EDUCATIVE COMMENTARY ON
JEE 2013 ADVANCED MATHEMATICS PAPERS

Contents

The year 2013 represents a drastic departure in the history of the JEE. Till 2012, the selection of the entrants to the IITs was entirely left to the IITs and for more than half a century they did this through the JEE which acquired a world wide reputation as one of the most challenging tests for enrty to an engineering programme. Having cleared the JEE was often a passport for many lucrative positions in all walks of life (many of them having litle to do with engineering). It is no exaggeration to say that the coveted positions of the IIT's was due largely to the JEE system which was renowned not only for its academic standards, but also its meticulous punctuality and its unimpeachable integrity.

The picture began to change since 2013. The Minsistry of Human Resources decided to have a common examination for not only the IITs, but all NIT's and other engineering colleges who would want to come under its umbrella. This common test would be conducted by the CBSE. Serious concerns were raised that this would result in a loss of autonomy of the IITs and eventually of their reputation. Finally a compromise was reached that the common entrance test conducted by the CBSE would be called the JEE (Main) and a certain number of top rankers in this examination would have a chance to appear for another test, to be called JEE (Advanced), which would be conducted solely by the IITs, exactly as they conducted their JEE in the past.

So, in effect, the JEE (Advanced) from 2013 takes the role of the JEE in the past except that the candidates appearing for it are selected by a procedure over which the IITs have no control. So, this arrangement is not quite the same as the JEE in two tiers which prevailed for a few years.

Academically, the JEE (Advanced) has the same status as the JEE in the past. So, from this year, the Educative Commentary to JEE Mathematics Papers will be confined only to JEE (Advanced). The numbering of the questions will be that in Code 1 of the question papers. As in the past, unless otherwise stated, all the references made are to the author's book *Educative JEE (Mathematics)* published by Universities Press, Hyderabad.

PAPER 1

Contents

SECTION I

Single Correct Choice Type

This section contains **ten** multiple choice questions. Each question has 4 choices out of which **ONLY ONE** is correct. A correct answer gets 2 points. No points if the question is not answered. No negative marking for an incorrect answer.

Q.41 Let complex numbers α and $\dfrac{1}{\overline{\alpha}}$ lie on circles $(x-x_0)^2+(y-y_0)^2=r^2$ and $(x-x_0)^2+(y-y_0)^2=4r^2$, respectively. If $z_0=x_0+iy_0$ satisfies the equation $2|z_0|^2=r^2+2$, then $\alpha =$

(A) $\dfrac{1}{\sqrt{2}}$ (B) $\dfrac{1}{2}$ (C) $\dfrac{1}{\sqrt{7}}$ (D) $\dfrac{1}{3}$

Answer and Comments: (C). Replacing r^2 by $2|z_0|^2-2$, the data becomes

$$\begin{aligned} |\alpha - z_0|^2 &= 2|z_0|^2-2 \qquad (1)\\ \text{and} \quad |\frac{1}{\overline{\alpha}} - z_0|^2 &= 8|z_0|^2-8 \qquad (2) \end{aligned}$$

Expanding the squares and canceling $|z_0|^2$, these become

$$\begin{aligned} |\alpha|^2 - 2\mathrm{Re}(\alpha\overline{z_0}) &= |z_0|^2-2 \qquad (3)\\ \text{and} \quad \frac{1}{|\alpha|^2} - 2\mathrm{Re}(\frac{\overline{z_0}}{\overline{\alpha}}) &= 7|z_0|^2-8 \qquad (4) \end{aligned}$$

respectively. This is a system of two equations in the two complex variables α and z_0. So it is tempting to solve it to get the values of α and z_0. But this approach will not work because both the sides of the equations are real while, each real complex number is determined by two real numbers, viz. its real and imaginary parts. So in essence, this is a system of two (real) equations in four real unknowns. So, it cannot be solved in general. Nor is it necessary. The problem does not ask for α but only for its absolute value $|\alpha|$.

If it were not for the nagging middle terms, this would have been a system of two equations in the two real unknowns $|\alpha|$ and $|z_0|$ and we could hope to solve it. The middle terms cannot be eliminated just by subtraction, because the coefficients of $\overline{z_0}$ are different. To make them equal we adopt a simple trick. We multiply both the numerator and the denominator of $\dfrac{\overline{z_0}}{\overline{\alpha}}$ by α to get

$$\mathrm{Re}(\frac{\overline{z_0}}{\overline{\alpha}}) = \mathrm{Re}(\frac{\alpha\overline{z_0}}{\alpha\overline{\alpha}}) = \mathrm{Re}(\frac{\alpha\overline{z_0}}{|\alpha|^2}) \tag{5}$$

The denominator of the last term is real and therefore can be taken out to give

$$2\mathrm{Re}(\frac{\overline{z_0}}{\overline{\alpha}}) = \mathrm{Re}(\frac{\alpha\overline{z_0}}{|\alpha|^2}) = \frac{1}{|\alpha|^2}\mathrm{Re}(\alpha\overline{z_0}) \tag{6}$$

Substituting this into (4) and multiplying throught by $\alpha|^2$ we get

$$1 - 2\mathrm{Re}(\alpha\overline{z_0}) = 7|\alpha|^2|z_0|^2 - 8|\alpha|^2 \tag{7}$$

We can *now* subtract (3) to get rid of the middle term. We then have

$$1 - |\alpha|^2 = 7|\alpha|^2|z_0|^2 - 8|\alpha|^2 - |z_0|^2 + 2 \tag{8}$$

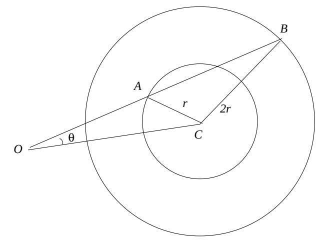
Bringing all terms on one side and factoring, this becomes

$$(7|\alpha|^2 - 1)(|z_0|^2 - 1) = 0 \tag{9}$$

Thus there are two possibilities. Either $|\alpha| = \dfrac{1}{\sqrt{7}}$ or else $|z_0| = 1$. But the latter possibility would give $r = 0$. This would mean that the two circles degenerate into a single point. So, we must discard it. (This seems to be the only reason why the variable r and the language of circles is used in the statement of the problem. Without r, the data could as well have been given in the form of equations (1) and (2). But then one would also have to consider the possibility $|z_0| = 1$. In that case we would have $\alpha = z_0$ and hence $|\alpha| = 1$.)

The language of the circles makes it tempting to try the problem geometrically. Note that the points α and $\dfrac{1}{\overline{\alpha}}$ always lie on the same ray from the origin. (This simply means that their ratio is a positive real number. This fact is also inherent in the trick we adopted above.)

As usual we let O be the origin, A, B and C the points representing the complex numbers α, $\frac{1}{\overline{\alpha}}$ and z_0 respectively. Then we have the accompanying diagram in which we let θ be the angle $\angle AOC = \angle BOC$. Applying the cosine rule to the two triangles OAC and OBC, we get



$$\begin{aligned} r^2 = 2|z_0|^2 - 2 &= |\alpha|^2 + |z_0|^2 - 2|\alpha||z_0|\cos\theta && (10) \\ \text{and} \quad 4r^2 = 8|z_0|^2 - 8 &= \frac{1}{|\alpha|^2} + |z_0|^2 - 2\frac{1}{|\alpha|}|z_0|\cos\theta && (11) \end{aligned}$$

Multiplying (11) by $|\alpha|^2$ and subtracting (10) from it gives (8) and then the solution can be obtained as before. There is a particular geometric significance to pairs of the form α and $\frac{1}{\overline{\alpha}}$. The points A and B represented by them are the reflections of each other w.r.t. the unit circle (see Exercise (8.21)). But this fact does not seem to simplify the present problem.

The data of the problem is somewhat clumsy. The solution hinges crucially on the fact that $|z_0| \neq 1$. If this were not so, all we could prove would be that *either* $|\alpha|$ *or* $|z_0|$ would have certain values. We would be even less lucky if the relationship between r and $|z_0|$ were of the form $r^2 = A|z_0|^2 + B$ for some real constants A and B other than 2 and -2 (as in the present problem). For, in that case, we could still get the analogue of Equation (8). But the vanishing expression in it may not factorise as a product of two factors. In that case the values of neither $|\alpha|$ nor $|z_0|$ can be determined uniquely. So, in this sense the present problem is accidentally solvable in a special case and the method adopted in it will not work in general. Such problems are tricky by nature. Also a good candidate who analyses the problem correctly but makes some numerical slip early in the solution gets stuck.

Q.42 Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$. Then the probability that the problem is solved correctly by at least one of them is

(A) $\frac{235}{256}$ (B) $\frac{21}{256}$ (C) $\frac{3}{256}$ (D) $\frac{253}{256}$

Answer and Comments: (A) In complete contrast with the last question, the present one is absolutely straightforward. Denote the probabilities of the four events by p_1, p_2, p_3, p_4 and p be the probability that at least one of them occurs. Then the complementary event is that none of the four events occurs and this happens with probability $q_1 q_2 q_3 q_4$ where

$q_i = 1 - p_i$ is the probability that the i-th event does not occur. So, we have,

$$p = 1 - q_1 q_2 q_3 q_4 = 1 - (1-p_1)(1-p_2)(1-p_3)(1-p_4) \qquad (1)$$

A direct substitution of the values of the p_i's gives the answer as $1 - \frac{1}{2}\frac{1}{4}\frac{3}{4}\frac{7}{8} = 1 - \frac{21}{256} = \frac{235}{256}$.

The antithetical contrast between the degree of difficulties of these two problems raises some concerns about the practice of staggering questions in different codes of the same question paper. The practice has a laudable intention of curbing copying. Also it stands on solid legal grounds because all candidates are finally asked exactly the same questions and have the freedom to approach them in any manner they like. But the human tendency is to first try the question that comes first. If the first question is as simple as the present one, it will undoubtedly boost a candidate's confidence. Even if he is unable to solve the subsequent more tough questions, he has at least bagged the points for the easy question. On the other hand, an intelligent candidate whose opeing question is a question like the last one, may waste considerable time trying it unsuccessfully and later may run out of time to attempt the easier question or may make a silly numerical slip while doing it hastily. Such inequities give credence to the criticism that success in JEE is more a consequence of strategy, which includes the right choice of the order in which to tackle the problems.

Q.43 Let $f : \left[\frac{1}{2}, 1\right] \longrightarrow I\!R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

(A) $(2e-1, 2e)$ (B) $(e-1, 2e-1)$ (C) $\left(\frac{e-1}{2}, e-1\right)$ (D) $\left(0, \frac{e-1}{2}\right)$

Answer and Comments: (D). A direct application of the fundamental theorem of calculus gives

$$f(1) - 1 = f(1) - f\left(\frac{1}{2}\right) = \int_{1/2}^{1} f'(x)dx \qquad (1)$$

for all $x \geq \frac{1}{2}$. The given ineqality would then imply that

$$f(1) - 1 < 2\int_{1/2}^{1} f(x)dx \qquad (2)$$

This gives a lower bound on the integral on the right. But it is of little help since $f(1)$ is not given. Moreover, (2) cannot give an upper bound on the integral. As we want both, this simple minded approach is not going to work. We therefore look for sharper inequalities. Since $f(x)$ is given to be positive, we can divide by it and get

$$\frac{f'(x)}{f(x)} < 2 \tag{3}$$

The trick now is to recognise the L.H.S. as the derivative of $\ln f(x)$ (which makes sense since $f(x)$ is given to be positive. Inequalities are preserved under integration. So, replacing the variable x in (3) by t and integrating both the sides of (3) over the interval $[1/2, x]$ (for $x > 1/2$), we get

$$\ln f(x) - \ln f(\frac{1}{2}) = \int_{1/2}^{x} \frac{f'(t)}{f(t)} dt < \int_{1/2}^{x} 2dt = 2(x - \frac{1}{2}) = 2x - 1 \tag{4}$$

Using $f(1/2) = 1$ and taking exponentials (which preserve inequalities) we get

$$f(x) < e^{2x-1} \tag{5}$$

for all $x > 1/2$. Integrating both the sides, we get

$$\int_{1/2}^{1} f(x)dx < \int_{1/2}^{1} e^{2x-1} dx = \frac{1}{2} e^{2x-1} \Big|_{1/2}^{1} = \frac{e-1}{2} \tag{6}$$

Since $f(x)$ is positive, so is its integral too. Hence the integral lies between 0 and $\frac{e-1}{2}$.

There is a slight reformulation of (3) which avoids logarithms. Simply multiply the two sides of the given inequality by e^{-2x} (which is always positive) and subtract to get

$$e^{-2x} f'(x) - 2e^{-2x} f(x) < 0 \tag{7}$$

The trick now is to recognise that the L.H.S. is simply the derivative of $e^{-2x} f(x)$. Replacing x by t and integrating over $[1/2, x]$ gives

$$e^{-2x} f(x) - e^{-1} f(1/2) < 0 \tag{8}$$

for all $x > 1/2$. Simplifying, we get (5). But (4) is certainly easier to coceive than (7). Moreover, once the idea to recast the given inequality strikes, the rest of the work is simple and not prone to errors. So this problem is a good combination of concepts and numerical work.

Q.44 The number of points in $(-\infty, \infty)$, for whixh $x^2 - x \sin x - \cos x = 0$ is

(A) 6 (B) 4 (C) 2 (D) 0

Answer and Comments: (D). The most direct approach would be to solve the equation and count the number of solutions. But that is ruled out here because the given equation is a combination of a quadratic and trigonometric equations. We could solve it as a quadratic in x. But the solution would be $x = \dfrac{\sin x \pm \sqrt{\sin^2 x + 4\cos x}}{2}$. The trigonometric expression does not admit any simplification. Even if it did, we would have to solve an equation in which x is equated with a trigonometric function of x and such equations are not easy to solve.

As a simplifying obervation we note that if we call the given function as $f(x)$, then it is an even function of x. So there are as many negative zeros as there are positive ones. Moreover $f(0) = -1 \neq 0$ and so we need to count only the zeros in $(0, \infty)$ and multiply it by 2. But that does not tackle the main difficulty.

Fortunately, the problem asks only the number of solutions and not what they are. And there are theorems which guarantee the existence (and sometimes the non-existence too) of zeros of functions which satisfy certain smoothness conditions such as continuity and differentiability. Two foremost such theorems the Intermediate Value Property (IVP) of continuous functions and Rolle's theorem (applicable to a function which is the derivative of some other function which takes equal values at the two end points of an interval).

Let us first try the IVP for the function $f(x)$. For large x, the dominating term is x^2 and so qualitatively $f(x)$ behaves like x^2. (To see this more precisely, note that $f(x) = x^2(1 - \frac{\sin x}{x} - \frac{\cos x}{x^2})$ and since $\sin x$ and $\cos x$ are both bounded, the second factor tends to 1 as $x \to \infty$. In particular we see that $f(x)$ is positive for all sufficiently large x, say for all $x \geq R$ for some $R > 0$. Since $f(0) = -1 < 0$ while $f(R) > 0$, the IVP implies that $f(x)$ vanishes at least once in $(0, R)$. To see if it vanishes more often, suppose a, b with $0 < a < b$ are two zeros of $f(x)$. Then by Rolle's theorem, $f'(c) = 0$ for some $c \in (a, b)$. But a direct calculation gives

$$f'(x) = 2x - \sin x - x\cos x + \sin x = x(2 - \cos x) \tag{1}$$

for all x. The second factor is always positive and so $f'(x) > 0$ for all $x > 0$. That would contradict that $f'(c) = 0$. Thus we see that $f(x)$ has precisely one zero in $(0, \infty)$ and hence precisely two zeros in $(-\infty, \infty)$.

Instead of appealing to Rolle's theorem, many candidates are more likely to conclude from (1) that $f(x)$ is strictly increasing in $(0, \infty)$ and hence it can have at most one zero. (This approach is hardly different since it is based on Lagrange's Mean Value Theorem which is just the next door neighbour of Rolle's theorem.) Also, those who miss that $f(x)$ is an even function can apply the derivative based argument for negative x and also the fact that for large negative x the dominating term in $f(x)$ is x^2 and directly conclude that $f(x)$ has precisely one zero in $(-\infty, 0)$.

A simple, familiar type of problem. Although we have given the justifications rather elaborately, in a multiple choice test, it is enough if the candidate sees something without necessarily substantiating it with logically perfect arguments.

Q.45 The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \dfrac{\pi}{2}\right]$ is

(A) $4(\sqrt{2}-1)$ (B) $2\sqrt{2}(\sqrt{2}-1)$ (C) $2(\sqrt{2}+1)$ (D) $2\sqrt{2}(\sqrt{2}+1)$

Answer and Comments: (B). We have to find the area between two curves of the form $y = f(x)$ and $y = g(x)$. We first have to see which of these two functions is greater and where. $g(x)$ equals $\cos x - \sin x$ when $\cos x \geq \sin x$ i.e. when $0 \leq x \leq \frac{\pi}{4}$. However, for $x \in [\frac{\pi}{4}, \frac{\pi}{2}]$, $\sin x \geq \cos x$ and so $g(x) = \sin x - \cos x$. But in both the cases, we have $g(x) \leq f(x)$ since both $\sin x$ and $\cos x$ are non-negative in $[0, \pi/2]$.

With these observations, we have that the desired area, say A, is

$$\begin{aligned}
\int_0^{\pi/2} |f(x) - g(x)| dx &= \int_0^{\pi/2} f(x) - g(x) dx \\
&= \int_0^{\pi/4} 2 \sin x dx + \int_{\pi/4}^{\pi/2} 2 \cos x dx \\
&= -2 \cos x \Big|_0^{\pi/4} + 2 \sin x \Big|_{\pi/4}^{\pi/2} \\
&= 2 - \frac{2}{\sqrt{2}} + 2 - \frac{2}{\sqrt{2}} \\
&= 4 - 2\sqrt{2} \qquad (1)
\end{aligned}$$

Rewriting 4 as $2\sqrt{2}\sqrt{2}$ we see that the answer tallies with (B).

Another very straightforward problem on integration. No tricky substitutions, no labourious partial fractions. The trigonometric facts needed are also very elementary. The only conceptual part is the realisation that because of the absolute value sign, the interval $[0, \pi/2]$ needs to be suitably split.

Q.46 A curve passes through the point $(1, \pi/6)$. Let the slope of the curve at each point (x, y) be $\dfrac{y}{x} + \sec\left(\dfrac{y}{x}\right)$. Then the equation of the curve is

(A) $\sin\left(\dfrac{y}{x}\right) = \log x + \dfrac{1}{2}$ (B) $\operatorname{cosec}\left(\dfrac{y}{x}\right) = \log x + \dfrac{1}{2}$

(C) $\sin\left(\dfrac{2y}{x}\right) = \log x + \dfrac{1}{2}$ (D) $\cos\left(\dfrac{2y}{x}\right) = \log x + \dfrac{1}{2}$

Answer and Comments: (A). Since the slope of a curve at a point

(x, y) is $\frac{dy}{dx}$, the data translates into a differential equation, viz.

$$\frac{dy}{dx} = \frac{y}{x} + \sec\left(\frac{y}{x}\right) \tag{1}$$

The standard method for solving differential equations of this type is to substitute $y = vx$. We then have $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and (1) is reduced to

$$x\frac{dv}{dx} = \sec v \tag{2}$$

which can be solved by separation of variables as

$$\int \cos v \, dv = \int \frac{dx}{x} \tag{3}$$

i.e.

$$\sin v = \log x + c \tag{4}$$

for some arbitrary constant c. Putting back in terms of y this becomes

$$\sin\left(\frac{y}{x}\right) = \log x + c \tag{5}$$

We are given that $y = \pi/6$ when $x = 1$. Putting these values in (5), we get $c = \sin(\pi/6) = \frac{1}{2}$. Hence the equation of the curve is

$$\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2} \tag{6}$$

(A clever student can omit the last step, since out of the four choices (A) is the only one where $\sin\left(\frac{y}{x}\right)$ occurs. The trigonometric functions in the other three options are quite different. Whether this is just a carelessness on the part of the paper-setters or whether it was done intentionally to reward such a student by saving his precious time is a moot question.)

