EFFICIENT GENERATION OF IDEALS IN A DISCRETE HODGE ALGEBRA

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ABSTRACT. Let \( R \) be a commutative Noetherian ring and \( D \) be a discrete Hodge algebra over \( R \) of dimension \( d > \dim(R) \). Then we show that

(i) the top Euler class group \( E_d(D) \) of \( D \) is trivial.

(ii) if \( d > \dim(R) + 1 \), then \((d-1)\)-st Euler class group \( E_{d-1}(D) \) of \( D \) is trivial.

1. INTRODUCTION

Let \( R \) be a commutative Noetherian ring. An \( R \)-algebra \( D \) is called a discrete Hodge algebra over \( R \) if \( D = R[X_1, \ldots, X_n]/I \), where \( I \) is an ideal of \( R[X_1, \ldots, X_n] \) generated by monomials. Typical examples are \( R[X_1, \ldots, X_n] \), \( R[X,Y]/(XY) \) etc. In [V], Vorst studied the behaviour of projective modules over discrete Hodge algebras. He proved [V, Theorem 3.2] that every finitely generated projective \( D \)-module is extended from \( R \) if for all \( k \), every finitely generated projective \( R[X_1, \ldots, X_k] \)-module is extended from \( R \).

Later Mandal [M 2] and Wiemers [Wi] studied projective modules over discrete Hodge algebra \( D \). In [Wi], Wiemers proved the following significant result. Let \( P \) be a projective \( D \)-module of rank \( \geq \dim(R) + 1 \). Then (i) \( P \simeq Q \oplus D \) for some \( D \)-module \( Q \) and (ii) \( P \) is cancellative, i.e. \( P \oplus D \simeq P' \oplus D \) implies \( P \simeq P' \).

When \( D = R[X,Y]/(XY) \), above results of Wiemers are due to Bhatwadekar and Roy [B-R]. Very recent, inspired by results of Bhatwadekar and Roy, Das and Zinna [D-Z 3] studied the behaviour of ideals in \( R[X,Y]/(XY) \) and proved the following result on efficient generation of ideals. Assume \( \dim(R) \geq 1 \), \( D = R[X,Y]/(XY) \) and \( I \subset D \) is an ideal of height \( n = \dim(D) \). Assume \( I/I^2 \) is generated by \( n \) elements. Then any given set of \( n \) generators of \( I/I^2 \) can be lifted to a set of \( n \) generators of \( I \). In particular, the top Euler class group \( E_n(D) \) of \( D \) is trivial.

As \( R[X,Y]/(XY) \) is the simplest example of a discrete Hodge algebra over \( R \), motivated by above discussions, one can ask the following question.

**Question 1.1.** Let \( R \) be a commutative Noetherian ring of dimension \( \geq 1 \) and \( D \) be a discrete Hodge algebra over \( R \) of dimension \( n > \dim(R) \). Let \( I \subset D \) be an ideal of height \( n \). Suppose that \( I = (f_1, \ldots, f_n) + I^2 \). Do there exist \( g_1, \ldots, g_n \in I \) such that

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\( I = (g_1, \cdots, g_n) \) with \( f_i - g_i \in I^2 \)? In other words, Is the top Euler class group \( E^n(D) \) of \( D \) trivial? (For definition of Euler class groups, see \([B-RS 2]\) and \([B-RS 3]\).)

We answer Question 1.1 affirmatively and prove the following more general result ((3.1) below).

**Proposition 1.2.** Let \( R \) be a commutative Noetherian ring of dimension \( \geq 1 \) and \( D \) be a discrete Hodge algebra over \( R \) of dimension \( n > \text{dim}(R) \). Let \( P \) be a projective \( D \)-module of rank \( n \) which is extended from \( R \) and \( I \) be an ideal in \( D \) of height \( \geq 2 \). Suppose that there is a surjection \( \alpha : P/IP \to I/I^2 \). Then \( \alpha \) can be lifted to a surjection \( \beta : P \to I \). In particular, the \( n \)-th Euler class group \( E^n(D) \) of \( D \) is trivial.

The above result can be extended to any rank \( n \) projective \( D \)-module when \( R \) contains \( \mathbb{Q} \). Here is the precise statement.

**Theorem 1.3.** Let \( R \) be a commutative Noetherian ring containing \( \mathbb{Q} \) of dimension \( \geq 2 \) and \( D \) be a discrete Hodge algebra over \( R \) of dimension \( n > \text{dim}(R) \). Let \( I \) be an ideal in \( D \) of height \( \geq 3 \) and \( P \) be any rank \( n \) projective \( D \)-module. Suppose that there is a surjection \( \alpha : P/IP \to I/I^2 \). Then \( \alpha \) can be lifted to a surjection \( \beta : P \to I \).

After studying the top rank case, one is tempted to go one step further and inquire the following question.

**Question 1.4.** Let \( R \) be a commutative Noetherian ring of dimension \( \geq 3 \) and \( D \) be a discrete Hodge algebra over \( R \) of dimension \( d > \text{dim}(R) \). Let \( I \) be an ideal in \( D \) of height \( d - 1 \) and \( P \) be a projective \( D \)-module of rank \( d - 1 \). Suppose that \( \alpha : P/IP \to I/I^2 \) is a surjection. Can \( \alpha \) be lifted to a surjection \( \beta : P \to I \)?

We answer Question 1.4 affirmatively when \( R \) contains \( \mathbb{Q} \) (see (4.3) below) as follows.

**Theorem 1.5.** Let \( R \) be a commutative Noetherian ring containing \( \mathbb{Q} \) of dimension \( \geq 3 \) and \( D \) be a discrete Hodge algebra over \( R \) of dimension \( d > \text{dim}(R) \). Let \( I \) be an ideal in \( D \) of height \( \geq n \geq \max\{\text{dim}(R) + 1, d - 1\} \). Suppose that \( \alpha : P/IP \to I/I^2 \) is a surjection. Then there exists a surjection \( \beta : P \to I \) which lifts \( \alpha \). As a consequence, if \( d \geq \text{dim}(R) + 2 \), then \((d - 1)\)-st Euler class group \( E^{d-1}(D) \) of \( D \) is trivial.

