Euler Class Group of a Noetherian Ring

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by
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CERTIFICATE

Certified that the work contained in the thesis entitled **Euler class group of a Noetherian ring**, by **Manoj Kumar Keshari**, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

(Prof. S. M. Bhatwadekar) (Manoj Kumar Keshari)

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Chapter 0

Introduction

Let A be a commutative Noetherian ring of dimension n and let P be a projective A-module. P is said to have a unimodular element if there exists a surjective A-linear map $\phi : P \twoheadrightarrow A$ (in other words $P \xrightarrow{\sim} Q \oplus A$). A classical theorem of Serre *([SM], Theorem 4.2.1)* asserts that if P is a projective A-module of rank $> n$, then P has a unimodular element. This result is the best possible in the general. A standard example to show this is :

Let $A = \mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2-1) = \mathbb{R}[x,y,z]$ = the coordinate ring of the real 2-sphere. Let $P = A^3/A\alpha$, where $\alpha = (x,y,z) \in A^3$. Then P is a projective A-module of rank 2 and is associated to the tangent bundle of the real 2-sphere. We have $P \oplus A \xrightarrow{\sim} A^3$ but it is well known that $P \not\simeq A^2$, hence P does not have a unimodular element. Hence Serre's result is not valid in general if rank $P = \dim\,(A)$.

Therefore it is natural to ask the following:
Main Question: Let A be a commutative Noetherian ring of dimension n and P be a projective A-module of rank n. Can we associate an invariant to P, the vanishing of which would ensure that P has a unimodular element?

Let A be a smooth affine domain over a field k. Let $F^nK_0(A)$ denote the subgroup of $K_0(A)$ generated by the images of the residue fields of all the maximal ideals of A. Let P be a projective A-module of rank n, then the n^{th} *Chern class* of P, $C_n(P) = \sum(-1)^i\,(\wedge^i P^*)$ (where P^* is the dual of P) is an element of $F^nK_0(A)$. It is easy to see that if $P \xrightarrow{\sim} Q \oplus A$ then $C_n(P) = 0$.

In the above setting, if k is an algebraically closed then Murthy *([Mu], Theorem 3.8)*, proved that P has a unimodular element if and only if $C_n(P) = 0$. Thus, the only obstruction for P to have a unimodular element is the possible non-vanishing of its "top Chern class" $C_n(P)$. However, if k is not algebraically closed, then the vanishing of the invariant top Chern class is not sufficient, as is shown by the example of the projective module associated to the tangent bundle of the real 2-sphere.

It is natural to ask, whether in the case of affine domains A of dimension ≥ 2 over arbitrary base fields, if one can attach a different invariant to a projective A-module P of rank $=$ dim (A), the vanishing of which would ensure that P has a unimodular element. To tackle this question, Nori defined the notion of the "Euler class group" of a smooth affine variety $X =$ Spec (A) over an infinite field, attached to any projective A-module P of rank $=$ dim (A), an element in this group, called the "Euler class" of P and asked whether the vanishing of the Euler class of P would ensure that P has a unimodular element. In *([B-RS 1])* S.M.Bhatwadekar and Raja Sridharan settled this question of Nori in the affirmative for projective modules of trivial determinant. In *([B-RS 1])* an explicit description of the Euler class group is given, which appeared amenable for plausible generalization to arbitrary Noetherian rings. Indeed such a generalization is possible. In order to answer the Main Question, in *([B-RS 3])*, to any Noetherian ring A of dimension $n \geq 2$ containing the field of rational numbers, an abelian group $E(A)$ is attached, defined roughly as follows:

First, one takes the free abelian group on the pairs (J, w_J), where $J \subset A$ is an ideal of height $n =$ dim (A) such that J/J^2 is generated by n elements and w_J a set of n generators of J/J^2. The group $E(A)$ is a quotient of this group by the subgroup generated by (J, w_J), where $J = (a_1, \ldots, a_n)$ and w_J is the induced set of generators of J/J^2. It is proved in *([B-RS 3])* that if A is a Noetherian ring containing the field of rationales, then the group $E(A)$ detects the obstruction for a projective A-module P of rank n with trivial determinant to have a unimodular element, thus answering the Main Question in the affirmative.

The aim of this thesis is to give a self contained account of the proof of this result and give some applications. The layout of this thesis is as follows: In chapter 1, we recall some basic definitions and some well known theorems. In chapter 2, we prove some preliminary results. In chapter 3, we prove some addition and subtraction principles which are the main ingredients for the proofs of the main theorems. In chapter 4, we define the notion of

Euler class group $E(A)$ and show how to attach to the pair (P, χ) (where P is a projective A-module of rank n and $\chi : A \xrightarrow{\sim} \wedge^n(P)$ an isomorphism), an element $e(P, \chi)$ of $E(A)$ called the Euler class of (P, χ). We show that P has a unimodular element if and only if $e(P, \chi)$ vanishes. In chapter 5, we use the above result to prove some theorems about projective modules over real affine varieties. In the last chapter, we define the notion of the weak Euler class group $E_0(A)$, which is obtained as a certain canonical quotient of $E(A)$. We also define the weak Euler class of a projective A-module of rank $n = \dim$ (A). It is proved that if A is a Noetherian ring of even dimension n, and P is a projective A-module of rank n with trivial determinant, then the weak Euler class $e(P)$ of P vanishes in $E_0(A)$ if and only if $[P] = [Q \oplus A]$ in $K_0(A)$, for some projective A-module Q of rank $n - 1$.

Chapter 1

Some Basic Definitions

In this thesis we assume that all rings are commutative Noetherian with unity and all modules are finitely generated unless otherwise stated. We assume that the multiplicative closed sets with respect to which we localize do not contain 0. We begin with a few definitions and subsequently state some basic and useful results without proof.

Definition 1.1 Let A be a ring. The supremum of the lengths r, taken over all strictly increasing chains $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals of A, is called the *Krull dimension* of A, or simply the dimension of A, and denoted by dim (A).

For a prime ideal $\mathfrak{p}$ of A, the supremum of the lengths r, taken over all strictly increasing chains $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r = \mathfrak{p}$ of prime ideals of A, is called the the height of $\mathfrak{p}$ and denoted by ht $(\mathfrak{p})$. Note that for a Noetherian ring A, ht $(\mathfrak{p}) < \infty$.

For an ideal $I \subset A$, the infimum of the heights of $\mathfrak{p}$, taken over all prime ideals $\mathfrak{p} \subset A$ such that $I \subset \mathfrak{p}$, is defined to be *height* of I, denoted by ht (I).

For a prime ideal $\mathfrak{p}$ of A, the supremum of the lengths r, taken over all strictly increasing chains $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals of A starting from $\mathfrak{p}$, is called the the *coheight* of $\mathfrak{p}$ and denoted by coht $(\mathfrak{p})$.

For an ideal $I \subset A$, the supremum of the coheights of $\mathfrak{p}$, taken over all prime ideals $\mathfrak{p} \subset A$ such that $I \subset \mathfrak{p}$, is defined to be *coheight* of I, denoted by coht (I).

It follows from the definitions that

ht $(\mathfrak{p}) = \dim (A_{\mathfrak{p}})$, coht $(\mathfrak{p}) = \dim (A/\mathfrak{p})$ and ht $(\mathfrak{p})$ + coht $(\mathfrak{p}) \leq \dim (A)$.

Definition 1.2 An A-module P is said to be *projective* if it satisfies one of the following equivalent conditions:

(i) Given A-modules M, N and an A-linear surjective map $\alpha : M \twoheadrightarrow N$, the canonical map from $\mathrm{Hom}_A(P, M)$ to $\mathrm{Hom}_A(P, N)$ sending θ to $\alpha\,\theta$ is surjective.

(ii) Given an A-module M and a surjective A-linear map $\alpha : M \twoheadrightarrow P$, there exists an A-linear map $\beta : P \to M$ such that $\alpha\,\beta = 1_P$.

(iii) There exists an A-module Q such that $P \oplus Q \simeq A^n$ for some positive integer n, i.e. $P \oplus Q$ is free.

Lemma 1.3 *(***Nakayama Lemma***) Let A be a ring and M be a finitely generated A-module. Let $I \subset A$ be an ideal such that $IM = M$. Then, there exists $a \in I$ such that $(1 + a)M = 0$. In particular, if I is contained in Jacobson radical of A, then $(1 + a)$ is a unit and hence $M = 0$.*

Corollary 1.4 *Let A be a ring and M be a finitely generated A-module. Let I be an ideal contained in the Jacobson radical of A and N be a submodule of M. If $N + IM = M$ then $N = M$.*

Corollary 1.5 *Let A be a local ring with $\mathfrak{m}$ its maximal ideal. Let M be a finitely generated A-module. Then $\mu(M)$ (the minimum number of generators of M) $= \dim_{A/\mathfrak{m}}(M/\mathfrak{m}M)$.*

Lemma 1.6 *Let I be an ideal of A contained in the Jacobson radical of A. Let P, Q be projective A-modules such that projective A/I-modules P/IP and Q/IQ are isomorphic. Then P and Q are isomorphic as A-modules.*

Proof Let $\overline{\alpha} : P/IP \xrightarrow{\sim} Q/IQ$ be an isomorphism. Then, since P is projective, $\overline{\alpha}$ can be lifted to an A-linear map $\alpha : P \to Q$. We claim that α is an isomorphism.

Since $\overline{\alpha}$ is surjective, $Q = \alpha(P) + IQ$. Hence as I is contained in the Jacobson radical of A, by Nakayama lemma, we get $Q = \alpha(P)$. Hence α is surjective.

Since Q is projective, there exists an A-linear map $\beta : Q \to P$ such that $\alpha\,\beta = \mathrm{id}_Q$. Let $\overline{\beta} : Q/IQ \to P/IP$ be the map induced by β. Then we have $\overline{\alpha}\,\overline{\beta} = \mathrm{id}_{Q/IQ}$. As $\overline{\alpha}$ is an isomorphism, we get that $\overline{\beta}$ is also an isomorphism and in particular $\overline{\beta}$ is surjective. Therefore $P = \beta(Q) + IP$. Hence as before, we see that β is surjective. Now, injectivity of α follows from the fact that $\alpha\beta = \mathrm{id}$. ∎

Corollary 1.7 *Let A be a local ring. Then every projective A-module is free.*

Proof Let $\mathfrak{m}$ be the maximal ideal A and let $k = A/\mathfrak{m}$ be the residue field of A. Let P be a projective A-module and let $n = \dim_k (P/\mathfrak{m}P)$. Now applying (1.6) to the projective modules P and A^n, we see that $P \xrightarrow{\sim} A^n$. ∎

Definition 1.8 (**Zariski Topology**) For an ideal $I \subset A$, we denote by $V(I)$, the set of all prime ideals of A containing I. For $f \in A$, we denote by $D(f)$, the set of all prime ideals of A not containing the element f. The *Zariski topology* on Spec (A) is the topology for which all the closed sets are of the form $V(I)$, for some ideal I of A or equivalently the basic open sets are of the form $D(f),\ f \in A$.

Definition 1.9 Let P be a projective A-module. In view of (1.7), we define the rank function rank_P as follows:

rank_P : Spec $(A) \to \mathbb{Z}$ is the function defined by $\text{rank}_P(\mathfrak{q})$ = rank of the free $A_{\mathfrak{q}}$-module $P \otimes_A A_{\mathfrak{q}}$. If rank_P is a constant function taking the value n then we define the rank of P to be n and denote it by $\text{rank}(P)$.

Remark 1.10 rank_P is a continuous function (with the discrete topology on $\mathbb{Z}$ and Zariski topology on Spec A). Moreover, rank_P is a constant function for every finitely generated projective A-module P if A has no non trivial idempotent elements.

Remark 1.11 As in corollary (1.7), one can show that if A is a semi-local ring and P is a projective A-module of constant rank n, then P is free of rank n.

Definition 1.12 Given a projective A-module P and an element $p \in P$, we define $\mathcal{O}_P(p) = \{\alpha(p) \,|\, \alpha \in P^*\}$. We say that p is *unimodular* if $\mathcal{O}_P(p) = A$. The set of all unimodular elements of P is denoted by $\text{Um}(P)$. If $P = A^n$, then we write $\text{Um}_n(A)$ for $\text{Um}(A^n)$.

Remark 1.13 $\mathcal{O}_P(p)$ is an ideal of A and p is unimodular if and only if there exists $\alpha \in P^*$ such that $\alpha(p) = 1$. An element $(a_1, \dots, a_n) \in A^n$ is unimodular if and only if there exists elements $b_1, \dots, b_n \in A$ such that $\sum_{i=1}^n a_i\, b_i = 1$. If $(a_1, \dots, a_n)$ is unimodular then we say that the row $[a_1, \dots, a_n]$ is a *unimodular row*.

We now state the classical stability theorem of Serre.

Theorem 1.14 (**Serre**) *Let A be a Noetherian ring of dimension n and P be a projective A-module of rank $> n$. Then $P \simeq Q \oplus A$. ([SM], p. 41).*

Definition 1.15 Let A be a ring. Let $\mathrm{GL}_n(A)$ be the subset of $\mathrm{M}_n(A)$ consisting of matrices having determinant equal to a unit in A. Let $\mathrm{SL}_n(A)$ be the subset of $\mathrm{M}_n(A)$ consisting of matrices of determinant 1. Let e_{ij}, $i \neq j$ denote the $n \times n$ matrix with 1 in the (i,j) coordinate and having zeros elsewhere and $\mathrm{E}_{ij}(a) = I_n + a\, e_{ij}$, $a \in A$. We denote by $\mathrm{E}_n(A)$ the subgroup of $\mathrm{SL}_n(A)$ generated by matrices of the type $\mathrm{E}_{ij}(a)$, $a \in A$.

Let A be a ring. Then $\mathrm{GL}_n(A)$ acts on $\mathrm{Um}_n(A)$. If two rows $f, g \in \mathrm{Um}_n(A)$ are conjugate under this action, we shall write $f \sim g$. This defines an equivalence relation on $\mathrm{Um}_n(A)$. The equivalence classes of $\mathrm{Um}_n(A)$ under $\sim$ are just the orbits of the $\mathrm{GL}_n(A)$ action. The next proposition shows how to associate a projective module to a unimodular row.

Proposition 1.16 *The orbits of* $\mathrm{Um}_n(A)$ *under the* $\mathrm{GL}_n(A)$ *action are in* $1-1$ *correspondence with the isomorphism classes of A-modules P for which $P \oplus A \simeq A^n$. Under this correspondence* $(1, 0, \ldots, 0)$ *corresponds to the free module* A^{n-1}.

Proof To any $[b_1, \ldots, b_n] \in \mathrm{Um}_n(A)$, we can associate $P = P(b_1, \ldots, b_n)$, the kernel of $[b_1, \ldots, b_n] : A^n \twoheadrightarrow A$. Such P is a typical module for which $P \oplus A \simeq A^n$. Suppose $\beta : P(b_1, \ldots, b_n) \xrightarrow{\sim} P(c_1, \ldots, c_n)$ is an isomorphism, for another $[c_1, \ldots, c_n] \in \mathrm{Um}_n(A)$. Then we can complete the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P(b_1, \ldots, b_n) \arrow[r] \arrow[d, "\beta"'] & A^n \arrow[r, "{[b_1,\ldots,b_n]}"] \arrow[d, dashed] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P(c_1, \ldots, c_n) \arrow[r] & A^n \arrow[r, "{[c_1,\ldots,c_n]}"] & A \arrow[r] & 0
\end{tikzcd}$$

with a suitable isomorphism $A^n \xrightarrow{\sim} A^n$ (note that the rows are split exact). If $\sigma \in \mathrm{GL}_n(A)$ denotes the matrix of this isomorphism, we will have $[b_1, \ldots, b_n] = [c_1, \ldots, c_n]\,\sigma$ and hence $[b_1, \ldots, b_n] \sim [c_1, \ldots, c_n]$. Conversely, if this equation holds for some $\sigma \in \mathrm{GL}_n(A)$, then the automorphism $A^n \xrightarrow{\sim} A^n$ defined by σ induces an isomorphism of the two kernels : $P(b_1, \ldots, b_n) \simeq P(c_1, \ldots, c_n)$. ∎

We now state some well known theorems on unimodular rows.

Theorem 1.17 *(***Swan, Towber***) Let A be a commutative ring and $v = [a^2, b, c] \in A^3$ be a unimodular row. Then v can be completed to a matrix in* $\mathrm{SL}_3(A)$ *([S-T], Theorem 2.1).*

Theorem 1.18 (Suslin) *Let A be a commutative ring and $[x_0, x_1, \dots, x_n] \in A^{n+1}$ be a unimodular row. Let $r_0, \dots, r_n$ be positive integers such that the product $r_0 r_1 \dots r_n$ is divisible by $n!$. Then the unimodular row $[x_0^{r_0}, x_1^{r_1}, \dots, x_n^{r_n}]$ is completable to a matrix in* $\mathrm{GL}_{n+1}(A)$ *([SM],Theorem 5.3.1).*

Theorem 1.19 (Ravi A. Rao) *Let A be a Noetherian ring of dimension n. If $1/n! \in A$, then any unimodular row $v \in \mathrm{Um}_{n+1}(A[X])$ is extended from A, i.e. $v \sim_{\mathrm{GL}_{n+1}(A[X])} v(0)$ i.e. there exists a matrix in $\mathrm{GL}_{n+1}(A[X])$ which takes v to $v(0)$ ([Ra], Corollary 2.5).*

The following proposition is analogous to (1.16).

Proposition 1.20 *For a projective A-module P, the following are equivalent:*

(i) For any projective A-module Q, if $P \oplus A \tilde{\to} Q \oplus A$, then $P \tilde{\to} Q$.

(ii) Given a unimodular element $(p, a) \in P \oplus A$, there exists an automorphism Δ of $P \oplus A$ such that $\Delta(p, a) = (0, 1)$.

Proof **(i) $\Rightarrow$ (ii).** Since (p, a) is unimodular element of $P \oplus A$, there exists an element $\alpha \in (P \oplus A)^*$ such that $\alpha(p, a) = 1$. Let $Q = \ker(\alpha)$. Then we get the following short exact sequence of A-modules:

$$0 \to Q \to P \oplus A \xrightarrow{\alpha} A \to 0$$

Let $\beta : A \to P \oplus A$ be an A-linear map such that $\beta(1) = (p, a)$. Then $\alpha \beta = 1_A$. Hence, the cyclic submodule $A(p, a)$ of $P \oplus A$ is isomorphic to A and $P \oplus A = Q \oplus A(p, a)$. Therefore, by assumption, there exists an isomorphism $\sigma : Q \tilde{\to} P$.

Let $\Delta : Q \oplus A(p, a) \to P \oplus A$ be an endomorphism of $P \oplus A$ defined by $\Delta(q, 0) = (\sigma(q), 0)$ for $q \in Q$ and $\Delta(p, a) = (0, 1)$. Then as σ is an isomorphism and $A(p, a) \tilde{\to} A$, it follows that Δ is an automorphism of $P \oplus A$ which sends (p, a) to (0,1).

(ii) $\Rightarrow$ (i). Let $\psi : Q \oplus A \tilde{\to} P \oplus A$ be an isomorphism and let $\psi(0, 1) = (p, a)$. Then as ψ is an isomorphism, (p, a) is a unimodular element of $P \oplus A$. Therefore, by assumption, there exists an automorphism Δ of $P \oplus A$ such that $\Delta(p, a) = (0, 1)$. Hence the isomorphism $\Delta \psi : Q \oplus A \tilde{\to} P \oplus A$ sends the element $(0, 1)$ of $Q \oplus A$ to $(0, 1)$ of $P \oplus A$. Note that $Q \tilde{\to} (Q \oplus A)/A(0, 1)$ and $P \tilde{\to} (P \oplus A)/A(0, 1)$. Hence $P \tilde{\to} Q$. This proves the result. ∎

Definition 1.21 Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be homomorphisms of A-modules. The *fiber product* of M_1 and M_2 over N is a triple (M, g_1, g_2) where M is an A-module, $g_1 : M \to M_1$ and $g_2 : M \to M_2$ are A-linear maps such that $f_1\, g_1 = f_2\, g_2$ and the triple is universal in the sense that given any other triple (M', g_1', g_2') of this kind with $f_1\, g_1' = f_2\, g_2'$, there is a unique homomorphism $h : M' \to M$ such that $g_1\, h = g_1'$ and $g_2\, h = g_2'$.

Example 1.22 Let A be a commutative ring and s, t be elements in A such that $As + At = A$. Then

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_s \arrow[d] \\ A_t \arrow[r] & A_{st} \end{tikzcd}$$

is a fiber product diagram of commutative rings. Further, if M is an A-module, then

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M_s \arrow[d] \\ M_t \arrow[r] & M_{st} \end{tikzcd}$$

is a fiber product diagram of A-modules.

Lemma 1.23 *Let A be a commutative ring and let $s, t \in A$ be such that $(s, t) = A$. Suppose M and M' are two A-modules. Let $f_1 : M_s \to M_s'$ be an A_s-linear map and $f_2 : M_t \to M_t'$ be an A_t-linear map such that $(f_1)_t = (f_2)_s$.*

(1) Then there is an A-linear map $f : M \to M'$ such that $(f)_s = f_1$ and $(f)_t = f_2$.

(2) Further, if f_1 and f_2 are injective (respectively surjective, isomorphisms) then so is f.

$$\begin{tikzcd}
M \arrow[rr] \arrow[dr, "f"] \arrow[dd] & & M_s \arrow[dr, "f_1"] \arrow[dd] & \\
& M' \arrow[rr] \arrow[dd] & & M_s' \arrow[dd] \\
M_t \arrow[rr] \arrow[dr, "f_2"'] & & M_{st} \arrow[dr, "f_0"] & \\
& M_t' \arrow[rr] & & M_{st}'
\end{tikzcd}$$

Definition 1.24 For a projective A-module P, we write (P) for the isomorphism class of P. The *Grothendieck group* $K_0(A)$ is an additive abelian group generated by the symbols (P) with certain natural relations. To be precise, we let:

$G =$ free abelian group generated by $(P) : P$ is a projective A-module,
$H =$ subgroup of G generated by $(P \oplus Q) - (P) - (Q) : P, Q$ are projective A-modules,
$K_0(A) = G/H$, and $[P] =$ image of (P) in $K_0(A)$.
Thus we have $[P \oplus Q] = [P] + [Q]$ in $K_0(A)$.