Another straightforward problem. In problems on geometric applications of differential equations, the data given often involves such things as the intercept made by the tangent or the angle it makes with some other curve. This requires the candidate to do some work to translate the data in terms of derivatives. In the present problem, the property of the curve given is directly in terms of the slope and so little geometric knowledge is tested.

Q.47 The value of $\cot\left(\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right)$ is

(A) $\frac{23}{25}$ (B) $\frac{25}{23}$ (C) $\frac{23}{24}$ (D) $\frac{24}{23}$

Answer and Comments: (B). There is a well-known closed form expression for the second sum, viz. $n(n+1)$. Hence the given expression, say S, becomes

$$S = \cot\left(\sum_{n=1}^{23} \cot^{-1}(n^2+n+1)\right) \tag{1}$$

Now, however, there is no standard formula for the sum. As the expression S is of the type of the cotangent of a sum of angles which are given as cotangent inverses of some numbers, we should try the problem by first obtaining an expression for the cotangent of a sum of two angles whose cotangents are given. Specifically, let $\alpha = \cot^{-1} x$ and $\beta = \cot^{-1} y$ where x, y are some real numbers. We need to express $\cot(\alpha+\beta)$ in terms of x and y. This can be done using the well-known formula for the tangent of a sum of two angles. Thus

$$\begin{aligned}\cot(\alpha+\beta) &= \frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta} \\ &= \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta} \\ &= \frac{xy-1}{x+y} \qquad (2)\end{aligned}$$

and hence,

$$\cot^{-1} x + \cot^{-1} y = \cot^{-1}\left(\frac{xy-1}{x+y}\right) \tag{3}$$

Normally, when we apply such a formula, we are given x and y. Here, however, the trick is to choose x and y in such a way that the expression $\dfrac{xy-1}{x+y}$ equals n^2+n+1. A tempting choice is to let $x = n+1$ and $y = -n$. But then the expression becomes $-n^2-n-1$ which is close to, but not quite what we want. This can be rectified by replacing y by $-y$. That gives

$$\cot^{-1} x - \cot^{-1} y = \cot^{-1}\left(\frac{-1-xy}{x-y}\right) = \cot^{-1}\left(\frac{xy+1}{y-x}\right) \tag{4}$$

which is valid for all real x and y. We are now in a position to put $y = n+1$ and $x = n$ to get

$$\cot^{-1}(n^2+n+1) = \cot^{-1}(n) - \cot^{-1}(n+1) \tag{5}$$

As a result, the series $\displaystyle\sum_{n=1}^{23} \cot^{-1}(n^2+n+1)$ becomes a telescoping series. Specifically,

$$\sum_{n=1}^{23} \cot^{-1}(n^2+n+1) = \sum_{n=1}^{23} \cot^{-1} n - \cot^{-1}(n+1)$$

$$= \cot^{-1} 1 - \cot^{-1}(24) \tag{6}$$

Hence the given expression S equals $\cot\left(\cot^{-1} 1 - \cot^{-1} 24\right)$. Using (4) again with $y = 24$ and $x = 1$, we get

$$S = \cot\left(\cot^{-1} \frac{25}{23}\right) = \frac{25}{23} \tag{7}$$

Equation (5) is the crucial step in the solution. Problems where a series, say $\sum_{k=1}^{n} a_k$ is summed by recasting its terms so that it becomes a telescopic series are generally tricky. In the present problem, however, even if a candidate is unable to think of telescopic series, he can still salvage the situation from the given possible answers if he is good at guessing. The question asks for the cotangent of the sum of 23 angles each having a certain form. Obviosly, there is nothing great about the number 23. So, more generally, we can consider the cotangent of the sum of n such angles where n is any positive integer. So we let

$$S_n = \cot\left(\sum_{k=1}^{n} \cot^{-1}(k^2 + k + 1)\right) \tag{8}$$

(Note that since we are now using n for the number of terms, we have replaced the index variable n in the original sum by k.) The problem asks for the value of S_{23} and gives $\frac{23}{25}, \frac{25}{23}, \frac{23}{24}$ and $\frac{24}{23}$ as possible answers. From these we make a guess about the possible values of S_n for any n. The simplest guesses that come to the mind are

$$S_n = \frac{n}{n+2} \tag{9}$$

$$S_n = \frac{n+2}{n} \tag{10}$$

$$S_n = \frac{n}{n+1} \tag{11}$$

$$\text{and} \quad S_n = \frac{n+1}{n} \tag{12}$$

respectively. But by a direct computation from (8), $S_1 = \cot(\cot^{-1} 3) = 3$. This is not consistent with (9), (11) and (12). So none of them is a correct guess. This does not conclusively prove that (10) is a right guess or that these are the only possible guesses. But, having made a guess which has a good chance of being correct, we can try to prove it by induction on n. If we are successful then the guess is established. And, this is as valid a proof as any.

It is a routine matter to prove (10) by induction on n. Instead of proving it in this form it is more concenient to prove an equivalent assertion

$$\sum_{k=1}^{n} \cot^{-1}(k^2+k+1) = \cot^{-1}\left(\frac{n+2}{n}\right) \tag{13}$$

Call the L.H.S. as T_n. The case $n = 1$ is true since both the sides equal $\cot^{-1} 3$. Now, for the inductive step,

$$T_{n+1} - T_n = \cot^{-1}[(n+1)^2 + (n+1) + 1] = \cot^{-1}(n^2+3n+3) \tag{14}$$

On the other hand, using (4), we also have

$$\begin{aligned}\cot^{-1}\left(\frac{n+3}{n+1}\right) - \cot^{-1}\left(\frac{n+2}{n}\right) &= \cot^{-1}\left(\frac{(n+2)(n+3)+n(n+3)}{(n+2)(n+1)-n(n+3)}\right)\\ &= \cot^{-1}(n^2+3n+3) \qquad (15)\end{aligned}$$

(14) and (15) complete the inductive step and so we have now *proved* our guess, viz.

$$\cot\left(\sum_{k=1}^{n} \cot^{-1}(k^2+k+1)\right) = \frac{n+2}{n} \tag{16}$$

for all positive integers. (The given question is a special case with $n = 23$.) Of course, we could also prove this by recasting the sum on the left as a telescopic series as we did above. But now the advantage we have is that the proof is not a tricky one but a very routine one. The price we are paying is that the earlier approach truly 'explained' why (16) holds. An inductive proof cannot do that. Yet another example that leads credence to the general criticism that although induction can *prove* a result, it does not truly *explain* it, a far more popular example being the identity

$$1+2+3+\ldots+n = \frac{n(n+1)}{2} \tag{17}$$

whose true explanation comes not by induction but by writing the terms of the L.H.S. in a reverse order and adding the corressponding terms of the two sums (e.g. 1 and n, 2 and $n-1$ etc. till n and 1 at the end.)

A problem of this type has not been asked in the JEE earlier. But such tricky problems appear in the literature catering to other competitive examinations such as the various olympiads. Those who have seen such problems before, will find the present problem relatively easy. For others, it is a fairly challenging problem. And it would have been even more so if the answers were not given because in that case the alternate solution we have given based on guessing and proving the guess by induction would not be available. To form a guess one would have to calculate S_n for some small values of n and see if some pattern emerges. This is usually not an easy task. But luckily, in the present problem it is. Direct calculations, using repeated applications of (3) yield, $S_1 = 3$, $S_2 = \dfrac{3 \times 7 - 1}{3 + 7} = 2$, $S_3 = \dfrac{2 \times 13 - 1}{2 + 13} = \dfrac{5}{3}$, $S_4 = \dfrac{\frac{5}{3} \times 21 - 1}{\frac{5}{3} + 21} = \dfrac{102}{68} = \dfrac{3}{2}$. So the guess $S_n = \dfrac{n+2}{n}$ emerges.

Q.48 For $a > b > c > 0$, the distance between $(1, 1)$ and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$. Then

(A) $a + b - c > 0$	(B) $a - b + c < 0$
(C) $a - b + c > 0$	(D) $a + b - c < 0$

Answer and Comments: (A) and (C). Since $a > 0$ and $b - c > 0$, (A) is always correct, regardless of any other information. Same holds for (C) since $a - b > 0$ and $c > 0$. This point probably eluded the paper-setters as otherwise the question would have been put in the section where more than one answers may be correct.

To see what happens when the additional hypothesis is used, by a direct calculation, the point of intersection of the two given lines comes out to be $\left(\dfrac{-c}{a+b}, \dfrac{-c}{a+b}\right)$. Its distance from the point $(1, 1)$ is $\sqrt{2\left(\dfrac{a+b+c}{a+b}\right)^2}$. This is given to be less than $2\sqrt{2}$. This gives

$$(a + b + c)^2 < 4(a + b)^2 \tag{1}$$

Taking positive square roots of both the sides,

$$a + b + c < 2(a + b) \tag{2}$$

which gives $a + b > c$, which is same as (A). Thus the additional data does not give anything new. This makes the problem confusing because normally no piece of data is redundant. Also, as said before, the problem is incorrect since more than one answer is correct.

Q.49 Perpendiculars are drawn from points on the line $\dfrac{x+2}{2} = \dfrac{y+1}{-1} = \dfrac{z}{3}$ to the plane $x + y + z = 3$. The feet of the perpendiculars lie on the line

(A) $\dfrac{x}{5} = \dfrac{y-1}{8} = \dfrac{z-2}{7}$	(B) $\dfrac{x}{2} = \dfrac{y-1}{3} = \dfrac{z-2}{-5}$
(C) $\dfrac{x}{4} = \dfrac{y-1}{3} = \dfrac{z-2}{-7}$	(D) $\dfrac{x}{2} = \dfrac{y-1}{-7} = \dfrac{z-2}{5}$

Answer and Comments: (D). Call the given line as L_1 and let L_2 be the orthogonal projection of L_1 onto the given plane. We are given the parametric equations of L_1 and the equation of P and are asked to find the parametric equations of L_2

A typical point, say P, on L_1 is of the form

$$P = (-2 + 2t, -1 - t, 3t) \tag{1}$$

for some real t. Let Q be the orthogonal projection of P (i.e. the foot of the perpendicular from P) on the given plane. Then the vector $\overrightarrow{PQ}$ is perpendicular to the plane $x + y + z = 3$ and hence is of the form

$$\overrightarrow{PQ}= \lambda\vec{i} + \lambda\vec{j} + \lambda\vec{k} \tag{2}$$

for some λ (which will depend on P and hence on t). From (1) and (2) we get the coordinates of Q as

$$Q = (-2 + 2t + \lambda, -1 - t + \lambda, 3t + \lambda) \tag{3}$$

Since Q lies on the plane $x + y + z = 3$ we have

$$4t + 3\lambda = 6 \tag{4}$$

which gives $\lambda = 2 - \frac{4}{3}t$. Putting this into (3) we get

$$Q = (\frac{2}{3}t, -\frac{7}{3}t + 1, \frac{5}{3}t + 2) \tag{5}$$

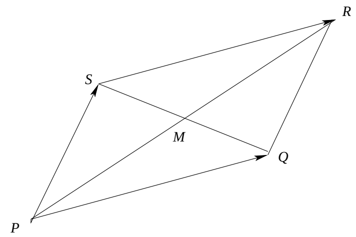
This gives a typical point on the line L_2. Hence the direction numbers of L_2 are proportional to $2, -7, 5$. This alone is sufficient to identify the correct answer as (D) because the paper-setters have not taken care to include at least one false answer with the same proportional direction numbers. Had they done so, we would also have to identify some point on L_2 and check on which of the lines in the possible answers it lies. Anyway, to complete the solution honestly, we see that by putting $t = 0$ we get $(0, 1, 2)$ as a point on L_2. And, this is also true of (D). (The paper-setters could have made the problem a little more trying by giving (D) in some other form where $(0, 1, 2)$ would lie on the line for some different value of the parameter t, than $t = 0$.)

Q.50 Let $\overrightarrow{PR}= 3\vec{i} + \vec{j} - 2\vec{k}$ and $\overrightarrow{SQ}= \vec{i} - 3\vec{j} - 4\vec{k}$ determine the diagonals of a parallelogram $PQRS$ and $\overrightarrow{PT}= \vec{i} + 2\vec{j} + 3\vec{k}$ be another vector. Then the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}$, $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ is

(A) 5 (B) 20 (C) 10 (D) 30

Answer and Comments: (D). To determine the volume, we need to identify the three vectors representing adjacent sides. Out of these, one, viz. $\overrightarrow{PT}$ is already given. The other two represent the adjacent sides of the parallelogram $PQRS$.

To find these sides, let M be the midpoint of both the diagonals. From the accompanying diagram we see that $\overrightarrow{PQ}$=$\overrightarrow{PM}$ + $\overrightarrow{MQ}$= $\frac{1}{2}(\overrightarrow{PR} + \overrightarrow{SQ}) = 2\vec{i} - \vec{j} - 3\vec{k}$. Similarly $\overrightarrow{PS}$=$\overrightarrow{PM}$ + $\overrightarrow{MS}$=$\overrightarrow{PM}$ − $\overrightarrow{SM}$= $\frac{1}{2}(\overrightarrow{PR} - \overrightarrow{SQ}) = \vec{i} - 2\vec{j} + \vec{k}$.



The desired volume, say V, is the absolute value of the scalar triple product $\overrightarrow{PT} \cdot \overrightarrow{PQ} \times \overrightarrow{PS}$, or equivalently the absolute value of the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$

A direct computation gives the value of this determinant as $-5-10-15 = -30$. Hence $V = 30$.

A straightforward problem. The only catch is to determine the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PS}$. This is easy once you draw the picture of the parallelogram $PQRS$ and consider its centre. The computation in the rest of the work is simple. So, this is a good problem for screening tests. It is not of the standard expected for an advanced test.

SECTION II

Multiple Correect Choice Type

This section contains **five** multiple choice questions. Each question has four choices out of which **ONE** or **MORE** are correct. There are 4 points if all correct and no other answers are marked, no points if no answers are marked and −1 point in all other cases.

Q.51 Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$. Then S_n can take value(s)

(A) 1056 (B) 1088 (C) 1120 (D) 1332

Answer and Comments: (A) and (D). The given sum is a sum of first $4n$ perfect squares, but some of them with a + sign and the others with a − sign. To see which sign holds where we need to find out when the expression $k(k+1)/2$ is even and when it is odd. As this number is simply the sum of the first k positive integers, it is very easy to list its values. The first few values of this expression are

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 36,\ 45\ \ldots$$

Our interest is not so much in these numbers *per se* but only in their parities, i.e. whether they are even or odd. By inspection, we see that the parity repeats in a cycle of 4. More precisely, $\dfrac{k(k+1)}{2}$ is even when k is of the form $4m$ or $4m-1$ for some positive integer m and odd when it is of the form $4m+1$ or $4m+2$. (Having guessed this, one can prove it easily by observing that $\dfrac{k(k+1)}{2}$ is even only when $k(k+1)$ is divisible by 4 and this can happen only in three ways : (i) k is divisible by 4 or (ii) $k+1$ is divisible by 4 or (iii) both k and $k+1$ are divisible by 2. But obviously the last possibility can never hold. In fact, this reasoning could have been given even before experimenting with the sequence above. But it is always more fun to observe some pattern from a few values. That is how a genius like Ramanujan worked many times, except that the patterns he saw would be far from obvious to others.)

So, now we know exactly which of the $4n$ terms of the sum S_n are positive and which are negative. Since these signs recur in cycles of 4, to evaluate S_n we break its $4n$ terms into groups of 4 each. Thus,

$$\begin{aligned} S_n = (-1-4+9+16) \quad &+ \quad (-25-36+49+64) \\ &+ \quad (-81-100+121+144)+\ldots \end{aligned} \tag{1}$$

or, more systematically,

$$S_n = T_1 + T_2 + \ldots + T_{n-1} + T_n \tag{2}$$

where we let

$$T_i = -(4i-3)^2 - (4i-2)^2 + (4i-1)^2 + (4i)^2 \tag{3}$$

for $i = 1, 2, 3, \ldots, n$. By a direct calculation, $T_1 = 20, T_2 = 52, T_3 = 84$. This suggests that the T_i's form an A.P. with commond difference 32. But before venturing such a guess, we should calculate T_4 or perhaps T_5 as well. But that is too laborious. So instead of guessing the value of T_i, we find it directly from (3) by expanding the squares. We then have,

$$T_i = 32i - 12 \tag{4}$$

which tallies with the values obtained above and also vindicates our guess that it is an A.P. with common difference 32. (Incidentally, this also shows that pattern recognition is not always the best tool. Sometimes a direct calculation is easier.)

Combining (2) and (4), we have

$$S_n = \sum_{i=1}^{n} 32i - 12 = 16n(n+1) - 12n = 16n^2 + 4n = 4n(4n+1) \qquad (5)$$

To finish the solution we now have to check which of the four given numbers are of the form $4n(4n+1)$ for some positive integer n. As all the four numbers are divisible by 4, none of them can be eliminated on this elementary ground. Dividing the numbers by 4, the problem becomes to find which of the numbers 264, 272, 280 and 333 is of the form $n(4n+1)$ for some posive integer n. We could do this by solving four quadratic equations of the form $4n^2 + n = 264$. But there is a simpler way. Since $\sqrt{4n^2+n}$ lies between the consecutive integers $2n$ and $2n+1$, we place the square roots of the numbers 264, 272, 280 and 333 between two consecutive integers, the lower one being even. The square roots are slightly bigger than 16, 16, 16 and 18 respectively. These are the possible values of $2n$. So the only possibilities are $n = 8$ and $n = 9$ and we indeed see that for these values of n, $4n^2 + n$ equals 264 and 333 respetively. Hence the possible values of S_n among the given numbers are 1056 and 1332.

This is an excellent problem. The algebra needed in it is minimal, viz. the sum of terms in an A.P. (Actually only a sum of consecutive integers.) But besides this many ideas and skills are tested, e.g. pattern recognition, grouping of terms, placement of numbers between consecutive squares. In fact this would have been a good full length question. Allotting only 4 points to it is a gross injustice to the problem and also to those intelligent students who spend considerable time on it when their mediocre competitors can easly get 6 points by doing say Q. 42, Q. 44 and 45 within the same amount of time.

Q.52 For 3×3 matrices M and N, which of the following statements is (are) NOT correct ?

(A) N^TMN is symmetric or skew symmetric, according as M is symmetric or skew symmetric

(B) $MN - NM$ is skew symmetric for all symmetric matrices M and N

(C) MN is symmetric for all symmetric matrices M and N

(D) $(adj\ M)(adj\ N) = (adj\ MN)$ for all invertible matrices M and N.