Finally we derive an interesting consequence of above result as follows (see (4.6)).

**Theorem 1.6.** Let \( R \) be a commutative Noetherian ring containing \( \mathbb{Q} \) of dimension \( \geq 3 \) and \( D \) be a discrete Hodge algebra over \( R \) of dimension \( d > \text{dim}(R) \). Let \( I \) be a locally complete intersection ideal in \( D \) of height \( n \geq \max\{\text{dim}(R) + 1, d - 1\} \). Then \( I \) is set theoretically generated by \( n \) elements.

In Section 5, we give some partial answer to the following question.
**Lemma 2.5.** [B-RS 2, 2.13] Let $K$ and $I$ be two ideals of $R$. Suppose that $I = (f_1, \cdots, f_n) + I^2$, where $n \geq \dim(D/I) + 2$. Do there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$?

The above question has been settled in the affirmative by Mandal in [M 1] when $D$ is a polynomial algebra over $R$. Recently Fasel [Fa] has settled a conjecture of Murthy and proved the following result. Let $k$ be an infinite field of characteristic $\neq 2$ and $I \subset k[T_1, \cdots, T_m]$ be an ideal. Then we have $\mu(I) = \mu(I/I^2)$.

Therefore, we may ask the following natural question.

**Question 1.8.** Let $k$ be an infinite field of characteristic $\neq 2$ and $D$ be a discrete Hodge algebra over $k$. Let $I \subset D$ be an ideal. Is $\mu(I) = \mu(I/I^2)$?

2. **Preliminaries**

**Assumptions.** Throughout this paper, rings are assumed to be commutative Noetherian and projective modules are finitely generated and of constant rank. For a ring $A$, $\dim(A)$ will denote the Krull dimension of $A$.

We start with the following definition.

**Definition 2.1.** An $R$-algebra $D$ is said to be a **discrete Hodge algebra over $R$** if $D$ is isomorphic to $R[X_1, \cdots, X_n]/J$, where $J$ is an ideal of $R[X_1, \cdots, X_n]$ generated by monomials. A discrete Hodge algebra over $R$ is called **trivial** if it is a polynomial algebra over $R$. Otherwise, it is called a **non-trivial** discrete Hodge algebra.

**Definition 2.2.** We call an ideal $I$ of a ring $R$ to be efficiently generated if $\mu(I) = \mu(I/I^2)$, where $\mu(I)$ (resp. $\mu(I/I^2)$) stands for the minimal number of generators of $I$ (resp. $I/I^2$) as an $R$-module (resp. $R/I$-module).

**Definition 2.3.** Let $I$ be an ideal of a ring $R$. We say that $I$ is **set theoretically generated by $k$ elements** $f_1, \cdots, f_k$ in $R$ if $[f_1, \cdots, f_k] = \sqrt{I}$.

The next two results are standard. For proofs the reader may consult [B-RS 2].

**Lemma 2.4.** [B-RS 2, 2.11] Let $R$ be a ring and $J$ be an ideal of $R$. Let $K \subset J$ and $L \subset J^2$ be two ideals of $R$ such that $K + L = J$. Then $J = K + (e)$ for some $e \in L$ with $e(1-e) \in K$ and $K = J \cap J'$, where $J' + L = R$.

**Lemma 2.5.** [B-RS 2, 2.13] Let $A$ be a ring and $P$ be a projective $A$-module of rank $n$. Let $(\alpha, a) \in (P^* \oplus A)$. Then there exists an element $\beta \in P^*$ such that $\text{ht}(I_\alpha) \geq n$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq n$, then $\text{ht} I \geq n$. Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and $I$ is a proper ideal of $A$, then $\text{ht} I = n$.

The following lemma is proved in [D-K, Lemma 3.1].
**Lemma 2.6.** Let $R$ be a ring and $J \subset R$ be an ideal. Let $P$ be a projective $R$-module of rank $n \geq \dim(R/J) + 1$ and let $\alpha : P/JP \to J/J^2 f$ be a surjection for some $f \in R$. Given any ideal $K \subset R$ with $\dim(R/K) \leq n - 1$, the map $\alpha$ can be lifted to a surjection $\beta : P \to J'$ such that:

1. $J' + (J^2 \cap K)f = J$,
2. $J' = J \cap J'$ and $\text{ht}(J') \geq n$,
3. $(J^2 \cap K)f + J' = R$.

The following theorem is due to Mandal [M 3, Theorem 2.1].

**Theorem 2.7.** Let $R$ be a ring and $I \subset R[T]$ be an ideal containing a monic polynomial. Let $P$ be a projective $R$-module of rank $n \geq \dim(R[T]/I) + 2$. Suppose that there exists a surjection $\phi : P[T] \twoheadrightarrow I/(I^2 T)$. Then, there exists a surjection $\psi : P[T] \twoheadrightarrow I$ which lifts $\phi$.

We improve [D-Z 3, Lemma 2.9] in the following form to suit our needs. The proof is similar to the one given in [D, Lemma 4.9].

**Lemma 2.8.** Let $R$ be a ring and $I, J$ be two ideals in $R$ such that $J \subset I^2$. Let $P$ be a projective $R$-module and $K \subset R$ be an ideal. Suppose that we are given surjections $\alpha : P \to I/J$ and $\beta : P \to \overline{I}$ such that $\alpha \equiv \beta \mod J$, where bar denotes reduction modulo the ideal $K$. Then $\alpha$ can be lifted to surjection $\phi : P \to I/(JK)$.

The following result is implicit in the proof of [V, Theorem 3.2].

**Theorem 2.9.** Let $R$ be a ring and $r > 0$ be an integer. Assume that all projective modules of rank $r$ over polynomial extensions of $R$ are extended from $R$. Then all projective modules of rank $r$ over discrete Hodge $R$-algebras are extended from $R$.

The following result is due to Das and Zinna [D-Z 1, Theorem 3.12].

