansion of the determinant gives $f(\theta) = 2(1+\tan^2\theta) = 2\sec^2\theta$. We are asked to find the image of the interval $[0, \frac{\pi}{2})$ under this function. As the function is increasing throughout this interval and tends to ∞ as $\theta \to \frac{\pi}{2}$ from the left, the image is the set $[2\sec^2 0, \infty)$, i.e. the interval $[2,\infty)$.

A ridiculously simple problem, especially so on the backdrop of (B) which requires considerable work. Perhaps the paper-setters intended part (C) to come as a welcome relief after (B). But such good intentions rarely serve their purpose because the paper-setters do not have the freedom to allot marks proportional to the difficulty of a problem. As a result, a candidate who skips (B) and solves only (C) is wiser than someone who spends precious time with (B) and then has no time to attempt (C).

Finally, in (D) we have a product of two functions. Both these functions are increasing on $[0,\infty)$ and so it is tempting to conclude that their product is also increasing. This argument would be valid if both the functions were non-negative. Unfortunately, this is not the case in the present problem for the second function, viz. $3x - 10$.

The best method is to use calculus. For this we need to find $f'(x)$. This, in turn, can be done either by applying the product rule first and then simplifying the expression for $f'(x)$ or by first simplifying the expression for $f(x)$. The latter approach is slightly better here. Thus, we have

$$f(x) \quad = \quad 3x^{5/2} - 10x^{3/2} \qquad (7)$$

which is a difference of two increasing functions. If it were a sum, we would have nothing more to do. But as it stands, we must take the derivative and find where it is positive. Differentiating (7), we get

$$f'(x) = \frac{15}{2}x^{3/2} - 15x^{1/2} = \frac{15}{2}(x^{1/2}(x-2) \qquad (8)$$

The first factor in the last term is a positive constant while the second factor is always non-negative. Hence the sign of $f'(x)$ is determined by that of the last factor, viz. $x - 2$. Obviously it is non-negative for $x \in [2, \infty)$. So, this is the interval on which f is increasing.

A straightforward problem. The amount of work involved is just right unlike in (B).

Q.43 Match the statements given in **Column I** with the values in **Column II**.

Column I	**Column II**
(A) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}, \vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	**(p)** $\frac{\pi}{6}$
(B) If $\int_a^b (f(x) - 3x)\, dx = a^2 - b^2$, then the value of $f(\frac{\pi}{6})$ is	**(q)** $\frac{2\pi}{3}$
(C) The value of $\frac{\pi^2}{\ln 3}\int_{7/6}^{5/6} \sec(\pi x)\, dx$ is	**(r)** $\frac{\pi}{3}$
(D) The maximum value of $\left|\text{Arg}\left(\frac{1}{1-z}\right)\right|$ for $\|z\| = 1$, $z \neq 1$ is given by	**(s)** π
	(t) $\frac{\pi}{2}$

Answer and Comments: (A, **(q)**), (B, **(p)**), (C, **(s)**), (D, **(t)**).

The entries in **Column II** are all distinct. But we have to allow for the possibility that some statements in **Column I** have more than one correct answers. But the paper-setters have precluded this possibility by using the article 'the' in the drafting of all statements in **Column I**.

In (A), it is important to realise that the question *does not* ask the angle between the vectors $\vec{a}$ and $\vec{b}$. If that was the intention then there was no need to specify the third vector $\vec{c}$. Since $\vec{c} = \vec{a} + \vec{b}$, it may also appear redundant to state that the three vectors form a triangle because, after all, that is the very definition of vector addition. The catch is that in this definition, the vectors represent *directed* sides of the triangle. Specifically, if the vector $\vec{a}$ is the directed segment from P to Q (say), then we require that $\vec{b}$ should start at Q and go to some point, say R. Only then we say that $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$.