Answer and Comments: (C) and (D). We are asked to check the truth or falsity of each of the four statements. These statements are all about matrices. But that is just about the only commonality they have. The four statements are independent and the truth of one of them is not going to help in finding that of another. There is an inherent unfairness in such questions. Even if a candidate answers three of the statements correctly, his work goes down the drain because he earns a negative point. What makes things even worse in the present question is that the notations are not explained. Although N^T is one of the standard notations for the transpose of a matrix N, N' and N^t are equally common. It would have also been better to clarify exactly what the adjoint means in the last statement as otherwise it is likely to be confused with the Hermitian adjoint, i.e. the conjugate transpose.

Going back to the question, a matrix P is symmetric if $P^T = P$ and skew-symmetric if $P^T = -P$. In (A), to check the symmetry (or skew-symmetry) of N^TMN, we apply the product rule for transposes, viz. $(PQ)^T = Q^TP^T$ and also the fact that transposition is **self-reciprocative**, i.e. that $(P^T)^T = P$ for all P. This gives

$$(N^TMN)^T = N^TM^TN \tag{1}$$

If M is symmetric then $M^T = M$ and so the R.H.S. equals N^TMN which means N^TMN is symmetric. If M is skew-symmetric, then $M^T = -M$ and the R.H.S. of (1) equals $-N^TMN$ which means N^TMN is skew-symmetric. Hence Statement (A) is true. As for (B), a simple calculation (this time involving one more property, called **additivity** of transposition, viz. $(P+Q)^TP^T + Q^T)$ gives

$$\begin{aligned}(MN-NM)^T = (MN)^T - (NM)^T &= N^TM^T - M^TN^T \\ &= NM - MN \qquad (2)\end{aligned}$$

since M and N are given to be symmetric. Hence Statement (B) is also true.

Coming to Statement (C), we first have $(MN)^T = N^TM^T$ which equals NM as M and N are both symmetric. So we have $(MN)^T = NM$. But, for MN to be symmetric, we need $(MN)^T = MN$. So, unless M and N commute with each other the given statement will not hold. And in general, symmetry of two matrices does not imply that they commute with each other. So (C) is not true. (This is yet another lacuna in Multiple Choice Type questions. Strictly speaking, inability to prove a given statement following a certain approach does not mean it is false. Somebody else may be able to prove it using a cleverer argument. The acid test of falsity in mathematics for statements made about a class (in the present case, the class of all 3×3 matrices) is what is called a **counter-example**, i.e. a concrete example of some member(s) of the class for whom the claimed statement fails. The inability to prove the statement with a particular approach may *lead to* a counter-example, but it is by no means a valid substitute for a counterexample. In a test where candidates do not have to show reasoning, this distinction is masked and a scrupulous candidate who finds out a valid counter-example after some struggle, is wasting his time.)

However, for the sake of honesty, we give a counter-example. It is a little easier to conceive it for 2×2 matrices. Obviously, we must not choose M to be too nice a matrix such as the identity matrix as it commutes with all matrices. Instead, let us take $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This matrix has the property that pre-multiplication by it interchanges the rows while postmultiplication by it interchanges columns. That is, for any 2×2 matrix N, the matrix MN is obtained by inerchanging the rows of N and similarly NM is obtained by interchanging the columns of N. If we now take $N = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then $MN \neq NM$ and so there is some hope that we shall get a counterexample. But we have already got it since $MN = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ which is not symmetric even though M, N both are.

To get a counterexample of matrices of size 3, we merely put 1 on the diagonal add zeros as remaining entries. Thus $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ give a desired counterexample.

Statement (D) deals with a relatively less common concept about a square matrix, viz. its adjoint. If M is an $n \times n$ matrix, then *adj* M is obtained by taking the transpose of the matrix whose entries are the cofactors of the corresponding entries in M. Becasue of the properties of determinants, we have the relationship that

$$M(adj\ M) = (adj\ M)M = D^n I_n \tag{3}$$

where D is the determinant of the matrix M. If $D \neq 0$, then M is invertible and we get a formula for its inverse by dividing both the sides by D^n,

$$M(\frac{1}{D}\ adj\ M) = I_n \tag{4}$$

which means

$$M^{-1} = \frac{1}{D}\ (adj\ M) \tag{5}$$

where M^{-1} is the inverse of M. This gives a method (although a highly inefiicient one) for finding the inverse of a matrix with a non-zero determinant. Similarly, if the determinant, say Δ, of N is non-zero then M is invertible and we have

$$N^{-1} = \frac{1}{\Delta}\ (adj\ N) \tag{6}$$

When M, N are both invertible, so is MN and

$$(MN)^{-1} = N^{-1}M^{-1} \tag{7}$$

This is a very well-known result which is best proved by directly multiplying both the sides by MN. Since $D\Delta$ is the determinant of MN, analogously to (5) and (6) we also have

$$(MN)^{-1} = \frac{1}{D\Delta}(adj\ MN) \tag{8}$$

On the other hand, applying (5) and (6) to the factors on the right of (7), we also have

$$(MN)^{-1} = \frac{1}{\Delta D}(adj\ N)(adj\ M) \tag{9}$$

Note that D and Δ are scalars and commute with each other. The same may not be true of matrices. So from (8) and (9) we only get

$$adj\ (MN) = (adj\ N)(adj\ M) \tag{10}$$

which is not quite the satement (D). The two would be the same if M and N commute. But as this is not given to be the case, (D) is not true in general. Once again, the answer would not be complete without a valid counter-example. As in (C), to get such an example, M and N must not commute. Luckily it turns out that the counter-example above for (C) also works here.

Those who already know the result (10) will find this part of the question easy. But it is not such a well-known result as (7). So, many candidates will have to spend some time to derive (10).

Q.53 Let $f(x) = x \sin \pi x,\ x > 0$. Then for all natural numbers n, $f'(x)$ vanishes at

(A) a unique point in the interval $(n, n + 1/2)$

(B) a unique point in the interval $(n + 1/2, n + 1)$

(C) a unique point in the interval $(n, n + 1)$

(D) two points in the interval $(n, n + 1)$

Answer and Comments: (B) and (C). Unlike the last question, where the four statements required different work, in the present question, all the four parts deal with the vanishing of the same function (viz. $f'(x)$) but in different intervals. So, at least some of the work is common. Moreover, the statements themselves are inter-related as we shall see.

We begin by calculating $f'(x)$. Since $f(x) = x \sin \pi x$.

$$f'(x) = \sin \pi x + \pi x \cos \pi x \tag{1}$$

Note that since n is an integer, $f(x)$ vanishes at both the end-points of the interval $[n, n + 1]$. Hence by Rolle's theorem, $f'(x)$ vanishes at least once in $(n, n+1)$. But this is not strong enough to answer the question because Rolle's theorem does not say at how many points the derivative vanishes, nor does it give any clue about the location of these points, other than that they lie in the interior of the interval to which the theorem is applied. We need a more careful analysis of the behaviour of $f'(x)$ (and hence $f(x)$) on the interval $[n, n+1]$.

Let us first see what happens at the midpoint, viz. $n + 1/2$. By a direct calculation,

$$\begin{aligned} f'(n+1/2) &= \sin\left((2n+1)\frac{\pi}{2}\right) + (n+1/2)\pi\cos\left((2n+1)\frac{\pi}{2}\right) \\ &= (-1)^n + 0 = (-1)^n \end{aligned} \tag{2}$$

Thus $f'(x)$ does not vanish at the midpoint of the interval $(n, n+1)$. As a result, we see that if both the statements (A) and (B) are true then so is (D). On the other hand, if only one of them is true then (C) is true.

Thus the problem reduces to checking (A) and (B). Suppose $f'(x)$ vanishes at c. Then we have

$$\tan \pi c + \pi c = 0 \tag{3}$$

Geometrically, this means that the graph of the function

$$g(x) = \tan \pi x + \pi x \tag{4}$$

crosses the x axis at the point $x = c$. But since $g'(x) = \sec^2 \pi x + \pi$ this function is strictly increasing for all $x > 0$ except where $\tan \pi x$ is undefined, i.e. at odd multiples of $1/2$. These points are precisely the midpoints of the intervals we are dealing with.

More specifically, as x varies over $(n, n+1/2)$, $g(x)$ increases strictly from $n\pi$ to ∞. So, in the interval $(n, n+1/2)$, $g(x)$ and hence $f'(x)$ can never vanish. So Statement (A) is false. Similarly, on the interval $(n+1/2, n+1)$, $g(x)$ increases strictly from $-\infty$ to $(n+1)\pi$. So $g(x)$ and hence $f'(x)$ vanishes at a unique point of $(n, n+1/2)$. Hence (B) is true. As noted earlier, this also makes (C) true.

Instead of giving an analytical argument for the location of the zeros of the function $g(x)$ one can also argue from (4) that these are precisely the points where the graphs of $y = \tan \pi x$ and $y = -\pi x$ intersect each other. Both these graphs are very familiar and can be drawn almost instantaneously. One then sees effortlessly that the line $y = -\pi x$ cuts the curve $y = \tan \pi x$ only once for $x \in (n, n+1)$ and further this happens in the right half of the graph. Of course, an argument based on graphs cannot be a substitute for an anlytical argument because before you conclude anything from a graph, you need to know that the graph is correctly drawn at least in its vital features such increasing/decreasing behaviour or concavity, and in order to ensure this, the function whose graph is to be drawn needs to be subjected to the same analysis. In other words, before you reap the fruits of graphs you have to put in a lot of work which would as well give you those fruits directly! But when it comes to graphs of some very standard functions, this work has already been done (possibly by others) and there is nothing wrong in tapping its fruits.

This is a good problem based on the increasing/decreasing behaviour of trigonometric functions. But the essential idea is duplicated in Q.44.

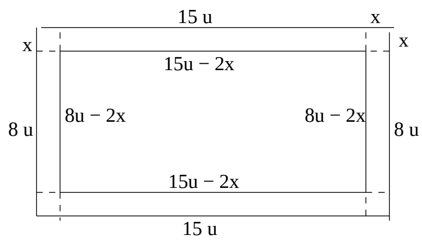
Q.54 A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are

(A) 24 (B) 32 (C) 45 (D) 60

Answer and Comments: (A) and (C). The data of the problem is given somewhat clumsily. The perimeter and the ratio of the sides determine a rectangle uniquely (upto congruence). So, the problem could have been stated more easily by saying that a rectangular sheet of paper whose sides are in the ratio 8 : 15 is to be converted .... . The mention of a fixed perimeter gives a wrong impression that the problem is of the type where something is to be maximised among all rectangles of a fixed perimeter. (A typical such problem is to find a rectangle of maximum area among all those that have a fixed perimeter.)

Although a good candidate will get the intended meaning, he may have to spend some time and in a test where time is severly scarce, the paper setters should not make unnecessary drains on a candidate's time.

Coming to the problem itself, let the sides of the rectangle be $8u$ and $15u$ where u is a constant which denotes some unit of measurement. Let x be the side of each of the squares cut off from the four corners. When the remaining portions of the four side strips are folded at right angles we get an open rectangular box with height x and sides $8u - 2x$ and $15u - 2x$.



The volume $V(x)$ of this box depends on x.

$$V = V(x) = (15u - 2x)(8u - 2x)x = 4x^3 - 46ux^2 + 120u^2x \tag{1}$$

We are given that the volume is maximum when each of the removed squares has area 25, i.e. side 5. In other words, $V(x)$ is maximum for $x = 5$. In particular this means that $\frac{dV}{dx} = 0$ for $x = 5$. By a direct calculation,

$$\frac{dV}{dx} = 12x^2 - 92ux + 120u^2 = 4(3x^2 - 23ux + 30u^2) \tag{2}$$

If this is to vanish for $x = 5$, we must have

$$30u^2 - 115u + 75 = 0 \tag{3}$$

which simplifies to

$$6u^2 - 23u + 15 = 0 \tag{4}$$

This is a quadratic in u with roots $\dfrac{23 \pm \sqrt{529 - 360}}{12}$, i.e. 3 and 5/6. Since one of the sides of the box is $8u - 10$ (for $x = 5$), we must have $u > 5/4$. So the value $u = 5/6$ is discarded. With $u = 3$, the sides of the sheet are 24 and 45.

Problems on finding maxima and minima often tend to be routine (and hence rank among the bread-and-butter problems of aspiring candidates). The present problem has a certain twist. We are not asked to maximise the volume of the resulting box (a very common problem about maxima and minima). Instead, we are *given* when the maximum occurs and are asked to find the lengths of the rectangle. In that sense this is an innovative problem. One wishes, however, that its formulation be simpler.

Q.55 A line l passing through the origin is perpendicular to the lines

$$l_1 : (3+t)\vec{i} + (-1+2t)\vec{j} + (4+2t)\vec{k}, -\infty < t < \infty$$
$$l_2 : (3+2s)\vec{i} + (3+2s)\vec{j} + (2+s)\vec{k}, -\infty < s < \infty$$

Then the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is (are)

(A) (7/3, 7/3, 5/3) (B) $((-1, -1, 0)$ (C) (1,1,1) (D) (7/9, 7/9, 8/9)

Answer and Comments: (B) and (D). We first identify the line l. Let $\mathbf{u}, \mathbf{u}_1$ and $\mathbf{u}_2$ be vectors in the directions of the lines l, l_1 and l_2 respectively. We are given that

$$\begin{aligned} \mathbf{u}_1 &= \vec{i} + 2\vec{j} + 2\vec{k} && (1) \\ \text{and} \quad \mathbf{u}_2 &= 2\vec{i} + 2\vec{j} + \vec{k} && (2) \end{aligned}$$

Since l is given to be perpendicular to both l_1 and l_2, $\mathbf{u}$ is perpendicular to both $\mathbf{u}_1$ and $\mathbf{u}_2$. So we can take $\mathbf{u}$ as $\mathbf{u}_1 \times \mathbf{u}_2$. This gives

$$\mathbf{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -2\vec{i} + 3\vec{j} - 2\vec{k} \tag{3}$$

Next, we have to find the point, say P_0, of intersection of the lines l and l_1. As l is given to pass through the origin, a typical point on l is of the form $(-2r, 3r, -2r)$. For this point to lie on l_1 we must have,

$$\frac{-2r-3}{1} = \frac{3r+1}{2} = \frac{-2r-4}{2} \tag{4}$$

which does have a (unique) solution, viz. $r = -1$. We thus see that

$$P_0 = (2, -3, 2) \tag{5}$$

Finally we have to determine which of the points on the line l_2 are at a distance $\sqrt{17}$ from P_0. Suppose P is such a point. Being on l_2, we have

$$P = (3+2s, 3+2s, 2+s) \tag{6}$$

for some real number s. If this is to be at a distance $\sqrt{17}$ from P_0, then we must have

$$(1+2s)^2 + (6+2s)^2 + s^2 = 17 \tag{7}$$

which simplifies to

$$9s^2 + 28s + 20 = 0 \tag{8}$$

and factors as $(s+2)(9s+10) = 0$. Hence the possible values of s are -2 and $-10/9$. (If this factorisation does not strike, then apply the quadratic formula. As the middle coefficient is even, the roots are $\dfrac{-14 \pm \sqrt{196-180}}{9}$, i.e. -2 and $-10/9$.) The corresponding points are $(-1, -1, 0)$ and $(7/9, 7/9, 8/9)$.

The problem is straigtforward. But there are too many steps in it. They all require extensive computations. Even one slip can be costly.

SECTION III

Integer Value Correct Type

This section contains **five** questions. The answer to each question is a single digit integer ranging from 0 to 9 (both inclusive). There are 4 points for a correct answer, no points if no answer and -1 point in all other cases.

Q.56 The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio $5 : 10 : 14$. Then $n =$

Answer and Comments: 6. We shall call $n+5$ as m. (The only reason to use $n+5$ instead of n which is simpler and direct is to make the answer lie between 0 to 9.) So we shall find m and then subtract 5 from it to get the answer. The classic controversey whether the first term of a binomial expansion should be treated as the 1-st term or the 0-th term does not, fortunately, affect the present problem because no matter which convention is adopted, the coefficients of the three consecutive terms of the expansion of $(1+x)^m$ are of the form

$$\binom{m}{r-1}, \binom{m}{r} \text{ and } \binom{m}{r+1} \tag{1}$$

for some positive integer r. We are given that the first two of these are in the raio $1:2$. This gives an equation in m and r, viz.

$$2\binom{m}{r-1} = \binom{m}{r} \tag{2}$$

If we expand out the binomial coefficients in terms of factorials, this becomes

$$\frac{2m!}{(r-1)!(m-r+1)!} = \frac{m!}{r!(m-r)!} \tag{3}$$

Canceling the common factors, this becomes

$$\frac{2}{m-r+1} = \frac{1}{r} \tag{4}$$

which simplifies to

$$m+1 = 3r \tag{5}$$

Similarly, we are given that the last two coefficients in (1) are in the ratio $5:7$. This gives

$$\frac{7m!}{r!(m-r)!} = \frac{5m!}{(r+1)!(m-r-1)!} \tag{6}$$

which reduces to

$$\frac{7}{m-r} = \frac{5}{r+1} \tag{7}$$

and hence to

$$5m = 12r + 7 \tag{8}$$

Eliminating r from (5) and (8), we have

$$5m = 4m + 4 + 7 \tag{9}$$

which gives $m = 11$ and hence $n = m - 5 = 6$.

Because of the multiple choice format, proofs of interesting binomial identities cannot be asked. The only way to cater to the binomial coefficients then is such very common and simple problems.

Q.57 A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then $k - 20 =$

Answer and Comments: 5. There is a minor grammatical error in the statement. If the numbers on two cards are consecutive then the cards are consecutively numbered and not consecutive numbered. Here the adverb 'consecutively' qualifies the adjective 'numbered', which, in turn, qualifies the noun 'cards'. Although the mistake is not mathematical and is certainly not going to confuse anybody, such mistakes do not go well with the high prestige of the IITs.

Coming to the problem itself, let the removed cards bear the numbers k and $k+1$. The sum of the numbers on all n cards is

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \tag{1}$$

The sum of the numbers on the removed cards is $2k+1$. Since the numbers on the remaining cards add up to 1224, we get an equation

$$n(n+1) = 4k + 2 + 2448 = 4k + 2450 \tag{2}$$

A single equation in two real unknowns has, in general, infinitely many solutions. (Equations of various plane curves are of this type. Indeed, if we let n and k take all possible real values, then the solutions of (2) are points on a parabola.) In the present case, however, the variables n and k are restricted to be positive integers (and moreover $k \leq n-1$). Such restrictions sometimes limit the possible solutions and then with some trial and error one can identify the correct solution. Adding 1/4 to both the sides we see that

$$(n + 1/2)^2 = 4k + 2450 + \frac{1}{4} \tag{3}$$

Multiplying both the sides by 4,

$$(2n+1)^2 = 16k + 9801 \tag{4}$$

So the R.H.S. is the square of some odd integer. Morever, since $9801 = (99)^2$, this odd integer must be greater than 99. At the same time it should not be too big as otherwise $16k$ and hence k would be large too. But we do not want k to be bigger than $n-1$. (It is not vital to the solution to recognise 9801 as a perfect square. But it simplifies the calculations.)

The first possibility to try is $2n+1 = 101$, i.e. $n = 50$. In this case, $16k = (101)^2 - (99)^2 = 200 \times 2 = 400$ and hence $k = 25$. The next case is $2n+1 = 103$. But already in this case, $16k = (103)^2 - (99)^2 = 202 \times 4 = 808$

which gives $k = 50.5$ which is fractional and, in any case bigger than $n-1$ since $n = 51$. There is no point in trying higher values of n because in that case, k (which equals $\frac{(2n+1)^2 - (99)^2}{16}$) would be even much bigger than n since the squares grow faster.