**Theorem 2.10.** Let $R$ be a ring of dimension $n \geq 2$. Let $R \hookrightarrow S$ be a subintegral extension and $L$ be a projective $R$-module of rank one. Then, the natural map $E^n(R, L) \to E^n(S, L \otimes_R S)$ is an isomorphism.

The following result follows from [Sw, Lemma 3.2].

**Lemma 2.11.** Let $R \hookrightarrow S$ be a subintegral extension and $\mathcal{J} \subset R[X_1, \cdots, X_m]$ be an ideal generated by monomials. Then $R[X_1, \cdots, X_m]/\mathcal{J} \to S[X_1, \cdots, X_m]/\mathcal{J}$ is also subintegral.

The following result is from [D-Z 2, Proposition 2.13] for $d \geq 2$. By patching argument, it can be proved for $d = 1$. 
Proposition 2.12. Let $A$ be a ring of dimension $d \geq 1$. Let $I$ be an ideal of $A[T]$ of height $\geq 2$ and $P$ be a projective $A[T]$-module of rank $n \geq d + 1$. Suppose that there exists a surjection $\phi : P/IP \twoheadrightarrow I/I^2$. Then $\phi$ can be lifted to a surjection $\Psi : P \twoheadrightarrow I$.

The following result is due to Wiemers [Wi, Corollary 4.3].

Theorem 2.13. Let $R$ be a ring of dimension $d$ and $D$ be a discrete Hodge algebra over $R$. Let $P$ be a projective $D$-module of rank $> d$. Then

1. $P = D \oplus Q$ for some projective $D$-module $Q$.
2. $P$ is cancellative, i.e. if $P \oplus D \xrightarrow{\sim} P' \oplus D$, then $P \xrightarrow{\sim} P'$.

It is not hard to see that, adapting the same proof of [D-RS, Theorem 4.2], we can extend [D-RS, Theorem 4.2] in the following form.

Theorem 2.14. Let $R$ be a ring containing $\mathbb{Q}$ with $\dim(R) = n \geq 3$ and $I \subseteq R[T]$ be an ideal of height $\geq 3$. Let $L$ be a projective $R$-module of rank 1 and $P$ be a projective $R[T]$-module of rank $n$ whose determinant is $L[T]$. Assume that we are given a surjection $\psi : P \twoheadrightarrow I/(I^2T)$. Assume further that $\psi \otimes R(T)$ can be lifted to a surjection $\psi' : P \otimes R(T) \twoheadrightarrow I R(T)$. Then, there exists a surjection $\Psi : P \twoheadrightarrow I$ such that $\Psi$ is a lift of $\psi$.

3. Main Theorems: Codimension Zero Case

We begin with the following result which is motivated by [D-Z 3, Theorem 4.2].

Proposition 3.1. Let $R$ be a ring of dimension $d \geq 1$ and $D$ be a discrete Hodge algebra over $R$ of dimension $n > \dim(R)$. Let $P$ be a projective $D$-module of rank $n$ which is extended from $R$ and $I$ be an ideal in $D$ of height $\geq 2$. Suppose that there is a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. Then $\alpha$ can be lifted to a surjection $\beta : P \twoheadrightarrow I$.

Proof. If $D$ is a trivial discrete Hodge algebra over $R$, then we are done by (2.12). So we assume that $R$ is a non-trivial discrete Hodge algebra over $R$. Let ‘prime’ denote reduction modulo the nil radical $N$ of $D$. Assume $\alpha \otimes D'$ can be lifted to a surjection $\alpha_1 : P \otimes D' \twoheadrightarrow I \otimes D'$. Then $\alpha_1$ can be lifted to a surjection $\alpha_2 : P_{1+N} \twoheadrightarrow I_{1+N}$. Since $1 + N$ consists of units of $D$, $\alpha_2$ is a lift of $\alpha$. Therefore, we may assume that $D$ is reduced.

Let $D = R[X_1,\cdots,X_m]/J$, where $J$ is an ideal of $R[X_1,\cdots,X_m]$ generated by square-free monomials. We prove the result using induction on the number of variables $m$. If $m = 1$, then $D$ is just $R[X_1]$ and the result follows from (2.12).

Let us assume that $m \geq 2$. We can assume that $J = K + X_m L$, where $K$ and $L$ are monomial ideals of $R[X_1,\cdots,X_{m-1}]$. Then $D = R[X_1,\cdots,X_m]/(K,X_m L)$.

Case 1. $n \geq 3$. Given $\alpha : P/IP \twoheadrightarrow I/I^2$, applying (2.6), $\alpha$ can be lifted to a surjection $\gamma_1 : P \twoheadrightarrow I'$ such that (1) $I' = I \cap J$, (2) $I + J = D$, (3) $\text{ht}(J) \geq n$. 


follows that assume that \( J \in D \) and we are done. So assume \( \operatorname{ht}(J) = n \). Let \( \gamma : P \to J/J^2 \) be the surjection induced from \( \gamma_1 \).

Let \( x_m \) and \( L \) be the images of \( X_m \) and \( \mathcal{L} \) in \( D \), respectively. We shall use ‘tilde’ when we move modulo \( (x_m) \) and ‘bar’ when we move modulo \( L \). We first go modulo \( x_m \) and consider the surjection \( \tilde{\gamma} : \tilde{P} \to \tilde{J}/\tilde{J}^2 \). Note that \( \tilde{J} \) is an ideal of \( \tilde{D} = R[X_1, \ldots, X_{m-1}]/K \) of height equal to dimension of \( \tilde{D} \). For this, we observe that

\[
\dim(\tilde{D}[X_m]) + \operatorname{ht}(\tilde{X}_m\mathcal{L}) = \dim(D),
\]

where \( \tilde{X}_m\mathcal{L} \) is the image of \( X_m\mathcal{L} \) in \( \tilde{D}[X_m] \).

By induction hypothesis on \( m \), there exists a surjection \( \phi : \tilde{P} \to \tilde{J} \) which is a lift of \( \tilde{\gamma} \). Therefore, it follows from (2.8) that \( \gamma \) can be lifted to a surjection \( \psi : P \to J/(J^2x_m) \).