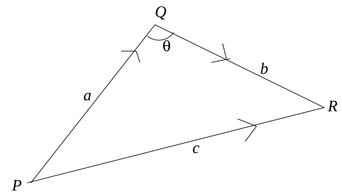
So, in the present problem, we are given a triangle, say PQR for which

$$\overrightarrow{PQ} = \vec{a} = \hat{j} + \sqrt{3}k \qquad (1)$$

$$\overrightarrow{QR} = \vec{b} = -\hat{j} + \sqrt{3}k \qquad (2)$$

and $$\overrightarrow{PR} = \vec{c} = 2\sqrt{3}k \qquad (3)$$

Now let θ be the internal angle at Q of the triangle PQR. Note that the two (directed) arms of this angle are the vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$. Out of these, the second vector is $\vec{b}$. But the first vector is $-\vec{a}$ and *not* $\vec{a}$. So the internal angle θ is not the angle between the vectors $\vec{a}$ and $\vec{b}$, but rather the angle between the vectors $-\vec{a}$ and $\vec{b}$.



Once this point is understood, finding θ is a simple calculation based on the dot product. Indeed we have

$$\cos\theta = -\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} \qquad (4)$$

which, by a direct calculation comes out as $-\dfrac{3-1}{2\times 2} = -\dfrac{1}{2}$. Therefore the desired angle is $\dfrac{2\pi}{3}$ (and not $\dfrac{\pi}{3}$ which would be the angle between the vectors $\vec{a}$ and $\vec{b}$).

This is a simple but good problem testing the conceptual understanding about the vectorial representation of the sides of a triangle.

The data in (B) is not proper. The symbols a and b are normally used for some constants whose values are fixed in a particular problem. With this interpretation the given equation viz.

$$\int_a^b (f(x) - 3x)\,dx = a^2 - b^2 \qquad (5)$$

means nothing. It would mean a lot if this equality were to hold for *all* real values of a and b. The paper-setters ought to have made this explicit. With this interpretation, we are free to give any values to a and b. Let us put $a = 0$ and b to be a variable x. Replacing the dummy variable of integration by t we now have

$$\int_0^x (f(t) - 3t)\,dt = -x^2 \qquad (6)$$

Differentiating both the sides w.r.t. x (using the second form of the fundamental theorem of calculus), we get

$$f(x) - 3x = -2x \qquad (7)$$

which means $f(x) = x$ for all $x \in I\!R$. In particular $f(\frac{\pi}{6}) = \frac{\pi}{6}$.

A simple and good problem. Q.22 was also based on the second fundamental theorem of calculus. But in that problem, a lot more work had to be done, viz. solving a certain differential equation. By contrast, in the present problem, once the conceptual part is understood, the computation needed is very little. However, as stated earlier, the fact that the given equality holds for all a, b ought to have been made clear.

Call the definite integral in (C) as I. By a direct calculation, we have

$$\begin{aligned} I &= \int_{7/6}^{5/6} \sec(\pi x)\ dx \\ &= \frac{1}{\pi} \ln(\tan(\pi x) + \sec(\pi x))\Big|_{7/6}^{5/6} \\ &= \frac{1}{\pi} \ln\left(\frac{\tan(\frac{5\pi}{6}) + \sec(\frac{5\pi}{6})}{\tan(\frac{7\pi}{6}) + \sec(\frac{7\pi}{6})}\right) \end{aligned} \tag{8}$$

We now use the identities

$$\tan(\pi \pm \theta) = \pm \tan\theta \tag{9}$$

$$\text{and} \quad \sec(\pi \pm \theta) = -\sec\theta \tag{10}$$

with $\theta = \frac{\pi}{6}$. Since $\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$ and $\sec(\frac{\pi}{6}) = \frac{2}{\sqrt{3}}$, (8) becomes

$$\begin{aligned} I &= \frac{1}{\pi} \ln\left(\frac{-\tan(\frac{\pi}{6}) - \sec(\frac{\pi}{6})}{\tan(\frac{\pi}{6}) - \sec(\frac{\pi}{6})}\right) \\ &= \frac{1}{\pi} \ln\left(\frac{-3/\sqrt{3}}{-1/\sqrt{3}}\right) \\ &= \frac{1}{\pi} \ln 3 \end{aligned} \tag{11}$$

Therefore the value of $\frac{\pi^2}{\ln 3} I$ is simply π.

A straightforward problem on integration of trigonometric functions. Instead of evaluating the integral I as above, we could have made a substitution $x = 1 + t$ and converted it to the integral, say J given by

$$J = \int_{1/6}^{-1/6} \sec(\pi + \pi t)\ dt = \int_{-1/6}^{1/6} \sec(\pi t)\ dt \tag{12}$$

which has a slight advantage because the integrand is an even function of t and the interval of integration is symmetric about the origin. But this simplification is largely cosmetic. The gain is not worth the risk of numerical mistakes that usually accompany such conversions. When the integrand of I has an obvious anti-derivative, it is best to evaluate it directly.