Proposition 1.25 *Let A be a ring. Let P and Q be projective A-modules. Then the following are equivalent :*

(1) $[P] = [Q] \in K_0(A)$,
(2) There exists a projective A-module T such that $P \oplus T \simeq Q \oplus T$,
(3) There exists a positive integer t such that $P \oplus A^t \simeq Q \oplus A^t$.

Definition 1.26 A projective A-module P is said to be *stably free* if $[P] = [A^n]$ in $K_0(A)$ for some n.

Theorem 1.27 *(**Bass Cancellation Theorem**) Let A be a Noetherian ring of dimension n and P be a projective A-module of rank $> n$. Suppose that $P \oplus Q \xrightarrow{\sim} P' \oplus Q$ for some projective A-modules P' and Q. Then $P \xrightarrow{\sim} P'$ i.e. if rank $P =$ rank $P' > \dim(A)$ and $[P] = [P']$ in $K_0(A)$ then $P \xrightarrow{\sim} P'$ ([SM], p. 42).*

Chapter 2

Some Preliminary Results

We begin with some lemmas on general position that are proved using prime avoidance arguments.

Lemma 2.1 *Let A be a Noetherian ring, $I \subset A$ be an ideal and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be prime ideals of A. Let $I = (a_1, \ldots, a_n) \not\subseteq \bigcup_1^r \mathfrak{p}_i$. Then there exists $b_2, \ldots, b_n \in A$ such that $c = a_1 + b_2\, a_2 + \ldots + b_n\, a_n \notin \bigcup_1^r \mathfrak{p}_i$.*

Proof Without loss of generality, we may assume that there are no inclusion relations between the various prime ideals $\mathfrak{p}_i$. We prove the lemma by induction on the number of prime ideals. Suppose by induction, we have chosen $c_2, \ldots, c_n \in A$ such that $d_1 = a_1 + c_2\, a_2 + \ldots + c_n\, a_n \notin \bigcup_1^{r-1} \mathfrak{p}_i$. If $d_1 \notin \mathfrak{p}_r$, we set $c = d_1$. We assume therefore, that $d_1 \in \mathfrak{p}_r$. Since $I \not\subseteq \mathfrak{p}_r$, it follows that one of the elements $a_2, \ldots, a_n \notin \mathfrak{p}_r$. Without loss of generality, we assume that $a_2 \notin \mathfrak{p}_r$. We choose an element $g \in A$ such that $g \in \bigcap_1^{r-1} \mathfrak{p}_i$ and $g \notin \mathfrak{p}_r$. Such a choice of g is possible, since there are no inclusion relations between the various prime ideals $\mathfrak{p}_i$. The element $c = d_1 + g\, a_2$ is of the form $a_1 + e_2\, a_2 + \ldots + e_n\, a_n$ and $c \notin \bigcup_1^r \mathfrak{p}_i$. ■

Lemma 2.2 *Let A be a Noetherian ring and $I = (a_1, \ldots, a_n) \subset A$ be an ideal of height $\geq n$. Then there exists an elementary matrix $\theta \in \mathrm{E}_n(A)$ such that $[a_1, \ldots, a_n]\, \theta = [b_1, \ldots, b_n]$, $I = (b_1, \ldots, b_n)$ and $\mathrm{ht}\ (b_1, \ldots, b_i) \geq i,\ 1 \leq i \leq n$.*

Proof By lemma (2.1), we find elements $b_2, \ldots, b_n \in A$ such that the element $d_1 = a_1 + b_2\, a_2 + \ldots + b_n\, a_n$ does not belong to the minimal prime ideals of A. Hence ht $(d_1) \geq 1$. The element d_1 is contained in only finitely many height one prime ideals of A, say $\mathfrak{p}_1, \dots, \mathfrak{p}_r$. Note that $I = (d_1, a_2, \dots, a_n)$. Applying lemma (2.1) once more, we find $c_1, c_3, \dots, c_n \in A$ such that the element $d_2 = a_2 + c_1\, d_1 + c_3\, a_3 + \dots + c_n\, a_n$ does not belong to any $\mathfrak{p}_i$ for $1 \leq i \leq r$. Hence ht $(d_1, d_2) \geq 2$. Note that $I = (d_1, d_2, a_3, \dots, a_n)$. Proceeding as above, we obtain a set of generators $d_1, \dots, d_n$ of I with the required properties. We note that the transformations we have performed are all elementary. Hence a matrix θ exists with the required property. ■

Lemma 2.3 *Let A be a Noetherian ring and $[a_1, \dots, a_n, a] \in A^{n+1}$. Then there exists $[b_1, \dots, b_n] \in A^n$ such that* ht $(I_a) \geq n$, *where $I = (a_1 + a\, b_1, \dots, a_n + a\, b_n)$ i.e. if $\mathfrak{p} \in$* Spec (A), *$I \subset \mathfrak{p}$ and $a \notin \mathfrak{p}$, then* ht $(\mathfrak{p}) \geq n$. *In particular if the ideal $(a_1, \dots, a_n, a)$ has height $\geq n$ then* ht $(I) \geq n$. *Further, if $(a_1, \dots, a_n, a)$ is an ideal of height $\geq n$ and I is a proper ideal of A, then* ht $I = n$.

Proof The only prime ideals of A which survive in A_a are those which do not contain a. If every minimal prime ideal of A contains a, then a is a nilpotent element and every prime ideal contains a and there is nothing to prove. Assume that $a \in A$ is not a nilpotent element. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals of A which do not contain a. Applying (2.1), we can find $b_1 \in A$ such that $(a_1 + a\, b_1)A_a \not\subset \bigcup_1^r \mathfrak{p}_i$. Assume that $b_1, \dots, b_{n-1} \in A$ are chosen so that ht $(a_1 + a\, b_1, \dots, a_{n-1} + a\, b_{n-1})A_a \geq n-1$. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the minimal prime ideals of $(a_1 + a\, b_1, \dots, a_{n-1} + a\, b_{n-1})$ which do not contain a. Applying (2.1), we can find $b_n \in A$ such that $a_n + ab_n \notin \bigcup_1^s \mathfrak{q}_i$ and hence ht $(a_1 + a\, b_1, \dots, a_n + a\, b_n)A_a \geq n$.

Now assume that ht $(a_1, \dots, a_n, a) \geq n$. We show that ht $(I) \geq n$, where $I = (a_1 + a\, b_1, \dots, a_n + a\, b_n)$. Assume ht $(I) = r < n$. Let $\mathfrak{p}$ be a prime ideal of A containing I such that ht $(\mathfrak{p}) = r$. If $a \notin \mathfrak{p}$, then $I_a \subset \mathfrak{p}_a$ and ht $\mathfrak{p}_a = r$, a contradiction. If $a \in \mathfrak{p}$, then $(I, a) = (a_1, \dots, a_n, a) \in \mathfrak{p}$, a contradiction as ht $(a_1, \dots, a_n, a) \geq n$ and ht $(\mathfrak{p}) = r < n$. Hence we have ht $(I) \geq n$.

Assume that ht $(a_1, \dots, a_n, a) \geq n$ and I is a proper ideal, then ht $(I) \leq n$ since I is generated by n elements. Hence ht $(I) = n$. ■

Lemma 2.4 *Let $[a_0, a_1, \dots, a_n]$ be a unimodular row. If $[a_1, \dots, a_n]$ is also unimodular, then $[a_0, a_1, \dots, a_n]$ can be taken to $[1, 0, \dots, 0]$ by an elementary transformation.*

Proposition 2.5 *Let A be a Noetherian ring of dimension d. If $n \geq d + 2$, then* $\mathrm{E}_n(A)$ *acts transitively on* $\mathrm{Um}_n(A)$

Proof Let $(a_1, \dots, a_n) \in \mathrm{Um}_n(A)$. By (2.3), there exist $b_1, \dots, b_{n-1} \in A$ such that $(a_1 + b_1\, a_n, \dots, a_{n-1} + b_{n-1}\, a_n)$ is a unimodular row. Hence by (2.4), $(a_1, \dots, a_n)$ can be taken to $(1, 0, \dots, 0)$ by elementary transformations. ∎

Lemma 2.6 *Let A be a Noetherian ring of dimension n and $[a_0, a_1, \dots, a_n]$ be a unimodular row. Then we can elementarily transform $[a_0, a_1, \dots, a_n]$ to $[b_0, \dots, b_n]$ such that (1) $\mathrm{ht}\ (b_1, \dots, b_n) \geq n$ and (2) if $J \subset A$ is an ideal of height n, then we can choose the elementary transformations so that in addition we have $(b_1, \dots, b_n) + J = A$.*

Proof Since $\dim\ (A/J) = 0$, by (2.5), we may perform elementary transformations to obtain $[b_0, b_1, \dots, b_n]$ such that $[b_0, b_1, \dots, b_n] = [0, \dots, 0, 1]$ modulo J. Further, adding suitable multiple of b_0 to $b_1, \dots, b_n$, we may assume by (2.3) that $\mathrm{ht}\ (b_1, \dots, b_n) \geq n$, and in addition that $(b_1, \dots, b_n) + J = A$. ∎

Lemma 2.7 *Let A be a Noetherian ring and $J \subset A$ be an ideal. Let $J_1 \subset J$ and $J_2 \subset J^2$ be two ideals of A such that $J_1 + J_2 = J$. Then $J = J_1 + (e)$ for some $e \in J_2$ and $J_1 = J \cap J'$, where $J_2 + J' = A$.*

Proof Since J/J_1 is an idempotent ideal of A/J_1 it is generated by an idempotent element. Let $J/J_1 = (\overline{e})$. Since $J_1 + J_2 = J$, we can assume that $e \in J_2$. Since $\overline{e}$ is an idempotent element, we have $e - e^2 \in J_1$. Take $J' = J_1 + (1 - e)$. Then $J_2 + J' = A$ since $e \in J_2$. We claim that $J \cap J' = J_1$.

Let $x \in J \cap J'$. Then $x = y + ez = y_1 + (1 - e)z_1$, where $y, y_1 \in J_1$ and $z, z_1 \in A$. This implies $ez - (1 - e)z_1 \in J_1$. But $e - e^2 \in J_1$ so $e^2 z \in J_1$ and hence $ez \in J_1$. Hence $x \in J_1$. This proves $J \cap J' = J_1$. ∎

Corollary 2.8 *Let A be a Noetherian local ring. Let $J \subset A$ be an ideal such that $J = (f_1, \dots, f_n) + J^2$. Then $J = (f_1, \dots, f_n)$.*

Lemma 2.9 *Let A be a Noetherian ring and $I \subset A$ an ideal. Let $f_1, \dots, f_n \in I$ and $J = (f_1, \dots, f_n)$. Then $I = J$ if and only if $I = J + I^2$ and $V(J) = V(I)$ in* Spec (A).

Proof This follows from (2.7), however we give an independent proof. In order to show that $J = I$, it is enough to show that $J_{\mathfrak{p}} = I_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}\ (A)$. If $\mathfrak{p} \not\supset J$, then $\mathfrak{p} \not\supset I$ and $J_{\mathfrak{p}} = I_{\mathfrak{p}} = A_{\mathfrak{p}}$. If $\mathfrak{p} \supset J$, then by hypothesis $\mathfrak{p} \supset I$. Since $I = (f_1, \dots, f_n) + I^2$ by (2.8), we have $J_{\mathfrak{p}} = I_{\mathfrak{p}}$. This proves the lemma. ∎

Lemma 2.10 (**Mohan Kumar**) *Let A be a Noetherian ring and I an ideal of A. Let I/I^2 is generated by n elements as an A/I-module. Let x be any element of A. Then the ideal $(I,x) \subset A$ is generated by $n+1$ elements ([MK 1]).*

Proof Let $a_1, \ldots, a_n$ be elements of I such that they generate I modulo I^2. In the ring $A/(a_1, \ldots, a_n)$, the ideal $\overline{I} = I/(a_1, \ldots, a_n)$ has the property that $\overline{I} = \overline{I^2}$. Hence $\overline{I}$ is generated by an idempotent. Let $h \in I$ be any lift of this idempotent. We see that $I = (a_1, \ldots, a_n, h)$ and $h(1-h) \in (a_1, \ldots, a_n)$. So $(I,x) = (a_1, \ldots, a_n, h, x)$. We claim that the ideal $J = (a_1, \ldots, a_n, h + (1-h)x) \subset (I,x)$ is actually equal to (I,x).

By multiplying $h + (1-h)x$ by h we have $h^2 \in J$, since $h(1-h) \in J$. Since $h = h^2 + h(1-h)$ we have $h \in J$. Since $h + (1-h)x \in J$ we have $x \in J$. Thus $J = (I,x)$ which proves the claim. ■

Remark 2.11 Implicit in the above proof is a proof of the following assertion. Let A be a ring, $e \in A$ be an idempotent. Then for any $x \in A$, the ideals (e,x) and $(e + (1-e)x)$ are equal.

The following is a theorem of Eisenbud and Evans (*[E-E]*) and this is a version proved in (*[P], p. 1420*). This was proved in (2.3) when P is free.

Lemma 2.12 *Let A be a Noetherian ring and P be a projective A-module of rank n. Let $(\alpha, a) \in (P^* \oplus A)$. Then there exists an element $\beta \in P^*$ such that* ht $(I_a) \geq n$, *where $I = (\alpha + a\beta)(P)$. In particular if the ideal $(\alpha(P), a)$ has height $\geq n$ then* ht $I \geq n$. *Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and I is a proper ideal of A, then* ht $I = n$.

Lemma 2.13 *Let A be a Noetherian ring of dimension d and let P be a projective A-module of rank $n > d$. Let $J \subset A$ be an ideal and let $\overline{\alpha} : P/JP \twoheadrightarrow J/J^2$ be a surjection. Then $\overline{\alpha}$ can be lifted to a surjection from P to J.*

Proof Let $\delta : P \to J$ be a lift of $\overline{\alpha}$. Then $\delta(P) + J^2 = J$ and hence, by (2.7), there exists $c' \in J^2$ such that $\delta(P) + (c') = J$. Now applying (2.12) to the element (δ, c') of $P^* \oplus A$, we see that there exists $\gamma \in P^*$ such that the height of the ideal $N_{c'} > n$, where $N = (\delta + c'\gamma)(P)$. Since dim $(A) = d$ and $n > d$, it follows that $(c')^r \in N$ for some positive integer r. As $N + (c') = J$ and $c' \in J^2$, we have $N = J$, by (2.9). Since $\delta + c'\gamma$ is also a lift of $\overline{\alpha}$, we get the result. ■

Corollary 2.14 *(**Moving Lemma**) Let A be a Noetherian ring of dimension $n \geq 2$ and let P be a projective A-module of rank n. Let $J \subset A$ be an ideal of height n and let $\overline{\alpha} : P/JP \twoheadrightarrow J/J^2$ be a surjection. Then there exists an ideal $J' \subset A$ and a surjection $\beta : P \twoheadrightarrow J \cap J'$ such that:*

(i) $J + J' = A$, (ii) $\beta \otimes A/J = \overline{\alpha}$, (iii) ht $(J') \geq n$, and

(iv) Further given finitely many ideals $J_1, J_2, \ldots, J_r$ of height $\geq 1, J'$ can be chosen with the additional property that J' is comaximal with $J_1, J_2, \ldots, J_r$.

Proof Let $K = J^2 \cap J_1 \ldots \cap J_r$. Then, by the assumption, ht $(K) \geq 1$. Therefore there exists an element $a \in K$ such that ht $(Aa) = 1$ and hence dim $(A/Aa) \leq n - 1$. By (2.13), the surjection $\overline{\alpha}$ can be lifted to a surjection $\delta : P/aP \to J/Aa$.

Let $\theta \in \mathrm{Hom}_A(P, J)$ be a lift of δ. Then, as $J/Aa = \delta(P/aP)$, we have $\theta(P) + Aa = J$. Applying (2.12) to the element (θ, a) of $P^* \oplus A$, we see that there exists $\psi \in P^*$ such that ht $\widetilde{J}_a \geq n$, where $\widetilde{J} = (\theta + a\psi)(P)$. But $(\theta(P), a) = J$ has height n, and $\widetilde{J}$ is a proper ideal ($\widetilde{J} \subset J$), hence by (2.12) ht $(\widetilde{J}) = n$. Since $\widetilde{J} + Aa = J$ and $a \in J^2$, by (2.7) there exists an ideal J' of A such that $\widetilde{J} = J \cap J'$ and $Aa + J' = A$. Now, setting $\beta = \theta + a\psi$, we get

(i) $\beta : P \twoheadrightarrow \widetilde{J} = J \cap J'$,

(ii) $\beta \otimes A/J = \overline{\alpha}$,

(iii) ht $(J') \geq n$ since ht $(\widetilde{J}_a) \geq n$, and

(iv) J' is comaximal with $J_1, \ldots, J_r$, since $Aa + J' = A$. ■

Lemma 2.15 *Let A be a ring and J be a proper ideal of A. Let $J = (a, b) = (c, d)$. Suppose $[a, b] = [c, d]$ modulo J^2. Then there exists an automorphism $\triangle$ of A^2 such that (1) $[a, b]\,\triangle = [c, d]$ and (2) det $(\triangle) = 1$.*

Proof We have $a - c$, $b - d \in J^2$. So we can write $a - c = a\,a_1 + b\,a_2$ and $b - d = a\,a_3 + b\,a_4$, where $a_i \in J$ for $1 \leq i \leq 4$. Let $u = 1 - a_1$, $v = -a_2$, $w = -a_3$, and $x = 1 - a_4$. Then we have the following equation

$$\begin{pmatrix} u & v \\ w & x \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix},$$

Now, we see that $u\,x - v\,w = 1 - f$, for some $f \in J$. There exists $t_1, t_2 \in A$ such that $f = d\,t_2 - c\,t_1$. The endomorphism Δ of A^2 given by

$$\begin{pmatrix} u + b\,t_2 & v - a\,t_2 \\ w + b\,t_1 & x - a\,t_1 \end{pmatrix}$$

is an automorphism of determinant 1 with $[a, b]\Delta = [c, d]$. $\blacksquare$

Lemma 2.16 *Let A be a Noetherian ring of dimension n and $J \subset A$ be an ideal of height n. Let P and P_1 be projective A-modules of rank n. Let $\alpha : P \to J$ and $\beta : P_1 \to J$ be maps such that $\alpha \otimes A/J$ and $\beta \otimes A/J$ are surjective. Let $\psi : P \to P_1$ be a homomorphism such that $\beta\psi = \alpha$. Then $\psi \otimes A/J : P/JP \to P_1/JP_1$ is an isomorphism.*

Proof By Nakayama lemma it is enough to prove that if $K = \sqrt{J}$ then $\overline{\psi} : P/KP \to P_1/KP_1$ is an isomorphism. Let "bar" denote reduction modulo K. Note that $\overline{\alpha}$ and $\overline{\beta}$ are surjections. We prove that $\overline{\alpha}$ and $\overline{\beta}$ are isomorphisms. Since $\overline{\beta\psi} = \overline{\alpha}$, it will follow that $\overline{\psi}$ is an isomorphism. Since A/K is semi-local, P/KP and P_1/KP_1 are free A/K-modules of rank n by (1.7). Hence in order to prove that $\overline{\alpha}$ and $\overline{\beta}$ are isomorphisms, it is enough to prove that J/KJ is a free A/K-module of rank n. Note that $J/KJ = \oplus J/\mathfrak{m}_i J$, where $\mathfrak{m}_i$ are the maximal ideals containing K. We prove that $J/\mathfrak{m}_i J$ is a free $A/\mathfrak{m}_i$-module of rank n. Since $\alpha \otimes A/J$ is surjective, J/J^2 is generated by n elements. Hence by (2.8), $J_{\mathfrak{m}_i}$ is generated by n elements. Since ht $(J) = n$, $J_{\mathfrak{m}_i}$ cannot be generated by less than n elements. Hence $\mu(J_{\mathfrak{m}_i}) = n$. Hence by (1.5), $J/\mathfrak{m}_i J$ is a free $A/\mathfrak{m}_i$-module of rank n. This proves the lemma. $\blacksquare$

Lemma 2.17 *Let A be a Noetherian ring and P a finitely generated projective A-module. Let $P[T]$ denote the projective $A[T]$-module $P \otimes_A A[T]$. Let $\alpha(T) : P[T] \twoheadrightarrow A[T]$ and $\beta(T) : P[T] \twoheadrightarrow A[T]$ be two surjections such that $\alpha(0) = \beta(0)$. Suppose further that the projective $A[T]$-modules* ker $\alpha(T)$ *and* ker $\beta(T)$ *are extended from A. Then there exists an automorphism $\sigma(T)$ of $P[T]$ with $\sigma(0) = id$ such that $\beta(T)\,\sigma(T) = \alpha(T)$.*

Proof First we show that there exists an automorphism $\theta(T)$ of $P[T]$ such that $\theta(0) =$ id and $\alpha(T)\,\theta(T) = \alpha(0) \otimes A[T]$. Let $Q =$ ker $(\alpha(T))$ and $L =$ ker $(\alpha(0))$. Since Q is extended from A, there exists an isomorphism $\mu : L[T] \xrightarrow{\sim} Q$. Now since the rows of the following diagram

$$\begin{tikzcd} 0 \arrow[r] & L[T] \arrow[r] \arrow[d, "\mu"] & P[T] \arrow[r, "\alpha(0)\otimes A[T]"] \arrow[d, dashed, "\rho(T)"] & A[T] \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & Q \arrow[r] & P[T] \arrow[r, "\alpha(T)"'] & A[T] \arrow[r] & 0 \end{tikzcd}$$

are split, we can find an automorphism $\rho(T)$ of $P[T]$ such that the above diagram is commutative. We have $\alpha(T)\,\rho(T) = \alpha(0) \otimes A[T]$ and hence $\alpha(0)\,\rho(0) = \alpha(0)$. Consider an automorphism $\theta(T) = \rho(T)\,(\rho(0) \otimes A[T])^{-1}$ of $P[T]$. Then $\alpha(T)\,\theta(T) = (\alpha(0) \otimes A[T])\,(\rho(0) \otimes A[T])^{-1} = (\alpha(0) \otimes A[T])$ and $\theta(0) = \mathrm{id}$.