Summing up, the only possibile solution to (4) is $n = 50$ which gives $k = 25$. So $k - 20 = 5$.

This is an excellent problem which requires some non-standard thinking to arrive at $k = 25$ as a possible answer. The tragedy is that even more thinking is needed to show that there cannot be any other solution. And those who are unable to do this extra work are unfairly protected by the multiple choice format. So, this question also deserves to be a full length question where the candidate has to give complete reasoning.

Q.58 Of the three independent eents E_1, E_2 and E_3, the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of the events E_1, E_2 or E_3 occurs satisfy the equations $(\alpha - 2\beta)p = \alpha\beta$ and $(\beta - 3\gamma)p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval $(0, 1)$.

Then $\frac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3} =$

Answer and Comments : 6. Let p_1, p_2, p_3 be the probabilities of the occurrences of E_1, E_2, E_3 respectively. We are asked to find the ratio p_1/p_3. We are given two equations involving the variables α, β, γ and p. But these are auxiliary variables which can all be expressed in terms of p_1, p_2, p_3. When we do so, we shall get a system of two equations in the three unknowns p_1, p_2, p_3. In general such a system has no unique solution. However, the problem does not ask us to find the values of p_1, p_2, p_3 but only that of the ratio p_1/p_3 and this may sometimes be possible even without finding p_1, p_2, p_3 individually, as for example, when the equations in the system are homogeneous. (As an illustration, the single equation $x^2 - 5xy + 6y^2 = 0$ cannot determine the two variables x and y but it does determine x/y as either 2 or 3, these being the roots of the quadratic $\alpha^2 - 5\alpha + 6 = 0$ which results if we divide the given equation by y^2.)

So, let us begin by expressing α, β, γ and p in terms of the probabilities p_1, p_2, p_3. Since we shall often deal with non-occurrence of some of these events, it is convenient to work in terms of the complementary probabilities q_1, q_2, q_3 of the events E_1, E_2, E_3 respectively, i.e. $q_i = 1 - p_i$ for $i = 1, 2, 3$. With these notations, we have

$$\begin{aligned} \alpha &= p_1 q_2 q_3 && (1) \\ \beta &= q_1 p_2 q_3 && (2) \\ \gamma &= q_1 q_2 p_3 && (3) \\ \text{and } p &= q_1 q_2 q_3 && (4) \end{aligned}$$

We are given two equations, relating these variables. The first one, viz.,

$$(\alpha - 2\beta)p = \alpha\beta \tag{5}$$

translates as

$$(p_1q_2q_3 - 2q_1p_2q_3)q_1q_2q_3 = p_1q_2q_3q_1p_2q_3 \tag{6}$$

Canceling the common factors (which are all non-zero), we get

$$p_1q_2 - 2p_2q_1 = p_1p_2 \tag{7}$$

or, going back to the p's,

$$p_1p_2 = p_1(1 - p_2) - 2p_2(1 - p_1) \tag{8}$$

Note that p_3 is not involved and so this is an equation in the two variables p_1 and p_2 only. As said earlier a single equation like this cannot determine p_1 and p_2. But it could determine p_1/p_2 if it were homogeneous. On the face of it, an equation like (8) cannot be homogeneous because it involves not only the first degree terms p_1 and p_2, but also the second degree term p_1p_2. Perhaps things may improve if we take the help of the second of the given equations, viz.

$$(\beta - 3\gamma)p = 2\beta\gamma \tag{9}$$

which translates as

$$(q_1p_2q_3 - 3q_1q_2p_3)q_1q_2q_3 = 2q_1p_2q_3q_1q_2p_3 \tag{10}$$

Canceling the factor $q_1^2q_2q_3$, this becomes

$$p_2q_3 - 3p_3q_2 = 2p_2p_3 \tag{11}$$

which, in terms of the original variables becomes

$$2p_2p_3 = p_2(1 - p_3) - 3p_3(1 - p_2) \tag{12}$$

which is analogous to (8) and suffers from the same difficulty, viz. that it is an equation in p_2 and p_3 but on the face of it, is not homogeneous.

Summing up, neither (8) nor (12) can be solved independently. Nor is it going to be of much help to try to solve them together as a system. We can eliminate p_2 from them. But that would again give only one equation in p_1 and p_3 which will be even more complicated than either (8) or (12).

So, we are stuck. The only way out is to take a more careful look at (8). If we do so, we see that the coefficient of p_1p_2 on the R.H.S. is $-1+2$ which is simply 1, the same as the coefficient of p_1p_2 on the L.H.S. Hence, the term p_1p_2 gets canceled and the equation becomes homogeneous, in fact a very simple one, viz.

$$p_1 - 2p_2 = 0 \tag{13}$$

Similarly, by another stroke of luck, if we expand the R.H.S. of (12) it becomes a homogeneous equation in p_2 and p_3, viz.

$$p_2 - 3p_3 = 0 \tag{14}$$

From (13) and (14) we get $p_1 = 6p_3$ and hence $p_1/p_3 = 6$ which answers the question.

The problem is weird. Although posed as a problem in probability, it is essentially a problem in algebra about a system of two equations in three unknowns. But again, these two equations are tackled separately and by very similar methods. The problem could as well have given only (5) and asked the candidates to find the value of p_1/p_2. Giving (9) and expecting the candidates to find p_2/p_3 by the same method as for (5) is a purposeless duplication of work. It can also be misleading. Whenever a system of several equations is given, it is comparatively rare that each single equation independently leads to a part of the solution. Far more commonly, the entire system has to be used simultaneously. A candidate who tries to do so in the present problem by working with both the equations together will be in a soup.

But an even more serious objection is that like Q.41, the problem is accidentally solvable, i.e. it happens to be solvable for some particular values of the parameters but will not be so if these values are changed. For example, if in the data of the problem instead of the equation (5) we had

$$(\alpha - 2\beta)p = \lambda\alpha\beta \tag{15}$$

for some constant λ, then instead of (8) we would get

$$\lambda p_1p_2 = p_1(1 - p_2) - 2p_2(1 - p_1) \tag{16}$$

which is not homogeneous for any $\lambda \neq 1$. Similarly, (9) would not lead to a homogeneous equation in p_2 and p_3 if its R.H.S. were $\mu\beta\gamma$ for some other μ. Only for $\mu = 2$, it happens to be so. So, the problem is solvable in an exceptional case and not by a method which would work in general.

Accidental sussess in solving a problem is not unheard of in mathematics. In fact, in some branches it is very common. Take, for example, the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \sec\left(\frac{y}{x}\right) \tag{17}$$

which we encountered in the solution to Q. 46 above. The substitution $y = vx$ served to reduce it to the equation

$$x\frac{dv}{dx} = \sec v \tag{18}$$

which was then solved by separating the variables. If (17) were

$$\frac{dy}{dx} = \lambda\frac{y}{x} + \sec\left(\frac{y}{x}\right) \tag{19}$$

where λ is some constant other than 1, we could still apply the same substitution and reduce it to

$$x\frac{dv}{dx} = (\lambda - 1)v + \sec v \tag{20}$$

We can still cast this into the separate variables form. But now to find the solution we would have to integrate $\int \frac{dv}{(\lambda-1)v + \sec v}$ and this cannot be done in a closed form unless $\lambda = 1$. As another example, there is no formula to evaluate the length of an ellipse whose major and minor axes are in a given ratio λ. But if $\lambda = 1$, then the ellipse becomes a circle and its perimeter can be evaluated.

But in both these illustrations, the problems do have solutions for all values of the parameter λ. It is just that for $\lambda = 1$, the solution can be expressed in a simple form. So, in these illustrations it is not the solvability of a problem *per se* but only our ability to cast the solution in a certain form that is accidental. Contrast this with Equation (16) above where if $\lambda \neq 1$ then it becomes

$$p_1 - 2p_2 = (\lambda - 1)p_1p_2 \tag{21}$$

This equation has infinitely many solutions for p_1 and p_2. (If we repalce p_1 and p_2 by x and y respectively, then the solutions are the points on a hyperbola in the xy-plane.) But the ratio p_1/p_2 is not the same for all these solutions. That happens only for $\lambda = 1$. (Geometrically, in this case the hyperbola degenerates into a straight line through the origin.) So, it is not as if the solution exists as a function of λ and we are unable to evaluate it except when $\lambda = 1$. The fact is that (16) simply does not *have* a solution for p_1/p_2 except when $\lambda = 1$. So the present problem is an example of a truly accidentally solvable problem.

Examples of accidentally solvable problems arise in trigonometric optimisations as pointed out in Comment No. 9 of Chapter 14. Suppose, for example, that we want to solve the equation

$$\cos A + \cos B + \cos C = \lambda \tag{22}$$

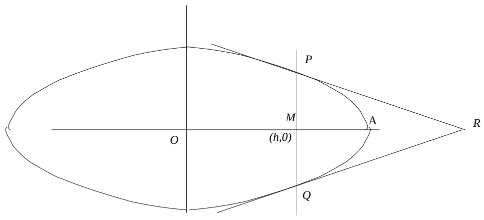
where A, B, C are the angles of a triangle. Since we have $A+B+C=\pi$, here we have a system of two equations in three unknowns and there is no unique solution in general. But if λ happens to be $3/2$, then there is a unique solution where the triangle is equilateral. This happens because of the inequality $\cos A + \cos B + \cos C \leq 3/2$ for any triangle ABC with equality holding only for an equilateral triangle.

But in general, one does not expect accidental solvability in a problem like the present one where some system of algebraic equations is given. In that sense this problem is a tricky one.

Q.59 A vertical line passing through the point $(h, 0)$ intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ is the area of the trinagle PQR , $\Delta_1 = \max\{\Delta(h) : 1/2 \leq h \leq 1\}$ and $\Delta_1 = \min\{\Delta(h) : 1/2 \leq h \leq 1\}$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$

Answer and Comments: 9. Normally, one would ask to find Δ_1 and Δ_2. Instead we are asked to find the value of a weird linear combination of both. All so as to make the answer a single digit integer.

Let us first find $\Delta(h)$ for $0 < h < 2$. Call $(h, 0)$ as M. P and Q are located symmetrically w.r.t. the x-axis and so M is the midpoint of PQ. Also the point R lies on the x-axis. To find it, it is enough to take the tangent at either P or Q.



To find P, we put $x = h$ in the equation of the ellipse to get

$$y^2 = 3\left(1 - \frac{h^2}{4}\right) \tag{1}$$

We take

$$P = \left(h, \frac{\sqrt{3}}{2}\sqrt{4-h^2})\right) \tag{2}$$

The equation of the tangent at P is

$$\frac{xh}{4} + \frac{y\sqrt{12-3h^2}}{6} = 1 \tag{3}$$

This gives

$$R = \left(\frac{4}{h}, 0\right) \tag{4}$$

which further gives

$$\Delta(h) = MR.MP = \left(\frac{4}{h} - h\right)\frac{\sqrt{12-3h^2}}{2} \tag{5}$$

We have to find the maximum and the minimum of this function over the interval $[1/2, 1]$. It is clear that as h increases, $\sqrt{12-h^2}$ decreases. The first factor, viz. $\frac{4}{h} - h$ also decreases since both $4/h$ and $-h$ decrease as h increases. This is also obvious geometrically, because as h increases, MP decreases and so does MR since both M and R move closer to A, the right extremeity of the major axis of the ellipse. So the maximum of (5) occurs at $h = 1/2$ while its minimum occurs at $h = 1$. This gives

$$\begin{aligned} \Delta_1 &= \Delta(1/2) = (8 - \frac{1}{2})\frac{\sqrt{45}}{4} = \frac{45}{8}\sqrt{5} && (6) \\ \text{and} \quad \Delta_2 &= \Delta(1) = 3\frac{\sqrt{9}}{2} = \frac{9}{2} && (7) \end{aligned}$$

Therefore,

$$\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45 - 36 = 9 \tag{8}$$

This is a good problem on maxima and minima because the function $\Delta(h)$ is decreasing on the interval $[1/2, 1]$. No derivatives are needed to find its maximum and the minimum. Students who find maxima and minima mechanically by differentiation will pay a price.

Q.60 Consider the set of eight vectors $V = \{a\vec{i} + b\vec{j} + c\vec{k} : a, b, c \in \{-1, 1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is

Answer and Comments: 5. The vectors in the set V are the position vectors of the eight vertices $(\pm1, \pm1, \pm1)$ of a cube centred at the origin. Note that whenever $\mathbf{v}$ is in V, so is $-\mathbf{v}$. We have to choose three non-coplanar vectors from V. Let us first consider the set S of all ordered triples of members of V that are non-coplanar. In symbols,

$$S = \{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in V \times V \times V : \mathbf{u}, \mathbf{v}, \mathbf{w} \text{ are non} - \text{coplanar}\} \tag{1}$$

Let us see in how many ways such an ordered triple be formed. The first vector $\mathbf{u}$ can be any of the eight members of V. Having chosen $\mathbf{u}$, we must not let $\mathbf{v} = \pm\mathbf{u}$ as otherwise $\mathbf{u}$ and $\mathbf{v}$ would lie along the same line. But we can let $\mathbf{v}$ to be any of the remaining six vectors in V. Having chosen $\mathbf{u}$ and $\mathbf{v}$, the vectors $-\mathbf{u}$ and $-\mathbf{v}$ lie in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$. But the remaining four vectors in V are outside this plane. So any one of them can be taken as $\mathbf{w}$.

Summing up, every ordered triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ can be chosen in $8 \times 6 \times 4$ ways. But our problem is not about such ordered triples but only about (unordered) sets $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ of three non-coplanar vectors. Every ordered triple in S gives rise to such a set. But since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are all distinct, $6(= 3!)$ different ordered triples give rise to the same set. Hence the number of ways to pick three non-coplanar vectors from V is $\dfrac{8 \times 6 \times 4}{6} = 32$. If this is to equal 2^p we must have $p = 5$.

This is a very good problem involving only simple geometric concepts and virually no calculations. There is also a built in hint to the problem that the number of ways to choose three non-coplanar vectors from V is a power of 2.

PAPER 2

Contents

SECTION I

One or more options Correct Type

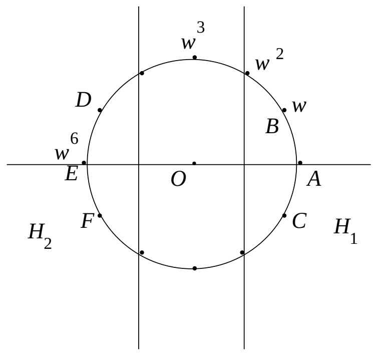
This section contains **eight** multiple choice questions. Each question has 4 choices out of which **ONE OR MORE** are correct. A correct answer gets 3 points. No points if the question is not answered. -1 point in all other cases.

Q.41 Let $w = \dfrac{\sqrt{3}+i}{2}$ and $P = \{w^n = n = 1, 2, 3, \ldots\}$. Further $H_1 = \{z \in \mathbb{C} : \text{Re } z > 1/2\}$ and $H_2 = \{z \in \mathbb{C} : \text{Re } z < -1/2\}$ where $\mathbb{C}$ is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1 O z_2 =$

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{6}$ (C) $\dfrac{2\pi}{3}$ (D) $\dfrac{5\pi}{6}$

Answer and Comments: (C) and (D). The set P consists of all powers of w with positive exponents. (This is probably the reason it is denoted by P, rather than S which is a more common notation for sets, P generally being reserved for points in a line or plane.) The sets H_1 and H_2 are vertical half planes bounded by the lines Re $z = \pm 1/2$ but not including the boundaries.

On the face of it P may be an infinite set and in that case there would be in general infinitely many choices for z_1 and z_2 and so the problem could have infinitely many answers. But if we recast w in its polar form, it becomes $w = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = e^{i\pi/6}$. By DeMoivre's rule we see that $w^6 = -1$ and $w^{12} = 1$. Hence there are only 12 distinct powers of w. They all lie on the unit circle. Three of them (corresponding to the points A, B, C) lie in H_1 while those corresponding to D, E, F lie in H_2.



As there are only three possibilities for z_1 and z_2 each, in all there are nine possible values for the angle $\angle z_1Oz_2$ and they are all multiples of $\pi/6$. But they are not all distinct. There is little point in enumerating all possibilities. From the figure we see at once that the angle z_1Oz_2 can be either $4\pi/6, 5\pi/6$ or $6\pi/6$. The first two possibilities give options (C) and (D) respectively. The third possibility arises in three cases, viz. when z_1 and z_2 are diametrically opposite. But it is not listed as an option. (We often also have to consider the orientation of an angle. If we do, then the angle $\angle z_1Oz_2$ is taken to be the angle through which the ray Oz_1 has to be rotated counterclockwise so that it falls on Oz_2. If we follow this convention then some of the angles would come out to be $7\pi/6$ (e.g. when z_1 is at A and z_2 at F) or $4\pi/3$ (when z_1 is at C and z_2 at F). But none of these values is listed as a possible option.)

This is a good problem on complex numbers, especially their geometric representations in the Argand diagram. Once the candidate correctly sees the location of the various powers of w, the rest can be done by inspection.

Q.42 If $3^x = 4^{x-1}$, then $x =$

(A) $\dfrac{2\log_3 2}{2\log_3 2 - 1}$ (B) $\dfrac{2}{2 - \log_2 3}$

(C) $\dfrac{1}{1 - \log_4 3}$ (D) $\dfrac{2\log_2 3}{2\log_2 3 - 1}$

Answer and Comments: (A), (B) and (C). This is a problem involving logarithms with different bases. In such problems it is preferable to take one suitable number as the common base for all logarithms and convert the other logarithms to those with this standard base using the general formula for conversion, viz.

$$\log_a b = \frac{\log_c b}{\log_c a} \tag{1}$$

for all positive a, b, c. A speical consequence of this is

$$\log_a b = \frac{1}{\log_b a} \tag{2}$$

It really does not matter which common base we select as lon as it is a positive real number other than 1. In fact, sometimes it is preferable to take some arbitray base (which need not even be specified and can be suppressed from notation). In that case (1) would simply become

$$\log_a b = \frac{\log b}{\log a} \tag{3}$$

This way we can mechanically convert all logs.

But just as in geometric problems where coordinates are to be applied, in theory any frame of reference would give the answer, in a particular problem, a particular frame simplifies the calculations considerably, the same is true of logarithms. Generally we take a base which already occurs in the statement of the problem. In the present problem, many of the terms involve 3 either as a base or as the number whose log is taken (in which case we apply (2) above to change the expression to a log with base 3). So, we take 3 as a base for all logarithms and suppress it from the notation.