Before tackling the last part (D) it is important to know some facts about the argument of a complex number. Every non-zero complex number z can be written in the form

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta) \tag{13}$$

where $r = |z| > 0$. Because of the periodicity of the trigonometric functions, the number θ is not unique. Any two values of it differ by an integral multiple of 2π. Any θ satisfying (13) is called *an* argument of z and denoted by $\arg(z)$. Sometimes $\arg(z)$ is defined not as a particular number but as the entire set of all real numbers which satisfy (13). This approach has the advantage that identities like

$$\arg(z_1z_2) = \arg(z_1) + \arg(z_2) \tag{14}$$

$$\text{and} \quad \arg(-z) = \pi + \arg(z) \tag{15}$$

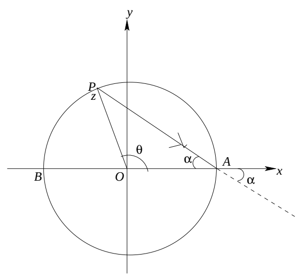
hold true as equalities of sets (see Comment No. 11 of Chapter 6). Out of the infinitely many values of the arguments of a non-zero complex number z, one is chosen and called the principal argument of z denoted by $\text{Arg}(z)$. Usually, this is taken to lie in the semi-open interval $(-\pi, \pi]$. The identities above no longer hold for principal arguments. But they do hold modulo 2π. For example if $\text{Arg}(z_1)$ and $\text{Arg}(z_2)$ are very close to π, then $\text{Arg}(z_1z_2)$ does not equal $\text{Arg}(z_1) + \text{Arg}(z_2)$ but equals $\text{Arg}(z_1) + \text{Arg}(z_2) - 2\pi$.

Geometrically, if P is the point in the Argand diagram which represents a complex number z, then $\text{Arg}(z)$ is the angle the vector $\overrightarrow{OP}$ makes with the positive x-axis (with the understanding that if P is below the x-axis then this angle is to be negative. More generally, if the points P_1, P_2 represent two (distinct) complex numbers z_1 and z_2 respectively, then $\text{Arg}(z_2 - z_1)$ is the angle the vector $\overrightarrow{P_1P_2}$ makes with the positive x-axis.

These identities and this interpretation give an easy answer to the problem (D). As a special case of (14), the arguments of z and $1/z$ are negatives of each other. Therefore,

$$\text{Arg}\left(\frac{1}{1-z}\right) = -\text{Arg}(1-z) \tag{16}$$

To view the R.H.S. geometrically, let P be the point on the unit circle which represents z. Let $A = (1, 0)$. Then $\text{Arg}(1-z)$ is the angle α shown in the figure. For P above the x-axis α is to be taken negative. For P below the x-axis, α is positive. In either case, we have



$$\text{Arg}\left(\frac{1}{1-z}\right) = -\alpha \tag{17}$$

Thus we see that $\text{Arg}\left(\frac{1}{1-z}\right)$ is positive if P is on the upper semi-circle and negative if P is on the lower semi-circle. It is 0 if P is at $B = (-1, 0)$. Further, it is obvious that as P moves closer to A along the upper semi-circle, $-\alpha$ keeps increasing from 0 to $\frac{\pi}{2}$. (As P approaches A along the lower semi-circle, $\text{Arg}\left(\frac{1}{1-z}\right)$ decreases from 0 to $-\frac{\pi}{2}$. Thus we see that it approaches different limits along the two paths. This is not paradoxical because the argument function itself has discontinuities.)

Thus it is tempting to say that the maximum asked is $\frac{\pi}{2}$. But that is not quite correct. As $z \neq 1$, the Argument never actually attains the value $\frac{\pi}{2}$, although it does attain values arbitrarily close to it (and less than it). Technically, here $\frac{\pi}{2}$ is the *supremum* and not the maximum. The distinction between a supremum and maximum is highly important from a conceptual point of view. It is unfortunate if the paper-setters missed it. Or perhaps they thought this distinction to be hair splitting at the JEE level. Unfortunately, there are some students even at the JEE level who are aware of this distinction, because of good teaching and keen thinking. Such students are likely to be baffled by the mistake. Because of their faith in JEE, they do not entertain the possibility that there could be a mistake in the question. And even if they do, they have no way of expressing themselves. They believe that there is some hidden catch in the problem and spend their precious time to find it. And once again, students who are blissfully unaware of the subtle distinction between a supremum and a maximum are the winners.