Similarly, we have an automorphism $\delta(T)$ of $P[T]$ such that $\beta(T)\,\delta(T) = \beta(0) \otimes A[T]$ and $\delta(0) = \mathrm{id}$. Consider the automorphism $\sigma(T) = \delta(T)\,(\theta(T))^{-1}$ of $P[T]$. As $\alpha(0) = \beta(0)$ we have $\beta(T)\,\sigma(T) = (\beta(0) \otimes A[T])(\theta(T))^{-1} = (\alpha(0) \otimes A[T])(\theta(T))^{-1} = \alpha(T)$ and $\sigma(0) = \mathrm{id}$. This proves the lemma. ■

Lemma 2.18 *Let A be a ring (not necessarily commutative) and $S \subset A$ be a multiplicative closed set which is contained in the center of A. Let $u(T) \in A_S[T]$ be a unit such that $u(0) = 1$. Then there exists $s \in S$ such that $u(sT)$ is a unit of $A[T]$.*

Lemma 2.19 (Quillen) *Let A be a ring and $s, t \in A$ be such that $As + At = A$. Let $\sigma(T) \in \mathrm{GL}_n(A_{st}[T])$ be such that $\sigma(0) = \mathrm{id}$. Then $\sigma(T) = (\psi_2(T))_t\,(\psi_1(T))_s$, where $\psi_1(T) \in \mathrm{GL}_n(A_t[T])$ such that $\psi_1(0) = \mathrm{id}$ and $\psi_1(T) = \mathrm{id}$ modulo (s) and $\psi_2(T) \in \mathrm{GL}_n(A_s[T])$ such that $\psi_2(0) = \mathrm{id}$ and $\psi_2(T) = \mathrm{id}$ modulo (t).*

Proof Since $\sigma(0) = \mathrm{id}$, $\sigma = \mathrm{id} + T\tau(T)$. Therefore by (2.18), we can choose large enough k_1 such that for all $k \geq k_1$ and for all $\lambda \in A$, $\sigma(\lambda s^k T) \in \mathrm{GL}_n(A_t[T])$ and $\sigma(\lambda s^k T) = \mathrm{id}$ modulo (sT). Hence we can write $\sigma(\lambda s^k T) = (\psi_1(T))_s$ where $\psi_1(T) \in \mathrm{GL}_n(A_t[T])$ and $\psi_1(T) = \mathrm{id}$ modulo (sT).

Let X and Y be variables. Write $\delta(X,T,Y) = \sigma((X+Y)T)\,\sigma(XT)^{-1}$. Clearly $\delta(X,T,Y) \in \mathrm{GL}_n(A_{st}[X,T,Y])$, $\delta(X,T,0) = \mathrm{id}$ and $\delta(X,0,Y) = \mathrm{id}$. Hence $\delta(X,T,Y) = \mathrm{id} + YT\,\widetilde{\tau}(X,T,Y)$. We can choose large enough k_2 such that for all $k \geq k_2$ and for all $\mu \in A$, $\delta(X,T,t^k\mu Y) \in \mathrm{GL}_n(A_s[X,T,Y])$ and is identity modulo (tTY). Hence we can write $\delta(X,T,t^k\mu Y) = (\psi_2(X,T,Y))_t$ where $\psi_2(X,T,Y) \in \mathrm{GL}_n(A_s[X,T,Y])$ and

$\psi_2(X, T, Y) = \text{id}$ modulo (tT).

Take $k \geq \max(k_1, k_2)$. Since $As + At = A$ we have $\lambda s^k + \mu t^k = 1$ for some $\lambda, \mu \in A$. Now $\sigma(T) = \sigma(T)\,\sigma(\lambda s^k T)^{-1}\,\sigma(\lambda s^k T)$. We have $\sigma(\lambda s^k T) = (\psi_1(T))_s$ and $\sigma(T)\,\sigma(\lambda s^k T)^{-1} = \sigma((\lambda s^k + \mu t^k)T)\,\sigma(\lambda s^k T)^{-1} = \delta(\lambda s^k, T, \mu t^k) = (\psi_2(\lambda s^k, T, 1))_t = (\psi_2(T))_t$. Hence we have $\sigma(T) = (\psi_2(T))_t\,(\psi_1(T))_s$. This proves the lemma. ■

Definition 2.20 Let A be a ring and let M, N be A-modules. Suppose $f, g : M \xrightarrow{\sim} N$ be two isomorphisms. We say that f is *isotopic to* g if there is an isomorphism $\phi : M[X] \xrightarrow{\sim} N[X]$ such that $\phi(0) = f$ and $\phi(1) = g$. A matrix $\theta \in \mathrm{GL}_n(A)$ is said to be isotopic to identity if the corresponding automorphism of A^n is isotopic to identity, i.e. there exists a matrix $\alpha(X) \in \mathrm{GL}_n(A[X])$ such that $\alpha(0) = \text{id}$ and $\alpha(1) = \theta$.

Corollary 2.21 *Let A be a ring and $s, t \in A$ be such that $As + At = A$. Let $\theta \in \mathrm{GL}_n(A_{st})$ be isotopic to identity. Then θ splits as $\theta = (\theta_1)_t\,(\theta_2)_s$, where $\theta_1 \in \mathrm{GL}_n(A_s)$ such that $\theta_1 = \text{id}$ modulo (t) and $\theta_2 \in \mathrm{GL}_n(A_t)$ such that $\theta_2 = \text{id}$ modulo (s).*

Example 2.22 Elementary automorphisms are isotopic to identity. If $\sigma = \prod(1 + \lambda e_{ij})$ is an elementary automorphism of A^n then $\gamma(T) = \prod(1 + \lambda T e_{ij})$ is an automorphism of $(A[T])^n$ (infact elementary) such that $\gamma(0) = \text{id}$ and $\gamma(1) = \sigma$.

Definition 2.23 Let P be a projective A-module of rank n. Let $\wedge^n(P)$ denote the n^{th} exterior power of P. Then $\wedge^n(P)$ is a projective A-module of rank 1 and is called the determinant of P. An A-linear endomorphism α of P gives rise, in a natural way, to an endomorphism $\wedge^n(\alpha)$ of $\wedge^n(P)$. Since rank of $\wedge^n(P) = 1$, we have $\mathrm{End}_A(\wedge^n(P)) = A$ and hence $\wedge^n(\alpha) \in A$. Note that α is an automorphism if and only if $\wedge^n(\alpha)$ is an invertible element of A.

Let P, α be as in the above paragraph. We define the determinant of α to be $\wedge^n(\alpha)$ and denote it by $\det(\alpha)$. We denote the group of automorphisms of P of determinant 1 by $\mathrm{SL}(P)$.

Definition 2.24 Let P be a projective A-module. Given an element $\phi \in P^*$ and an element $p \in P$, we define an endomorphism ϕ_p as the composite $P \xrightarrow{\phi} A \xrightarrow{p} P$.

If $\phi(p) = 0$, then $\phi_p^2 = 0$ and $1 + \phi_p$ is a unipotent automorphism of P and hence is an element of $\mathrm{SL}(P)$.

By a *transvection*, we mean an automorphism of P of the form $1 + \phi_p$, where $\phi\,(p) = 0$ and either ϕ is unimodular in P^* or p is unimodular in P. We denote by $\mathrm{E}(P)$ the subgroup of $\mathrm{SL}(P)$ generated by all the transvections of P.

Remark 2.25 When $P = A^n$, a transvection is an element of $\mathrm{SL}_n(A)$ of the form $1 + vw^t$, where $v, w \in \mathrm{M}_{n\times 1}(A)$ and $wv = 0$ and either v or w is unimodular. For example $e_{ij}(\lambda) = 1 + \lambda\, e_i e_j^t$ is a transvection. Hence $\mathrm{E}_n(A)$ is a subgroup of $\mathrm{E}(A^n)$.

The following lemma is proved in (*[B-R]*).

Lemma 2.26 *Let A be a Noetherian ring, $I \subset A$ ideal of A and P a projective A-module. Then any transvection of P/IP can be lifted to a (unipotent) automorphism of P.*

Proof Let $\phi' \in (P/IP)^*$ and $p' \in P/IP$ be such that $\phi'(p') = 0$. Assume that p' is unimodular. Let $p \in P$ (resp. $\theta \in P^*$) be a lift of p' (resp. ϕ'). Then we have $\theta(p) = a$ for some $a \in I$. Since p' is unimodular, there exists a $\psi \in P^*$ such that $\psi(p) = 1 + b$ for some $b \in I$, as P is projective. Set $\phi = (1 + b)\theta - a\psi$. Then ϕ is a lift of ϕ' and $\phi(p) = 0$. Consequently, $1 + \phi_p$ is an automorphism of P lifting $1 + \phi'_{p'}$.

Now assume that ϕ' is unimodular. Then there exists $q \in P$ such that $\theta(q) = 1 + b$, for some $b \in I$. Set $p_1 = (1 + b)p - aq$. Then $\theta(p_1) = 0$. Consequently, $1 + \theta_{p_1}$ is an automorphism of P lifting $1 + \phi'_{p'}$. ∎

Chapter 3

Some Addition and Subtraction Principles

In *([MK 2])*, Mohan Kumar proved the following theorems

Theorem 3.1 *(*Addition principle*) Let A be a reduced affine ring of dimension n over k, where k is algebraically closed. Let I and J be two comaximal ideals of height n which are generated by n elements. Then $I \cap J$ is also generated by n elements.*

Theorem 3.2 *(*Subtraction principle*) Let A be a reduced affine ring of dimension n over k, where k is algebraically closed. Let I and J be two comaximal ideals of height n. Assume that I and $I \cap J$ are generated by n elements. Then J is also generated by n elements.*

Theorem 3.3 *Let A be a reduced affine ring of dimension n over an algebraically closed field. Let P be a projective A-module of rank n. If P maps onto an ideal J of height n which is generated by n elements, then P has a unimodular element.*

In this chapter we prove some addition and subtraction principles. These are modeled upon those proved by Mohan Kumar. Roughly the idea is to consider ideals J together with sets of generators of J/J^2 and formulate the addition and subtraction principles using this data.

Theorem 3.4 (Addition Principle) *Let A be a Noetherian ring of dimension $n \geq 2$. Let J_1 and J_2 be two comaximal ideals of height n and $J_3 = J_1 \cap J_2$. Suppose $J_1 = (a_1, \ldots, a_n)$ and $J_2 = (b_1, \ldots, b_n)$. Then $J_3 = (c_1, \ldots, c_n)$ where $a_i - c_i \in J_1^2$ and $b_i - c_i \in J_2^2$.*

Proof We have $J_1 = (a_1, \ldots, a_n)$ and $J_2 = (b_1, \ldots, b_n)$. Since $J_1 + J_2 = A$ we have that $[\overline{a}_1, \ldots, \overline{a}_n]$ is a unimodular row over A/J_2. Since dim $(A/J_2) = 0$, by (2.5), there exists an elementary matrix $\overline{\sigma} \in \mathrm{E}_n(A/J_2)$ such that $[\overline{a}_1, \ldots, \overline{a}_n]\,\overline{\sigma} = [1, 0, \ldots, 0]$.

Let $\sigma \in \mathrm{E}_n(A)$ be a lift of $\overline{\sigma}$. Let $[a_1, \ldots, a_n]\,\sigma = [\widetilde{a}_1, \ldots, \widetilde{a}_n]$. Then $\widetilde{a}_1 = 1$ modulo J_2 and $\widetilde{a}_2, \ldots, \widetilde{a}_n \in J_2$. Hence adding suitable multiples of a_n to $a_1, \ldots, a_{n-1}$, we can assume that (1) $a_1 = 1$ modulo J_2, (2) if $K = (a_1, \ldots, a_{n-1})$, then ht $(K) = n-1$ and (3) $K + J_2 = A$. Let $S = 1 + K$. Then $S \cap J_2 \neq \varnothing$. Hence $[b_1, \ldots, b_n] \in A_S^n$ is a unimodular row.

Claim : $[b_1, \ldots, b_n]$ can be taken to $[0, \ldots, 0, 1]$ by an element of $\mathrm{SL}_n(A_S)$.

Assume the claim. Then, there exists an element $s \in S$ and an automorphism Γ of A_s^n of determinant 1 such that $[b_1, \ldots, b_n]\,\Gamma = [0, \ldots, 0, 1]$. Since $S \cap J_2 \neq \varnothing$, without loss of generality, we may assume that $s \in J_2$. Hence we have $(J_3)_s = (a_1, \ldots, a_n)_s$.

Let $s = 1 + t$, for some $t \in K$. Then $(J_3)_t = (b_1, \ldots, b_n)_t$.

Since $t \in K$, we have that $[a_1, \ldots, a_{n-1}] \in A_t^{n-1}$ is unimodular row. Hence by (2.4), $[a_1, \ldots, a_n]$ can be taken to $[0, \ldots, 0, 1]$ by an elementary transformation Δ of A_t^n. Hence, we have $[a_1, \ldots, a_n]\,\Delta_s\,(\Gamma^{-1})_t = [b_1, \ldots, b_n]$.

Let $\Phi = \Gamma_t\,\Delta_s\,(\Gamma^{-1})_t$. Then $[a_1, \ldots, a_n]\,(\Gamma^{-1})_t\,\Phi = [b_1, \ldots, b_n]$. Since Δ_s is an elementary automorphism, Φ is isotopic to identity automorphism of A_{st}^n. Hence by (2.21), there exists a splitting $\Phi = (\Phi_1)_t\,(\Phi_2)_s$, where Φ_2 is an automorphism of A_t^n which is identity modulo the ideal (s) and Φ_1 is an automorphism of A_s^n which is identity modulo the ideal (t). Let

$$[a_1, \ldots, a_n]\,\Gamma^{-1}\,\Phi_1 = [a_1', \ldots, a_n'] : A_s^n \twoheadrightarrow (J_3)_s,$$

$$[b_1, \ldots, b_n]\,\Phi_2^{-1} = [b_1', \ldots, b_n'] : A_t^n \twoheadrightarrow (J_3)_t.$$

These two surjections patch up to give a surjection $\psi = [g_1, \ldots, g_n] : A^n \twoheadrightarrow J_3$. Since s is unit modulo J_1, the homomorphism $A \to A/J_1$ factors through A_s. Similarly, the homomorphism $A \to A/J_2$ factors through A_t. Now, since ϕ_1 is identity modulo the ideal $(t) \subset J_1$ and ϕ_2 is identity modulo J_2, it follows that $[g_1, \ldots, g_n] \otimes A/J_1$ and $[g_1, \ldots, g_n] \otimes A/J_2$ differ from $[a_1, \ldots, a_n] \otimes A/J_1$ and $[b_1, \ldots, b_n] \otimes A/J_2$ by an element of $\mathrm{SL}_n(A/J_1)$ and $\mathrm{SL}_n(A/J_2)$ respectively. Since $\dim(A/J_i) = 0$ for $i = 1, 2$, $\mathrm{SL}_n(A/J_i) = \mathrm{E}_n(A/J_i)$.

Hence using (2.26), we can alter $[g_1, \dots, g_n]$ by an element of $\mathrm{SL}_n(A)$, to get a surjection $\theta : A^n \twoheadrightarrow J_3$, say $J_3 = (c_1, \dots, c_n)$, such that $a_i - c_i \in J_1^2$ and $b_i - c_i \in J_2^2$. This proves the theorem.

Proof of the claim First we assume that $n \geq 3$. Let "bar" denote modulo K. Then $[\overline{b}_1, \dots, \overline{b}_n]$ is a unimodular row in $\overline{A}\,^n_S$. Since dim $(A/K) = 1$, by (2.5), $[\overline{b}_1, \dots, \overline{b}_n]$ can be taken to $[0, \dots, 0, 1]$ by an elementary transformation, say $\overline{\psi} \in \mathrm{E}_n(\overline{A}_S)$. Taking a lift $\psi \in \mathrm{E}_n(A_S)$ of $\overline{\psi}$ by (2.26), we see that $[b_1, \dots, b_n]$ can be taken to $[d_1, \dots, d_{n-1}, 1 + d_n]$ by an elementary transformation, where $d_i \in K_S$. Since $1 + d_n$ is a unit in A_S, $[d_1, \dots, d_{n-1}, 1 + d_n]$ can be taken to $[0, \dots, 0, 1 + d_n]$ by an elementary transformation. Now $[0, \dots, 0, 1 + d_n]$ can be taken to $[0, \dots, 0, 1]$ by an elementary automorphism of A_S^n by (2.4). This proves the claim.

When $n = 2$. Given $[b_1, b_2]$ is a unimodular row in A_S^2. Let $a_1, a_2 \in A_S$ be chosen so that $a_1\, b_1 + a_2\, b_2 = 1$. Consider the matrix $\gamma = \begin{bmatrix} a_1 & -b_2 \\ a_2 & b_1 \end{bmatrix}$.
Then $[b_1, b_2]\, \gamma = [1, 0]$ and det $(\gamma) = 1$. Hence the claim is proved. ■

Theorem 3.5 (Subtraction Principle) *Let A be a Noetherian ring of dimension $n \geq 2$. Let J and J_1 be two comaximal ideals of height n. Let $J_2 = J \cap J_1$. Assume that $J_2 = (a_1, \dots, a_n)$ and $J_1 = (b_1, \dots, b_n)$ with $a_i - b_i \in J_1^2$. Then $J = (c_1, \dots, c_n)$ with $a_i - c_i \in J^2$.*

Proof Let $\sigma \in \mathrm{E}_n(A)$. Suppose $[b_1, \dots, b_n]\, \sigma = [\widetilde{b}_1, \dots, \widetilde{b}_n]$ and $[a_1, \dots, a_n]\, \sigma = [\widetilde{a}_1, \dots, \widetilde{a}_n]$. Then, since $b_i - a_i \in J_1^2$, we have $\widetilde{b}_i - \widetilde{a}_i \in J_1^2$. Therefore, without loss of generality we can perform elementary transformations on $[b_1, \dots, b_n]$.

We have $(a_1, \dots, a_n) = J \cap (b_1, \dots, b_n)$. Let "bar" denote modulo J. Then $[\overline{b}_1, \dots, \overline{b}_n]$ is a unimodular row over A/J. Since dim $(A/J) = 0$, by (2.5), there exists an elementary transformation $\overline{\sigma} \in \mathrm{E}_n(A/J)$ such that $[\overline{b}_1, \dots, \overline{b}_n]\, \overline{\sigma} = [1, 0, \dots, 0]$.

After changing by elementary transformation we can assume, as in (3.4), that $b_1 = 1$ modulo J, $b_i \in J$, $i = 2, \dots, n$ and ht $(K) = n - 1$, where $K = (b_1, \dots, b_{n-1})$. Then $K + J = A$. Let $S = 1 + K$. Consider the natural mapping from $A \to A_S$. Since $S \cap J \neq \varnothing$, we have $(a_1, \dots, a_n)_S = (b_1, \dots, b_n)_S$.

Claim There exists $\tau \in \mathrm{GL}_n(A_S)$ such that $[a_1, \dots, a_n]\,\tau = [b_1, \dots, b_n]$.

Assume the claim. Then, there exists an element $b = 1 + a \in S$, $a \in K$ and $\tau \in \mathrm{GL}_n(A_b)$ such that $[a_1, \dots, a_n]\,\tau = [b_1, \dots, b_n]$. Moreover, since $S \cap J \neq \varnothing$, we can assume that $b \in J$.

Let $\beta : (A_b)^n \twoheadrightarrow J_b = A_b$ defined by $\beta(e_1) = 1$ and $\beta(e_i) = 0$, $i = 2, \dots, n$.

$\alpha : (A_a)^n \twoheadrightarrow J_a = (a_1, \dots, a_n)$ defined by $\alpha(e_i) = a_i$.

Since $[b_1, \dots, b_{n-1}] \in (A_{ab})^{n-1}$ is a unimodular row, $[b_1, \dots, b_n]$ can be taken to $[1, 0, \dots, 0]$ by an elementary transformation $\Delta \in \mathrm{E}_n(A_{ab})$.

Define $\delta : (A_{ab})^n \twoheadrightarrow J_{ab} = A_{ab} = (J_1)_{ab}$ by $\delta(e_i) = b_i$. Hence, we have $\alpha_b\,\tau_a = \delta$ and $\delta\,\Delta = \beta_a$. From these two relations, we get $\alpha_b\,\tau_a\,\Delta = \beta_a$. Let $\widetilde{\Delta} = \tau_a\,\Delta\,\tau_a^{-1}$. Then we have $\alpha_b\,\widetilde{\Delta}\,\tau_a = \beta_a$. Hence $\alpha_b\,\widetilde{\Delta} = \beta_a\,\tau_a^{-1} = (\beta\,\tau^{-1})_a$.