Now, coming to the given equation, viz.

$$3^x = 4^{x-1} \tag{4}$$

if we take logs of both the sides (w.r.t. 3 as the base), we get a linear equation in x, viz.

$$x = (x-1)\log 4 \tag{5}$$

whose solution is

$$x = \frac{\log 4}{\log 4 - 1} \tag{6}$$

The problem now is to see which of the given four numbers equals x. Since $2\log 2 = \log 4$ we see directly that (A) is a correct answer. In (B) we convert $\log_2 3$ to $\dfrac{1}{\log_3 2} = \dfrac{1}{\log 2}$. Then (B) becomes $\dfrac{2\log 2}{2\log 2 - 1}$ which is the same as (A) and hence is also a correct answer. In (C) too we first write $\log_4 3$ as $\dfrac{1}{\log 4}$ and see that it also equals $\dfrac{\log 4}{\log 4 - 1}$. Hence (C) is also correct. In (D), we write $\log_2 3$ as $\dfrac{1}{\log 2}$. Then (D) becomes $\dfrac{2}{2 - \log 2}$. If this were equal to x, then we would have

$$\frac{\log 4}{\log 4 - 1} = \frac{2}{2 - \log 2} \tag{7}$$

or equivalently,

$$\log 4 \log 2 = 2 \tag{8}$$

By the multiplicative property of logarithms, $\log 4 = \log 2 + \log 2 = 2\log 2$. Hence (8) gives $(\log 2)^2 = 1$ which means $\log 2 = \pm 1$. As we are taking logarithms with base 3, this means $2 = 3$ or $2 = 1/3$. As both the possibilities are false, so is (D).

In the very old JEE (when there was no calculus and no coordinate geometry), considerable weitage used to be given to algebra. Invariably there would be one full length question on logarithms (just their algebraic aspects and not properties of the logarithm and exponential functions). Such problems looked complicated but were often simple if all logs were converted to a common base. The present problem is a reminder of those good old days. In those days, however, you would have to justify why (D) is not a correct answer. Proving equality of two numbers is mostly a matter of algebraic manipulations. Proving that they are not equal has to be done by an indirect method as we did, i.e. by reaching a contradiction if they are assumed to be equal. And, often this demands a different kind of reasoning.

Q.43 Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be an $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq O$ when $n =$

(A) 57 (B) 55 (C) 58 (D) 56

Answer and Comments: (B), (C) and (D). Note that ω can be taken either as $e^{2\pi i/3}$ or $e^{4\pi i/3}$. (These are the complex numbers w^4 and w^8 where w is the complex number in Q.41. That may not be of much relevance here. But it is a good habit to relate new things to the old ones wherever easily possible.) Each is the square, the complex conjugate and also the reciprocal of the other. As a result, we can arbitrarily call one of them as ω and the other as ω^2.

Note that since $\omega^3 = 1$, the powers of ω recur in a cycle of length 3. (This is analogous to and can in fact, be derived from the fact that the powers of the complex number w in Q.41 recur in a cycle of 12.) This means any power, say ω^m equals either 1 or ω or ω^2, depending upon what residue is left when m is divided by 3 (or, in a slightly sophisticated language introduced in Comment No. 15 of Chapter 4, depending on the congruence class of m modulo 3).

Let us now turn to the given problem. We are given that the (i, j)-th entry of P is ω^{i+j}. Note that this makes the matrix symmetric. Therefore P^2 is the same as PP^T. It is a little easier to calculate a typical (i, j)-th entry, say q_{ij}, of PP^T. It is simply the 'inner product' of the i-th and the

j-th rows of P. In other words,

$$q_{ij} = p_{i1}p_{j1} + p_{i2}p_{j2} + \ldots + p_{ik}p_{jk} + \ldots + p_{in}p_{jn} = \sum_{k=1}^{n} p_{ik}p_{jk} \qquad (1)$$

We are given that $p_{ik} = \omega^{i+k}$ and $p_{jk} = \omega^{j+k}$. Substituting these values we get

$$q_{ij} = \sum_{k=1}^{n} \omega^{i+j+2k} = \omega^{i+j} \sum_{k=1}^{n} (\omega^2)^k \qquad (2)$$

Using the formula for the sum of a geometric progression, we further get

$$q_{ij} = \frac{\omega^{i+j+2}(1-\omega^{2n})}{1-\omega^2} \qquad (3)$$

So, for q_{ij} to vanish we must have $\omega^{2n} = 1$. This happens if and only if $2n$ and hence n is a multiple of 3. For such values of n all the entries of P^2 are zero. For other values of them, none of them is zero. Therefore among the four given values of n, $P^2 = O$, the zero matrix only for $n = 57$.

The only property of ω needed was that its powers recur in a cycle of length 3. The analogous property of w in Q.41 was also crucially needed there to ensure that the set of powers of w was finite. Thus there is a duplication of ideas in Q.41 and Q.43. Those who do Q.41 first will have a slightly easier time with the present question. Because of the practice of permuting questions in different codes, this might give a slight advantage to some candidates.

Although the computations needed in calculating P^2 are not so laborious, the very idea of forming the square of an $n \times n$ matrix can be intimidating to some candidates. They are likely to evaluate P^2 for some lower values of n such as $n = 1, 2, 3$ and guess that P^2 vanishes precisely when n is a multiple of 3. While such a pattern recognition deserves to be rewarded, it cannot be a substitute for a full proof. Yet another example where the MCQ form masks the difference between those who only guess correctly and those who can also prove the guess.

Q.44 The function $f(x) = 2|x| + |x+2| - \Big||x+2| - 2|x|\Big|$ has a local minimum or a local maximum at $x =$

(A) -2 (B) $\frac{-2}{3}$ (C) 2 (D) $\frac{2}{3}$

Answer and Comments: (A) and (B). Functions defined in terms of the absolute values of expressions are continuous wherever those expressions are continuous but they fail to be differentiable at the points where the expressions change signs. So the maxima and minima of such functions cannot be found using derivatives. Instead, one has to carefully see where the function is increasing and where it is decreasing. A well-drawn graph gives the answer instantaneously. But the catch is that it often takes considerable work to draw an accurate graph in the first place! One has to first get rid of the absolute value signs and see which expressions the functions assume over various intervals.

The graph of the prototype function $|x|$ and hence more genrally, of a function of the form $|x + a|$ is known to everybody. The formula for the function changes definition at the point $-a$. When several such proptotypes are added, we have to take all points where such a change of formula occurs and then on each of the subintervals determined by two such adjacent points sketch the graph of the function (which, fortunately, is often easy since these expressions are linear). The local maxima and minima of the function lie at some of the points where this graph has a 'kink'. (Of course, not every kink represents a local maximum or a local minimum.)

Now, coming to the given function , viz.

$$f(x) = 2|x| + |x+2| - \Big||x+2| - 2|x|\Big| \tag{1}$$

from the first two terms we see that the function will change its formula at the points -2 and 0. But, there is also the third term which is also in the form of the absolute value of a function of x. So, we shall also have to see at what point(s) it changes its behaviour.

Luckily, in the present problem the last term is not totally unrelated to the first two terms. In fact, if we let

$$\begin{array}{rrcl} & u(x) & = & 2|x| \qquad (2) \\ \text{and} & v(x) & = & |x+2| \qquad (3) \end{array}$$

then the last term is simply $|v(x) - u(x)|$. It equals $v(x) - u(x)$ or $u(x) - v(x)$ depending upon which of $u(x)$ and $v(x)$ is greater. This simplifies the formula for $f(x)$ as

$$f(x) = \begin{cases} 2u(x) = 4|x| & \text{when } |x+2| \geq 2|x| \\ 2v(x) = 2|x+2| & \text{when } |x+2| < 2|x| \end{cases} \tag{4}$$

We thus have to see which possibility holds when in (4). For this we need to split the real line at the points -2 and 0 and consider the intervals $(-\infty, -2), (-2, 0)$ and $(0, \infty)$ separately.

For $x < -2$, $|x+2| = -(x+2)$ and $|x| = -x$. So here $|x+2| \geq 2|x|$ if and only if $-(x+2) \geq -2x$ which is equivalent to $x + 2 \leq 2x$ i.e. $2 \leq x$. But in the inerval $(-\infty, -2)$ this can never happen.

For $-2 \leq x \leq 0$, $|x+2| = x+2$ and $|x| = -x$. So the inequality $|x+2| \geq 2|x|$ reduces to $x + 2 \geq -2x$ which holds if and only if $x \geq -2/3$.

Finally, for $x \in (0, \infty)$, we have $|x+2| = x+2$ and $|x| = x$ and so the inequality $|x+2| \geq 2|x|$ becomes $x+2 \geq 2x$, i.e. $x \leq 2$.

Summing up, we see that the inequality $|x+2| \geq 2|x|$ holds for $x \in [-2/3, 0] \cup [0, 2] = [-2/3, 2]$ while the opposite inequality $|x+2| < 2|x|$ holds for $x \in (-\infty, -2/3) \cup (2, \infty)$.

Putting this into (4) we now have

$$f(x) = \begin{cases} 2u(x) = 4|x| & \text{when } -2/3 \leq x \leq 2 \\ 2v(x) = 2|x+2| & \text{otherwise} \end{cases} \tag{5}$$

We are not yet done. To evaluate $f(x)$ we still have to tackle the nagging absolute values $|x|$ and $|x+2|$ which change expressions at 0 and -2 respectively. All in all we now have to split the real line into five intervals by the points $-2, -2/3, 0$ and 2 and get an expression for $f(x)$ on each that does not involve absolute values.

For $x \leq -2$, we have $f(x) = 2|x+2| = -2x - 4$.

For $-2 \leq x \leq -2/3$ too, we have $f(x) = 2|x+2| = 2x + 4$.

For $-2/3 \leq x \leq 0$, we have $f(x) = 4|x| = -4x$.

For $0 \leq x \leq 2$ also we have $f(x) = 4|x| = 4x$.

Finally, for $x \geq 2$, we have $f(x) = 2|x+2| = 2x + 4$.

Since we have finally got rid of all absolute value signs, we now have all the information to draw an accurate graph of $f(x)$. But that is hardly necessary. The hard (or rather, the tedious) work we have done is already sufficient to give us the answer, viz. the points of local maxima and minima of $f(x)$. Note that on each of the five intervals $(-\infty, -2), (-2, -2/3), (-2/3, 0), (0, 2)$ and $(2, \infty)$, the function $f(x)$ is linear with slopes $-2, 2, -4, 4$ and 2 respectively. A local maximum occurs when the slope changes from positive to negative. In the present problem, this happens only at $-2/3$. A local minimum occurs where the slope changes from negative to positive. In the present problem this happens at two points, viz. -2 and 0. Out of these, only $-2/3$ is listed as a possible answer.

The essential idea in the problem is merely how a function defined in terms of absolute values changes its behaviour. In the present problem, such changes have come too many times and that makes the problem extremely tedious and error prone without testing the intelligence of a candidate. As said at the beginning, a well-drawn graph gives the answer instantaneously, if somebody else has drawn it. This point needs to be noted by the paper-setters. The logistics of the JEE does stipulate a certain minimum distance between two candidates so that a candidate cannot read what his neighbour has written. But graphs can be seen even from greater distances. Even if only its shape is seen that will give a free hint to the neighbour.

Q.45 For $a \in I\!R$ (the set of all real numbers), $a \neq -1$,

$$\lim_{n\to\infty} \frac{(1^a + 2^a + \ldots + n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+\ldots+(na+n)]} = \frac{1}{60}$$

Then $a =$

(A) 5 (B) 7 (C) $\frac{-15}{2}$ (D) $\frac{-17}{2}$

Answer and Comments: (B). A perceptive reader will hardly fail to notice, right at the start, a dirty trick played by the paper-setters. For large n, $n+1$ behaves the same way as n in the sense that the ratio $\frac{n+1}{n}$ tends to 1 as $n \to \infty$. The same holds for any power of $n+1$. Hence the limit in the question will be the same as the limit, say L, defined by

$$L = \lim_{n\to\infty} \frac{(1^a + 2^a + \ldots + n^a)}{n^{a-1}[(na+1)+(na+2)+\ldots+(na+n)]} \tag{1}$$

To see this formally, divide and multiply the denominator in the given limit by n^{a-1} and use the fact that $\lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^{a-1} = 1^{a-1} = 1$.) So from now onwards, we shall deal with the limit L rather than the one in the statement of the ptoblem.

The sum in the denominator allows a closed form expression, viz. $n^2 a + \frac{n(n+1)}{2}$. Obviously this tends to ∞ as $n \to \infty$. But sine the expression is a polynomial in n of degree 2, when it is divided by n^2 it does tend to a finite limit, viz. $a + \frac{1}{2}$. So, multiplying and dividing the denominator of the R.H.S. of (1) by n^2, we get

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{(1^a + 2^a + \ldots + n^a)}{(n^{a+1})\,\frac{(a+1/2)n^2+(1/2)n}{n^2}} \\ &= \lim_{n\to\infty} \frac{(1^a + 2^a + \ldots + n^a)}{n^{a+1}} \times \lim_{n\to\infty} \frac{n^2}{(a+1/2)n^2+(1/2)n} \end{aligned} \tag{2}$$

provided both the limits exist. Let us call these limits as L_1 and L_2 respectively. We already know that

$$L_2 = \frac{1}{a+1/2} = \frac{2}{2a+1} \tag{3}$$

Let us now concentrate on the first limit L_1. There is a fairly well-known result that if a is a positive integer, then the sum $1^a + 2^a + \ldots + n^a$ is a polynomial of degree $a+1$ in n with leading coefficient $\frac{1}{a+1}$. (See the end of Comment No. 2 of Chapter 2 and also Exercise (17.11). This result is usually stated, without proof, after derving the formulas for the sums $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$ which equal $\frac{n(n+1)}{2}$, $\frac{n(n+1)(2n+1)}{6}$ and $\frac{n^2(n+1)^2}{4}$ respectively.

What if we just *assume* this result to be true for all real a? In that case the first limit L_1 will be simply $L_1 = \frac{1}{a+1}$ and from (2) and (3), we get an equation for a, viz.

$$\frac{1}{a+1} \times \frac{2}{2a+1} = \frac{1}{60} \tag{4}$$

This results into a quadratic for a, viz.

$$2a^2 + 3a - 119 = 0 \tag{5}$$

whose roots are seen (either by factoring the L.H.S. as $(2a+17)(a-7)$ or by the quadratic formula) to be 7 and $-17/2$.

As both these values are among the given options, an unscrupulous candidate is most certain to feel vindicated and leave the problem there. But a scrupulous candidate would be wary. His conscience would prick him at unwarrantedly extending the result about the sum $1^a + 2^a + \ldots + n^a$ from positive integral values of a to all values of a. There has to be a proof for it. And indeed there is if we recast the sum as the Riemann sum of a suitable function over a suitable interval. With a slight rewriting, we have

$$\begin{aligned} \frac{(1^a + 2^a + \ldots + n^a)}{n^{a+1}} &= \frac{1}{n}\left[\left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \ldots + \left(\frac{n}{n}\right)^a\right] \\ &= \sum_{k=1}^{n} \left(\frac{k}{n}\right)^a \frac{1}{n} \end{aligned} \tag{6}$$

If we consider the function $f(x) = x^a$ defined over the interval $[0, 1]$ and partition this interval into n equal parts (so that the length of each part is $1/n$), then the R.H.S. of (6) is nothing but a Riemann sum (or more specifically, the upper Riemann sum for $a > 0$) of $f(x)$ for this partition. All Riemann sums of a function $f(x)$ over an interval $[a, b]$ converge to the definite integral $\int_a^b f(x)dx$ as the maximum size of the intervals in the partition tends to 0. In particular we see that as n tends to ∞ the sum on the R.H.S. of (6) tends to the integral $\int_0^1 x^a dx$. Since $\frac{x^{a+1}}{a+1}$ is an anti-derivative of x^a, this integral is simply

$$\int_0^1 x^a dx = \frac{1}{a+1} x^{a+1}\Big|_0^1 = \frac{1}{a+1} \tag{7}$$

The scrupulous candidate will now rest assured that his guess stands substantiated. He would also note happily that the statement of the question stipulates $a \neq -1$. This is consistent with the fact that (7) is not valid for $a = -1$. Since neither $-17/2$ nor 7 is an excluded value, both are correct answers.

It will take scrupulousness of an even higher order to realise that there is still something missing here. For (7) to be valid, not only do we need that $\dfrac{x^{a+1}}{a+1}$ is an antiderivative of x^a, we also need that the integral $\displaystyle\int_0^1 x^a dx$ exists in the first place. The first requirement is satisfied by all $a \neq -1$. But the second one requires that the integrand viz. x^a be bounded on the interval $[0, 1]$ (one of the starting assumptions about Riemann integrals). If $a < 0$, this requirement fails and so the integral in (7) does not exist. For $-1 < a < 0$, the integral can still be evaluated if we regard it as an improper integral, see Comment No. 16 of Chapter 18. But for $a < -1$ even this consolation is not available and the integral in (7) simply does not exist. Since $-17/2 < -1$, we have no option but to discard it as a right answer. (The failure of (7) for $a < -1$ can be seen more easily by noting that while the L.H.S. is the integral of a non-negative function, the R.H.S. is negative for $a < -1$.)

The incorrigible optimist may still not give up his fight to salvage $-17/2$ as a valid answer. "It is true," he will argue, "that the integral in (7) does not exist for $a < -1$. But after all that integral was only a tool to evaluate the limit L_1 in (2). Just because a particular tool fails does not mean that no other tool will work. Could it not happen that even though (7) fails for $a < -1$, the limit L_1 still exists and can be evaluated by some other method?"

Unfortunately, this optimism turns out to be misplaced too. To see how, let us recast the limit L_1 slightly. We are assuming $a < -1$. Let us write p for $-a$. Then $p > 1$. With this change of notation, we have

$$\begin{aligned} L_1 &= \lim_{n\to\infty} \frac{\dfrac{1}{1^p} + \dfrac{1}{2^p} + \ldots + \dfrac{1}{n^p}}{n^{1-p}} \\ &= \lim_{n\to\infty} \left(\frac{1}{1^p} + \frac{1}{2^p} + \ldots + \frac{1}{n^p} \right) n^{p-1} \qquad (8) \end{aligned}$$

It can be shown that the first factor in the R.H.S. of (8) tends to a finite non-zero limit as $n \to \infty$. (This is a consequence of a well-known result that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for all $p > 1$.) So whether L_1 exists or not now depends on the behaviour of the second factor, viz. n^{p-1}. But as $p > 1$, $n^{p-1} \to \infty$ as $n \to \infty$. So, the R.H.S. of (8) tends to ∞ and so L_1 does not exist (as a finite real number).

As all efforts to vindicate $-17/2$ as a possible answer have failed, we conclude that the only valid answer is 7.

We evaluated the limit L_2 (or, rather, its reciprocal) by using a closed form expression for the sum $(na+1)+(na+2)+\ldots+(na+n)$. If one wants, L_2 can also be evaluated by recasting it as a definite integral. We first rewrite $\dfrac{(na+1)+(na+2)+\ldots+(na+n)}{n^2}$ as

$$\left[(a+\frac{1}{n})+(a+\frac{2}{n})+\ldots+(a+\frac{n}{n})\right]\frac{1}{n}=\sum_{k=1}^{n}\left(a+\frac{k}{n}\right)\frac{1}{n} \tag{9}$$

We now consider the function $g(x)=a+x$ on the interval $[0,1]$. If this interval is partitioned into n equal parts, then the sum in (9) is precisely the upper Riemann sum of $g(x)$ for this partition. Therefore its limit is the integral $\displaystyle\int_0^1 g(x)dx=\int_0^1 a+x\;dx$ which comes out to be $a+\dfrac{1}{2}$. Its reciprocal, viz. $\dfrac{2}{2a+1}$ is the limit L_2.

But it is foolish to apply this method when there is a simple closed form expression for the sum involved. In the case of L_1, however, there is no such expression for the sum $1^a+2^a+\ldots+n^a$ and so the use of definite integrals becomes necessary.