The problem can also be solved analytically, that is without resorting to the geometric intuition. Let us first write the complex number $\frac{1}{1-z}$ in the standard cartesian form $u + iv$. As $|z| = 1$, we let $z = e^{i\theta} = \cos\theta + i\sin\theta$ for some $\theta \in (-\pi, \pi]$. Then,

$$\begin{aligned}
\frac{1}{1-z} &= \frac{1}{1-\cos\theta - i\sin\theta} \\
&= \frac{1-\cos\theta + i\sin\theta}{(1-\cos\theta)^2 + \sin^2\theta} \\
&= \frac{1-\cos\theta}{2-2\cos\theta} + i\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} \\
&= \frac{1}{2} + i\frac{\cot\frac{\theta}{2}}{2}
\end{aligned} \tag{18}$$

whence Now let $\alpha = \text{Arg}\left(\frac{1}{1-z}\right)$. Then

$$\cos\alpha = \frac{1}{|\text{cosec}\,\frac{\theta}{2}|} \tag{19}$$

$$\text{and} \quad \sin\alpha = \frac{\cot\frac{\theta}{2}}{|\text{cosec}\,\frac{\theta}{2}|} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}|\text{cosec}\,\frac{\theta}{2}|} \tag{20}$$

These equations together imply that

$$\tan\alpha = \cot\left(\frac{\theta}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \tag{21}$$

From this it is tempting to conclude that

$$\alpha = \frac{\pi}{2} - \frac{\theta}{2} \tag{22}$$

This is also obvious geometrically from the diagram above. But this is valid only when $0 < \theta \leq \pi$, i.e. when z lies in the upper semicircle. When z is in the lower semicircle, $-\pi < \theta < 0$ and $|\text{cosec}\frac{\theta}{2}| = -\text{cosec}\frac{\theta}{2}$ and so (19) becomes $\cos\alpha = -\sin\frac{\theta}{2}$ while (20) becomes $\sin\alpha = -\cos\frac{\theta}{2}$. Together these two imply

$$\alpha = -\frac{\pi}{2} - \frac{\theta}{2} \tag{23}$$

Neither (22) nor (23) is valid for $\theta = 0$, because for this value of θ, $z = 1$ and so $\frac{1}{1-z}$ is undefined. All we can say from (22) and (23) is that α comes as close to $\frac{\pi}{2}$ but never attains the value $\frac{\pi}{2}$. So, once again $\frac{\pi}{2}$ is the supremum but not the maximum of $\text{Arg}\left(\frac{1}{1-z}\right)$.

This is basically a good problem, marred by a single wrong word. The paper-setters could have replaced 'maximum' by 'supremum'. But this is a risky option since many candidates may not know what a supremum means and those among them who blissfully interpret it as a maximum will get unfairly rewarded, since the test is a multiple choice one where no reasoning needs to be given. A safer option would have been to ask the candidates to find the limt of $\text{Arg}\left(\frac{1}{1-z}\right)$ as z tends to 1 along the upper semi-circle. (To make the problem more interesting, they could also be asked to find this limit as z approaches along the lower semi-circle. In a conventional examination, the candidates could further be asked to explain why these two limits are different.)

CONCLUDING REMARKS

As compared with JEE 2010, most of the problems in 2011 are fairly simple and the work expected from the candidates is reasonable (possible exceptions are Q.22 and Q.42(B) and also Q. 37 and Q.41 if done honestly). Q. 18 and Q. 23 have been asked in the past JEE's. Those familiar with their answers will get an unfair advantage over those who start from the scratch.

In order to make the problems easy, a few problems have become almost trivial. These include Q. 2, 14 and 27 in coordinate geometry and Q.5, 9 and 38 on vectors.

There is considerable duplication of ideas and work. Q.5 and Q.9, in addition to being trivial, are almost identical except for variation in numerical data. One also wonders if ω, the complex cube root of unity is such an important concept to warrant three questions (Q.15, 25 and 41) based on it (one of them containing a mistake). It also makes little sense to involve the roots of a quadratic in three problems (Q.1, 16 and 42(B)).