Since Δ is an elementary automorphism, we have that $\widetilde{\Delta}$ is isotopic to identity. Hence $\widetilde{\Delta} = (\widetilde{\Delta}_1)_b\,(\widetilde{\Delta}_2)_a$ by (2.21), where $\widetilde{\Delta}_1$ is an automorphism of $(A_a)^n$ which is identity modulo the ideal (b) and $\widetilde{\Delta}_2$ is an automorphism of $(A_b)^n$ which is identity modulo the ideal (a). Hence we have $\alpha_b\,(\widetilde{\Delta}_1)_b\,(\widetilde{\Delta}_2)_a = (\beta\,\tau^{-1})_a$ and hence we get $(\alpha\,\widetilde{\Delta}_1)_b = (\beta\,\tau^{-1}\,\widetilde{\Delta}_2^{-1})_a$. The surjections

$$\alpha\,\widetilde{\Delta}_1 = [c'_1, \dots, c'_n] : (A_a)^n \twoheadrightarrow J_a$$

$$\beta\,\tau^{-1}\,\widetilde{\Delta}_2^{-1} = [d'_1, \dots, d'_n] : (A_b)^n \twoheadrightarrow J_b$$

patch up to give $J = (c_1, \dots, c_n)$ such that $c_i = c'_i$ in A_a and $c_i = d'_i$ in A_b. Now we show that $c_i - a_i \in J^2$. Since $b = 1 + a \in J$, the map $A \to A/(b)$ factors through A_a. Since $\widetilde{\Delta}_1 = \mathrm{id} \pmod{(b)}$, from the equation $\alpha\,\widetilde{\Delta}_1 = [c'_1, \dots, c'_n]$, it follows by going modulo J, that $c_i - a_i \in J^2$.

Proof of the claim To prove the claim, replace A by A_S. Then $K = (b_1, \dots, b_{n-1})$ is an ideal of height $n-1$ such that $K \subset J(A)$. Given $J = (a_1, \dots, a_n) = (b_1, \dots, b_n)$ such that $a_i - b_i \in J^2$ and $\mathrm{ht}\,(J) = n$. To show that there exists $\tau \in \mathrm{GL}_n(A)$ such that $[a_1, \dots, a_n]\,\tau = [b_1, \dots, b_n]$.

Let $b_i = a_i + d_i$, $d_i \in J^2$. Then $d_i = \sum_{j=1}^n \lambda'_{ij}\,a_j$, $\lambda'_{ij} \in J$. Hence there exists $\sigma \in \mathrm{M}_n(A)$ such that $\sigma = \mathrm{id} \pmod{J}$, and $[a_1, \dots, a_n]\,\sigma = [b_1, \dots, b_n]$. Similarly, there exists $\theta \in \mathrm{M}_n(A)$ such that $\theta = \mathrm{id} \pmod J$ and $[b_1, \dots, b_n]\,\theta = [a_1, \dots, a_n]$.

Let $\sigma = (\lambda_{ij})$ and $\theta = (\mu_{ij})$. Then, we have $[b_1, \dots, b_n]\,\theta\,\sigma = [b_1, \dots, b_n]$. Let $\gamma = \sum_{j=1}^{n} \lambda_{nj}\,\mu_{jn}$. From the above equation, we get $b_n = c_1\,b_1 + \dots + c_{n-1}\,b_{n-1} + \gamma\,b_n$, for some $c_1, \dots, c_{n-1} \in A$. Hence we have $(1-\gamma)\,b_n \in (b_1, \dots, b_{n-1}) = K$. Since ht $(K) = n-1$ and K is generated by $n-1$ elements, then any minimal prime ideal of K is also of height $n-1$. Hence b_n does not belong to any minimal prime ideal of K. Hence $(1-\gamma) \in \sqrt{K}$. This implies that $(\gamma) + \sqrt{K} = A$ and so $(\gamma) + K = A$. But $K \subset J(A)$, hence $(\gamma) = A$. This shows that $\gamma \in A$ is a unit. Hence $[\lambda_{n1}, \dots, \lambda_{nn}]$ is a unimodular row. In fact $[\lambda_{n1}, \dots, \lambda_{nn}] \in \mathrm{Um}_n(A, J)$, i.e. it is a lift of the unimodular row $[0, \dots, 0, 1]$ in $\mathrm{Um}_n(A/J)$.

Assume that $n \geq 3$. Let "bar" denote modulo K. Since dim $(A/K) = 1$, by (2.5), the unimodular row $[\overline{\lambda}_{n1}, \dots, \overline{\lambda}_{nn}]$ can be taken to $[1, 0, \dots, 0]$ by an elementary transformation $\overline{\sigma}_1$. Since $K \subset J(A)$, $1+K$ are units in A. We first show that $[\lambda_{n1}, \dots, \lambda_{nn}]$ can be taken to $[1, 0, \dots, 0]$ by elementary transformation. To see this, first take an elementary lift, say $\sigma_1 \in \mathrm{E}_n(A)$ of $\overline{\sigma}_1$. Assume $[\lambda_{n1}, \dots, \lambda_{nn}]\,\sigma_1 = [1+u_1, \dots, u_n]$, where $u_i \in K$. Since $1+u_1$ is a unit, $[1+u_1, \dots, u_n]$ can be taken to $[1, 0, \dots, 0]$ by an elementary transformation. Hence, the row $[\lambda_{n1}, \dots, \lambda_{nn}]$ is completable to an elementary matrix over A. Let $\Delta \in \mathrm{E}_n(A)$ be such that

$$\Delta = \begin{bmatrix} \Delta_1 & & * \\ & & \vdots \\ \lambda_{n1} & \dots & \lambda_{nn} \end{bmatrix}_{n \times n}$$

Consider the map from $\mathrm{E}_n(A) \to \mathrm{E}_n(A/J)$. Let "tilde" denote modulo J. Then

$$\widetilde{\Delta} = \begin{bmatrix} \widetilde{\Delta}_1 & & * \\ & & \vdots \\ 0 & \dots & 1 \end{bmatrix}_{n \times n}$$

where $\widetilde{\Delta}_1 \in \mathrm{SL}_{n-1}(A/J) = \mathrm{E}_{n-1}(A/J)$. Let $\delta \in \mathrm{E}_{n-1}(A)$ be a lift of $\widetilde{\Delta}_1$. Then the inverse $\delta^{-1} \in \mathrm{E}_{n-1}(A)$. Then

$$\begin{bmatrix} \delta^{-1} & & 0 \\ & & \vdots \\ 0 & \dots & 1 \end{bmatrix} \Delta = \begin{bmatrix} \theta' & & * \\ & & \vdots \\ \lambda_{n1} & \dots & \lambda_{nn} \end{bmatrix}$$

where $\theta' \in \mathrm{M}_{n-1}(A)$ is such that $\theta' = \mathrm{id} \pmod{J}$. Hence, after changing by elementary transformation, we can assume that $\Delta \in \mathrm{E}_n(A)$ is such that $\Delta_1 = \mathrm{id} \pmod{J}$. Let $[a_1, \ldots, a_n]\,\Delta = [a'_1, \ldots, a'_{n-1}, b_n]$, where $a_i - a'_i \in J^2$. Then $(a_1, \ldots, a_n) = (a'_1, \ldots, a'_{n-1}, b_n) = (b_1, .., b_n)$. Let $a'_i = c_i + d_i\, b_n$, $c_i \in K$ and $d_i \in A$. Consider the matrix

$$\Gamma = \begin{bmatrix} 1 & \ldots & 0 & -d_1 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 1 & -d_{n-1} \\ 0 & \ldots & 0 & 1 \end{bmatrix}_{n \times n}$$

Then Γ is an elementary matrix and $[a'_1, \ldots, a'_{n-1}, b_n]\,\Gamma = [c_1, \ldots, c_{n-1}, b_n]$, where $a'_i - c_i \in (b_n)$. Hence $[a_1, \ldots, a_n]\,\Gamma\,\Delta = [c_1, \ldots, c_{n-1}, b_n]$ and so $(c_1, \ldots, c_{n-1}, b_n) = (b_1, \ldots, b_n)$. Since $c_i - a_i \in J^2 + A\, b_n$, we have $c_i - b_i \in J^2 + A\, b_n$. Let "bar" denote modulo (b_n). Since $K + A\, b_n = J$ we have $\overline{J} = (\overline{b}_1, \ldots, \overline{b}_{n-1}) = (\overline{c}_1, \ldots, \overline{c}_{n-1})$ and $\overline{b}_i - \overline{c}_i \in \overline{J}^{\,2}$. Hence there exists a $\psi \in \mathrm{M}_{n-1}(\overline{A})$ such that $\psi = \mathrm{id} \pmod{\overline{J}}$ and $[\overline{b}_1, \ldots, \overline{b}_{n-1}]\,\psi = [\overline{c}_1, \ldots, \overline{c}_{n-1}]$.

Let $\psi = (\overline{s_{ij}})$, where $\overline{s_{ij}} \in A/(b_n)$. Then we have

$$h_j = \sum_{i=1}^{n-1} b_i\, s_{ij} \;-\; c_j \in (b_n), \quad \text{for } 1 \leq j \leq n-1 \,.$$

Let $h_j = f_j\, b_n$. Since $\psi = \mathrm{id} \pmod{J}$, we have $s_{ii} = 1 + t_{ii}\, b_n + d_{ii}$ for $1 \leq i \leq n-1$, where $t_{ii} \in A$, $d_{ii} \in K$ and $s_{ij} = t_{ij}\, b_n + d_{ij}$ for $1 \leq i, j \leq n-1$, $i \neq j$, where $t_{ij} \in A$, $d_{ij} \in K$. Hence we have the following relations

$$\sum_{i=1}^{n-1} b_i\,(\delta_{ij} + d_{ij}) - c_j = b_n(f_j - \sum_{i=1}^{n-1} b_i t_{ij})$$

where δ_{ij} is the *Kronecher delta function.* Let us denote by

$$g_j = f_j - \sum_{i=1}^{n-1} b_i t_{ij}, \quad 1 \leq j \leq n-1.$$

Consider the matrix

$$\alpha = \begin{bmatrix} 1 + d_{11} & \ldots & d_{1,n-1} & -g_1 \\ \vdots & & \vdots & \vdots \\ d_{n-1,1} & \ldots & 1 + d_{n-1,n-1} & -g_{n-1} \\ 0 & \ldots & 0 & 1 \end{bmatrix}_{n \times n}$$

Then we have $[b_1, \ldots, b_n]\,\alpha = [c_1, \ldots, c_{n-1}, b_n]$ and $\alpha \in \mathrm{GL}_n(A)$, since det $(\alpha) = 1 + x$ for some $x \in K$. But $K \subset J(A)$, hence $1 + x$ is a unit in A. Thus the claim is proved.

When $n = 2$, then the claim follows from (2.15). Hence the theorem is proved.

$\blacksquare$

Remark 3.6 In fact one can prove by the method of (*[MK 2], p. 248*) that the subtraction principle (3.5) implies the addition principle (3.4) as follows.

Let I and J be two comaximal ideals of height n. Let $I = (a_1, \ldots, a_n)$ and $J = (b_1, \ldots, b_n)$. We want to show that $I \cap J = (c_1, \ldots, c_n)$ with $a_i - c_i \in I^2$ and $b_i - c_i \in J^2$. We can find $x_1, \ldots, x_n \in I \cap J$ which generate $I \cap J$ modulo $(I \cap J)^2$ such that $a_i - x_i \in I^2$ and $b_i - x_i \in J^2$. Using (2.14), we may further assume that $(x_1, \ldots, x_n) = I \cap J \cap K$, where K is a height n ideal, comaximal with $I \cap J$. Since $I = (a_1, \ldots, a_n)$ and $a_i - x_i \in I^2$, by the subtraction principle, we have $J \cap K = (d_1, \ldots, d_n)$ where $x_i - d_i \in (J \cap K)^2$. Since $J = (b_1, \ldots, b_n)$ and $b_i - x_i \in J^2$, $b_i - d_i \in J^2$, again by the subtraction principle $K = (g_1, \ldots, g_n)$ where $d_i - g_i \in K^2$. Hence $x_i - g_i \in K^2$. Applying subtraction principle to the ideal $I \cap J$ and K, we get $I \cap J = (c_1, \ldots, c_n)$ where $x_i - c_i \in (I \cap J)^2$. Hence $a_i - c_i \in I^2$ and $b_i - c_i \in J^2$. This proves the addition principle.

Now we state the subtraction principle in the general case *([B-RS 3], Theorem 3.3)*. Notice that when P is free, this reduces to (3.5).

Theorem 3.7 (Subtraction Principle) *Let A be a Noetherian ring of dimension $n \geq 2$. Let P be a projective A-module of rank n with trivial determinant. Let $\chi : A \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. Let J, J' be two ideals of A. Let "bar" denote reduction modulo J'. Assume*

(i) ht $J \geq n$, ht $J' = n$ *and* $J + J' = A$.

(ii) $\alpha : P \twoheadrightarrow J \cap J'$ and $\beta : A^n \twoheadrightarrow J'$ be two surjections.

(iii) $\overline{\alpha} : \overline{P} \twoheadrightarrow J'/J'^2$ and $\overline{\beta} : \overline{A^n} \twoheadrightarrow J'/J'^2$ be surjections induced from α and β respectively.

(iv) There exists an isomorphism $\delta : \overline{A^n} \xrightarrow{\sim} \overline{P}$ such that $\overline{\alpha}\delta = \overline{\beta}$, and $\wedge^n(\delta) = \overline{\chi}$.

Then, there exists a surjection $\theta : P \twoheadrightarrow J$ such that $\theta \otimes A/J = \alpha \otimes A/J$.

Taking $J = A$ in the above theorem, we obtain the following:

Corollary 3.8 *Let A be a Noetherian ring of dimension $n \geq 2$. Let P be a projective A-module of rank n with trivial determinant. Let $\chi : A \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. Let $J' \subset A$ be an ideal of height n. Let "bar" denote reduction modulo J'. Assume*

(i) $\alpha : P \twoheadrightarrow J'$ and $\beta : A^n \twoheadrightarrow J'$ be two surjections.

(ii) $\overline{\alpha} : \overline{P} \twoheadrightarrow J'/J'^2$ and $\overline{\beta} : \overline{A^n} \twoheadrightarrow J'/J'^2$ be surjections induced from α and β respectively.

(iii) There exists an isomorphism $\delta : \overline{A^n} \xrightarrow{\sim} \overline{P}$ such that $\overline{\alpha}\delta = \overline{\beta}$ and $\wedge^n(\delta) = \overline{\chi}$.

Then P has a unimodular element.

Chapter 4

The Euler Class Group of a Noetherian Ring

For the rest of this thesis, we assume that all rings considered contain the field $\mathbb{Q}$ of rational numbers. We make this assumption as we need to apply (4.7) to show that the "Euler class" of a projective module is well defined. In general, one can define the Euler class group of A with respect to any rank one projective A-module L. However, we'll define it with respect to A only.

Let A be a Noetherian ring with dim $(A) = n \geq 2$. We define the *Euler class group of* A, denoted by $E(A)$, as follows:

Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let α and β be two surjections from $(A/J)^n$ to J/J^2. We say that α and β are *related* if there exists an automorphism σ of $(A/J)^n$ of determinant 1 such that $\alpha\sigma = \beta$. It is easy to see that this is an equivalence relation on the set of generators of J/J^2. If $\alpha : (A/J)^n \twoheadrightarrow J/J^2$ is a surjection, then by $[\alpha]$, we denote the equivalence class of α. We call such an equivalence class $[\alpha]$ a *local orientation* of J.

Since dim $(A/J) = 0$ and $n \geq 2$, we have $\mathrm{SL}_n(A/J) = \mathrm{E}_n(A/J)$ and therefore, by (2.26), the canonical map from $\mathrm{SL}_n(A)$ to $\mathrm{SL}_n(A/J)$ is surjective. Hence if a surjection $\alpha : (A/J)^n \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : A^n \twoheadrightarrow J$, and α is equivalent to $\beta : (A/J)^n \twoheadrightarrow J/J^2$, then β can also be lifted to a surjection from A^n to J. For, let $\alpha\,\sigma = \beta$, for some $\sigma \in \mathrm{SL}_n(A/J)$. Then there exists $\widetilde{\sigma} \in \mathrm{SL}_n(A)$ which is a lift of σ by

(2.26). Then, $\theta\widetilde{\sigma} : A^n \twoheadrightarrow J$ is a lift of β.

A local orientation $[\alpha]$ of J is called a *global orientation* of J if the surjection $\alpha : (A/J)^n \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : A^n \twoheadrightarrow J$.

We shall also, from now on, identify a surjection α with the equivalence class $[\alpha]$ to which α belongs.

Let $\mathcal{M} \in A$ be a maximal ideal of height n and $\mathcal{N}$ be a $\mathcal{M}$-primary ideal such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements. Let $w_{\mathcal{N}}$ be a local orientation of $\mathcal{N}$. Let G be the free abelian group on the set of pairs $(\mathcal{N}, w_{\mathcal{N}})$, where $\mathcal{N}$ is a $\mathcal{M}$-primary ideal and $w_{\mathcal{N}}$ is a local orientation of $\mathcal{N}$.

Let $J = \cap \mathcal{N}_i$ be the intersection of finitely many ideals $\mathcal{N}_i$, where $\mathcal{N}_i$ is $\mathcal{M}_i$-primary, $\mathcal{M}_i \subset A$ being distinct maximal ideals of height n. Assume that J/J^2 is generated by n elements. Let w_J be a local orientation of J. Then, w_J gives rise, in a natural way, to a local orientation $w_{\mathcal{N}_i}$ of $\mathcal{N}_i$. We associate to the pair (J, w_J), the element $\sum(\mathcal{N}_i, w_{\mathcal{N}_i})$ of G. By abuse of notation, we denote the element $\sum(\mathcal{N}_i, w_{\mathcal{N}_i})$ by (J, w_J).

Let H be the subgroup of G generated by set of pairs (J, w_J), where J is an ideal of height n which is generated by n elements and w_J is a global orientation of J. We define the *Euler class group* of A, $E(A) = G/H$. Thus $E(A)$ can be thought of as the quotient of the group of local orientations by the subgroup generated by global orientations.

One of the aims of this chapter is to prove theorem (4.2) which states that if (J, w_J) is zero in $E(A)$ i.e. $(J, w_J) \in H$, then J is generated by n elements and w_J is a global orientation of J. This is proved as follows: first assume that $J = J_1 \cap J_2$ where J_1 and J_2 are two comaximal ideals of height n, which are generated by n elements. Assume w_J be a local orientation of J which is induced by generators of J_1 and J_2. Then $(J, w_J) = 0$ in $E(A)$. By (3.4), J is generated by n elements and w_J is a global orientation of J. Now assume $J_2 = J \cap J_1$ where J and J_1 are comaximal ideals of height n. Assume $J_1 = (a_1, \ldots, a_n)$ and $J_2 = (b_1, \ldots, b_n)$ where $a_i - b_i \in J_1^2$. Assume w_J is a local orientation of J which is induced by the generators of J_2. Then $(J, w_J) = 0$ in $E(A)$. By (3.5) J is generated by n elements and w_J is a global orientation of J. Hence the addition and subtraction principles are actually special cases of (4.2). Using these two special cases and a formal group theoretic lemma (4.1), we prove the theorem. Using this we show (4.10) that $E(A)$ detects the obstruction for a projective module of trivial determinant to have a unimodular element.

Lemma 4.1 *Let F be the free abelian group with basis $(e_i)_{i\in I}$. Let $\sim$ be an equivalence relation on $(e_i)_{i\in I}$. Define $x \in F$ to be "*reduced*" if $x = e_1 + \ldots + e_r$ and $e_i \neq e_j$ if $i \neq j$. For $x \in F$ with $x = e_1 + \ldots + e_r$, define "*support*" of x to be the set $\{e_1, \ldots, e_r\}$ and denote it by* supp (x). *Define $x \in F$ to be "*nicely reduced*" if $x = e_1 + \ldots + e_r$ and $e_i \neq e_j$ if $i \neq j$ and such that no e_i belongs to the equivalence class of other e_j for $i, j = 1, \ldots, r$ and $i \neq j$. Let $S \subset F$ be such that :*

(1) Every element of S is nicely reduced.

(2) Let x, $y \in F$ be nicely reduced such that $x + y$ is also nicely reduced. Then if any two of x, y and $x + y$ belongs to S so does the third one.

(3) Let $x \in F$, $x \notin S$ and x is nicely reduced and let $J \subset I$ be a finite set. Then, there exists $y \in F$ satisfying the following properties:

(i) y is nicely reduced, (ii) $x + y \in S$ and (iii) $y + e_j$ is nicely reduced $\forall\, j \in J$.

Let H be the subgroup of F generated by S. Then if $x \in H$ is nicely reduced, then $x \in S$.

Remark Let x, y be elements of F with positive coefficients and $z = x + y$ be nicely reduced. Then x and y are nicely reduced.

Proof Let

$$y_1 + \ldots + y_r + x = z_1 + \ldots + z_s \qquad (*)$$

where $y_i, z_j \in S$, $1 \leq i \leq r$, $1 \leq j \leq s$. If $z_1 + \ldots + z_s$ is nicely reduced, then by previous remark $y_1 + \ldots + y_r$ is also nicely reduced. Hence $z_1 + \ldots + z_s \in S$ and $y_1 + ... + y_r \in S$ by assumption (2). Then $x \in S$ by assumption (2).

Now assume that $z_1 + \ldots + z_s$ is not nicely reduced. Given an equality of the type $(*)$, we associate a non-negative integer $n(*)$ in the following manner: For a basis element e_i of F, we associate a number $n(e_i(*))$ as follows: $n(e_i(*)) + 1$ is the cardinality of the set $\{t \,|\, e_i + z_t$ is not nicely reduced for $1 \leq t \leq s\}$ and let $n(*) = \sum n(e_i(*))$ for the equation $(*)$, where the sum is over those e_i's which belong to the set $\bigcup_1^s \mathrm{supp}(z_t)$. We note that

$n(*) = 0$ if and only if $z_1 + \ldots + z_s$ is nicely reduced.