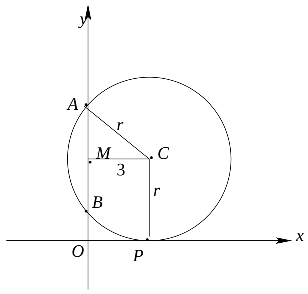
The problem is a good test of the ability to apply Riemann integrals to evaluate the limits of certain sums. But what is even more subtly tested in the problem is the ability to realise that $-17/2$ is not a valid answer. Normally, the multiple choice format and occasionally the carelessness on the part of the paper-setters tends to reward the unscrupulous candidates over the scrupulous ones. The present problem is a pleasant (and a rare) exception and the paper-setters deserve to be commended for coming up with it. The cheap twist they have given by unnecessarily putting $(n+1)^{a-1}$ instead of n^{a-1} can be forgiven.

Q.46 Circles touching x-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on y-axis is (are)

(A) $x^2+y^2-6x+8y+9=0$ (B) $x^2+y^2-6x+7y+9=0$
(C) $x^2+y^2-6x-8y+9=0$ (D) $x^2+y^2-6x-7y+9=0$

Answer and Comments:(A) and (C).

The desired circle touches the x-axis at a point $(\pm 3, 0)$. So its centre must be at a point $(\pm 3, \pm r)$ where r is also the radius of the circle. There are four such circles with centres in the four quadrants. One of them is shown here. Let it cut the y-axis at A and B and let M be the mid-point of AB. We are given that $AM = \sqrt{7}$ and $MC = OP = 3$.



From the right angled triangle AMC, we get

$$r^2 = 7 + 9 = 16 \tag{1}$$

which gives $r = 4$. Therefore the four desired circles have equations of the form

$$(x \pm 3)^2 + (y \pm 4)^2 = 16 \tag{2}$$

i.e.

$$x^2 + y^2 \pm 6x \pm 8y + 9 = 0 \tag{3}$$

Out of the given options, (A) and (C) are of this type. We have solved the problem in a purely geometric manner. Another approach is to take the equation of the circle in the form

$$x^2 + y^2 \pm 6x \pm 2ry + 9 = 0 \tag{4}$$

and then to calculate the y-intercept as a function of r by first finding the points of intersection of the circle and the y-axis. Yet another approach is to start with the general equation of a circle, viz.

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{5}$$

and determine g, f, c by writing a system of three equations in three unknowns using the data of the problem. We have formulated the problem in terms of only one unknown viz. r by incorporating the data in the choice of the variables and using symmetry considerations.

A very simple problem. Choosing an approach (such as the one we have taken) which minimises the computations is the only part which needs some brain.

Q.47 Two lines $L_1 : x = 5, \dfrac{y}{3-a} = \dfrac{z}{-2}$ and $L_2 : x = \alpha, \dfrac{y}{-1} = \dfrac{z}{2-\alpha}$ are coplanar. Then α can take the value(s)

(A) 1 (B) 2 (C) 3 (D) 4

Answer and Comments: (A) and (D). For two lines to be coplanar, they must either intersect or be parallel to each other. Let us consider the two possibilities separately.

If L_1 and L_2 have a common point, say (x_0, y_0, z_0), then we must have $x_0 = 5 = \alpha$. But this value of α makes the other equations in L_1 and L_2 to be

$$\begin{aligned} \frac{y}{-2} &= \frac{z}{-2} \qquad (1) \\ \text{and} \quad \frac{y}{-1} &= \frac{z}{-3} \qquad (2) \end{aligned}$$

respectively. The only common solution is $y_0 = z_0 = 0$. Hence the lines L_1 and L_2 meet at the point $(5, 0, 0)$ and hence are coplanar if $\alpha = 5$. So this is a possible value. But it is not listed in the options.

The second possibility is that L_1 and L_2 are parallel to each other. In this case their direction numbers are proportional, i.e. there is some λ such that

$$(0, 3-\alpha, -2) = \lambda(0, -1, 2-\alpha) \qquad (3)$$

which splits into

$$\begin{aligned} 3-\alpha &= -\lambda \qquad (4) \\ \text{and} \quad -2 &= \lambda(2-\alpha) \qquad (5) \end{aligned}$$

Eliminating λ we get a quadratic in α viz.

$$2 = (3-\alpha)(2-\alpha) \qquad (6)$$

which becomes $\alpha^2 - 5\alpha + 4 = 0$. Thus the possible values of α are 1 and 4. Both are listed as possible options.

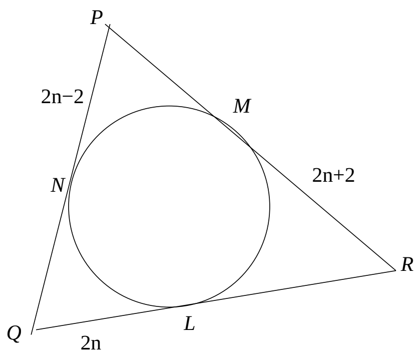
Another very simple problem. Once the idea of the two possibilities for coplanarity strikes, the calculations are very simple. But the exclusion of the value $\alpha = 5$ (which is a valid possibility) from the four options can cause some confusion to some candidates, because this is in sharp contrast with the questions in Section 2 of Paper 1 where too, more than one options could be correct, but where all correct possibilities were listed. Technically, the wording of the present question is not wrong. But a better formulation would have been, "Determine which of the following values of α will make the lines L_1 and L_2 coplanar". Similarly, in the last question, although there are four circles which fit the description, only two of them are listed as possible answers. So a fomulation like "Determine which of the following four circles has the property that it touches the x-axis at a distance 3 from the origin and .... ". These formulations cannot be confusing because they do not give the impression that all correct possibilities are listed, much the same way that a question like "Determine which of the following four integers is (are) prime. " cannot be confusing because everybody knows that there are infinitely many primes and hence not all of them can occur in the given set.

Q.48 In a triangle PQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L, M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)

(A) 16 (B) 18 (C) 24 (D) 22

Answer and Comments: (B) and (D). Like most other problems on solution of triangles, the present problem requires a diagram even to understand it.

It is not given whether the consecutive even integers are in an ascending or in a descending order. That would affect the lengths of the individual sides and hence also the equation we shall get from the data $\cos P = \frac{1}{3}$ and that may lead to additional answers. So, we must consider both the possibilities. We first choose the first possibility. (Later we shall prove that the other possibility cannot hold.)



So we take PN, QL and RM to be $2n-2, 2n, 2n+2$ respectively for some positive integer n. By properties of tangents we have

$$\begin{aligned} NQ &= QL = 2n & (1) \\ LR &= RM = 2n+2 & (2) \\ \text{and } MP &= PN = 2n-2 & (3) \end{aligned}$$

We now get the lengths of the sides PQ, QR and PR as $4n-2, 4n+2$ and $4n$ respectively. Had the three consecutive integers been in the descending order, then QL and QN would remain $2n$. But PN and PM would each equal $2n+2$ while RL and RM would equal $2n-2$ each. But this would make $QR = 4n-2$ and $PQ = 4n+2$ contradicting that QR is the longest side of the triangle (since P is given to be the largest angle of the triangle PQR.) So we are justified in assuming, as we have, that the three even integers are in an ascending order.

We are given that $\cos P = \frac{1}{3}$. Using the cosine formula we get

$$\frac{PQ^2 + PR^2 - QR^2}{2PQ.PR} = \frac{1}{3} \tag{4}$$

Substituting the lengths of the sides found above, this gives an equation in n, viz.

$$\frac{(4n-2)^2 + (4n)^2 - (4n+2)^2}{2(4n-2)(4n)} = \frac{1}{3} \tag{5}$$

which can be simplified as

$$\frac{16n^2 - 32n}{16n(2n-1)} = \frac{1}{3} \tag{6}$$

and further as

$$3(n-2) = 2n - 1 \tag{7}$$

which is a linear equation in n with $n = 5$ as its only solution. With this value, the lengths of the sides PQ, QR and PR become $18, 22$ and 20 respectively. Out of these only 18 and 22 are given as possible answers.

This is a very simple problem requiring nothing beyond a very elementary property of tangents to a circle and the cosine formula. The purpose of stipulating that P is the largest angle of the triangle seems to be only to ensure that the three even integers are in an ascending order as shown above. A scrupulous candidate will have to spend some time to eliminate the other possibility.

Now the fact is that whenever we talk of three consecutive integers, we, in real life as well as often in mathematics, generally take them to be in ascending order. It is only when this assumption leads to something absurd that we think of the other possibility. In the present problem, an unscrupulous candidate who starts with an ascending order without justification, will blissfully get the right answer. If the paper-setters really wanted to test a candidate's ability to rule out the wrong alternative, they should have given the data in such a way that only the descending order of the even integers would give the right answer. (For example, instead of the lengths of PN, QL and RM, they could have given that the lengths of RM, QL and PN are consecutive even integers.) As the problem now stands, it rewards the unscrupulous over the scrupulous.

Note, incidentally, that if the data had merely said that the largest angle of the triangle is $\cos^{-1}\left(\frac{1}{3}\right)$ (instead of further saying that this angle is P), then the solution would be independent of whether the order is ascending or descending. For no matter which order is followed, the lengths of the three sides would come to be $4n$, $4n+2$ and $4n-2$ and by equating the angle opposite to the side with length $4n+2$ (the longest side) with $\cos^{-1}\left(\frac{1}{3}\right)$ we would get the same equation as (5).

When a problem is designed in a hurry, it is human to overlook such fine points. But the JEE paper-setters have plenty of time to consider them. Perhaps, before finalising a question it should be given to a couple of members of the team who have weird minds, i.e. who are quick to think of unusual or exceptional possibilities without getting swayed by the majority thinking.

SECTION II

Paragraph Type

This section contains **4 paragraphs** each describing theory, experiment, data etc. **Eight questions** relate to four paragraphs with two questions on each paragraph. Each question of a paragraph has **only one correct answer** among the four choices (A), (B), (C) and (D). There are **3 points** for marking only the correct answer, no points if no answer is marked and -1 point in all other cases.

Paragraph for Questions 49 and 50

Let $S = S_1 \cap S_2 \cap S_3$, where

$S_1 = \{z \in \mathbb{C} : |z| < 4\}$, $S_2 = \{z \in \mathbb{C} : Im\left[\dfrac{z - 1 + \sqrt{3}i}{1 - \sqrt{3}i}\right] > 0\}$ and

$S_3 = \{z \in C : Re\ z > 0\}$.

Q.49 Area of $S =$

(A) $\dfrac{10\pi}{3}$ (B) $\dfrac{20\pi}{3}$ (C) $\dfrac{16\pi}{3}$ (D) $\dfrac{32\pi}{3}$

Q.50 $\min\{|1 - 3i - z| : z \in S\} =$

(A) $\dfrac{2-\sqrt{3}}{2}$ (B) $\dfrac{2+\sqrt{3}}{2}$ (C) $\dfrac{3-\sqrt{3}}{2}$ (D) $\dfrac{3+\sqrt{3}}{2}$

Answers and Comments: (B) and none respectively. The answers to both the questions require us to correctly identify the set S which is given as the intersection of three subsets of $\mathbb{C}$, the complex plane. The set S_1 is the disc of radius 4 centred at the origin O while S_3 is the half plane to the right of the y-axis. But the set S_2 is a little tricky to identify. For this, let us first write

$$\frac{z - 1 + \sqrt{3}i}{1 - \sqrt{3}i} = \frac{z}{1 - \sqrt{3}i} - 1 \tag{1}$$

and observe that the imaginary part of a complex number does not change if any real number is added to it. As a result, we get

$$S_2 = \{z \in \mathbb{C} : Im\left(\frac{z}{1 - \sqrt{3}i}\right) > 0\} \tag{2}$$

and further, after multiplying and dividing by $1+\sqrt{3}i$,

$$S_2 = \{z \in C : Im\left(\frac{z(1+\sqrt{3}i)}{4}\right) > 0\} \tag{3}$$

As a further simplification we observe that even though the imaginary part of a complex number changes when it is multiplied by a positive real number, its sign does not. So we could multiply by 4 and get rid of the denominator above. Instead, we multiply it only by 2 and get

$$S_2 = \{z \in \mathbb{C} : Im\left(z\left(\frac{1+\sqrt{3}i}{2}\right)\right) > 0 \tag{4}$$

The reason we have chosen to multiply by 2 and not by 4 is that the number $\frac{1+\sqrt{3}i}{2}$ is a very nice number. Let us call it α. Then $|\alpha| = 1$ and casting it in the polar form we get

$$\alpha = \cos(\pi/3) + i\sin(\pi/3) = e^{i\pi/3} \tag{5}$$

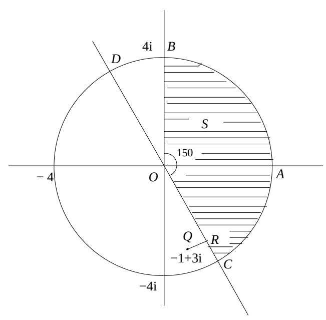
Thus we see that $\alpha^6 = 1$. In fact α is simply w^2 where w is the complex number in Q.41. (As we saw already, the complex number ω in Q. 43 was also related to w. It appears that the paper-setters have an obsession with this number.) In the present problem, however, it is not so much the powers of α that are relevant. Instead, what matters is the effect of multiplying a complex number z by α. Geometrically, if the point P in the plane represents the complex number z, then αz is represented by the point which is obtained from P by a counterclockwise rotation through $60°$ $(= \pi/3)$ Or, in terms of the arguments of complex numbers, we have

$$\arg(\alpha z) = \arg(z) + \pi/3 \tag{6}$$

Now, the imaginary part of a complex number is positive if and only if its argument lies between 0 and π. This fact, coupled with (4) gives us yet another formulation of the set S_2, viz.

$$\begin{aligned} S_2 &= \{z \in C : Im\ (\alpha z) > 0\} \\ &= \{z \in C : 0 < \arg(\alpha z) < \pi\} \\ &= \{z \in C : -\pi/3 < \arg z < 2\pi/3\} \end{aligned} \tag{7}$$

Thus we see that S_2 is the portion of the complex plane bounded by the rays OC and OD with arguments $-\pi/3$ and $2\pi/3$ respectively. These two rays are parts of the same straight line. The intersection $S = S_1 \cap S_2 \cap S_3$ is thus the shaded region shown in the figure. It consist of a sector of the disc bounded by OC and OB. Clearly the angle of this sector is $5\pi/6 = 150°$.



Having sketched S, it is now easy to answer both the questions. As S is a sector of angle $5\pi/6$ of a disc with radius r its area is simply $\frac{5\pi/6}{2\pi} \times 16\pi = \frac{20\pi}{3}$.

For the second question, let Q be the point which represents the complex number $1-3i$. Note that Q lies outside S. The question then amounts to finding a point, say R, in S which is closest to Q (among all points of S). Obviously such a point cannot be in the interior of S, for if it were there would always be some points in a small neighbourhood of R which are even closer to Q than R. So R has to lie on the boundary of S. The boundary consists of three parts, viz. the segment OC, the segment OB and the arc of the circle $|z| = 4$ from C to B. From the figure it is clear that R has to lie on the segment OC. We first try R to be the foot of the perpendicular from Q to the line OC whose equation is

$$y = \tan(-\pi/3)x = -\sqrt{3}x \tag{8}$$

On the other hand the line QR has slope $\frac{1}{\sqrt{3}}$ and so its equation is

$$y + 3 = \frac{1}{\sqrt{3}}(x - 1) \tag{9}$$

So the foot of the perpendicular from Q to the line OC is $\left(\frac{3\sqrt{3}+1}{4}, -\frac{\sqrt{3}+9}{4}\right)$. Its distance from O is $\frac{3\sqrt{3}+1}{2}$ which is less than 4. So, the foot of the perpendicular from $(1, -3)$ to the line OC lies on the segment OC and hence in the set S. We are now justified in taking $R = \left(\frac{3\sqrt{3}+1}{4}, -\frac{\sqrt{3}+9}{4}\right)$. (This justification is important because if the foot of the perpendicular fell outside the segment OC, then R would be either O or C.)

By an easy calculation we now have,

$$\begin{aligned} QR &= \sqrt{\left(\frac{3\sqrt{3}-3}{4}\right)^2 + \left(\frac{3-\sqrt{3}}{4}\right)^2} \\ &= \sqrt{\frac{48-24\sqrt{3}}{16}} \\ &= \frac{\sqrt{12-6\sqrt{3}}}{2} \\ &= \frac{3-\sqrt{3}}{2} \end{aligned} \tag{10}$$

Thus we may say that (C) is the correct answer. But technically, it is not so. The inequalities used in the definitions of the sets S_1, S_2, S_3 are all strict and therefore they do not contain any of their boundary points. So, none of the boundary poins of S is in S. Hence unless the point Q happens to be in S (in which case the minimum distance would be 0), the distance has no minimum. The answer we have got is the infimum and not the minimum. It may be argued that at the JEE level the distinction between an infimum and a minimum is a too subtle one. But it is precisely this difference which is the very beginning of calculus. (The fact that the set of positive real numbers has an infimum but no minimum is the genesis of the concept of a limit.) Some well-trained students may be aware of this difference and if they are scrupulous, may get confused. The best way out would have been to include all the boundaries in the respective sets S_1, S_2 and S_3. This could have been done by replacing the strict inequalities so as to include possible equalities too.

Except for this, the first question is a good one as it tests the ability to correctly sketch the set S, and especially the set S_2. Once that is done the rest of the work involves very little computation. In the second question, however, there is a lot of computational work, if done honestly. Those candidates who simply apply the formula for the shortest distance of a point from a given line wihtout verifying that the foot of the perpendicular lies in S will still score and save precious time.

Paragraph for Questions 51 and 52

A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls.

Q.51 If 1 ball is drawn from each of the boxes B_1, B_2 abd B_3, the probability that all three drawn balls are of the same colour is

(A) $\frac{82}{648}$ (B) $\frac{90}{648}$ (C) $\frac{558}{648}$ (D) $\frac{566}{648}$

Q.52 If two balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, the probability that these two balls are drawn from box B_2 is

(A) $\frac{116}{181}$ (B) $\frac{126}{181}$ (C) $\frac{65}{181}$ (D) $\frac{55}{181}$

Answers and Comments: (A) and (D). The first problem is a straightforward problem where a certain probability is to be computed after correctly identifying the sample space. By making the denominators of all listed answers equal, the paper-setters have given an implied hint that the size of the sample space would be 648 (or possibly some multiple or submultiple of it).

The sample space, say S consists of all ordered triples of the form (x, y, z) where x, y, z stand for the balls drawn from boxes B_1, B_2, B_3 respectively. As the boxes contain 6, 9 and 12 balls respectively, we have

$$S = 6 \times 9 \times 12 = 648 \tag{1}$$

Let us now identify, the set, say T of all facourable cases. There are three mutually exclusive possibilities depending upon the common colour of the three balls drawn. Accordingly T splits as

$$T = T_1 \cup T_2 \cup T_3 \tag{2}$$

where

$$\begin{aligned} T_1 &= \{(x,y,z) \in S : x,y,z \text{ are all white}\} && (3)\\ T_2 &= \{(x,y,z) \in S : x,y,z \text{ are all red}\} && (4)\\ \text{and}\quad T_3 &= \{(x,y,z) \in S : x,y,z \text{ are all black}\} && (5) \end{aligned}$$

Clearly, we have

$$\begin{aligned} |T_1| &= 1 \times 2 \times 3 = 6\\ |T_2| &= 3 \times 3 \times 4 = 36\\ \text{and}\quad |T_3| &= 2 \times 4 \times 5 = 40 && (6) \end{aligned}$$

Hence

$$|T| = |T_1| + |T_2| + |T_3| = 82 \tag{7}$$

Hence the desired probability is $\frac{82}{648}$.