We now move to the ring \( \mathcal{D} = \frac{R[X_1, \ldots, X_{m-1}]}{(K, \mathcal{L})}[X_m] \) (i.e., go modulo \( L \)) and consider the surjection

\[
\tilde{\psi} : \tilde{P} \to \tilde{J}/(\tilde{J}^2X_m)
\]

Now observe that \( J \) is of the form \( J'/X_m\mathcal{L} \) for some ideal \( J' \) in \( \frac{R[X_1, \ldots, X_{m-1}]}{K}[X_m] \) containing \( X_m\mathcal{L} \). Observe that \( \operatorname{ht}(J') = \dim(\frac{R[X_1, \ldots, X_{m-1}]}{K}[X_m]) \). Therefore we may assume that \( J' \) contains a monic polynomial in \( X_m \). Since \( \mathcal{J} = J/L \cap J = J'/L \cap J' \), it follows that \( \mathcal{J} \) contains a monic in \( X_m \). Also \( n \geq \dim(\mathcal{D}/\mathcal{J}) + 2(= 2) \). By (2.7), there exists a surjection \( \theta : \tilde{P} \to \tilde{J} \) which lifts \( \tilde{\psi} \).

Therefore, it follows from (2.8) that there exists a surjection \( \delta : P \to J/(J^2x_mL) \) which is a lift of \( \psi \). As \( x_mL = 0 \) in \( D \), we obtain \( \delta : P \to J \) is a surjection which lifts \( \gamma \). Now we have

1. \( \gamma_1 : P \to I \cap J \) such that \( \gamma_1 \otimes D/I = \alpha \otimes D/I \),
2. \( \delta : P \to J \) with \( \delta \otimes D/J = \gamma_1 \otimes D/J = \gamma \).

Now by (2.13), \( P = D \oplus P' \). Also it follows that \( n \geq \dim(D/I) + 2 \) and \( n + \operatorname{ht}(J) \geq \dim(D) + 3 \). We can now use the subtraction principle [D-K, Proposition 3.2] to find a surjection \( \beta : P \to I \) which lifts \( \alpha \). This completes the proof in case \( n \geq 3 \).

**Case 2.** \( n = 2 \). In this case \( \dim(R) = 1 \) and hence by (2.13), \( P \simeq L \oplus D \) for some rank one projective \( D \)-module \( L \).

We have \( I = \alpha(P) + I^2 \). Applying (2.4), we can find \( f \in I \) such that \( I = (\alpha(P), f) \) with \( f(1 - f) \in \alpha(P) \) and therefore we have a surjection \( \alpha_{1-f} : P_{1-f} \to I_{1-f} \). Let \( \pi : P_f = L_f \oplus D_f \to D_f = I_f \) be the projection onto the second factor. Now consider the following surjections:

\[
\alpha_f(1-f) : P_f(1-f) \to I_f(1-f) = D_f(1-f)
\]

\[
\pi_{1-f} : P_f(1-f) \to I_f(1-f) = D_f(1-f)
\]
Now it is not hard to show that there exists \( \tau \in SL(P_{(1-f)}) \) such that \( \alpha_{(1-f)}\tau = \pi_{1-f} \). Therefore standard patching argument implies that there is a projective \( D \)-module \( Q \) of rank 2 such that \( Q \) maps onto \( I \). By (2.13), \( Q = \wedge^2(Q) \oplus D \). Also note that \( Q \) has determinant \( L \) and hence \( Q \simeq L \oplus D \).

By (2.13), \( L \oplus D \) is cancellative. We can now apply [B, Lemma 3.2] to find a surjection \( \beta : P \rightarrow I \) which lifts \( \alpha \).

**Corollary 3.2.** Let \( R \) be a ring of dimension \( \geq 1 \) and \( D \) be a discrete Hodge algebra over \( R \) of dimension \( n > \dim(R) \). Let \( I \) be an ideal in \( D \) of height \( \geq 2 \). Suppose that \( I = (f_1, \cdots, f_n) + I^2 \). Then there exist \( g_1, \cdots, g_n \) such that \( I = (g_1, \cdots, g_n) \) with \( f_i - g_i \in I^2 \) for \( i = 1, \cdots, n \).

**Corollary 3.3.** Let \( R \) be a ring of dimension \( \geq 1 \) and \( D \) be a discrete Hodge algebra over \( R \) of dimension \( n > \dim(R) \). Let \( L \) be any rank one projective \( D \)-module. Then the \( n \)-th Euler class group \( E^n(D, L) \) is trivial.

Proof. Let \( D = R[X_1, \cdots, X_m]/J \). Without loss of generality we can assume that \( D \) is reduced (see [B-RS 2, Corollary 4.6]). In particular, \( R \) is reduced. Let \( S \) be the seminormalization of \( R \) in its total quotient ring. Since \( S \) is seminormal, by [Sw, Theorem 6.1], every rank one projective \( S[X_1, \cdots, X_k] \)-module is extended from \( S \) for all \( k \). Therefore, it follows from (2.9) that \( L \otimes_R S \) is extended from \( S \).

Let us denote \( S[X_1, \cdots, X_m]/J \) by \( D_1 \). Since \( R \hookrightarrow S \) is a subintegral extension, by (2.11), \( D \hookrightarrow D_1 \) is also subintegral. As \( L \otimes_R S \) is extended from \( S \), by (3.1), it follows that \( E^n(D_1, L \otimes_R S) \) is trivial. Finally, using (2.10), we have \( E^n(D, L) \) is trivial. \( \square \)

The following result is due to Katz [Ka].

**Theorem 3.4.** Let \( R \) be a ring and \( I \subset R \) be an ideal. Let \( d \) be the maximum of the heights of maximal ideals containing \( I \), and suppose that \( d < \infty \). Then some power of \( I \) admits a reduction \( J \) satisfying \( \mu(J/J^2) \leq d \).

A result of Mandal from [M 2], can now be deduced.