Since $z_1 + \ldots + z_s$ is not nicely reduced (i.e. $n(*)$ is positive), there exist z_k, z_l, $1 \leq k, l \leq s$, $k \neq l$ such that $z_k + z_l$ is not nicely reduced. Without loss of generality, we can assume that $k = 1$, $l = 2$. Let $z_1 = e_1 + w_1$, $z_2 = e_1' + w_2$ and $e_1 \sim e_1'$. Since x is nicely reduced, at least one of e_1, $e_1' \in$ supp (y_i) for some $1 \leq i \leq r$. Without loss of generality, we can assume that $e_1 \in$ supp (y_1) and assume that $y_1 = e_1 + u_1$. The equation $(*)$ can be written as

$$u_1 + y_2 + \ldots + y_r + x = w_1 + z_2 + \ldots + z_s \tag{$*_1$}$$

If $e_1 \in S$, then by assumption (2) u_1, $w_1 \in S$. Then, we see that $n(*_1) < n(*)$. Hence by induction, we are through.

Now, we assume that $e_1 \notin S$. Then, by assumption (2) $w_1, u_1 \notin S$. Let J be the set $\{ i \in I |\ e_i \in \bigcup_1^s \text{supp } (z_t)\}$. Then $J \subset I$ is a finite set and $w_1 \in F$ is nicely reduced such that $w_1 \notin S$. By assumption (3), there exists $\theta \in F$ such that (i) θ is nicely reduced, (ii) $w_1 + \theta \in S$ and (iii) $\theta + e_j$ is nicely reduced $\forall j \in J$. Now we claim that $\theta + u_1 \in S$.

Proof of the claim : Let J' be the set $\{ i \in I |\ e_i \in \bigcup_1^s \text{supp}\,(z_t) \bigcup \text{supp}\,(\theta)\}$. Then $J' \subset I$ is a finite set. Then by assumption (3), there exists $\theta' \in F$ such that (i) θ' is nicely reduced, (ii) $e_1 + \theta' \in S$ and (iii) $\theta' + e_j$ is nicely reduced $\forall j \in J'$.

We have $w_1 + \theta \in S$, $e_1 + \theta' \in S$. As $w_1 + \theta + e_1 + \theta'$ is nicely reduced, by assumption (2) $w_1 + \theta + e_1 + \theta' \in S$. Hence $e_1 + w_1 + \theta + \theta' \in S$ and $e_1 + w_1 = z_1 \in S$. Then by assumption (2), $\theta + \theta' \in S$. We have $e_1 + u_1 = y_1 \in S$ and $e_1 + u_1 + \theta + \theta' \in S$ as it is nicely reduced. Since $e_1 + \theta' \in S$, we have $\theta + u_1 \in S$. Thus the claim is proved.

Now, the equation $(*_1)$ can be written as

$$(u_1 + \theta) + y_2 + \ldots + y_r + x = (w_1 + \theta) + z_2 + \ldots + z_s \tag{$*_2$}$$

where $\theta + u_1 \in S$ and $\theta + w_1 \in S$. Hence we see that $n(*_2) < n(*_1)$, since $n(e_1(*_2)) < n(e_1(*_1))$ and $n(e_i(*_2)) = n(e_i(*_1))$ or 0 for $i \neq 1$ according as $e_i \in \bigcup_1^s \text{supp}\,(z_i)$ or $e_i \in \text{supp}\,(\theta)$. Hence, by induction, the lemma follows. ■

Theorem 4.2 *Let A be a Noetherian ring of dimension $n \geq 2$. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements, and let $w_J : (A/J)^n \twoheadrightarrow J/J^2$ be a local orientation of J. Suppose that the image of (J, w_J) is zero in the Euler class group $E(A)$ of A. Then J is generated by n elements and w_J is a global orientation of J.*

Proof Let F be the free abelian group on the set of pairs $(\mathcal{N}, w_{\mathcal{N}})$ such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements (where $\mathcal{N}$ is $\mathcal{M}$-primary ideal and $\mathcal{M}$ is a maximal ideal of height n). Define an equivalence relation on the set of pairs $(\mathcal{N}, w_{\mathcal{N}})$ by $(\mathcal{N}, w_{\mathcal{N}}) \sim (\mathcal{N}_1, w_{\mathcal{N}_1})$ if $\sqrt{\mathcal{N}} = \sqrt{\mathcal{N}_1}$, i.e. $\mathcal{N}$, $\mathcal{N}_1$ both are $\mathcal{M}$-primary ideals of A. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements and $J = \cap \mathcal{N}_i$ be a reduced primary decomposition of J. Then denote $(J, w_J) = \sum (\mathcal{N}_i, w_i)$ where w_i is induced from w_J.

Let $S = \{(J, w_J) \in F \mid J = (a_1, \dots, a_n) \,\text{and}\, w_J = (\overline{a}_1, \dots, \overline{a}_n)\,\}$. We check that the conditions $1, 2$ and 3 of lemma (4.1) hold.

(1) If $(J, w_J) \in S$, then it is nicely reduced. Since if $J = \bigcap_{i=1}^r \mathcal{N}_i$, then each $\mathcal{N}_i$ is comaximal with the other $\mathcal{N}_j$, $j \neq i$ and $(J, w_J) = (\mathcal{N}_1, w_1) + \ldots + (\mathcal{N}_r, w_r)$.

(2) If (J, w_J) and $(J', w_{J'})$ are nicely reduced elements of F such that $(J, w_J) + (J', w_{J'})$ is also nicely reduced (i.e. $J + J' = A$). Then by the addition principle (3.4) and subtraction principle (3.5), it follows that if any two of (J, w_J), $(J', w_{J'})$ and $(J, w_J) + (J', w_{J'})$ belong to S, then so does the third.

(3) Similarly by (2.14), condition 3 of lemma (4.1) holds. Now applying (4.1), the theorem is proved. ∎

Lemma 4.3 *Let A be a Noetherian ring and P a projective A-module of rank n. Let $\lambda : P \twoheadrightarrow J_0$ and $\mu : P \twoheadrightarrow J_1$ be surjections, where $J_0, J_1 \subset A$ are ideals of height n. Then, there exists an ideal I of $A[T]$ of height n and a surjection $\alpha(T) : P[T] \twoheadrightarrow I$ such that $I(0) = J_0, \alpha(0) = \lambda$ and $I(1) = J_1, \alpha(1) = \mu$, where for $a \in A, I(a) = \{F(a) : F(T) \in I\}$.*

Proof Let $\alpha(T) = T\,\mu(T) + (1-T)\,\lambda(T)$, where $\lambda(T) = \lambda \otimes A[T]$ and $\mu(T) = \mu \otimes A[T]$. Then $\alpha(0) = \lambda$, and $\alpha(1) = \mu$.

Claim $(\alpha(T)(P[T]) + (T(1-T))) = (J_0A[T], T) \,\cap\, (J_1A[T], 1-T)$.

Clearly LHS $\subset$ RHS. Now let $G = Tf + g = (1-T)f_1 + g_1 \in$ RHS, where $f, f_1 \in A[T]$, $g \in J_0A[T]$, $g_1 \in J_1A[T]$. Then $T(f + f_1) = f_1 + g_1 - g$. Now $G = (1-T)f_1 + g_1 = T(1-T)(f + f_1) + (1-T)g + Tg_1$. We want to show that $(1-T)g + Tg_1 \in$ LHS. Now there exist $p(T), q(T) \in P[T]$ such that $\lambda(T)(p(T)) = g$ and $\mu(T)(q(T)) = g_1$. Hence $\alpha(T)((1-T)p(T)) = T(1-T)\mu(T)(p(T)) + (1-T)^2 g \in$ LHS and so $(1-T)^2 g \in$ LHS.

But $(1-T)^2g = (1-T)g - T(1-T)g$. Hence $(1-T)g \in \text{LHS}$. Similarly, taking $Tq(T)$ we can show that $Tg_1 \in \text{LHS}$. This proves the claim.

Now replacing $\alpha(T)$ by $\alpha(T) + T(1-T)\beta(T)$ for a suitable $\beta(T) \in P[T]^*$, we may assume, by (2.12), that $\alpha(P[T]) = I$ has height n. This proves the lemma. $\blacksquare$

Lemma 4.4 *Let A be a ring and $J \subset A$ an ideal. Let $B = A_{1+J}$. Then JB is contained in the Jacobson radical of B.*

Lemma 4.5 *Let A be a ring and $\mathfrak{p}_1 \subsetneqq \mathfrak{p}_2 \subsetneqq \mathfrak{p}_3$ be a chain of prime ideals of $A[T]$. Then we can not have $\mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A = \mathfrak{p}_3 \cap A$.*

Proof Let us assume contrary. By going modulo $\mathfrak{p}_1 \cap A$, we can assume that A is a domain and $\mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A = \mathfrak{p}_3 \cap A = 0$. Let $S = A - \{0\}$. Then $S^{-1}A[T]$, being principal ideal domain, is of dimension 1. But $S^{-1}\mathfrak{p}_1 \subsetneqq S^{-1}\mathfrak{p}_2 \subsetneqq S^{-1}\mathfrak{p}_3$. This is a contradiction. $\blacksquare$

Lemma 4.6 *Let A be a Noetherian ring and $I \subset A[T]$ be an ideal of height k. Then* ht $(I \cap A) \geq k-1$.

Proof First we assume that $I = \mathfrak{p}$ is a prime ideal. Then we claim that $\text{ht}(\mathfrak{p}) = \text{ht}(\mathfrak{p} \cap A)$ if $\mathfrak{p} = (\mathfrak{p} \cap A)[T]$ and $\text{ht}(\mathfrak{p}) = \text{ht}(\mathfrak{p} \cap A) + 1$ if $\mathfrak{p} \supsetneqq (\mathfrak{p} \cap A)[T]$.

Any prime chain $\mathfrak{q}_0 \subsetneqq \dots \subsetneqq \mathfrak{q}_r \subsetneqq (\mathfrak{p} \cap A)$ in A extends to a prime chain $\mathfrak{q}_0[T] \subsetneqq \dots \subsetneqq \mathfrak{q}_r[T] \subsetneqq (\mathfrak{p} \cap A)[T] \subset \mathfrak{p}$ in $A[T]$. Hence $\text{ht}(\mathfrak{p}) \geq$ the given values. Now let $\text{ht}(\mathfrak{p} \cap A) = r$. Then by the dimension theorem, $(\mathfrak{p} \cap A)$ is minimal over an ideal $\mathfrak{a} = (a_1, \dots, a_r)$. Then $(\mathfrak{p} \cap A)[T]$ is minimal over $\mathfrak{a}[T]$, so $\text{ht}((\mathfrak{p} \cap A)[T]) \leq r$. Thus we have $\text{ht}(\mathfrak{p}) = \text{ht}(\mathfrak{p} \cap A)$ in the case $\mathfrak{p} = (\mathfrak{p} \cap A)[T]$.

Now assume $(\mathfrak{p} \cap A)[T] \subsetneqq \mathfrak{p}$, say $f \in \mathfrak{p} - (\mathfrak{p} \cap A)[T]$. We will be done if we can show that $\mathfrak{p}$ is a minimal prime over $\mathfrak{a}[T] + fA[T]$, for then $\text{ht}(\mathfrak{p}) \leq r+1$. Let $\mathfrak{p}'$ be a prime between these. Then $\mathfrak{a} \subset (p' \cap A) \subset (\mathfrak{p} \cap A)$, so $(\mathfrak{p}' \cap A) = (\mathfrak{p} \cap A)$ since $(\mathfrak{p} \cap A)$ is minimal prime over $\mathfrak{a}$. In particular, $(\mathfrak{p} \cap A)[T] \subsetneqq \mathfrak{p}' \subset \mathfrak{p}$. By (4.5), we have $\mathfrak{p} = \mathfrak{p}'$.

Now we prove the lemma for any ideal $I \subset A[T]$. Let $\sqrt{I} = \bigcap_1^r \mathfrak{p}_i$ where $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are minimal primes over I. Then $\sqrt{(I \cap A)} = \bigcap_1^r (\mathfrak{p}_i \cap A)$. The prime ideals minimal over

$I \cap A$ occur among $\mathfrak{p}_1 \cap A, \ldots, \mathfrak{p}_r \cap A$. Choose $\mathfrak{p}_i$ such that $\mathrm{ht}(I \cap A) = \mathrm{ht}(\mathfrak{p}_i \cap A)$. Then $\mathrm{ht}(I \cap A) = \mathrm{ht}(\mathfrak{p}_i \cap A) \geq \mathrm{ht}(\mathfrak{p}_i) - 1 \geq \mathrm{ht}(I) - 1$. This proves the lemma. ■

Roughly, the aim of the next proposition (4.7) is to show that if $I \subset A[T]$ is an ideal of height n which is the surjective image of an extended projective module $P[T]$ of rank n, then there exists an ideal K of A of height $\geq n$ such that I is comaximal with $KA[T]$ and $I \cap KA[T]$ is generated by n elements. We construct K as follows : we choose K such that $I(0) \cap K$ is generated by n elements. Further, we choose K to be comaximal with $I \cap A$. This is achieved using (2.12). Then, using patching argument we show that $I \cap KA[T]$ is generated by n elements.

Proposition 4.7 *Let A be a Noetherian ring of dimension $n \geq 2$ such that $(n-1)!$ is invertible in A. Let P be a projective A-module of rank n with trivial determinant. Let $\chi : A \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. Suppose that $\alpha(T) : P[T] \twoheadrightarrow I$ is a surjection, where $I \subset A[T]$ is an ideal of height n. Then, there exists a homomorphism $\phi : A^n \to P$, an ideal $K \subset A$ of height $\geq n$ which is comaximal with $I \cap A$ and a surjection $\rho(T) : (A[T])^n \twoheadrightarrow I \cap KA[T]$ such that:*

(i) $\phi \otimes A/N$ is an isomorphism, where $N = (I \cap A)$ and $\wedge^n(\phi) = u\chi$ where $u = 1$ modulo $I \cap A$.

(ii) $(\alpha(0)\phi)(A^n) = I(0) \cap K$.

(iii) $\alpha(T)\phi(T) \otimes A[T]/I = \rho(T) \otimes A[T]/I$.

(iv) $\rho(0) \otimes A/K = \rho(1) \otimes A/K$.

Proof First we show the existence of ϕ satisfying (i) and (ii).

Since ht $(I) = n$, we have ht $(N) \geq n-1$ by (4.6), and hence dim $(A/N) \leq 1$. Now since P has trivial determinant, by Serre's theorem (1.14), there exists an isomorphism $\eta : (A/N)^n \xrightarrow{\sim} P/NP$. We can alter η by an automorphism of $(A/N)^n$ to obtain an isomorphism $\overline{\delta} : (A/N)^n \xrightarrow{\sim} P/NP$ such that $\wedge^n(\overline{\delta}) = \overline{\chi}$, where "bar" denotes reduction modulo N. Let $\delta : A^n \to P$ be a lift of $\overline{\delta}$. Since $NA_{1+N} \subset J(A_{1+N})$, by (1.6), $\delta_{1+N} : (A_{1+N})^n \to P_{1+N}$ is an isomorphism.

Let $J = I(0)$, where $I(0) = \{F(0) | F(T) \in I\}$ and $\beta = \alpha(0) : P \twoheadrightarrow J$. The equality $\delta(A^n) + NP = P$, shows that $(\beta\,\delta)(A^n) + NJ = J$. Since $NJ \subset J^2$, by (2.7) there exists $c \in NJ$ such that $(\beta\,\delta)(A^n) + (c) = J$. Therefore, applying (2.12) to $(\beta\,\delta, c)$ we see that there exists $\gamma \in (A^n)^*$ such that the ideal $(\beta\,\delta + c\gamma)(A^n)$ has height at least minimum of

n and ht (J). Since $(\beta\,\delta + c\gamma)(A^n) + (c) = J$ and $c \in J^2$, by (2.7), $(\beta\,\delta + c\gamma)(A^n) = J \cap K$, where K is either A or an ideal of height n which is comaximal with (c), and hence with N and J.

The next step of the proof is to show that there exists a map $\phi : A^n \to P$ which lifts $\overline{\delta}$ and such that $\beta\phi(A^n) = J \cap K$. This is achieved by altering δ by an element of $\mathrm{Hom}(A^n, NP)$.

Since $c \in NJ$, $c = \sum a_i\, d_i$, where $a_i \in N$ and $d_i \in J$. Any element of $(A^n)^*$ of the form $d\gamma$, where $d \in J$, has its image contained in J. Now, since $d_i \in J$ and $\beta : P \twoheadrightarrow J$ is surjective, there exists $\nu_i : A^n \to P$ such that $\beta\,\nu_i = d_i\,\gamma$. Let $\nu = \sum a_i\,\nu_i$. Then $c\gamma = \Sigma a_i\, d_i\,\gamma = \sum a_i\,\beta\,\nu_i = \beta\,\nu$, where $\nu = 0$ modulo N. Let $\phi = (\delta + \nu)$. Then ϕ is also a lift of $\overline{\delta}$ and hence $\wedge^n\phi = u\chi$, where $u = 1$ modulo N. Moreover ϕ has the property that

$$\begin{aligned}\beta\,\phi(A^n) &= (\beta\,\delta + \beta\,\nu)(A^n) = (\beta\,\delta + \beta \textstyle\sum a_i\,\nu_i)(A^n) = (\beta\,\delta + \sum a_i\,\beta\,\nu_i)(A^n)\\ &= (\beta\,\delta + \textstyle\sum a_i\, d_i\,\gamma)(A^n) = (\beta\,\delta + c\,\gamma)(A^n) = J \cap K.\end{aligned}$$

This proves (i) and (ii).

Since $K + N = A$, we have $I + KA[T] = A[T]$. Let $I' = I \cap KA[T]$. Then $I'(0) = J \cap K$ and $I'/I'^2 = I/I^2 \oplus KA[T]/K^2A[T]$.

Since $NA_{1+N} \subset J(A_{1+N})$, by (1.6), $\phi_{1+N} : (A_{1+N})^n \xrightarrow{\sim} P_{1+N}$ is an isomorphism. Further $I'_{1+N} = I_{1+N}$, as $K \cap (1+N) \neq \varnothing$. Therefore, the map $(\alpha(T)\,\phi(T))_{1+N}$: $(A_{1+N}[T])^n \to I'_{1+N}$ is surjective, where $\phi(T) = \phi \otimes A[T]$. Hence, there exists $a \in N$ such that the map $(\alpha(T)\,\phi(T))_{1+a} : (A_{1+a}[T])^n \to I'_{1+a}$ is surjective. We can assume that $1 + a \in K$, as $N + K = A$. Since $a \in N \subset I$, we have $I'_a = KA_a[T]$. Therefore, we get a surjection $(\beta\,\phi) \otimes A_a[T] : (A_a[T])^n \twoheadrightarrow I'_a$.

The elements $(\beta\,\phi) \otimes A_{a(1+aA)}[T]$ and $(\alpha(T)\,\phi(T))_{a(1+aA)}$ are surjections from $(A_{a(1+aA)}[T])^n \twoheadrightarrow A_{a(1+aA)}[T]$ and as $\alpha(0) = \beta$, they are equal modulo (T). Note that dim $(A_{a(1+aA)}) \leq n-1$ (for if $\mathfrak{m}$ is any maximal ideal of A, then either $a \in \mathfrak{m}$ or $(1 + aA) \cap \mathfrak{m} \neq \varnothing$). The kernels of the surjections $(\beta\,\phi) \otimes A_{a(1+aA)}$ and $(\alpha(T)\,\phi(T))_{a(1+aA)}$ are stably free modules given by unimodular rows. These are extended from $A_{a(1+aA)}$ by (1.19), since $(n-1)!$ is invertible in A. By (2.17), there exists an $\sigma(T) \in \mathrm{GL}_n(A_{a(1+aA)}[T])$ such that $\sigma(0) = \mathrm{id}$ and $(\alpha(T)\,\phi(T))_{a(1+aA)}\,\sigma(T) = (\beta\,\phi) \otimes A_{a(1+aA)}[T]$.

Let $(1+a\,a') = (1+a)\,(1+a\,a'')$ where $a'' \in A$ is chosen so that the following properties hold : (1) det $(\sigma(T))$ is a unit belonging to $A_{a\,(1+a\,a')}[T]$ and (2) $(\alpha(T)\ \phi(T))_{a\,(1+a\,a')}\ \sigma(T) = (\beta\ \phi) \otimes A_{a\,(1+a\,a')}[T]$. Let $b = (1+a\,a')$. Then $\sigma(T) \in \mathrm{GL}_n(A_{ab}[T])$ with $\sigma(0) = \mathrm{id}$. Since $\sigma(0) = \mathrm{id}$, by lemma (2.19), we see that $\sigma(T) = \tau(T)_a.\,\theta(T)_b$, where $\tau(T)$ is an $A_b[T]$-automorphism of $(A_b[T])^n$ such that $\tau(0) = \mathrm{id}$ and $\tau = \mathrm{id}$ modulo (a) and $\theta(T)$ is an $A_a[T]$-automorphism of $(A_a[T])^n$ such that $\theta(0) = \mathrm{id}$ and $\theta = \mathrm{id}$ modulo (b).

We have $((\alpha(T)\,\phi(T))_b.\,\tau(T))_a = (((\beta\,\phi) \otimes A_a[T])\,(\theta(T))^{-1})_b$. Hence, the surjections $(\alpha(T)\,\phi(T))_b.\,\tau(T) : (A_b[T])^n \twoheadrightarrow I'_b$ and $((\beta\,\phi) \otimes A_a[T])\,(\theta(T))^{-1} : (A_a[T])^n \twoheadrightarrow I'_a$ patch to yield a surjection $\rho(T) : (A[T])^n \twoheadrightarrow I'$ such that $\rho(0) = \beta\phi$.