The second question is about conditional probability. We let E be the event that the two balls drawn from a randomly selected box are one red and one white. Then E splits into three mutually exclusive events E_1, E_2 and E_3 where E_i is the event that these balls are from the box B_i, for $i = 1, 2, 3$. The numbers of ways to draw two balls (without replacement) from the boxes B_1, B_2 and B_3 are respectively, $\binom{6}{2}, \binom{9}{2}$ and $\binom{12}{2}$, i.e. 15, 36 and 66. The numbers of possible white-red combinations for B_1, B_2 and B_3 are $3, 6$ and 12 respectively. So we get

$$\begin{aligned} P(E) &= P(E_1) + P(E_2) + P(E_3)\\ &= \frac{3}{15} + \frac{6}{36} + \frac{12}{66}\\ &= \frac{1}{5} + \frac{1}{6} + \frac{2}{11} = \frac{66 + 55 + 60}{330} = \frac{181}{330} && (8) \end{aligned}$$

The question asks for the probability of E_2 given E. It is hardly necessary to apply Bayes theorem for this, because in this problem E_2 is a sub-event of E and we have already calculated $P(E_2)$ as $1/6$. So,

$$P(E_2|E) = \frac{P(E_2)}{P(E)} = \frac{1/6}{181/330} = \frac{55}{181} \tag{9}$$

Very simple problems, both conceptually and computationally. One wonders if these problems deserve to be asked in an advanced test. They properly belong to a screening test.

Paragraph for Questions 53 and 54

Let $f : [0,1] \longrightarrow I\!R$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x, x \in [0,1]$.

Q.53 Which of the following is true for $0 < x < 1$?

(A) $0 < f(x) < \infty$ (B) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(C) $-\frac{1}{4} < f(x) < 1$ (D) $-\infty < f(x) < 0$

Q. 54 If the function $e^{-x}f(x)$ assumes its minimum in the interval $[0,1]$ at $x = \frac{1}{4}$, which of the following is true ?

(A) $f'(x) < f(x),\ \frac{1}{4} < x < \frac{3}{4}$ (B) $f'(x) > f(x),\ 0 < x < \frac{1}{4}$

(C) $f'(x) < f(x),\ 0 < x < \frac{1}{4}$ (D) $f'(x) < f(x),\ \frac{3}{4} < x < 1$

Answers and Comments: (D) and (C). Problems on solving differential equations are very common. In the present problems, however, the function $f(x)$ is not given to satisfy a differential equation but only a differential inequality, i.e. an inequality about its derivatives. Normally, inequalities do not mix easily with derivatives because an inequality of the form $f'(x) < g(x)$ implies little about $f(x)$ except possibly when the inequality is of a very simple form (e.g. $g(x)$ is constant). However, as illustrated in Comment No. 13 of Chapter 19, the fact that the integrating factor of linear differential equations is an exponential function and hence always positive sometimes allows us to recast the inequality so that it becomes an inequality of a very simple form.

In first order differential equations, the purpose of an integrating factor is that when the expression involving a first order derivative is multiplied by the I.F., the product is the derivative of some easily recognisable function. In the present problem, the L.H.S. of the inequality involves a second order derivative. But the principle is the same. If we multiply it by e^{-x}, we see that it is exactly the second derivative of the product $e^{-x}f(x)$. (This can be seen by applying Leibnitz rule or by a direct computation.)

This observation is the key to the problem. (A hint for multiplying by e^{-x} is also provided by the statement of the second question.) Since e^{-x} is always positive, we can recast the given inequality as

$$e^{-x}(f''(x) - 2f'(x) + f(x)) \geq 1 \quad (1)$$

and hence as

$$\frac{d^2}{dx^2}(e^{-x}f(x)) \geq 1 \quad (2)$$

for $x \in [0,1]$. For simplicity call $e^{-x}f(x)$ as $h(x)$. Then (2) says that

$$h''(x) \geq 1 \tag{3}$$

for $0 \leq x \leq 1$. In particular we see that $h(x)$ is strictly concave upwards on the interval $[0,1]$. Hence its graph for $0 < x < 1$ lies strictly below the chord joining the points $(0, h(0))$ and $(1, h(1))$. Since $f(0) = f(1) = 0$, we have $h(0) = h(1) = 0$. So both the end-points of the chord lie on the x-axis. Thus we get

$$h(x) < 0 \tag{4}$$

for all $0 < x < 1$. Since $f(x) = e^x h(x)$ and e^x is always positive, we also have

$$f(x) < 0 \tag{5}$$

for all $0 < x < 1$.

The second question directly deals with the function $h(x) = e^{-x}f(x)$ we have already introduced. If it assumes its minimum on $[0,1]$ at $x = 1/4$, then since this point is an interior point, the minimum at it must be a local minimum. This means that $h'(1/4) = 0$. But by (3) $h'(x)$ is strictly increasing on $(0,1)$. In particular, we have

$$h'(x) < h'(1/4) = 0 \tag{6}$$

for $0 < x < 1/4$ and also

$$h'(x) > h'(1/4) = 0 \tag{7}$$

for $1/4 < x < 1$. A direct computation gives

$$h'(x) = e^{-x}(f'(x) - f(x)) \tag{8}$$

As the factor e^{-x} is always positive, from (6) we get

$$f'(x) < f(x) \tag{9}$$

for $0 < x < 1/4$. This is statement (C). As only one of the alternatives holds, we need not go further. Still, for the sake of completeness, from (7) we get

$$f'(x) > f(x) \tag{10}$$

for $1/4 < x < 1$. This makes both (A) and (D) false. Falsity of (B) already follows from the truth of (C).

Both the questions, especially the first one, are fairly tricky because the work involves not so much the function $f(x)$ directly but the associated function $e^{-x}f(x)$. Even after this idea strikes, the answers demand a good knowledge of concavity of functions. So, these are good problems fully deserving a place in an advanced test.

The crucial step in the solution was to get that the function $h(x)$ was strictly concave on the interval $[0,1]$. This merely requires that $h''(x) > 0$ for all $[0,1]$. So (3) was not used to its full capcacity. If the function $f(x)$ satisfied the inequality

$$f''(x) - 2f'(x) + f(x) \geq \lambda e^x, x \in [0,1] \tag{11}$$

where λ is any positive constant, then the same argument would give $h''(x) \geq \lambda$ (instead of (3)) for all $x \in [0,1]$ and this is good enough for strict concavity of $h(x)$. We can go still further and get rid of the exponential factor altogether. That is, suppose $f(x)$ satisfies

$$f''(x) - 2f'(x) + f(x) \geq \mu,\ x \in [0,1] \tag{12}$$

where μ is some positive constant. Since $e^x \leq e$ for all $x \in [0,1]$, we see that (11) holds if we take λ to be μ/e. This formularion of the problem would be really challenging because now there is no hint that multiplication by e^{-x} would do the trick.

Paragraph for Questions 55 and 56

Let PQ be a focal chord of of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$.

Q.55 Length of chord PQ is

(A) $7a$ (B) $5a$ (C) $2a$ (D) $3a$

Q.56 If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta =$

(A) $\frac{2}{3}\sqrt{7}$ (B) $-\frac{2}{3}\sqrt{7}$ (C) $\frac{2}{3}\sqrt{5}$ (D) $-\frac{2}{3}\sqrt{5}$

Answer and Comments: (B) and (D). The focus of the parabola is at $(a,0)$. So the equation of a focal chord would be of the form

$$y = m(x-a) \tag{1}$$

for some m. We can then find the points P and Q by solving (1) and $y^2 = 4ax$ together. Next we shall find the tangents at these points and their concurrency with the line $y = 2x + a$ will give us an equation for m. Solving it we get the value of m. Once that is known, the points P and Q would be both known and then we can answer both the questions.

But this approach would be two complicated. Note that our concern is not so much to determine the points P and Q, but only to determine the distance PQ and the angle POQ and there may be ways to do this without finding P and Q.

So, we begin by taking P and Q in the parametric form, i.e.

$$\begin{aligned} P &= (at_1^2, 2at_1) \\ \text{and } Q &= (at_2^2, 2at_2) \end{aligned} \tag{2}$$

where t_1 and t_2 are some real numbers. As the line PQ is given to pass through the focus $(a, 0)$, we have

$$\begin{vmatrix} a & 0 & 1 \\ at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \end{vmatrix} = 0 \tag{3}$$

which simplifies to

$$(t_1 - t_2) + t_1t_2(t_1 - t_2) = 0 \tag{4}$$

and hence to

$$t_1t_2 = -1 \tag{5}$$

since $t_1 \neq t_2$ as P, Q are distinct.

We now consider the tangents at the points P and Q. Their equations are

$$\begin{aligned} x - t_1y + at_1^2 &= 0 \\ \text{and } x - t_2y + at_2^2 &= 0 \end{aligned} \tag{6}$$

respectively. We are given that these two lines and the line $2x - y + a = 0$ are concurrent. This gives us one more equation, viz.

$$\begin{vmatrix} 2 & 1 & a \\ 1 & t_1 & at_1^2 \\ 1 & t_2 & at_2^2 \end{vmatrix} = 0 \tag{7}$$

which, after expansion and cancellation of the factor $t_1 - t_2$ becomes

$$-2t_1t_2 + (t_1 + t_2) - 1 = 0 \tag{8}$$

Combining this with (5) we get

$$t_1 + t_2 = 1 + 2t_1t_2 = -1 \tag{9}$$

We can now determine t_1 and t_2 and hence the points P and Q. But that is hardly necessary. Expressions for the distance PQ as well as for $\tan \angle POQ$ involve only symmetric functions of t_1 and t_2 and hence can be expressed in terms of the two elementary symmetric functions $t_1 + t_2$ and t_1t_2 whose values are known to us. Specifically,

$$\begin{aligned} PQ^2 &= (at_1^2 - at_2^2)^2 + (2at_1 - 2at_2)^2 \\ &= a^2\left[(t_1^2 + t_2^2)^2 - 4t_1^2t_2^2 + 4(t_1 - t_2)^2\right] \\ &= a^2\left[\left((t_1 + t_2)^2 - 2t_1t_2\right)^2 - 4t_1^2t_2^2 + 4\left((t_1 + t_2)^2 - 4t_1t_2\right)\right] \end{aligned} \tag{10}$$

Putting the values from (5) and (9) this is simply $25a^2$ from which we get that the distance PQ is $5a$. This answers Q.55.

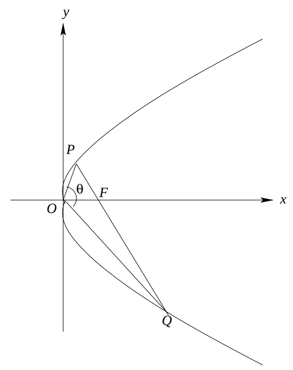
As for Q.56, the slopes of the lines OP and OQ are, respectively, $\dfrac{2}{t_1}$ and $\dfrac{2}{t_2}$. This gives

$$\begin{aligned}
\tan\theta &= \frac{\frac{2}{t_1} - \frac{2}{t_2}}{1 + \frac{4}{t_1 t_2}} \\
&= \frac{2(t_2 - t_1)}{4 + t_1 t_2} \\
&= \frac{2}{3}(t_2 - t_1) \qquad (11)
\end{aligned}$$

Here $t_2 - t_1$ is not a symmetric function of t_1 and t_2. But its square is. Squaring and then taking square roots, we get

$$t_2 - t_1 = \pm\sqrt{(t_2 - t_1)^2} = \pm\sqrt{(t_1 + t_2)^2 - 4t_1 t_2} = \pm\sqrt{5} \qquad (12)$$

Substituting this into (11) we get $\tan\theta = \pm\frac{2}{3}\sqrt{5}$. To determine which sign holds, it is fairly obvious from a diagram that θ is obtuse and hence the negative sign holds.



For those who want an analytical proof, we can do the sign determination from $\cos\theta$ which can be obtained by applying the cosine formula for the triangle OPQ. We have

$$\cos\theta = \frac{OP^2 + OQ^2 - PQ^2}{2OP.OQ} \qquad (13)$$

The denominator is always positive. So the sign of the numerator will tell us if θ is acute or obtuse. Thus we have, by a direct calculation,

$$\begin{aligned}
OP^2 + OQ^2 - PQ^2 &= a^2\left[t_1^4 + t_2^4 + 4t_1^2 + 4t_2^2 - (t_1^2 - t_2^2)^2 - 4(t_1 - t_2)^2\right] \\
&= a^2\left[2t_1^2 t_2^2 + 8t_1 t_2\right] = -6a^2 \qquad (14)
\end{aligned}$$

since $t_1 t_2 = -1$ by (5). Hence θ is obtuse. (It is interesting that the numerator of (13) is the same for all focal chords.)

Instead of determining $\tan\theta$ first and then making a sign distinction using (13), one can as well use (13) to find $\cos\theta$ directly and then $\tan\theta$. We have already calculated the numerator in (14). Let us now compute the denominator, say D, of (13).

$$\begin{aligned} D &= 2OP.OQ = 2a^2\sqrt{t_1^4 + 4t_1^2}\sqrt{t_2^4 + 4t_2^2} \\ &= 2a^2|t_1 t_2|\sqrt{(t_1^2+4)(t_2^2+4)} \\ &= 2a^2\sqrt{t_1^2 t_2^2 + 4(t_1^2 + t_2^2) + 16} \\ &= 2a^2\sqrt{17 + 4[(t_1+t_2)^2 - 2t_1 t_2]} \\ &= 2a^2\sqrt{29} \end{aligned} \tag{15}$$

using (9). So, we now get

$$\cos\theta = \frac{-6a^2}{2\sqrt{29}a^2} = -\frac{3}{\sqrt{29}} \tag{16}$$

and hence,

$$\tan^2\theta = \sec^2\theta - 1 = \frac{29}{9} - 1 = \frac{20}{9} \tag{17}$$

and hence $\tan\theta = -\frac{2}{3}\sqrt{5}$ since θ is known to be obtuse.

Both the questions are reasonable once the idea of using the parametric equations of a parabola strikes. The result (5) is fairly well-known for the parameters of the end points of a focal chord. Those who know it can shorten the work by taking the parameters as t_1 and $-\frac{1}{t_1}$. This is fine. But those who do the sign determination of $\tan\theta$ merely from the diagram are a bit unfairly rewarded.

SECTION III

Matching list Type

This section contains **4 multiple choice questions. Each question has matching lists.** Each question has two lists List I and List II. The entries in List I are marked P, Q, R and S while those in List II are numbered from 1 to 4. Each entry in List I matches with **one** entry in List II. There are **three points** if all the matchings are correctly made, **zero point** if no matchings are made and **minus one** point in all other cases.

Q.57 A line $y = mx + 3$ meets the y-axis at $E(0, 3)$ and the arc of the parabola $y^2 = 16x, 0 \leq y \leq 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$ The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum. Match the entries in List I with those in List II.

List I	**List II**
P. $m =$	1. $\frac{1}{2}$
Q. Maximum area of triangle EFG is	2. 4
R. $y_0 =$	3. 2
S. $y_1 =$	4. 1

Answer and Comments: (P,4), (Q,1), (R,2), (S,3). A straightforward approach would be to determine the vertices E, F, G and then the area of the triangle EFG as functions of m. E is given (redundantly) as $(0, 3)$. For finding F we have to solve the equations

$$\begin{aligned} y_0^2 &= 16x_0 && (1) \\ \text{and} \quad y_0 &= mx_0 + 3 && (2) \end{aligned}$$

simultaneously. There will be two solutions and we have to choose the one for which $0 \leq y_0 \leq 6$. But that would be complicated because we do not know m. So as in Q.55 and 56, we resort to the parametric equations of the parabola $y^2 = 16x$. So, let

$$F = (x_0, y_0) = P(t_0) = (4t_0^2, 8t_0) \tag{3}$$

where t_0 is some real number. The stipulation $0 \leq y_0 \leq 6$ is equivalent to

$$0 \leq t_0 \leq \frac{3}{4} \tag{4}$$

We can determine m in terms of t_0 using the fact that it is the slope of the line EF which gives

$$m = \frac{8t_0 - 3}{4t_0^2} \tag{5}$$

Since m can be written in terms of t_0, we can as well work out the problem in terms of the variable t_0. We already know E and F in terms of t_0. To find G, we note that the equation of the tangent to the parabola at the point $F(= P(t_0))$ is

$$y = 8t_0 + \frac{1}{t_0}(x - 4t_0^2) \tag{6}$$

This line is given to meet the y-axis at G. So,

$$G = (0, 4t_0) = (0, y_1) \tag{7}$$

In the triangle EFG clearly $EG = |4t_0 - 3| = 3 - 4t_0$ (by (4)) and the perpendicular distance of F from EG is simply its x-coordinate, i.e. $4t_0^2$. Hence the area, say A of the triangle EFG is given by

$$A = \frac{1}{2} \times (3 - 4t_0)4t_0^2 = 6t_0^2 - 8t_0^3 \tag{8}$$

We are given that t_0 is chosen so as to maximise A. So we regard A as a function of t_0 and maximise it for $0 \leq t_0 \leq \frac{3}{4}$. A straightforward calculation gives

$$\frac{dA}{dt_0} = 12t_0 - 24t_0^2 \tag{9}$$

which vanishes at $t_0 = 0$ and $t_0 = 1/2$. Further $\frac{dA}{dt_0}$ is positive for $0 < t_0 < 1/2$ and negative for $t_0 > 1/2$. Hence A ia maximum at $t_0 = 1/2$.

Now that t_0 is known, everything else can be determined. Thus from (5),

$$m = \frac{4-3}{1} = 1 \tag{10}$$

Similarly, from (3) $y_0 = 8t_0 = 4$ while from (7), $y_1 = 4t_0 = 2$. It only remains to determine the area of the triangle EFG for $t_0 = 1/2$. A direct substitution in (8) gives this as $\frac{6}{4} - \frac{8}{8} = \frac{1}{2}$.

The problem is simple once it strikes that instead of the given unknown m it will be easier to work with the parameter used in the parametric equations of a parabola. Exactly the same idea was the key to Q.55 and Q.56. This duplication (or, rather, triplication) should have been avoided. Perhaps like the complex roots of unity (which appeared in three problems) the paper-setters are obsessed with the parametric equations of a parabola!

Q.58 Match the entries in List I with those in List II.

List I	List II
P. $\left(\dfrac{1}{y^2}\left(\dfrac{\cos(\tan^{-1} y) + y\sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)}\right)^2 + y^4\right)^{1/2}$ takes value	1. $\dfrac{1}{2}\sqrt{\dfrac{5}{3}}$
Q. If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$, then possible value of $\cos\left(\dfrac{x-y}{2}\right)$ is	2. $\sqrt{2}$
R. If $\cos\left(\dfrac{\pi}{4} - x\right)\cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\dfrac{\pi}{4} + x\right)\cos 2x$, then the possible value of $\sec x$ is	3. $\dfrac{1}{2}$
S. If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}(x\sqrt{6})\right), x \neq 0$, then possible value of x is	4. 1

Answer and Comments: (P,4), (Q,3), (R,2), (R,4), (S,3). Unlike in the last question where all four items were based on a common theme, in the present problem, each item is independent of the others. They have a superficial commonality, viz. they all deal with trigonometric and inverse trigonometric functions. But each one has to be tackled separately.