**Corollary 3.5.** Let \( R \) be a ring of dimension \( \geq 1 \) and \( D \) be a discrete Hodge algebra over \( R \) of dimension \( n > \dim(R) \). Let \( I \subset D \) be an ideal of height \( \geq 2 \). Then \( I \) is set theoretically generated by \( n \) elements.

Proof. Using Katz (3.4), there exists \( k > 0 \) such that \( I^k \) has a reduction \( J \) with \( \mu(J/J^2) \leq n \). If \( \mu(J/J^2) \leq n - 1 \), then clearly \( J \) is generated by at most \( n \) elements. Therefore we assume that \( \mu(J/J^2) = n \). Since \( J \) is a reduction of \( I^k \), it is easy to see that \( \sqrt{J} = \sqrt{I} = J \) and \( \text{ht}(I) = \text{ht}(J) \). Applying (3.2), we see that \( J \) is generated by \( n \) elements. Therefore, \( I \) is set-theoretically generated by \( n \) elements. \( \square \)
We have the following variant of (3.1) for rings containing $\mathbb{Q}$.

**Proposition 3.6.** Let $R$ be a ring containing $\mathbb{Q}$ of dimension $\geq 2$ and $D$ be a discrete Hodge algebra over $R$ of dimension $n > \dim(R)$. Let $I$ be an ideal in $D$ of height $\geq 3$ and $P$ be any rank $n$ projective $D$-module whose determinant is extended from $R$. Suppose that there is a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. Then $\alpha$ can be lifted to a surjection $\beta : P \twoheadrightarrow I$.

Proof. We follow the proof of (3.1). The only thing which we need to show is that $\bar{\psi} : \bar{P} \twoheadrightarrow \bar{J}/(\bar{J}^2X_m)$ can be lifted to a surjection $\theta : \bar{P} \twoheadrightarrow \bar{J}$. Rest of the proof is same. To show this, we use (2.14) in place of (2.7). By (2.14), it is enough to show that $\bar{\psi} \otimes R(X_m)$ can be lifted to a surjection from $\bar{P} \otimes R(X_m) \twoheadrightarrow \bar{J} \otimes R(X_m)$. This is clearly true, since $\bar{J}$ contains a monic polynomial in $X_m$ and $\bar{P} = \bar{D} \oplus P'$ by (2.13). □

The following lemma is very crucial to generalize above result.

**Lemma 3.7.** Let $R$ be a reduced ring and $D$ be a discrete Hodge algebra over $R$. Let $L$ be a rank one projective $D$-module. Then there exists a ring $S$ such that

1. $R \hookrightarrow S \hookrightarrow Q(R)$,
2. $S$ is a finite $R$-module,
3. $R \hookrightarrow S$ is subintegral and
4. $L \otimes_R S$ is extended from $S$.

Proof. Let $R \hookrightarrow B \hookrightarrow Q(R)$ be the seminormalization of $R$. By Swan’s result [Sw, Theorem 6.1], rank one projective modules over polynomial extensions of $B$ are extended from $B$. Hence by (2.9), rank one projective modules over discrete Hodge algebras over $B$ are extended from $B$. In particular $L \otimes_R B$ is extended from $B$. By [Sw, Theorem 2.8], $B$ is direct limit of $B_\lambda$, where $R \hookrightarrow B_\lambda$ is finite and subintegral extension. Since $L$ is finitely generated, we can find a subring $S = B_\lambda$ for some $\lambda$ satisfying conditions (1–4). □

We now prove the general case of (3.6).

**Theorem 3.8.** Let $R$ be a ring containing $\mathbb{Q}$ of dimension $\geq 2$ and $D$ be a discrete Hodge algebra over $R$ of dimension $n > \dim(R)$. Let $I$ be an ideal in $D$ of height $\geq 3$ and $P$ be any rank $n$ projective $D$-module. Suppose that there is a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. Then $\alpha$ can be lifted to a surjection $\beta : P \twoheadrightarrow I$.

Proof. Without loss of generality, we may assume that $D$ is reduced. In particular, $R$ is reduced. Let $D = R[X_1, \cdots, X_m]/I$, where $\mathcal{J}$ is an ideal of $R[X_1, \cdots, X_m]$ generated by square free monomials. By (3.7), there exists an extension $R \hookrightarrow S$ such that

1. $R \hookrightarrow S \hookrightarrow Q(R)$,
2. $S$ is a finite $R$-module,
(3) $R \rightarrow S$ is subintegral and
(4) $\wedge^n(P) \otimes_R S$ is extended from $S$.

Let $E = S[X_1, \ldots, X_m]/J$. Since $\wedge^n(P) \otimes_R S$ is extended from $S$, by (3.6), the induced surjection $\alpha^*: P \otimes E \rightarrow IE/I^2E$ can be lifted to a surjection $\phi: P \otimes E \rightarrow IE$. By (2.13), $P = D \oplus Q$. In case $P = \wedge^n(P) \oplus D^{n-1}$, the rest of the proof is given in [D-Z 1, Theorem 3.12]. The proof of [D-Z 1, Theorem 3.12] works for $P = D \oplus Q$ also. Hence we are done.

4. Main Theorems: Codimension One Case:

The aim of this section is to give an affirmative answer to Question 1.4 mentioned in the introduction. We start with the following lemma which generalizes (2.12).

**Lemma 4.1.** Let $R$ be a ring containing $\mathbb{Q}$ of dimension $\geq 2$ and $I$ be an ideal of $R[X,Y]$ of height $\geq 3$. Let $P$ be a projective $R[X,Y]$-module of rank $\geq \dim(R) + 1$ whose determinant is extended from $R[X]$. Suppose that there exists a surjection $\phi: P \rightarrow I/I^2$. Then $\phi$ can be lifted to a surjection $\bar{\phi}: P \rightarrow I$.

Proof. If rank of $P$ is $\geq \dim(R) + 1$, then we are done by (2.12). So assume rank of $P = \dim(R) + 1$. Since $R$ contains $\mathbb{Q}$, using [B-RS 1, Lemma 3.3] and replacing $Y$ by $Y - \lambda$ for some $\lambda \in \mathbb{Q}$, we can assume that either $I(0) = R[X]$ or $\text{ht}(I(0)) = \text{ht}(I)$. If $I(0) = R[X]$, then by (2.8), we can lift $\phi$ to a surjection $\alpha: P \rightarrow I/I^2(Y)$.