Since $\theta(T) = \mathrm{id}$ modulo the ideal (b) and $b \in K$, it follows from the construction of $\rho(T)$ that $\rho(T)(e_i) - (\beta\phi \otimes A[T])(e_i) \in K^2 A[T]\ \ \forall i$ (where e_i are the coordinate functions of $A[T]^n$). Hence $\rho(0) \otimes A/K = \rho(1) \otimes A/K$. Further, using the fact that $\tau(T) = \mathrm{id}$ modulo the ideal (a), we see that $(\alpha(T)\,\phi(T)) \otimes A[T]/I = \rho(T) \otimes A[T]/I$. This proves (iii) and (iv) and hence the proposition is proved.

$$\begin{tikzcd}
(A[T])^n \arrow[rr] \arrow[dr, two heads, "\rho(T)"] \arrow[dd] & & (A_b[T])^n \arrow[dr, two heads, "(\alpha(T)\,\phi(T))_b\,\tau(T)"] \arrow[dd] & \\
& I' \arrow[rr] \arrow[dd] & & I'_b \arrow[dd] \\
(A_a[T])^n \arrow[rr] \arrow[dr, two heads, "((\beta\,\phi)\otimes A_a[T])\,(\theta(T))^{-1}"'] & & (A_{ab}[T])^n \arrow[dr] & \\
& I'_a \arrow[rr] & & I'_{ab}
\end{tikzcd}$$

■

Remark 4.8 Now we discuss (4.7) in the context of the Euler class group $E(A)$. Let $I \subset A[T]$ be an ideal of height n, where A is of dimension n. Suppose I is a surjective image of a projective $A[T]$-module $P[T]$, where P is a projective A-module of rank n having trivial determinant. Further assume that $I(0)$ and $I(1)$ are ideals of height n. Now tensoring the surjection from $P[T]$ to I and $\phi : A^n \to P$ given in (4.7) with $A[T]/I$ and composing, we get a 'local orientation' $w(T)$ of I i.e. a surjection $w(T) : (A[T]/I)^n \twoheadrightarrow I/I^2$, which in turn gives rise to local orientations $w(0) : (A/I(0))^n \twoheadrightarrow I(0)/I(0)^2$ and $w(1) : (A/I(1))^n \twoheadrightarrow I(1)/I(1)^2$ of $I(0)$ and $I(1)$ respectively.

The gist of (4.7) is that there exists an ideal $K \subset A$ of height n and a local orientation w_K of K, which is the class of the surjection $\rho(0) \otimes A/K : (A/K)^n \twoheadrightarrow K/K^2$, such that

$$(I(0), w(0)) + (K, w_K) = 0 = (I(1), w(1)) + (K, w_K)$$

in $E(A)$. Therefore $(I(0), w(0)) = (I(1), w(1))$ in $E(A)$.

In case when $K = A$, we get $\rho(T) : (A[T])^n \twoheadrightarrow I$, hence I is generated by n elements. Hence $w(0)$ and $w(1)$ are global orientations of $I(0)$ and $I(1)$ respectively. So $(I(0), w(0)) = 0 = (I(1), w(1))$ in $E(A)$.

Let P be a projective A-module of rank n with trivial determinant. Let $\chi : A \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. We call χ an *orientation of* P and $\chi(1)$ a generator of $\wedge^n(P)$. We write χ for $\chi(1)$. To the pair (P, χ), we associate an element $e(P, \chi)$ of $E(A)$ as follows:

Let $\lambda : P \twoheadrightarrow J_0$ be a surjection, where $J_0 \subset A$ is an ideal of height n. Let "bar" denote reduction modulo J_0. We obtain an induced surjection $\overline{\lambda} : P/J_0P \twoheadrightarrow J_0/J_0^2$. We choose an isomorphism $\overline{\gamma} : (A/J_0)^n \xrightarrow{\sim} P/J_0P$, such that $\wedge^n(\overline{\gamma}) = \overline{\chi}$.

Let w_{J_0} be a local orientation of J_0 given by $\overline{\lambda}\,\overline{\gamma} : (A/J_0)^n \twoheadrightarrow J_0/J_0^2$. Let $e(P, \chi)$ be the image in $E(A)$ of the element (J_0, w_{J_0}) of G. We say that (J_0, w_{J_0}) is *obtained* from the pair (λ, χ). We show that the assignment sending the pair (P, χ) to the element $e(P, \chi)$ of $E(A)$ is well defined.

Let $\mu : P \twoheadrightarrow J_1$ be another surjection, where $J_1 \subset A$ is an ideal of height n. Then by (4.3), there exists an ideal I of $A[T]$ of height n and a surjection $\alpha(T) : P[T] \twoheadrightarrow I$ such that $\alpha(0) = \lambda$ $\alpha(1) = \mu$, $I(0) = J_0$ and $I(1) = J_1$.

Then from the above discussion, we have $(J_0, w_{J_0}) = (J_1, w_{J_1})$ in $E(A)$, where $w_{J_0} = \overline{\lambda}\,\overline{\gamma}$ and $w_{J_1} = \overline{\mu}\,\overline{\gamma}$. Hence $e(P, \chi)$ does not depend on the choice of the surjection.

Now, let $\lambda : P \twoheadrightarrow J_0$ be a surjection, where $J_0 \subset A$ is an ideal of height n. If $\overline{\delta} : (A/J_0)^n \xrightarrow{\sim} P/J_0P$ is another isomorphism such that $\wedge^n(\overline{\delta}) = \overline{\chi}$, then $\overline{\delta}$ and $\overline{\gamma}$ differ by an element of $\mathrm{SL}_n(A/J_0)$. Hence there exists an $\overline{\sigma} \in \mathrm{SL}_n(A/J_0)$ such that $\overline{\delta} = \overline{\sigma}\,\overline{\gamma}$. This shows that $e(P, \chi)$ does not depend on the choice of $\overline{\gamma}$ and proves that $e(P, \chi)$ is well defined. We define the *Euler class* of (P, χ) to be $e(P, \chi)$. ■

Corollary 4.9 *Let A be a Noetherian ring of dimension $n \geq 2$. Let P be a projective A-module of rank n with trivial determinant and χ be an orientation of P. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let w_J be a local orientation of J. Suppose that $e(P, \chi) = (J, w_J)$ in $E(A)$. Then, there exists a surjection $\alpha : P \twoheadrightarrow J$ such that (J, w_J) is obtained from (α, χ).*

Proof We can regard w_J as a surjection : $(A/J)^n \twoheadrightarrow J/J^2$. We choose an isomorphism $\lambda : P/JP \xrightarrow{\sim} (A/J)^n$ such that $\wedge^n(\lambda) = (\overline{\chi})^{-1}$, where "bar" denotes modulo J. Consider the surjection $\overline{\alpha} = w_J\,\lambda : P/JP \twoheadrightarrow J/J^2$. By (2.14), there exists an ideal $J' \subset A$ and a surjection $\beta : P \twoheadrightarrow J \cap J'$ such that:

(*i*) $J + J' = A$, (*ii*) $\beta \otimes A/J = \overline{\alpha}$, and (*iii*) height $(J') \geq n$.

If $J' = A$, then $\beta : P \twoheadrightarrow J$ is such that $\beta \otimes A/J = \overline{\alpha}$. Hence β satisfies the required property. Otherwise, if ht $(J') = n$, then we have $e(P, \chi) = (J, w_J) + (J', w_{J'})$ in $E(A)$ (where $w_{J'}$ is obtained using P). By the assumption of the theorem, $e(P, \chi) = (J, w_J)$ in $E(A)$. Hence, we have $(J', w_{J'}) = 0$ in $E(A)$. Therefore, by (4.2), there exists a surjection $\gamma : A^n \twoheadrightarrow J'$ such that $w_{J'} = \gamma \otimes A/J'$. Now applying the subtraction principle (3.7), we get a surjection $\alpha : P \twoheadrightarrow J$ such that (J, w_J) is obtained from the pair (α, χ). This proves the corollary. ■

Corollary 4.10 *Let A be a Noetherian ring of dimension $n \geq 2$. Let P be a projective A-module of rank n with trivial determinant and χ be an orientation of P. Then $e(P, \chi) = 0$ if and only if P has a unimodular element. In particular if the determinant of P is trivial and P has a unimodular element then every generic section ideal J of P (i.e. an ideal J of height n which is a surjective image of P) is generated by n elements.*

Proof Let $\alpha : P \twoheadrightarrow J$ be a surjection, where J is an ideal of height n. Let $e(P, \chi) = (J, w_J)$ in $E(A)$, where (J, w_J) is obtained from the pair (α, χ). First assume that $e(P, \chi) = 0$. Then $(J, w_J) = 0$ in $E(A)$. Hence, by (4.2), there exists a surjection $\beta : A^n \twoheadrightarrow J$ such that $w_J = \beta \otimes A/J$. Now applying (3.8), we see that P has a unimodular element.

Now we assume that P has a unimodular element, i.e. $P = Q \oplus A$. Then $\alpha = (\theta, a)$ as an element of $P^* = Q^* \oplus A$. By performing an elementary automorphism of P, i.e. replacing θ by $\theta + a\,\theta'$, we may assume by (2.12), that ht $(\theta(Q)) = n - 1$. Let $K = \theta(Q)$. Note that since determinant of Q is trivial, without loss of generality we may assume that

χ is induced by an isomorphism $\chi' : A \stackrel{\sim}{\to} \wedge^{n-1}(Q)$.

Since dim $(A/K) \leq 1$, there exists an isomorphism $\gamma : (A/K)^{n-1} \stackrel{\sim}{\to} Q/KQ$ such that $\wedge^{n-1}(\gamma) = \chi'$ modulo K. The surjection $(\theta \otimes A/K)\,\gamma : (A/K)^{n-1} \twoheadrightarrow K/K^2$ can be lifted to a map $\delta : A^{n-1} \to K$ such that $\delta(A^{n-1}) + K^2 = K$. Let $\delta(A^{n-1}) = K'$. Then since $K' + K^2 = K$, by (2.7), $K = K' + (e)$, with $e \in K^2$ and $e^2 - e \in K'$. Therefore, by (2.10), $J = K + (a) = K' + (b)$, where $b = e + (1-e)a$. Now consider the surjection $(\delta, b) : A^n \twoheadrightarrow J$. As $e \in K^2$, we have that w_J is obtained by tensoring the surjection (δ, b) with A/J. Hence by definition, $e(P, \chi) = 0$ in $E(A)$. This proves the corollary. ■

Lemma 4.11 *Let A be a Noetherian ring of dimension $n \geq 2$. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let w_J be a local orientation of J. Suppose that $(J, w_J) \neq 0$ in $E(A)$. Then, there exists an ideal J_1 of height n which is comaximal with J and a local orientation w_{J_1} of J_1 such that $(J, w_J) + (J_1, w_{J_1}) = 0$ in $E(A)$. Further, given any element $f \in A$ such that ht $fA = 1$, J_1 can be chosen with the additional property that it is comaximal with (f).*

Proof Let $\alpha : (A/J)^n \twoheadrightarrow J/J^2$ be a surjection corresponding to w_J. Then by (2.14), there exists an ideal J_1 of height $\geq n$, which is comaximal with fJ, and a surjection $\beta : A^n \twoheadrightarrow J \cap J_1$ such that $\beta \otimes A/J = \alpha$. Since $(J, w_J) \neq 0$ in $E(A)$, J_1 is a proper ideal of height n. Let w_{J_1} be the local orientation of J_1 induced by β. Then $(J, w_J) + (J_1, w_{J_1}) = 0$ in $E(A)$. ■

Lemma 4.12 *Let A be a Noetherian ring of dimension $n \geq 2$. Then any element of the Euler class group $E(A)$ is of the form (J, w_J), where J is an ideal of A of height n such that J/J^2 is generated by n elements and w_J is a local orientation of J.*

Proof First we show that if $(J, w_J) \in E(A)$, $(J, w_J) \neq 0$ and $f \in A$ such that ht $fA = 1$ then $-(J, w_J) = (J_1, w_{J_1}) \in E(A)$ with $J_1 + fJ = A$. By (4.11), there exists an ideal J_1 of height n which is comaximal with fJ and a local orientation w_{J_1} of J_1 such that $(J, w_J) + (J_1, w_{J_1}) = 0$ in $E(A)$. Hence $-(J, w_J) = (J_1, w_{J_1})$. Hence any element $z \in E(A)$ is of the form $z = \sum_1^r (J_i, w_{J_i})$. It is enough to show that any element $z \in E(A)$ of the form $z = (J_1, w_{J_1}) + (J_2, w_{J_2})$ can be written as $z = (J', w_{J'})$ in $E(A)$ for some ideal J' of height n and a local orientation $w_{J'}$ of J'. Then, by induction, the result will follow.

Without loss of generality we may assume that both (J_1, w_{J_1}) and (J_2, w_{J_2}) are not zero in $E(A)$. Choosing $f \in J_1 \cap J_2$ and ht $(f) = 1$ and applying (4.11), we see that there exists an ideal J_0 of height n which is comaximal with J_1 and J_2 and a local orientation w_{J_0} of J_0 such that $(J_1, w_{J_1}) + (J_0, w_{J_0}) = 0$ in $E(A)$. Hence we have $z = -(J_0, w_{J_0}) + (J_2, w_{J_2})$. Now, $-(J_0, w_{J_0}) = (K_1, w_{K_1})$ in $E(A)$, where K_1 is comaximal with J_0 and J_2. We have $z = (K_1, w_{K_1}) + (J_2, w_{J_2})$, where $K_1 + J_2 = A$. Therefore we have $z = (J', w_{J'})$ in $E(A)$, where $J' = K_1 \cap J_2$ and $w_{J'}$ is a local orientation of J' induced from w_{K_1} and w_{J_2}. This proves the lemma. ∎

Let A be a Noetherian ring of dimension $n \geq 2$. Let $\mathfrak{a}$ be the nil radical of A. Let "bar" denote modulo $\mathfrak{a}$. Let $G(A)$ be the free abelian group on the set $(\mathcal{N}, w_{\mathcal{N}})$, where $\mathcal{N}$ is $\mathcal{M}$-primary ideal of height n such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements and $w_{\mathcal{N}}$ is a local orientation of $\mathcal{N}$. Similarly, we define $G(\overline{A})$. If $\mathcal{N}$ is $\mathcal{M}$-primary ideal then $\overline{\mathcal{N}} = (\mathcal{N} + \mathfrak{a})/\mathfrak{a}$ is also $\overline{\mathcal{M}}$-primary and if $\mathcal{N}/\mathcal{N}^2$ is generated by the images of $(a_1, \dots, a_n)$ then $\overline{\mathcal{N}}/(\overline{\mathcal{N}})^2$ is also generated by the images of $(a_1, \dots, a_n)$. Hence if $(\mathcal{N}, w_{\mathcal{N}}) \in G(A)$ then $(\overline{\mathcal{N}}, w_{\overline{\mathcal{N}}}) \in G(\overline{A})$.

Let $J \subset A$ be an ideal of height n with primary decomposition as $J = \cap \mathcal{N}_i$, where $\mathcal{N}_i$ is $\mathcal{M}_i$-primary, $\mathcal{M}_i$ a maximal ideal of A. Then $\overline{J} = (J + N)/N = J/J \cap N \subset \overline{A}$ is an ideal of height n with primary decomposition as $\overline{J} = \cap \overline{\mathcal{N}}_i$. The following diagram is commutative

$$\begin{array}{ccc} J/J^2 & \longrightarrow & \oplus \mathcal{N}_i/\mathcal{N}_i^2 \\ \big\downarrow & & \big\downarrow \\ \overline{J}/\overline{J^2} & \longrightarrow & \oplus \overline{\mathcal{N}}_i/\overline{\mathcal{N}}_i^2 \end{array}$$

Hence any surjection $w_J : (A/J)^n \twoheadrightarrow J/J^2$ induces a surjection $\overline{w}_J : (\overline{A}/\overline{J})^n \twoheadrightarrow \overline{J}/\overline{J}^{\,2} = (J + N)/(J^2 + N)$. From the above discussion, it follows that the assignment sending (J, w_J) to $(\overline{J}, \overline{w}_J)$ gives rise to a group homomorphism $\Phi : E(A) \to E(\overline{A})$.

As a consequence of (4.2), we have the following:

Corollary 4.13 *The homomorphism $\Phi : E(A) \to E(\overline{A})$ is an isomorphism.*

Proof Let $\overline{w}_{\overline{J}} : (\overline{A}/\overline{J})^n \twoheadrightarrow \overline{J}/\overline{J}^2$ be a surjection. Let $J \supset \mathfrak{a}$ be an ideal of height n such that $J/\mathfrak{a} = \overline{J}$. Then $\overline{w}_{\overline{J}}$ can be considered as a surjective map from $(A/J)^n$ to $J/(J^2 + \mathfrak{a})$. Let $\alpha : A^n \to J$ be an A-linear map which is a lift of $\overline{w}_{\overline{J}}$. Let $\alpha(A^n) = (f_1, \dots, f_n)$.

Then $(f_1,\dots,f_n)+J^2+\mathfrak{a}=J$. By (2.7), there exists an element $e\in J^2$ such that $(f_1,\dots,f_n)+\mathfrak{a}+Ae=J$, and $e(1-e)\in((f_1,\dots,f_n)+\mathfrak{a})$. Then $\overline{e}\in A/((f_1,\dots,f_n)+\mathfrak{a})$ is an idempotent.

Since $\mathfrak{a}$ is a nilpotent ideal and idempotent elements can be lifted modulo a nilpotent ideal, we can lift $\overline{e}\in A/((f_1,\dots,f_n)+\mathfrak{a})$ to an idempotent element of $A/(f_1,\dots,f_n)$. Let $f\in A$ be such that $\overline{f}\in A/(f_1,\dots,f_n)$ is a lift of $\overline{e}\in A/((f_1,\dots,f_n)+\mathfrak{a})$, i.e. $f-f^2\in(f_1,\dots,f_n)$ and $f-e\in((f_1,\dots,f_n)+\mathfrak{a})$. Let $J_1=(f_1,\dots,f_n,f)$, then $(f_1,\dots,f_n)+J_1^2=J_1$ and $J_1+\mathfrak{a}=(f_1,\dots,f_n,f)+\mathfrak{a}=(f_1,\dots,f_n,e)+\mathfrak{a}=J$ (as $f-e\in((f_1,\dots,f_n)+\mathfrak{a})$). Let "bar" denote modulo $\mathfrak{a}$. Then $\overline{J}_1=\overline{J}$ and $\overline{J}_1/\overline{J}_1^{\,2}=(\overline{f}_1,\dots,\overline{f}_n)$. Hence $(\overline{f}_1,\dots,\overline{f}_n)=\overline{J}/\overline{J}^{\,2}=J/(J^2+\mathfrak{a})$. This implies that $\overline{w}_{\overline{J}}$ is induced from $\overline{\alpha}$. Hence the map Φ is surjective.

Now we prove that the map Φ is injective. By (4.12), every element of $E(A)$ is of the form (J,w_J). Hence, it is enough to prove that for $(J,w_J)\in E(A)$ (where $J\subset A$ is an ideal of height n and $w_J:(A/J)^n\twoheadrightarrow J/J^2$ is a local orientation of J), if the image of $(J,w_J)=0$ in $E(\overline{A})$ then $(J,w_J)=0$ in $E(A)$. Assume that the image of $(J,w_J)=(\overline{J},\overline{w}_{\overline{J}})=0$ in $E(\overline{A})$. Then by (4.2), $\overline{w}_{\overline{J}}$ is a global orientation of J, i.e. there exists a surjection $\gamma:(A/\mathfrak{a})^n\twoheadrightarrow(J+\mathfrak{a})/\mathfrak{a}$ such that $\overline{w}_{\overline{J}}=\gamma\otimes\overline{A}/\overline{J}$.

We are given surjections $\alpha:A^n\twoheadrightarrow J/J^2$ (which is obtained by w_J by composing with the natural map $A^n\twoheadrightarrow(A/J)^n$) and $\beta:A^n\twoheadrightarrow(J+\mathfrak{a})/\mathfrak{a}=J/(J\cap\mathfrak{a})$ (which is obtained by γ by composing with natural map $A^n\twoheadrightarrow(A/\mathfrak{a})^n$) such that they induce the same surjective map from A^n to $J/(J^2+(J\cap\mathfrak{a}))$.

Since $J/(J^2\cap\mathfrak{a})$ is the fiber product of J/J^2 and $J/(J\cap\mathfrak{a})$ over $J/(J^2,J\cap\mathfrak{a})$, α and β patch to yield a map $\delta:A^n\to J/(J^2\cap\mathfrak{a})$.

$$\begin{tikzcd}
J/(J^2\cap\mathfrak{a}) \arrow[r] \arrow[d] & J/J^2 \arrow[d] \\
J/(J\cap\mathfrak{a}) \arrow[r] & J/(J^2,J\cap\mathfrak{a})
\end{tikzcd}$$

Let $\theta:A^n\to J$ be a lift of δ. Then θ is a lift of α and β. Hence we have

(i) $\theta(A^n)+J^2=J$, and (ii) $\theta(A^n)+(J\cap\mathfrak{a})=J$. Since $\mathfrak{a}$ is nilpotent, $\sqrt{\theta(A^n)}=\sqrt{J}$ are same by (ii). Hence $\theta(A^n)=J$ by (i) and (2.9).

Since θ is a lift of α, we get $(J, w_J) = 0$ in $E(A)$. Hence Φ is injective and hence an isomorphism. This proves the corollary. $\blacksquare$

Chapter 5

Some Results on $E(A)$

If A is an affine domain of dimension n over an algebraically closed field and P is a projective A-module of rank n and trivial determinant, then it follows from a result of Mohan Kumar (3.3) that if P maps onto an ideal J of height n which is generated by n elements, then P has a unimodular element and hence all its generic section ideals (i.e. ideals of height n which are surjective image of P) are generated by n elements (4.10). But this is not necessarily true if the base field is not algebraically closed. For example, all the *reduced* generic section ideals of the tangent bundle of the real 2-sphere are generated by 2 elements *([B-RS 2], (5.6(i)))*. There are however *non-reduced* generic section ideals of the tangent bundle which are not generated by 2 elements *([B-RS 1],(5.2))*.

This phenomenon is explained by the result (5.10) of this chapter, which asserts that for any n-dimensional real affine domain, a projective module of rank n with trivial determinant, all of whose generic section ideals are generated by n elements, has a unimodular element. To prove this result, we first prove some lemmas.