Let us begin with Item P. Let us write E for the expression and x for the fraction inside the parentheses. Then we have

$$E^2 = \frac{x^2}{y^2} + y^4 \tag{1}$$

To evaluate x, we evaluate its numerator, say u, and denominator, say v, separately. For u we call $\tan^{-1} y$ as θ. Then

$$\begin{aligned} u &= \cos\theta + \tan\theta\sin\theta \\ &= \cos\theta + \frac{\sin^2\theta}{\cos\theta} \\ &= \sec\theta = \sqrt{1+y^2} \end{aligned} \tag{2}$$

Similarly, for v, we write θ for $\sin^{-1} y$. (No harm in doing so because the earlier θ will not figure again. Otherwise, to play it safe, call $\sin^{-1} y$ as ϕ.) Then

$$\begin{aligned} v &= \cot\theta + \tan\theta \\ &= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \\ &= \frac{1}{\sin\theta\cos\theta} = \frac{1}{y\sqrt{1-y^2}} \end{aligned} \tag{3}$$

Substituting into (1),

$$\begin{aligned} E^2 &= \frac{u^2}{y^2v^2} + y^4 \\ &= (1+y^2)(1-y^2) + y^4 = 1 \end{aligned} \tag{4}$$

Hence E takes the value 1, as the other possible value -1 is not listed.

The condition in Item Q can be expressed in terms of complex numbers. Let $\alpha = \cos x + i\sin x = e^{ix}, \beta = \cos y + i \sin y = e^{iy}$ and $\gamma = \cos z + i \sin z = e^{iz}$. Then α, β, γ are complex numbers lying on the unit circle and the condition given means that the centroid of the triangle with vertices at α, β, γ is at the origin and hence coincides with its circumcentre. In such a case it can be shown easily that the triangle must be equilateral. But that is not necessary here. The question asks for *one possible value* of the expression $\cos\left(\frac{x-y}{2}\right)$. So, we are free to take x, y, z to be any three angles which represent the arguments of the vertices of an equilateral triangle with the unit circle as its circumcircle. For example, we can take $x = 0, y = \frac{2\pi}{3}$ and $z = \frac{4\pi}{3}$. Then $\frac{x-y}{2} = -\frac{\pi}{3}$ and its cosine is $\frac{1}{2}$. So this is one possible answer to Item Q.

For a simpler and a more direct solution, write $\cos z = -(\cos x + \cos y)$ and $\sin z = -(\sin x + \sin y)$. Squaring and adding we get $1 = 2 + 2\cos(x-y)$, which gives $\cos(x-y) = -1/2$. So we may take $x - y = 2\pi/3$ whence $\frac{x-y}{2} = \pi/3$ which also gives $1/2$ as the answer.

In Item R we are given that x satisfies a trigonometric equation and we are asked to find a possible value of $\sec x$. Let us first recast this equation as

$$\cos\left(\frac{\pi}{4} - x\right)\cos 2x - \cos\left(\frac{\pi}{4} + x\right)\cos 2x = \sin 2x \sec x(\cos x - \sin x) \tag{5}$$

The L.H.S. equals $\sqrt{2}\sin x \cos 2x$. The R.H.S., on the other hand, equals $2\sin x(\cos x - \sin x)$. Hence we have

$$\sqrt{2}\sin x \cos 2x = 2\sin x(\cos x - \sin x) \tag{6}$$

One possibility is $\sin x = 0$ which gives $\sec x = \pm 1$. 1 is listed as a possible value. In case $\sin x \neq 0$ we have,

$$\cos 2x = \sqrt{2}(\cos x - \sin x) \tag{7}$$

We factorise the L.H.S. as $(\cos x + \sin x)(\cos x - \sin x)$. As the second factor also appears on the R.H.S. we see that $\cos x = \sin x$ is a possibility which yields $\tan x = 1$ and hence $\sec x = \pm\sqrt{2}$ as possible values. Of these, $\sqrt{2}$ is listed in List - II.

Finally, if $\cos x - \sin x \neq 0$ we must have

$$\frac{1}{\sqrt{2}}(\sin x + \cos x) = 1 \tag{8}$$

which can be rewritten as $\sin(\frac{\pi}{4} + x) = 1$ which yields $x = \frac{\pi}{4}$ as yet another solution. But that is already included in $\sin x = \cos x$. So we get no new possible values for $\sec x$. Summing up, if x satisfies the equation given in Item R, then the possible values of $\sec x$ are ± 1 and $\pm\sqrt{2}$.

Finally, let us come to Item S. Let us call $\sin^{-1}\sqrt{1-x^2}$ as θ. Then $\cot\theta = \frac{\cos\theta}{\sin\theta} = \pm\frac{x}{\sqrt{1-x^2}}$. This is the L.H.S. of the given equation. As for the R.H.S. call $\tan^{-1}(x\sqrt{6})$ as ϕ. Then $\sin\phi = \frac{\tan\phi}{\sec\phi} = \pm\frac{x\sqrt{6}}{\sqrt{1+6x^2}}$. Hence the given equation becomes

$$\frac{x}{\sqrt{1-x^2}} = \pm\frac{x\sqrt{6}}{\sqrt{1+6x^2}} \tag{9}$$

It is given that $x \neq 0$. So, canceling it and squaring both the sides we get

$$1 + 6x^2 = 6(1 - x^2) \tag{10}$$

which gives $12x^2$ as 5 and hence x as $\pm\frac{1}{2}\sqrt{\frac{5}{3}}$. The positive value appears in List II.

An extremely long and laborious problem. Although the expressions reduce to something very simple, the reduction takes considerable work and makes the problem highly error prone. And because of the marking scheme, even a single mistake will earn a candidate a negative credit even though he has done the other three parts correctly. What makes the present problem even worse is that for Item R, there are two possible matching values in List II. (The key published by the IITs gives only $\sqrt{2}$ as a correct answer.) In the past, it was common for some entries in each List to have more than one matches in the other. In the present year, to make machine evaluation easier, the paper-setters have prepared four possible codes for these questions (which we have not shown since their purpose is not academic but purely administrative). And in each code, a permutation of the entries in List II indicates the matching. In each permutation, the entries appearing under the Items P, Q, R, S are all distinct. This gives the impression that every item in each list has exactly one correct answer. But this is nowhere stated explicitly in any of the instructions. If this intention of the paper-setters is assumed, then perhaps the rationale behind dropping 1 as a correct matching for Item R was that it is the only choice for Item P. So, if it is taken by Item R, then Item P would be matchless.

In any case, even if this confusion were not there, the time the problem demands for an honest solution is far in excess of what is justified by the maximum credit allotted to the question (which is a measly figure of 3 marks in a paper of 180 marks to be attempted in three hours.) Item R alone would take no less than three or four minutes if all possibilities are carefully considered.

Q.59 Consider the lines $L_1 : \dfrac{x-1}{2} = \dfrac{y}{-1} = \dfrac{z+3}{1}$, $L_2 : \dfrac{x-4}{1} = \dfrac{y+3}{1} = \dfrac{z+3}{2}$ and the planes $P_1 : 7x+y+2z = 3, P_2 : 3x+5y-6z = 4$. Let $ax+by+cz = d$ be the equation of the plane passing through the point of intersection of the lines L_1 and L_2 and perpendicular to the planes P_1 and P_2.

List I		List II	
P.	$a =$	1.	13
Q.	$b =$	2.	-3
R.	$c =$	3.	1
S.	$d =$	4.	-2

Answer and Comments: (P,3), (Q,2), (R,4), (S,1). A very straighforward problem. We first determine the point, say C, where L_1 and L_2 intersect. For this we consider their parametric equations and equating the corresponding coordinates, get

$$\begin{aligned} 1+2t &= 4+s && (1)\\ -t &= -3+s && (2)\\ \text{and} \quad -3+t &= -3+2s && (3) \end{aligned}$$

for some values of s and t. The last two equations give $s = 1$ and $t = 2$ which is consistent with (1). So the lines L_1 and L_2 do intersect and their common point is

$$C : (5, -2, -1) \tag{4}$$

The vectors perpendicular to the planes P_1 and P_2 are, respectively,

$$7\vec{i} + \vec{j} + 2\vec{k} \quad \text{and} \quad 3\vec{i} + 5\vec{j} - 6\vec{k} \tag{5}$$

The cross product, say $\mathbf{v}$, of these two vectors will be perpendicular to P since the desired plane P is to be perpendicular to both P_1 and P_2. Hence

$$\mathbf{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 1 & 2 \\ 3 & 5 & -6 \end{vmatrix} = -16\vec{i} + 48\vec{j} + 32\vec{k} \tag{6}$$

Dividing by 16 we might as well take $\mathbf{w} = -\vec{i} + 3\vec{j} + 2\vec{k}$ as a vector perpendicular to P. As P passes through the point $C : (5, -2, -1)$ an equation of P is

$$-(x-5) + 3(y+2) + 2(z+1) = 0 \tag{7}$$

i.e.

$$-x + 3y = 2z = -13 \tag{8}$$

This means a, b, c, d are proportional to $-1, 3, 2$ and -13 respectively. The data does not determine them uniquely. But we see that $1, -3, -2$ and 13 are proportional and match all entries in List II.

An extremely strightforward problem. Maybe this was designed as an antidote to the killer in the last question!

Q.60 Match the entries in List I with those in List II.

	List I		**List II**
P.	Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelepiped determined by vectors $2(\vec{a} \times \vec{b}), 3(\vec{b} \times \vec{c})$ and $(\vec{c} \times \vec{a})$ is	1.	100
Q.	Volume of parallelepiped determined by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelepiped determined by vectors $3(\vec{a} + \vec{b}), (\vec{b} + \vec{c})$ and $2(\vec{c} + \vec{a})$ is	2.	30
R.	Area of a triangle with adjacent sides determined by the vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by the vectors $(2\vec{a} + 3\vec{b})$ and $(\vec{a} - \vec{b})$ is	3.	24
S.	Area of a paralleogram with adjacent sides determined by the vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a} + \vec{b})$ and $\vec{a}$ is	4.	60

Answer and Comments: (P,3), (Q,4), (R,1), (S,2). The four items are independent but all have a common idea. The last two deal with areas of plane figures and hence are slightly simpler than the first two items which deal with volumes of parallelepipeds. So, let us begin with Item S. The area of a parallelogram is simply the length of the cross product of the vectors which represent its adjacent sides. So, in Item S we are given that

$|\vec{a} \times \vec{b}| = 30$ and are asked to find $|(\vec{a} + \vec{b}) \times \vec{a}|$. This can be done most easily using the distributive law for the cross product. Thus

$$|(\vec{a} + \vec{b}) \times \vec{a}| = |\vec{a} \times \vec{a} + \vec{b} \times \vec{a}| \tag{1}$$

Using the anti-commutativity of the cross product (which, in particular implies that $\vec{a} \times \vec{a} = \vec{0}$), we get

$$|(\vec{a} + \vec{b}) \times \vec{a}| = | - \vec{a} \times \vec{b}| = 30 \tag{2}$$

So, S matches with the second entry in List - II.

The calculations in Item R are similar except that the area of the triangle is half that of the parallelogram. So, here we are given $|\vec{a} \times \vec{b}| = 40$. Using the same properties as those used above we have

$$\begin{aligned} (2\vec{a} + 3\vec{b}) \times (\vec{a} - \vec{b}) &= -2(\vec{a} \times \vec{b}) - 3(\vec{a} \times \vec{b}) \\ &= -5(\vec{a} \times \vec{b}) \end{aligned} \tag{3}$$

Taking lengths and dividing by 2, the answer comes as 100 listed as the first item in List - II.

Items P and Q are handled similarly except that here we use the **scalar triple product** (also called the **box product** sometimes since it represents the volume of a (not necessarily rectangular) box) instead of the cross product. Naturally, we shall have to use the properties of the scalar triple product. The basic result is that the volume, say V, of a parallelepiped whoce adjacent sides are represented by the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is

$$V = |(\mathbf{u} \;\; \mathbf{v} \;\; \mathbf{w})| = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \tag{4}$$

If we resolve $\mathbf{u}, \mathbf{v}, \mathbf{w}$ along a right handed orthonormal basis (such as $\vec{i}, \vec{j}, \vec{k}$) then $(\mathbf{u} \;\; \mathbf{v} \;\; \mathbf{w})$ can be expressed as a 3×3 determinant whose rows are the coefficients of $\mathbf{u}, \mathbf{v}, \mathbf{w}$. As a result the basic properties of scalar triple products can be proved using properties of determinants. In particular they are linear in each argument and reverse their signs when any two vectors are interchanged and vanish if any two of the vectors are multiples of each other.

This information is sufficient to answe Item Q. By linearity in each argument, we can express $[(\vec{a} + \vec{b}) \;\; (\vec{b} + \vec{c}) \;\; (\vec{c} + \vec{a})]$ as a sum of eight scalar triple products of the form $[\vec{x} \;\; \vec{y} \;\; \vec{z}]$ where $\vec{x}$ is either $\vec{a}$ or $\vec{b}$, $\vec{y}$ is either $\vec{b}$ or $\vec{c}$ and $\vec{z}$ is either $\vec{c}$ or $\vec{a}$. Out of these eight terms we need to take only those where $\vec{x}, \vec{y}$ and $\vec{z}$ are all distinct. That leaves only two terms, viz. $[\vec{a} \;\; \vec{b} \;\; \vec{c}]$ and $[\vec{b} \;\; \vec{c} \;\; \vec{a}]$. By properties of determinants, these two are equal since they can be obtained from each other by a cyclic permutation. We can now find the volume W (say) of the parallelepiped with edges along

$3(\vec{a}+\vec{b}), \vec{b}+\vec{c}, (3\vec{c}+\vec{a})$.

$$\begin{aligned} W &= |3(\vec{a}+\vec{b}) \ \ (\vec{b}+\vec{c}) \ \ 2(\vec{c}+\vec{a})| \\ &= 6|[(\vec{a}+\vec{b}) \ \ (\vec{b}+\vec{c}) \ \ (\vec{c}+\vec{a})]| \\ &= 12|[\vec{a} \ \ \vec{b} \ \ \vec{c}]| \\ &= 60 \end{aligned} \tag{5}$$

since we are given that $|[\vec{a} \ \ \vec{b} \ \ \vec{c}]| = 5$.

Finally, we come to Item P which is a little more involved. The basic identity needed here is

$$[(\vec{a}\times\vec{b}) \ \ (\vec{b}\times\vec{c}) \ \ (\vec{c}\times\vec{a})] = [\vec{a} \ \ \vec{b} \ \ \vec{c}]^2 \tag{6}$$

This can be done by first using another identity to expand $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})$ (which is called the **vector triple product**) of the three vectors $\vec{b}\times\vec{c}$, $\vec{c}$ and $\vec{a}$. The identity says

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w} \tag{7}$$

and implies

$$\begin{aligned} (\vec{b}\times\vec{c})\times(\vec{c}\times a) &= [(\vec{b}\times\vec{c})\cdot\vec{a}]\vec{c} - [(\vec{b}\times\vec{c})\cdot\vec{c}]\vec{a} \\ &= [\vec{a} \ \ \vec{b} \ \ \vec{c}]\vec{c} - 0\vec{a} \end{aligned} \tag{8}$$

An immediate application of (6) gives that the desired volume in Item P is $2\times 3\times[\vec{a} \ \ \vec{b} \ \ \vec{c}]^2 = 24$. So, Item P matches with the third entery in List - II.

Even if you cannot think of (6), there are two clever ways to answer Item P. One is straight elimination. In all the codes given, distinct items in List - I match with distinct items in List - II. Since all entries in List - II other than 24 are already matched with the other items in List - I, we have to match P with 24. (This is how sometimes we overcome our ignorance in schools when asked to match entries. Suppose, for example, that we are given four countries and are asked to match them with their capitals which are also given. If three of the countries are, say, India, England and France, then we can correctly tell the capital of the fourth country even if it is some tinpot little country whose name we have never heard before!)

But let us not be sneaky in a non-mathematical way. Some mathematical sneaking can be done in Item P (and actually, all others). The nature of the question is such that the volume, say W, of the parallelepiped determined by the three vectors $2(\vec{a}\times\vec{b}), 3(\vec{b}\times\vec{c})$ and $(\vec{c}\times a)$ is of the form λV where V is the volume of the parallelepiped with edges $\vec{a}, \vec{b}$ and $\vec{c}$ and the multiplicative factor λ does not depend on the particular vectors $\vec{a}, \vec{b}, \vec{c}$. Therefore we would get the same answer if we choose these vectors as we like. Let us take $\vec{a} = \vec{i}, \vec{b} = \vec{j}$ and $\vec{c} = 2\vec{k}$. Then the volume V is 2. With this choice, the vectors $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$ and $\vec{c} \times \vec{a}$ come out to be simply, $\vec{k}, 2\vec{i}$ and $2\vec{j}$ respectively. Hence the volume of the parallelepiped with these as edges is $[\vec{k} \;\; 2\vec{i} \;\; 2\vec{j}] = 4$. So, if we multiply the first vector by 2 and the second by 3, the volume is $2 \times 3 \times 4 = 24$.

In fact, if this idea strikes then all the four parts of the question can be answered almost instantaneously. Again, perhaps the time saved is intended as a compensation for the extremely time consuming matchings in Q.58!

CONCLUDING REMARKS

Although JEE Advanced plays the same role as the JEE till last year, the major difference is the calibre of the candidates. In the earlier JEE anybody who is appearing for HSC exam or has cleared it and satisfies certain age limits could appear for JEE. Hardly 5% of the candidates were really good. About 10 to 15 % were mediocre but could get in because of coaching classes. The overwhelming majority was of the also ran type.

By contrast, in the present set-up the candidates were the toppers of the JEE Main. So, the percentage of good candidates could be presumed to be fairly high. It was expected that this fact would be reflected in the choice of the questions. So, there would be a few carefully chosen questions designed to test specific qualities and skills.

This expectation has not been met. In fact, the overall papers are hardly different from the those of the JEE in the past. Some of the questions are shockingly straightforward and their place in an advanced test is questionable. These include all three problems on calculating probability, the only question on differential equations and the only question on symmetric matrices. The computaional problems on coordinate geometry (Q. 49, 55 in Paper 1 and 46, 47, 59 in Paper 2) are also very straightforward. Some problems (e.g. Q. 41 and 58 in Papaer 1) are weird.

This is not to say that there are no good problems. We especially mention Q. 47, 51, 57 and 60 in Paper 1 and 42, 43, 45, 49, 53 and 60 in Paper 2. Q. 45 in Paper stands out as a rare example of a problem which punishes unscrupulous students.

There is considerable duplication. In Paper 2, there are three questions on roots of unity and three on parametric equations of a parabola. On the other hand some areas are totally eliminated. There are no questions testing continuity and differentiability. (And, of course, no questions on the integer part function.) There are no questions on inequalities. Q. 48 in Paper 1 does have inequality. But it suffers from multiplicity of answers and also redundancy of data. Binomial coefficients are paid only a lip service.

As it happens every year, there are many many questions where because of the Multiple Choice format, a candidte can score even without being able to justify the answer.

There is some talk going on that from the next year the JEE Advanced will revert to the old JEE where you had to justify your steps. Let us hope this happens and many of the evils will be cured.