Now assume that $\text{ht}(I(0)) = \text{ht}(I) \geq 3$. Let “bar” denote the reduction modulo $Y$ and consider $\bar{\phi}: \bar{P} \rightarrow \bar{I}/\bar{I}^2$. By (2.12), there exists a surjection $\beta: \bar{P} \rightarrow \bar{I}$ which lifts $\bar{\phi}$. Therefore, again by (2.8), we can lift $\bar{\phi}$ to a surjection $\alpha: P \rightarrow (I/I^2Y)$. Therefore, in any case, we can lift $\phi$ to a surjection $\alpha: P \rightarrow I/I^2Y$.

Consider the surjection $\alpha \otimes R(Y): P \otimes R(Y) \rightarrow I \otimes R(Y)/I^2 \otimes R(Y)$. Since $\dim(R(Y)) = \dim(R)$, by (2.12), $\alpha \otimes R(Y)$ can be lifted to a surjection $\delta: P \otimes R(Y) \rightarrow I \otimes R(Y)$. Using (1.14), we get a surjection $\bar{\phi}: P \rightarrow I$ which lifts $\alpha$ and hence lifts $\phi$.

**Proposition 4.2.** Let $R$ be a ring containing $\mathbb{Q}$ of dimension $\geq 3$ and $D$ be a discrete Hodge algebra over $R$ of dimension $d > \dim(R)$. Let $I$ be an ideal in $D$ of height $\geq 4$ and $P$ be a projective $D$-module of rank $n \geq \max\{\dim(R) + 1, d - 1\}$ whose determinant is extended from $R$. Suppose that $\alpha: P \rightarrow I/I^2$ is a surjection. Then there exists a surjection $\beta: P \rightarrow I$ that lifts $\alpha$.

Proof. As in the proof of (3.1), we can assume that $R$ is reduced and $D = R[X_1, \ldots, X_m]/\mathcal{I}$, where $\mathcal{I}$ is an ideal of $R[X_1, \ldots, X_m]$ generated by square free monomials. where $X_{i_1}^{l_1} \cdots X_{i_k}^{l_k} \in \mathcal{I}$ and $l_i \geq 1$ We prove the result using induction on $m$. If $m = 1$, then $D = R[X_1]$ and the result follows from (2.12).
Let us assume that \( m \geq 2 \). If \( D \) is a polynomial ring over \( R \), then we are done by (4.1). Now suppose that \( D \) is a non-trivial discrete Hodge algebra. Then we can assume that \( \mathcal{I} = (\mathcal{K}, X_m \mathcal{L}) \), where \( \mathcal{K} \) and \( \mathcal{L} \) are monomial ideals in \( R[X_1, \ldots, X_{m-1}] \). Then \( D = R[X_1, \ldots, X_m]/(\mathcal{K}, X_m \mathcal{L}) \).

Let \( x_m \) and \( L \) be the images of \( X_m \) and \( \mathcal{L} \) in \( D \) respectively. We shall use “tilde” when we move modulo \( (x_m) \) and “bar” when we move modulo \( L \). We first go modulo \( (x_m) \), i.e. to the discrete Hodge algebra \( \tilde{D} = R[X_1, \ldots, X_{m-1}]/\mathcal{K} \) and consider the surjection \( \tilde{\alpha} : \tilde{P} \twoheadrightarrow I/\tilde{I}^2 \). Note that \( \tilde{I} \) is an ideal of \( \tilde{D} \) of height \( \geq \dim(\tilde{D}) - 1 \). By induction hypothesis on \( m \), there exists a surjection \( \theta : \tilde{P} \twoheadrightarrow \tilde{I} \) which is a lift of \( \tilde{\alpha} \). Therefore, using (2.8), we can lift \( \alpha \) to a surjection \( \psi : P \twoheadrightarrow I/(I^2 x_m) \).

We now move modulo \( L \), i.e. \( \bar{D} = R[X_1, \ldots, X_{m-1}]/X_m \) := \( D_0[X_m] \) and consider the surjection \( \bar{\psi} : \bar{P} \twoheadrightarrow \bar{I}/(\bar{I}^2 X_m) \).

Observe that \( h(\bar{I}) \geq \dim(R) \geq 3 \). If \( \dim(D_0) < n \), then by (2.12), \( \bar{\psi} \) can be lifted to a surjection \( \theta : \bar{P} \twoheadrightarrow \bar{I} \). So assume \( \dim(D_0) = n \). Since \( \dim(R(X_m)) = \dim(R) \) and \( \bar{D} \otimes R(X_m) = R[X_1, \ldots, X_{m-1}]_{(k, L)} \), by (3.6), the surjection \( \bar{\psi} \otimes R(X_m) : \bar{P} \otimes R(X_m) \twoheadrightarrow \bar{I} \otimes R(X_m) \) can be lifted to a surjection \( \eta : \bar{P} \otimes R(X_m) \twoheadrightarrow \bar{I} \otimes R(X_m) \). By (2.14), there exists a surjection \( \theta : \bar{P} \twoheadrightarrow \bar{I} \) which lifts \( \bar{\psi} \).

Finally it follows from (2.8) that there exists a surjection \( \beta : P \twoheadrightarrow I/(I^2 x_m L) \) which lifts \( \psi \). As \( x_m L = 0 \) in \( D \), we obtain a surjection \( \beta : P \twoheadrightarrow I \) which lifts \( \alpha \).

Now we will answer Question 1.4.

**Theorem 4.3.** Let \( R \) be a ring of dimension \( \geq 3 \) containing \( \mathbb{Q} \) and \( D \) be a discrete Hodge algebra over \( R \) of dimension \( d > \dim(R) \). Let \( I \) be an ideal in \( D \) of height \( \geq 4 \) and \( P \) be a projective \( D \)-module of rank \( n \geq \max\{\dim(R) + 1, d - 1\} \). Suppose that \( \alpha : P \twoheadrightarrow I/\bar{I}^2 \) is a surjection. Then there exists a surjection \( \beta : P \twoheadrightarrow I \) which lifts \( \alpha \).