Let A be a Noetherian ring of dimension $n \geq 2$. Let $J \subset A$ be an ideal of height n and $w_J : (A/J)^n \twoheadrightarrow J/J^2$ be a local orientation of J. Let $\overline{b} \in A/J$ be a unit. Let $\sigma : (A/J)^n \xrightarrow{\sim} (A/J)^n$ be an automorphism with det $(\sigma) = \overline{b}$. Then $w_J\,\sigma$ is another local orientation of J, which we denote by $\overline{b}\,w_J$.

Lemma 5.1 *Let A, J be as above. Let w_J and $\widetilde{w}_J$ be two local orientations of J. Then $\widetilde{w}_J = \overline{b}\,w_J$ for some unit $\overline{b} \in A/J$.*

Proof We have two surjections $w_J : (A/J)^n \twoheadrightarrow J/J^2$ and $\widetilde{w}_J : (A/J)^n \twoheadrightarrow J/J^2$. We will define a map $\psi : (A/J)^n \to (A/J)^n$ such that $w_J\,\psi = \widetilde{w}_J$.

Let $\{e_i, i = 1, \dots, n\}$ be a basis of $(A/J)^n$. Given $\widetilde{w}_J(e_i) = \overline{a_i}$, $w_J(e_i) = \overline{b_i}$. Let $\overline{a_i} = \sum \overline{c_{ij}}\,\overline{b_j}$. Define $\psi(e_i) = \sum_j c_{ij}\,e_j$. Then $w_J\,\psi = \widetilde{w}_J$. Now, by (2.16) ψ is an isomorphism. Let det $(\psi) = \overline{b}$. Then $\widetilde{w}_J = \overline{b}\,w_J$. This proves the lemma. ■

Lemma 5.2 *Let A be a ring and $J \subset A$ be an ideal which is generated by two elements a_1, a_2. Let $a \in A$ be a unit modulo J and $b \in A$ be such that $ab = 1$ modulo J. Suppose that the unimodular row $(b, a_2, -a_1)$ is completable to a matrix in* $\mathrm{SL}_3(A)$. *Then, there exists a matrix $\tau \in \mathrm{M}_2(A)$ with* det $(\tau) = a$ *modulo J such that $[a_1, a_2]\tau^t = [b_1, b_2]$, where b_1, b_2 generate J.*

Proof Choose a completion $\sigma \in \mathrm{SL}_3(A)$ of the unimodular row $(b, a_2, -a_1)$. Suppose that second and third rows of σ are $(d, \lambda_{11}, \lambda_{12})$ and $(e, \lambda_{21}, \lambda_{22})$ respectively.

Let $\gamma : A^3 \twoheadrightarrow J$ be a surjection given by $\gamma(e_1) = 0$, $\gamma(e_2) = a_1$, $\gamma(e_3) = a_2$. The vectors $(b, a_2, -a_1)$, $(d, \lambda_{11}, \lambda_{12})$ and $(e, \lambda_{21}, \lambda_{22})$ generate A^3. Hence their images under γ generate J. Hence $J = (b_1, b_2)$, where $b_1 = a_1\,\lambda_{11} + a_2\,\lambda_{12}$ and $b_2 = a_1\,\lambda_{21} + a_2\,\lambda_{22}$. Let $\tau = (\lambda_{ij}) \in \mathrm{M}_2(A)$. Since $\sigma \in \mathrm{SL}_3(A)$ and $a_1, a_2 \in J$, we get $\det(\tau) = a$ modulo J. Further $[a_1, a_2]\,\tau^t = [b_1, b_2]$. This proves the lemma. ■

Lemma 5.3 *Let A be a Noetherian ring of dimension $n \geq 2$, $J \subset A$ an ideal of height n and $w_J : (A/J)^n \twoheadrightarrow J/J^2$ a surjection. Suppose that w_J can be lifted to a surjection $\alpha : A^n \twoheadrightarrow J$. Let $a \in A$ be a unit modulo J. Let θ be an automorphism of $(A/J)^n$ with determinant $\overline{a^2}$. Then, the surjection $w_J\theta : (A/J)^n \twoheadrightarrow J/J^2$ can be lifted to a surjection $\gamma : A^n \twoheadrightarrow J$.*

Proof Let $P = A^{n-2}$ and $\alpha : P \oplus A^2 \twoheadrightarrow J$ be map defined by $(a_3, \dots, a_n, a_1, a_2)$ such that $w_J = \alpha \otimes A/J$. Let $J' = (a_3, \dots, a_n)$ and let "tilde" denote reduction modulo J'. Then $\widetilde{\alpha} : \widetilde{A}^2 \twoheadrightarrow \widetilde{J}$ is defined by $\widetilde{\alpha}(0,1,0) = \widetilde{a_1}, \widetilde{\alpha}(0,0,1) = \widetilde{a_2}$.

Since $a \in A$ is a unit modulo J, there exists an element $b \in A$ such that $ab = 1(\bmod J)$. Then $(\widetilde{b^2}, \widetilde{a_2}, -\widetilde{a_1})$ is a unimodular row, which is completable to an invertible matrix in $\mathrm{SL}_3(\widetilde{A})$, by (1.17). Hence by (5.2), there exists a matrix $\widetilde{\tau} \in \mathrm{M}_2(\widetilde{A})$ such that $[\widetilde{a_1}, \widetilde{a_2}]\,\widetilde{\tau} = [\widetilde{b_1}, \widetilde{b_2}]$, where $\widetilde{J} = (\widetilde{b_1}, \widetilde{b_2})$ and det $(\widetilde{\tau}) = \widetilde{a^2}(\bmod J)$.

Define a surjection $\gamma' : P \oplus A^2 \twoheadrightarrow J$ by setting $\gamma' = \alpha$, on P and $\gamma'(0,1,0) = b_1$, $\gamma'(0,0,1) = b_2$. Define $\theta' : (A/J)^n \xrightarrow{\sim} (A/J)^n$ by $\begin{pmatrix} I_{n-2} & 0 \\ 0 & \overline{\tau} \end{pmatrix}$.

Then, det $\theta' = \overline{a}^2$. It follows that $w_J\,\theta' = \gamma' \otimes A/J$. Hence $w_J\theta'$ can be lifted to a surjection $A^n \twoheadrightarrow J$.

Since dim $(A/J) = 0$, we have $\mathrm{SL}_n(A/J) = \mathrm{E}_n(A/J)$ and the canonical map from $\mathrm{SL}_n(A) \twoheadrightarrow \mathrm{SL}_n(A/J)$ is surjective. Now since det $\theta =$ det $\theta' = \overline{a^2}$, it follows that $w_J\theta$ can also be lifted to a surjection $A^n \twoheadrightarrow J$. ■

Lemma 5.4 *Let A be a Noetherian ring of dimension $n \geq 2$, $J \subset A$ an ideal of height n and w_J be a local orientation of J. Let $\overline{a} \in A/J$ be a unit. Then $(J, w_J) = (J, \overline{a^2} w_J)$ in $E(A)$.*

Proof If $(J, w_J) = 0$ in $E(A)$, then by (4.2), w_J can be lifted to a surjection from $A^n \to J$. By (5.3), $\overline{a^2} w_J$ can also be lifted to a surjection from $A^n \to J$. Hence $(J, \overline{a^2} w_J) = 0$ in $E(A)$. Hence the result follows in this case.

Now assume that $(J, w_J) \neq 0$ in $E(A)$ where $w_J : (A/J)^n \twoheadrightarrow J/J^2$ is a surjection. By (2.14), there exists an ideal J_1 of height n which is comaximal with J and a surjection $\alpha : A^n \twoheadrightarrow J \cap J_1$ such that $\alpha \otimes A/J = w_J$ (If $J_1 = A$, then $(J, w_J) = 0$ in $E(A)$). Let $w_{J_1} = \alpha \otimes A/J_1$.

Let $x + y = 1$, $x \in J$, $y \in J_1$. If we set $b = a^2(1-x) + x$, then $b = a^2 \pmod{J}$ and $b = 1 \pmod{J_1}$. Applying (5.3), there exists a surjection $\gamma : A^n \twoheadrightarrow J \cap J_1$ such that $\gamma \otimes A/J = \overline{a^2}\, w_J$, $\gamma \otimes A/J_1 = w_{J_1}$.

From the surjection α, we get $(J, w_J) + (J_1, w_{J_1}) = 0$ in $E(A)$ and from the surjection γ, we get $(J, \overline{a^2}\, w_J) + (J_1, w_{J_1}) = 0$ in $E(A)$. Thus $(J, w_J) = (J, \overline{a^2}\, w_J)$ in $E(A)$. This proves the lemma. ■

Lemma 5.5 *Let A be a Noetherian ring of dimension $n \geq 1$ and $J \subset A$ be an ideal of height n. Let $f \neq 0 \in A$ such that JA_f is a proper ideal of A_f. Assume $JA_f = (a_1, \ldots, a_n)$ where $a_i \in J$. Then there exists $\sigma \in \mathrm{SL}_n(A_f)$ such that $[a_1, \ldots, a_n]\,\sigma = [b_1, \ldots, b_n]$ where $b_i \in J$ and* ht $(b_1, \ldots, b_n) = n$.

Proof Let I be the set $\{\sigma \in \mathrm{SL}_n(A_f) : [a_1, \ldots, a_n]\sigma = [b_1, \ldots, b_n],\ b_i \in J\}$. Then $I \neq \varnothing$ since id $\in I$. For $\sigma \in I$, if $[a_1, \ldots, a_n]\sigma = [b_1, \ldots, b_n] \in A^n$, $b_i \in J$ let $N(\sigma)$ denote ht $(b_1, \dots, b_n)$. Then it is enough to prove that there exists $\sigma \in I$ such that $N(\sigma) = n$. This is proved by showing that for any $\sigma \in I$ with $N(\sigma) < n$, there exists $\sigma_1 \in I$ such that $N(\sigma_1) > N(\sigma)$.

Let $\sigma \in I$ be such that $N(\sigma) < n$. Let $[a_1, \dots, a_n]\sigma = [b_1, \dots, b_n] \in A^n$, $b_i \in J$. Then by (2.3), there exists $[c_1, \dots, c_{n-1}] \in A^{n-1}$ such that ht $(J'_{b_n}) \geq n-1$, where $J' = (b_1 + c_1 b_n, \dots, b_{n-1} + c_{n-1} b_n)$. The transformation τ sending $[b_1, \dots, b_n]$ to $[b_1 + c_1 b_n, \dots, b_{n-1} + c_{n-1} b_n, b_n]$ is elementary. Hence $\sigma\tau \in I$. Note that $N(\sigma) = N(\sigma\tau)$. Hence, if necessary, we can replace σ by $\sigma\tau$ and assume that if a prime ideal $\mathfrak{p}$ of A contains $(b_1, \dots, b_{n-1})$ and does not contain b_n then we have ht $(\mathfrak{p}) \geq n-1$. Now we claim that $N(\sigma) =$ ht $(b_1, \dots, b_{n-1})$.

We have $N(\sigma) \leq n-1$. Since $N(\sigma) = \text{ht}\,(b_1, \dots, b_n)$ we have ht $(b_1, \dots, b_{n-1}) \leq N(\sigma) \leq n-1$. Let $\mathfrak{p}$ be a minimal prime ideal of $(b_1, \dots, b_{n-1})$ such that ht $(\mathfrak{p}) =$ ht $(b_1, \dots, b_{n-1})$. If $b_n \notin \mathfrak{p}$ then ht $(\mathfrak{p}) \geq n-1$. Hence, we have the inequalities $n-1 \leq \text{ht}\,(b_1, \dots, b_{n-1}) \leq N(\sigma) \leq n-1$. This implies that $N(\sigma) = \text{ht}\,(b_1, \dots, b_{n-1}) = n-1$. If $b_n \in \mathfrak{p}$ then ht $(b_1, \dots, b_{n-1}) = \text{ht}\,(b_1, \dots, b_n) = N(\sigma)$. This proves the claim.

Let K denote the set of minimal prime ideals of $(b_1, \dots, b_{n-1})$. Let $K_1 = \{\mathfrak{p} \in K : b_n \in \mathfrak{p}\}$ and let $K_2 = K - K_1$. Note that $K_1 \neq \varnothing$. For, if $K_1 = \varnothing$, no minimal prime ideal of $(b_1, \dots, b_{n-1})$ contains b_n. Then ht $(b_1, \dots, b_n) >$ ht $(b_1, \dots, b_{n-1})$. But ht $(b_1, \dots, b_n) = N(\sigma) = \text{ht}\,(b_1, \dots, b_{n-1})$. This proves that $K_1 \neq \varnothing$. Now we claim that $f \in \mathfrak{p}$ for all $\mathfrak{p} \in K_1$.

For, if $f \notin \mathfrak{p}$ for some $\mathfrak{p} \in K_1$, then since $\mathfrak{p}$ is minimal prime ideal of $(b_1, \dots, b_{n-1})$, we have ht $(\mathfrak{p}) \leq n-1$. Since $b_n \in \mathfrak{p} \in K_1$, we have $\mathfrak{p}$ is minimal prime ideal of $(b_1, \dots, b_n)$ and $f \notin \mathfrak{p}$. Hence ht $(\mathfrak{p}A_f) =$ ht $(\mathfrak{p}) \leq n-1$. But $(b_1, \dots, b_n)A_f \subset \mathfrak{p}A_f$ and ht $(b_1, \dots, b_n)A_f = \text{ht}\,(a_1, \dots, a_n)A_f = n$. This is a contradiction. This proves that $f \in \mathfrak{p}$ for all $\mathfrak{p} \in K_1$.

If $\mathfrak{p} \in K$ then ht $(\mathfrak{p}) \leq n-1$, since $\mathfrak{p}$ is a minimal prime ideal of $(b_1, \dots, b_{n-1})$. If $\mathfrak{p} \in K_2$ then $b_n \notin \mathfrak{p}$ and hence ht $(\mathfrak{p}) \geq n-1$. Therefore, ht $(\mathfrak{p}) = n-1$ for all $\mathfrak{p} \in K_2$. Since $\bigcap_{\mathfrak{p} \in K_2} \mathfrak{p} \not\subset \bigcup_{\mathfrak{p} \in K_1} \mathfrak{p}$, choose $x \in \bigcap_{\mathfrak{p} \in K_2} \mathfrak{p}$ such that $x \notin \bigcup_{\mathfrak{p} \in K_1} \mathfrak{p}$. Since $f \in \mathfrak{p}$ for all $\mathfrak{p} \in K_1$, we have $xf \in \bigcap_{\mathfrak{p} \in K} \mathfrak{p}$. This implies that $(xf)^r \in (b_1, \dots, b_{n-1})$ for some positive integer r.

Let $(xf)^r = \sum_1^{n-1} d_i b_i$. Consider the matrix $\theta \in \mathrm{E}_n(A)$ which takes $[c_1, \dots, c_n]$ to $[c_1, \dots, c_{n-1}, c_n + \sum_1^{n-1} d_i c_i]$. Let θ_1 be the matrix : diagonal $(1, \dots, 1, f^r)$. Then $\theta_1 \in \mathrm{GL}_n(A_f)$. Let $\widetilde{\theta} = \theta_1\, \theta_f\, \theta_1^{-1}$. Then $\widetilde{\theta} \in \mathrm{SL}_n(A_f)$. Let $\sigma_1 = \sigma\widetilde{\theta}$. Then

$$
\begin{aligned}
&[a_1, \dots, a_n]\, \sigma_1 \;=\; [b_1, \dots, b_n]\, \widetilde{\theta} \;=\; [b_1, \dots, b_n]\, \theta_1\, \theta_f\, \theta_1^{-1} \\
&= [b_1, \dots, b_{n-1}, f^r b_n]\, \theta_f\, \theta_1^{-1} \;=\; [b_1, \dots, b_{n-1}, f^r b_n + \textstyle\sum_1^{n-1} d_i b_i]\, \theta_1^{-1} \\
&= [b_1, \dots, b_{n-1}, f^r b_n + f^r x^r]\, \theta_1^{-1} \;=\; [b_1, \dots, b_{n-1}, b_n + x^r].
\end{aligned}
$$

Now we claim that no minimal prime ideal of $(b_1, \dots, b_{n-1})$ contains $b_n + x^r$. For, if $\mathfrak{p} \in K_1$, then $b_n \in \mathfrak{p}$ but $x \notin \mathfrak{p}$. Hence $b_n + x^r \notin \mathfrak{p}$ for all $\mathfrak{p} \in K_1$. If $\mathfrak{p} \in K_2$ then $b_n \notin \mathfrak{p}$ but $x \in \mathfrak{p}$. Hence $b_n + x^r \notin \mathfrak{p}$ for all $\mathfrak{p} \in K_2$. This proves the claim. Hence ht $(b_1, \dots, b_{n-1}, b_n + x^r) >$ ht $(b_1, \dots, b_{n-1})$. Note that $\sigma_1 \in I$ and $N(\sigma_1) = \mathrm{ht}\,(b_1, \dots, b_{n-1}, b_n + x^r) > \mathrm{ht}\,(b_1, \dots, b_{n-1}) = N(\sigma)$. This completes the proof of the lemma. ■

Lemma 5.6 *Let A be an affine domain over a field k of dimension $n \geq 2$ and f be a non-zero element of A. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Suppose that $(J, w_J) \neq 0$ in $E(A)$, but the image of $(J, w_J) = 0$ in $E(A_f)$. Then, there exists an ideal J_2 of height n such that $(J_2)_f = A_f$ and $(J, w_J) = (J_2, w_{J_2})$ in $E(A)$.*

Proof Since $(J, w_J) \neq 0$ in $E(A)$ and $(J, w_J) = 0$ in $E(A_f)$, $f \in A$ is not a unit. By (4.11), we can choose an ideal J_1 of height n, which is comaximal with J and (f) such that $(J, w_J) + (J_1, w_{J_1}) = 0$ in $E(A)$. We have $(J_1, w_{J_1}) \neq 0$ in $E(A)$. Since the image of $(J, w_J) = 0$ in $E(A_f)$, it follows that the image of $(J_1, w_{J_1}) = 0$ in $E(A_f)$. By (4.2), w_{J_1} induces a global orientation of $(J_1)_f$. Hence, $(J_1)_f = (b_1, \dots, b_n)$ and $w_{J_1} \otimes A_f$ is induced by the set of generators $b_1, \dots, b_n$ of $(J_1)_f$ mod $(J_1)_f^2$. Choose k large enough such that $f^{2k}\, b_i \in J_1$, $1 \leq i \leq n$. Since f is a unit modulo J_1, by (5.4), $(J_1, w_{J_1}) = (J_1, \overline{f^{2kn}}\, w_{J_1})$ in $E(A)$. Hence, without loss of generality, we can assume that $b_i \in J_1$. Now, by lemma (5.5), we get $\sigma \in \mathrm{SL}_n(A_f)$ such that if $[b_1, \dots, b_n]\sigma = [c_1, \dots, c_n]$, then $c_i \in J_1$ and $c_1, \dots, c_n$ generate an ideal of height n in A. We claim that $(c_1, \dots, c_n) = J_1 \cap J_2$ where $(J_2)_f = A_f$.

Let $(c_1, \dots, c_n) = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_r \cap \mathfrak{a}_{r+1} \dots \cap \mathfrak{a}_t$ be a reduced primary decomposition, where $\mathfrak{a}_i$ is $\mathfrak{m}_i$-primary ideal. Assume that $f \in \mathfrak{m}_i$ for $r+1 \leq i \leq t$ and $f \notin \mathfrak{m}_i$ for $1 \leq i \leq r$. Then $(c_1, \dots, c_n)_f = (J_1)_f = \bigcap_1^r (\mathfrak{a}_i)_f$. We observe that $J_1 = \bigcap_1^r \mathfrak{a}_i$. This follows easily from the fact that f is a unit modulo J_1 and modulo $\mathfrak{a}_i$, $1 \leq i \leq r$. Write $J_2 = \bigcap_{r+1}^t \mathfrak{a}_i$. Then we have $(c_1, \dots, c_n) = J_1 \cap J_2$ and $(J_2)_f = A_f$.

Note that $A/J_1 \xrightarrow{\sim} A_f/(J_1)_f$. The image of $\sigma \in \mathrm{SL}_n(A_f)$ in $\mathrm{SL}_n(A_f/(J_1)_f)$ gives rise to an element in $\mathrm{SL}_n(A/J_1)$ and hence the n generators of $(J_1)_f/(J_1)_f^2$ gives rise to n generators of J_1/J_1^2.

Now, $J_1 \neq (c_1, \dots, c_n)$, since $(J_1, w_{J_1}) = (J_1, \overline{f^{2kn}} w_{J_1}) \neq 0$ in $E(A)$. Hence J_2 is a proper ideal of height n. Since $(f)+J_1 = A$ and $(J_2)_f = A_f$ we have $J_1+J_2 = A$. Since det $(\sigma) = 1$, $\overline{f^{2kn}} w_{J_1}$ is given by the set of generators $c_1, \dots, c_n$ of J_1 modulo J_1^2. Therefore, $(J_1, \overline{f^{2kn}} w_{J_1})+(J_2, w_{J_2}) = 0$ in $E(A)$, where w_{J_2} is given by the set of generators $c_1, \dots, c_n$ of J_2 modulo J_2^2. Since $(J, w_J) + (J_1, w_{J_1}) = 0$ in $E(A)$, it follows that $(J, w_J) = (J_2, w_{J_2})$ in $E(A)$. This proves the lemma. ■

Remark 5.7 The hypothesis A is an affine domain ensures that dim $(A_f) = n$ and hence $E(A_f)$ is defined.

Lemma 5.8 *Let A be an affine domain over a field k of dimension $n \geq 2$ and P a projective A-module of rank n having trivial determinant. Let $f \in A$ be a non-zero element. Suppose that the projective A_f-module P_f has a unimodular element. Then there exists a surjection $\alpha : P \twoheadrightarrow J$, where $J \subset A$ is an ideal of height n such that $J_f = A_f$.*

Proof Suppose P has a unimodular element then by (4.10), $e(P, \chi) = 0$ in $E(A)$. Let J be an ideal of height n and generated by n elements such that $f \in J$. Then, for some w_J, $(J, w_J) = 0 = e(P, \chi)$ in $E(A)$. By (4.9), there exists a surjection $\alpha : P \twoheadrightarrow J$ and $J_f = A_f$.