Q. 6 and Q.31 on evaluation of definite integrals are based on the same trick (on which many problems in the past JEE's were also based). One of them could have been replaced by the integral of a periodic function. There is already a full question (Q.17) on the A.M.-G.M. inequality. There was really no need to add it as an appendage to Q.30 which is basically a question on limits of exponential functions. Both Q.22 and Q.43(B) are based on the second form of the fundamental theorem of calculus.

There are also cases of internal duplication of work within the same question. Although there is only one question (Q.42(B)) on the sign determination of a quadratic expression, in order to get the solution, this determination has to be carried out for four different quadratics! And as if this is not enough, logarimms are thrown in unnecessarily, even though there aleady are two problems (Q.3 and Q.6) involving elementary properties of logarighms. Q.39 is a good problem testing if a candidate knows how to decide if a given point belongs to a certain given region. But this has to be done for four different points. Nothing is gained by this quadriplication.

Possibly as a result of such duplications, some areas have been totally missed. The mandatory integer part function and the the absolute value function are conspicuous by their absence! But more seriously, there are no questions on combinatorics, binomial identities, number theory and solution of triangles. Admittedly some of these areas are not suitable for a multiple choice test. Still, in recent years the paper-setters have occasionally managed to incorporate them. Probability has been paid only a lip service. The questions on it (Q.12, 13 and Q.32) are too standard. Probability, combinatorics, differential equations (and also heights and distances which is no longer in the JEE syllabus) are areas where some problems based on real life situations can be asked. Apart from bringing a welcome relief to the drab, abstract settings of the vast majority of the other problems, these problems serve a valuable purpose of testing the ability to convert a real life problem to a mathematical one.

This is certainly not to suggest that there are no good problems in the entire two papers. Considering the constraints the paper-setters work under (e.g. the mandatory multiple choice format, the denial of the freedom to allot credit proportional to the degree of dicciculty of a problem), they have come up with a large number of good problems. All the three problems on matrices (Q.8, 40 and 41) are good. So are all the six problems involving complex numbers (Q.15, 19, 25, 41, 42(A) and 43(D)). Although not entirely unprecedented in the past JEE's, the idea of a recurrence relation satisfied by the roots of a quadratic appearing in Q.1 is a novel one. Similarly, although the sums of progressions have been asked many times, the idea of the ratio of two such sums being independent of the number of terms summed, appearing in Q.21 requires some thinking, which is not common. Q.4 deserves to be especially mentioned for its trick which shifts the focus from trigonometry to surds. Although most questions on vectors are routine computations, Q.43(A) stands out as it tests the correct understanding of the way three vetors represent the three sides of a triangle.

But probably the best problem in the entire two paers is Q.39 where a candidate is to decide which of the given points lie in the smaller region of a given circular region, formed by a given chord of the circle. It tests the knowledge of how a curve divides the plane into two regions of which it is the common boundary.

Sadly many of these good problems are marred by mistakes. As already pointed out, the mistakes in Q.21 (sum of an A.P.), Q.41 (wrong definition of ω) and Q.43(D) (use of 'maximum' instead of 'supremum') make them mathematically incorrect while the mistake in Q.9 (skew-symmetric non-singular matrices) makes the question vacuous. Relatively less damaging mistake is the inconsistency of the data in Q.22 and and the unclear data in Q.43(B), where it is not specified if the given relationship holds for all a and b as is necessary to assume for getting a solution.

There are also instances (Q.10 and 34) where some of the alternatives given are controversial, that is where their truth depends on certain conventions which are adopted sometimes but not always.

Finally, as in the past JEE's there are problems where the answers can be arrived at by a sneaky method. This has been commented in the individual questions.

The ideal solution to correct most of these evils is to go back to the two stage JEE where the final selection is based on a conventional paper. But even within the present framework, some improvement is possible if some members of the paper-setting team are intentionally kept away while the question paper is drafted. After the draft is ready, these members should take a critical look at it. They are more likely to notice errors than those who set the questions since the minds of the latter are already channelised. (For example, even if they have written $\omega = e^{i\pi/3}$, they will read it as $\omega = e^{2\pi i/3}$, because that is what they had in mind.)