Proof. Without loss of generality we may assume that \( D \) is reduced. Using (2.6), we can lift \( \alpha \) to a surjection \( \alpha' : P \twoheadrightarrow I \cap I_1 \) such that \( I + I_1 = D \) and \( \dim(I_1) \geq n \).

If \( h(I_1) > n \), then \( I_1 = D \) and hence \( \alpha' \) is the required surjective lift of \( \alpha \). Assume \( h(I_1) = n \). The map \( \alpha' \) induces a surjection \( \alpha_1 : P \twoheadrightarrow I_1/I_1^2 \). If we can show that \( \alpha_1 \) can be lifted to a surjection \( \Delta : P \twoheadrightarrow I_1 \), then by subtraction principle [D-K, Proposition 3.2], we can find a surjection \( \Delta_1 : P \twoheadrightarrow I \) which lifts \( \alpha \). Therefore it is enough to show that \( \alpha_1 \) has a surjective lift \( \Delta \). Now replacing \( I_1 \) by \( I \) and \( \alpha_1 \) by \( \alpha \), we assume that \( h(I) = n \).

By (3.7), there exists an extension \( R \hookrightarrow S \) such that

1. \( R \hookrightarrow S \hookrightarrow Q(R) \),
2. \( S \) is a finite \( R \)-module,
(3) \( R \twoheadrightarrow S \) is subintegral and
(4) \( \wedge^n(P) \otimes_R S \) is extended from \( S \).

Let \( C \) be the conductor ideal of \( R \) in \( S \). Then \( \text{ht}(C) \geq 1 \). Since \( \text{ht}(I) = n \geq \max\{\dim(R) + 1, d - 1\} \) and \( \text{ht}(C) \geq 1 \), it follows that \( \text{ht}(I^2 \cap C) \geq 1 \). Therefore, we can choose an element \( b \in I^2 \cap C \) such that \( \text{ht}(b) = 1 \). Let \( "\text{bar}" \) denote reduction modulo the ideal \( (b) \). Consider the surjection \( \pi : \overline{P} \twoheadrightarrow \overline{I}/\overline{I}^2 \) and note that \( \dim(\overline{R}) < \dim(R) \).

Now applying (3.8), we can find a surjection \( \gamma' : \overline{P} \twoheadrightarrow \overline{I} \) which lifts \( \pi \). Choose a lift \( \gamma : P \twoheadrightarrow I \) of \( \gamma' \). Since \( b \in I^2 \), \( \gamma \) is a lift of \( \alpha \) and hence \( (\gamma(P), b) = I \). Since \( \text{hh}(I) = n \) and \( b \in I^2 \), applying (2.5) and replacing \( \gamma \) by \( \gamma + b \delta \) for some \( \delta \in P^* \), we can assume that \( \text{ht}(\gamma(P)) = n \).

Applying (2.4), there exists an ideal \( I' \) of height \( \geq n \) such that \( I' + bD = D \) and \( \gamma(P) = I \cap I' \). If \( \text{ht}(I') > n \), then \( I' = D \) and hence \( \gamma \) is the required surjective lift of \( \alpha \). Assume that \( \text{ht}(I') = n \) and consider the surjection \( \theta : P \twoheadrightarrow I'/I'^2 \) induced from \( \gamma : P \twoheadrightarrow I \cap I' \).

Consider the surjection \( \theta \otimes_R S : P \otimes S \twoheadrightarrow I'/I'^2 \otimes S \). Since \( \wedge^n(P \otimes_R S) \) is extended from \( S \), by (4.2), \( \theta \otimes S \) can be lifted to a surjection \( \Theta : P \otimes S \twoheadrightarrow I' \otimes S \). Now we need to show that we get a surjection \( \eta : P \twoheadrightarrow I' \) which lifts \( \theta \). In the case of \( P = \wedge^n(P) \oplus D^{n-1} \), this is proved in [D-Z 2, Lemma 5.1]. Note that \( P = D \oplus P' \), by (2.13). The proof of [D-Z 2, Lemma 5.1] works in this case also, so we do not repeat it here. Therefore we have a surjection \( \eta : P \twoheadrightarrow I' \) which lifts \( \theta \). Applying subtraction principle [D-K, Proposition 3.2], we can find a surjection \( \beta : P \twoheadrightarrow I \) which lifts \( \alpha \). \( \square \)

The following result is immediate from (4.3).

**Corollary 4.4.** Let \( R \) be a ring of dimension \( d \geq 3 \) containing \( \mathbb{Q} \) and \( D = \frac{R[X_1, X_2, X_3]}{2} \) be a discrete Hodge algebra over \( R \). Let \( I \) be an ideal in \( D \) of height \( \geq 4 \) and \( P \) be a projective \( D \)-module of rank \( n \geq \dim(R) + 1 \). Suppose that \( \alpha : P \twoheadrightarrow I/I^2 \) be a surjection. Then there exists a surjection \( \beta : P \twoheadrightarrow I \) which lifts \( \alpha \).

The following theorem is due to Ferrand and Szpiro [Sz].

**Theorem 4.5.** Let \( R \) be a ring and \( I \subset R \) be a locally complete intersection ideal of height \( r \geq 2 \) and \( \dim(R/I) \leq 1 \). Then there is a locally complete intersection ideal \( J \subset R \) of height \( r \) such that

1. \( \sqrt{I} = \sqrt{J} \) and
2. \( J/J^2 \) is free \( R/J \)-module of rank \( r \).

As an application of (4.3), we improve a result of Mandal [M 2, Corollary 2.2], albeit with a stronger hypothesis on ideals.
Theorem 4.6. Let $R$ be a ring of dimension $\geq 3$ containing $\mathbb{Q}$ and $D$ be a discrete Hodge algebra over $R$ with $\dim(D) = d > \dim(R)$. Let $I$ be a locally complete intersection ideal in $D$ of height $n = \max\{\dim(R) + 1, d - 1\}$. Then there exist $f_1, \cdots, f_n \in I$ such that $\sqrt{I} = \sqrt{(f_1, \cdots, f_n)}$. In other words, $I$ is set theoretically generated by $n$ elements.

Proof. By (4.5), there is a locally complete intersection ideal $J$ such that $\sqrt{I} = \sqrt{J}$ and $J/J^2$ is a free $R/J$-module of rank $n$. Applying (4.3), we see that $J$ is generated by $n$ elements. Therefore, $I$ is set theoretically generated by $n$ elements. \qed

5. Some Auxiliary Results

After answering Question 1.1 and Question 1.4, it is natural to ask the following more general question.

Question 5.1. Let $R$ be a commutative Noetherian ring of dimension $\geq 1$ and $D$ be a discrete Hodge algebra over $R$ of dimension $> \dim(R)$. Let $I \subset D$ be an ideal of height $> \dim(R)$. Suppose that $I = (f_1, \cdots, f_n) + I^2$, where $n \geq \dim(D/I) + 2$. Do there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots , g_n)$ with $f_i - g_i \in I^2$?

The above question has been answered affirmatively by Mandal when $D$ is a polynomial algebra over $R$ ([M 1]). Using [D-RS, Theorem 4.2] and following the proofs of (3.1) and (4.2), we can obtain the following result which gives a partial answer to the above question.

Theorem 5.2. Let $R$ be a ring of dimension $d \geq 2$ containing $\mathbb{Q}$ and $D = \frac{R[X_1, \cdots , X_m]}{(J_1, \ldots, J_0)}$, where $J_1, J_2$ are two ideals of $R[X_1, \cdots , X_{m-1}]$ generated by monomials. Let $I$ be an ideal in $D$ of height $> d$. Suppose that $I = (f_1, \cdots , f_n) + I^2$ with $n \geq \dim(D/I) + 2$. Then $I = (g_1, \cdots , g_n)$ with $f_i - g_i \in I^2$ in each of the following cases:

1. $n \geq \max\{\dim(D/J_1), \dim(D/J_2)\}$ and $\text{ht}\left(\frac{I + J_2}{J_2}\right) \geq 2$.
2. $n = \max\{\dim(D/J_1) - 1, \dim(D/J_2) - 1\}$ and $\text{ht}\left(\frac{I + J_2}{J_2}\right) \geq 3$.

As an application of (5.2), we give some explicit examples.

Example 5.3. Let $R$ be a ring of dimension $d \geq 4$ containing $\mathbb{Q}$ and $D = \frac{R[X_1, \cdots , X_4]}{(X_2J)}$ where $J = (X_1X_2, X_2X_3, X_1X_3)$. Let $I \subset D$ be an ideal of height $n \geq d + 1$. Suppose that $I = (f_1, \cdots , f_n) + I^2$. Then there exist $g_1, \cdots , g_n \in I$ such that $I = (g_1, \cdots , g_n)$ with $f_i - g_i \in I^2$. In other words, the $n$-th Euler class group $E^n(D)$ is trivial.

Proof. Using (3.1) and (4.2), we can assume that $n = d + 1$. We have $\dim(D/J) = d + 2$, i.e., $n = d + 1 = \dim(D/J) - 1$ and $\text{ht}\left(\frac{I + J}{J}\right) \geq 3$. Also note that $n = d + 1 \geq 5 \geq \dim(D/I) + 2$. Now the result follows from (5.2(2)). \qed

The following result follows from (5.2).
**Example 5.4.** Let $R$ be a ring of dimension $d \geq 3$ containing $\mathbb{Q}$ and $D = R[X_1, \ldots, X_m]/(X_m, J)$ where $J = (X_iX_j | 1 \leq i \neq j \leq m - 1)$. Let $I \subseteq D$ be an ideal such that $\text{ht}(I/J) \geq 3$. Suppose that $I = (f_1, \ldots, f_n) + I^2$ with $n \geq \max\{d + 1, \dim(D/I) + 2\}$. Then there exist $g_1, \ldots, g_n \in I$ such that $I = (g_1, \ldots, g_n)$ with $f_i - g_i \in I^2$. □

Now using (4.4) and following the proof of (5.2), we can derive the following.

**Example 5.5.** Let $R$ be a ring of dimension $d \geq 4$ containing $\mathbb{Q}$ and $D = R[X_1, \ldots, X_4]/(J_1, X_4X_2)$ where $J_1, J_2$ are two ideals in $R[X_1, X_2, X_3]$ generated by monomials and $\text{ht}(J_1 + J_2) \geq 2$. Let $I \subseteq D$ be an ideal of height $n \geq d + 1$. Suppose that $I = (f_1, \ldots, f_n) + I^2$. Then there exist $g_1, \ldots, g_n \in I$ such that $I = (g_1, \ldots, g_n)$ with $f_i - g_i \in I^2$. In other words, the $n$-th Euler class group $E^n(D)$ is trivial.

Proof. Since $\dim(D) \leq d + 3$, the case $n \geq d + 2$ is covered by (3.1) and (4.2). Let us assume that $n = d + 1$. Then $n = d + 1 = \dim(D/J_2) - 1$ and $2n \geq \dim(D) + 2$. Now the result follows from (5.2). □

The following result follows from (5.2).

**Example 5.6.** Let $R$ be a ring of dimension $d \geq 4$ containing $\mathbb{Q}$ and $D = R[X_1, \ldots, X_5]/(X_5, J)$ where $J = (X_1X_2X_3, X_1X_2X_4, X_2X_3X_4)$. Let $I \subseteq D$ be an ideal such that $\text{ht}(I) = n \geq d + 1$. Suppose that $I = (f_1, \ldots, f_n) + I^2$ with $n \geq d + 2$. Then there exist $g_1, \ldots, g_n \in I$ such that $I = (g_1, \ldots, g_n)$ with $f_i - g_i \in I^2$. In other words, the $n$-th Euler class group $E^n(D)$ is trivial. □

**References**