Now assume that P has no unimodular element. Let $e(P, \chi) = (J, w_J)$, where J is an ideal of height n such that J/J^2 is generated by n elements. Now $(J, w_J) \neq 0$ in $E(A)$ but its image in $E(A_f) = 0$. Therefore, by (5.6), there exists an ideal J_2 of height n such that $(J_2)_f = A_f$ and $(J, w_J) = (J_2, w_{J_2})$ in $E(A)$. Then $e(P, \chi) = (J_2, w_{J_2})$. Hence by (4.9), there exists a surjection $\alpha : P \twoheadrightarrow J_2$ such that (J_2, w_{J_2}) is obtained from (α, χ). This proves the lemma. ■

Definition Let A be a Noetherian ring of dimension n and P a projective A-module of rank n. Let $\alpha : P \twoheadrightarrow J$ be a surjection. We say that α is a *generic* surjection, if J has height n. In this case J is said to be a *generic surjection ideal* or *generic section ideal* of P.

Lemma 5.9 *Let A be an affine domain over a field k of dimension $n \geq 2$ and P a projective A-module of rank n having trivial determinant. Let $f \in A$ be a non-zero element. Assume that every generic surjection ideal of P is generated by n elements. Then, every generic surjection ideal of P_f is generated by n elements.*

Proof Let $\beta : P_f \twoheadrightarrow \widetilde{J}$ be a generic surjection. Let $J' = \widetilde{J} \cap A$. Then $J'A_f = \widetilde{J}$ and $(f) + J' = A$. Let χ be a generator of $\wedge^n(P)$, and let $(J'_f, w_{J'_f})$ be obtained from (β, χ_f). Since $\overline{f} \in A/J'$ is a unit, by (5.4), we may replace $w_{J'_f}$ by $\overline{f^{2m}}\, w_{J'_f}$, for some large integer m, and assume that $w_{J'_f}$ is given by a set of generators of J'/J'^2 which induce $w_{J'}$. The element $e(P, \chi) - (J', w_{J'})$ of $E(A)$ is zero in $E(A_f)$. If $e(P, \chi) - (J', w_{J'}) = 0$ in $E(A)$ then $e(P, \chi) = (J', w_{J'})$. Hence there exists a surjection from P to J' by (4.9). By assumption, J' is generated by n elements. Hence $\widetilde{J}$ is generated by n elements. Hence assume otherwise. By (4.12), $e(P, \chi) - (J', w_{J'}) = (J_2, w_{J_2})$ in $E(A)$. By (5.6), we can assume that $(J_2)_f = A_f$. Since $J' + Af = A$ we have $J' + J_2 = A$. Hence $e(P, \chi) = (J', w_{J'}) + (J_2, w_{J_2}) = (J' \cap J_2, w_{J' \cap J_2})$ in $E(A)$, where $w_{J' \cap J_2}$ is obtained from $w_{J'}$ and w_{J_2}. By (4.9), there exists a surjection $\gamma : P \twoheadrightarrow J' \cap J_2$. By hypothesis, $J' \cap J_2$ is generated by n elements, as it is a generic surjection ideal of P. Hence $J'_f = (J' \cap J_2)_f$ is generated by n elements. This proves the lemma. ■

Theorem 5.10 *Let A be an affine domain over $\mathbb{R}$ of dimension $n \geq 2$ and P a projective A-module of rank n having trivial determinant. Assume that for every generic surjection $\alpha : P \twoheadrightarrow J$, the generic surjection ideal J is generated by n elements. Then P has a unimodular element.*

Proof To any generic surjection $\alpha : P \twoheadrightarrow J$, we associate an integer $N(P, \alpha)$, which is equal to the number of real maximal ideals containing J (if $\mathcal{M}$ is a maximal ideal of A containing J, then it is called *real* if the quotient field $A/\mathcal{M}$ is isomorphic to $\mathbb{R}$, otherwise it is called a *complex* maximal ideal and in this case $A/\mathcal{M}$ is isomorphic to $\mathbb{C}$). Let $t(P)$ = min $N(P, \alpha)$, where α varies over all generic surjections of P.

Case 1. Suppose that $t(P) = 0$. Let $\alpha : P \twoheadrightarrow J$ be a generic surjection with $N(P, \alpha) = 0$. This means that J is contained only in complex maximal ideals. By assumption, J is generated by n elements. These n elements give rise to $\widetilde{\omega}_J$, a local orientation of J, such that the element $(J, \widetilde{\omega}_J) = 0$ in $E(A)$. Let χ be a generator of $\wedge^n(P)$ and $e(P, \chi) = (J, w_J)$ in $E(A)$. Then, by (5.1), $(J, w_J) = (J, \overline{u}\,\widetilde{\omega}_J)$ in $E(A)$, where $\overline{u} \in A/J$ is a unit. Since J is contained only in complex maximal ideals, $\overline{u}$ is a square. It follows now from (5.4), that $e(P, \chi) = (J, w_J) = (J, \overline{u}\,\widetilde{\omega}_J) = (J, \widetilde{\omega}_J) = 0$ in $E(A)$. Therefore, by (4.10), P has a unimodular element.

Case 2. Suppose that $t(P) = 1$. Let $\alpha : P \twoheadrightarrow J$ be a generic surjection with $N(P, \alpha) = 1$. This means that J is contained only in one real maximal ideal. By assumption, J is generated by n elements. These n elements give rise to $\widetilde{\omega}_J$, a local orientation of J, such that the element $(J, \widetilde{\omega}_J) = 0 = (J, -\widetilde{\omega}_J)$ in $E(A)$. Let χ be a generator of $\wedge^n(P)$ and $e(P, \chi) = (J, w_J)$ in $E(A)$. Let $(J, w_J) = (J, \overline{u}\,\widetilde{\omega}_J)$ in $E(A)$, where $\overline{u} \in A/J$ is a unit. Then, since J is contained only in one real maximal ideal, it follows as in case 1 that either $\overline{u} \in A/J$ is a square or $-\overline{u}$ is a square. Therefore, it follows that either $(J, w_J) = (J, \widetilde{\omega}_J)$ or $(J, w_J) = (J, -\widetilde{\omega}_J)$ in $E(A)$. In any case, $(J, w_J) = 0$ in $E(A)$ and hence, by (4.10), P has a unimodular element.

Case 3. Now we show that under the assumption of the theorem $t(P) \leq 1$ and hence the theorem will be proved. Suppose $N(P, \alpha) = r \geq 2$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the real maximal ideals containing J. Let $f \in A$ be chosen so that f belongs to only the real maximal ideals $\mathfrak{m}_2, \ldots, \mathfrak{m}_r$ (Such an f exists, for choose a set of generators $h_1, \ldots, h_k$ of $\mathfrak{m}_2 \cap \ldots \cap \mathfrak{m}_r$. Take $f = h_1^2 + \ldots + h_k^2$). Then $N(P_f, \alpha_f) = 1$ and hence $t(P_f) \leq 1$. Since for every generic surjection $\alpha : P \twoheadrightarrow J$, J is generated by n elements, it follows from (5.9), that for every generic surjection $\beta : P_f \twoheadrightarrow J'_f$, J'_f is generated by n elements. Hence, by cases 1 and 2, P_f has a unimodular element. Therefore, by (5.8), there exists a surjection $\gamma : P \twoheadrightarrow J_1$, where J_1 is an ideal of height n such that $(J_1)_f = A_f$. Since $\mathfrak{m}_2, \ldots, \mathfrak{m}_r$ are the only real maximal ideals containing f, it follows that $N(P, \gamma) = r - 1$. Repeating this process, we see that $t(P) \leq 1$. This proves the theorem. ■

Chapter 6

The Weak Euler Class Group of a Noetherian Ring

Let A be a Noetherian ring of dimension $n \geq 2$. We define the *weak Euler class group* $E_0(A)$ of A as follows:

Let S be the set of ideals $\mathcal{N} \subset A$ such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements (where $\mathcal{N}$ is $\mathcal{M}$-primary ideal for some maximal ideal $\mathcal{M}$ of height n). Let G be the free abelian group on the set S.

Let $J = \cap \mathcal{N}_i$ be the intersection of finitely many ideals $\mathcal{N}_i$, where $\mathcal{N}_i$ is $\mathcal{M}_i$-primary and $\mathcal{M}_i$ being distinct maximal ideals of height n. Assume that J/J^2 is generated by n elements. We associate to J, the element $\sum \mathcal{N}_i$ of G. By abuse of notation, we denote this element of G by (J). Let H be the subgroup of G generated by elements of the type (J), where $J \subset A$ is an ideal of height n generated by n elements.

Definition : The weak Euler class group of A is defined as $E_0(A) = G/H$.

Let P be a projective A-module of rank n with trivial determinant and $\lambda : P \twoheadrightarrow J_0$ be a surjection, where $J_0 \subset A$ is an ideal of height n. We define $e(P) = (J_0)$ in $E_0(A)$. We show that this assignment is well defined.

Let $\mu : P \twoheadrightarrow J_1$ be another surjection, where J_1 is an ideal of height n. Then, by (4.3), there exists a surjection $\alpha(T) : P[T] \twoheadrightarrow I$, where $I \subset A[T]$ is an ideal of height n, with $\alpha(0) = \lambda$ and $\alpha(1) = \mu$. Now, using (4.7), there exists an ideal K of height n comaximal with $I \cap A$ such that $I \cap KA[T]$ is generated by n elements. Therefore $J_0 \cap K$

and $J_1 \cap K$ are generated by n elements. Hence $(J_0) = (J_1)$ in $E_0(A)$.

We note that there is a canonical surjective homomorphism from $E(A)$ to $E_0(A)$ obtained by forgetting orientations.

The aim of this chapter is to prove theorem (6.8) i.e. if A is Noetherian ring of even dimension n then (J) is zero in $E_0(A)$ if and only if J is a surjective image of a stably free A-module of rank n. This is proved along the same lines as (4.2) : first we prove some addition and subtraction principles (6.7), and then using the group theoretic lemma (4.1), we prove the theorem.

Lemma 6.1 *Let A be a ring and P a projective A-module of rank n. Let α be any element of P^*. Let $p_0, p_1, \ldots, p_n$ be $n+1$ elements of P. Let $w_i \in \wedge^n(P)$ be defined as follows : $w_0 = \alpha(p_0)\,(p_1 \wedge p_2 \wedge \ldots \wedge p_n)$ $w_i = \alpha(p_i)\,(p_0 \wedge \ldots \wedge p_{i-1} \wedge p_{i+1} \wedge \ldots \wedge p_n)$, $1 \leq i \leq n$. Then $\sum_{i=0}^{n}(-1)^i w_i = 0$.*

Proof Let e denote the element $(1,0) \in A \oplus P$. The map $x \mapsto e \wedge x$ is an isomorphism from $\wedge^n(P) \xrightarrow{\theta} \wedge^{n+1}(A \oplus P)$. Let $w = \sum_{i=0}^{n}(-1)^i w_i$. Now consider the map $\gamma : P \to A \oplus P$ defined by $\gamma(p) = (\alpha(p), p)$. We obtain an induced map $\wedge^{n+1}\gamma : \wedge^{n+1}P \to \wedge^{n+1}(A \oplus P)$. We get $\wedge^{n+1}\gamma(p_0 \wedge \ldots \wedge p_n) = e \wedge w + p_0 \wedge \ldots \wedge p_n$. But $\wedge^{n+1}(P) = 0$ hence $e \wedge w = 0$. But the map θ is an isomorphism, hence $w = 0$.

■

Lemma 6.2 *Let A be a Noetherian ring and P a projective A-module of rank n. Suppose that we are given the following short exact sequence*

$$0 \to P_1 \to A \oplus P \xrightarrow{(b,-\alpha)} A \to 0.$$

Let $(a_0, p_0) \in A \oplus P$ be such that $a_0 b - \alpha(p_0) = 1$. Let $q_i = (a_i, p_i) \in P_1$, $1 \leq i \leq n$. Then,
(1) The map $\delta : \wedge^n(P_1) \to \wedge^n(P)$ given by

$\delta(q_1 \wedge \ldots \wedge q_n) = a_0(p_1 \wedge \ldots \wedge p_n) + \sum_{i=1}^{n}(-1)^i a_i(p_0 \wedge \ldots \wedge p_{i-1} \wedge p_{i+1} \wedge \ldots \wedge p_n)$

is an isomorphism.

(2) $\delta(bq_1 \wedge \ldots \wedge q_n) = p_1 \wedge \ldots \wedge p_n$.

Proof Let $e = (1,0)$, $f = (a_0, p_0)$ in $A \oplus P$. Then $A \oplus P = Af \oplus P_1$ and as in (6.1), $f \wedge q_1 \wedge \ldots \wedge q_n = e \wedge w$ in $\wedge^{n+1}(A \oplus P)$, where $w = a_0(p_1 \wedge \ldots \wedge p_n) + \sum_{i=1}^{n}(-1)^i a_i(p_0 \wedge \ldots \wedge p_{i-1} \wedge p_{i+1} \wedge \ldots \wedge p_n)$.

Since the map $x \mapsto e \wedge x$ is an isomorphism from $\wedge^n(P)$ to $\wedge^{n+1}(A \oplus P)$, result (1) follows. Since $q_i = (a_i, p_i) \in P_1$, we have $ba_i = \alpha(p_i), 1 \leq i \leq n$. Moreover, $ba_0 = 1 + \alpha(p_0)$. Therefore, (2) follows from (6.1). ∎

Lemma 6.3 *Let A be a Noetherian ring and P a projective A-module of rank n. Suppose that we are given the following exact sequence*

$$0 \to P_1 \to A \oplus P \overset{(b,-\alpha)}{\longrightarrow} A \to 0.$$

Then, (i) The map $\beta : P_1 \to A$ given by $\beta(q) = c$, where $q = (c, p)$, has the property that $\beta(P_1) = \alpha(P)$.
(ii) The map $\Phi : P \to P_1$ given by $\Phi(p) = (\alpha(p), bp)$ has the property that $\beta\,\Phi = \alpha$ and $\delta \wedge^n (\Phi)$, where δ is as in (6.2), is scalar multiplication by b^{n-1}.

Proof Let $c \in \beta(P_1)$, then there exists $q = (c, p) \in P_1$ such that $\beta(q) = c$. Since $q \in P_1$, we have $bc = \alpha(p)$. Also there exists $q_0 = (a_0, p_0) \in A \oplus P$ such that $a_0 b - \alpha(p_0) = 1$. Now $\alpha(a_0 p - c p_0) = c$, hence $c \in \alpha(P)$. Conversely, let $c = \alpha(p)$, $p \in P$, then $bc = \alpha(bp)$. This shows that $(c, bp) \in P_1$, and hence $c \in \beta(P_1)$. This proves the first part.

The map $\delta : \wedge^n(P_1) \to \wedge^n(P)$ is given by

$$\delta(q_1 \wedge \ldots \wedge q_n) = a_0 (p_1 \wedge \ldots \wedge p_n) + \sum_{i=1}^{n} (-1)^i a_i (p_0 \wedge \ldots \wedge p_{i-1} \wedge p_{i+1} \wedge \ldots \wedge p_n)$$

where $q_i = (a_i, p_i) \in P_1, 1 \leq i \leq n$, and $(a_0, p_0) \in A \oplus P$. We have

$$\delta \wedge^n (\Phi)(p_1 \wedge \ldots \wedge p_n) \;=\; \delta((\alpha(p_1), bp_1) \wedge \ldots \wedge (\alpha(p_n), bp_n))$$

$$= a_0 b^n (p_1 \wedge \ldots \wedge p_n) + \textstyle\sum_{i=1}^{n} (-1)^i \alpha(p_i) b^{n-1} (p_0 \wedge \ldots \widehat{\wedge p_i} \ldots \wedge p_n)$$

$$= b^{n-1}((1 + \alpha(p_0))(p_1 \wedge \ldots \wedge p_n) + \textstyle\sum_{i=1}^{n} (-1)^i \alpha(p_i)(p_0 \wedge \ldots \widehat{\wedge p_i} \ldots \wedge p_n))$$

$$= b^{n-1}(p_1 \wedge \ldots \wedge p_n + \alpha(p_0)(p_1 \wedge \ldots \wedge p_n) + \textstyle\sum_{i=1}^{n} (-1)^i \alpha(p_i)(p_0 \wedge \ldots \widehat{\wedge p_i} \ldots \wedge p_n))$$

$$= b^{n-1}(p_1 \wedge \ldots \wedge p_n), \text{ by (6.1).}$$ ∎

Lemma 6.4 *Let A be a Noetherian ring of dimension $n \geq 2$. Let P be a projective A-module of rank n with trivial determinant and χ be an orientation of P. Let $\alpha : P \twoheadrightarrow J$*

be a surjection, where $J \subset A$ be an ideal of height n and let (J, w_J) be obtained from (α, χ). Let $a, b \in A$ be such that $ab = 1$ modulo J and let P_1 be the kernel of the surjection $(b, -\alpha) : A \oplus P \twoheadrightarrow A$. Let $\beta : P_1 \twoheadrightarrow J$ be as in (6.3) and χ_1 be the orientation of P_1 given by $\delta^{-1}\chi : A \overset{\sim}{\to} \wedge^n(P_1)$ (where δ is as in (6.2)). Then $(J, \overline{a^{n-1}} w_J)$ is obtained from (β, χ_1).

Proof We have an exact sequence $\quad 0 \to P_1 \to A \oplus P \overset{(b,-\alpha)}{\longrightarrow} A \to 0.$
The map $\beta : P_1 \twoheadrightarrow J$ is defined by $\beta(q) = c$, where $q = (c, p)$. By (6.3), we have $\beta(P_1) = \alpha(P) = J$. Since P_1 is stably isomorphic to P, determinant of P_1 is trivial. Let $\delta : \wedge^n(P_1) \overset{\sim}{\to} \wedge^n(P)$ be an isomorphism defined in (6.2). Let $\chi_1 = \delta^{-1} \chi : A \overset{\sim}{\to} \wedge^n(P_1)$ be an orientation of P_1. Then by (6.3), the map $\Phi : P \to P_1$ given by $\Phi(p) = (\alpha(p), b\,p)$ has the property that $\beta\,\Phi = \alpha$ and $\delta \wedge^n (\Phi)$ is a scalar multiplication by b^{n-1}. By (2.16), the map $\Phi \otimes A/J : P/JP \overset{\sim}{\to} P_1/JP_1$ is an isomorphism. Let $\overline{\gamma} : (A/J)^n \overset{\sim}{\to} P/JP$ be an isomorphism such that $\wedge^n(\overline{\gamma}) = \overline{\chi}$ and $w_J = \overline{\alpha}\,\overline{\gamma}$. Consider the commutative diagram

$$\begin{tikzcd}
P_1/JP_1 \arrow[r, two heads, "\overline{\beta}"] & J/J^2 \\
(A/J)^n \arrow[u, "\overline{\Phi}\,\overline{\gamma}"] \arrow[ur, two heads, "w_J"'] &
\end{tikzcd}$$

Since $\wedge^n(\overline{\Phi}\,\overline{\gamma}) = \overline{\delta}^{\,-1}\,\overline{b^{n-1}}\,\overline{\chi} = \overline{b^{n-1}}\,\overline{\chi}_1$. Hence $\overline{\chi}_1 = \overline{a^{n-1}}\,\wedge^n(\overline{\Phi}\,\overline{\gamma})$ since $ab = 1$ modulo J. Let θ be an automorphism of $(A/J)^n$ of determinant $\overline{a^{n-1}}$. Now consider the isomorphism $\overline{\Phi}\,\overline{\gamma}\theta : (A/J)^n \overset{\sim}{\to} P_1/JP_1$. Then $(J, \widetilde{w}_J)$ is obtained from (β, χ_1), where $\widetilde{w}_J = \overline{\beta}\,\overline{\Phi}\,\overline{\gamma}\theta = \overline{\alpha}\,\overline{\gamma}\theta = w_J\,\theta = \overline{a^{n-1}} w_J$. Hence $(J, \overline{a^{n-1}}\,w_J)$ is obtained from (β, χ_1). This proves the lemma. ∎

Lemma 6.5 *Let A be a Noetherian ring of even dimension n. Let P be a stably free A-module of rank n and χ be a generator of $\wedge^n(P)$. Suppose that $e(P, \chi) = (J, w_J)$ in $E(A)$, where J is an ideal of height n and w_J is a local orientation of J. Then there exists an ideal J' of height n which is generated by n elements and a local orientation $w_{J'}$ of J', such that $(J, w_J) = (J', w_{J'})$ in $E(A)$. Moreover, J' can be chosen to be comaximal with any given ideal of height n.*

Proof By the Bass Cancellation theorem (1.27), we have $P \oplus A \simeq A^{n+1}$. We have $P = A^{n+1}/(a_0, \ldots, a_n)$ for some unimodular row $(a_0, \ldots, a_n)$ in A^{n+1}. We can assume, by (2.3), that $J' = (a_1, \dots, a_n)$ is an ideal of height n. Further, we can assume that J' is comaximal with any given ideal of height n.

Let $\overline{e}_i$ be the image of the basis vector e_i of A^{n+1} in P. Then there exists a surjective map $\psi : P \twoheadrightarrow J'$ defined by $\psi(\overline{e}_0) = 0$, $\psi(\overline{e}_i) = a_{i+1}$ if i is odd and $\psi(\overline{e}_i) = -a_{i-1}$ if i is even. The map ψ is well defined since we have $\sum_0^n a_i \overline{e}_i = 0$ in P and $\psi(\sum_0^n a_i \overline{e}_i) = 0$. Computing $e(P, \chi)$ using ψ, we see that $(J, w_J) = (J', w_{J'})$.

$\blacksquare$

Lemma 6.6 *Let A be a Noetherian ring of even dimension n. Let P be a projective A-module of rank n having trivial determinant and χ_P be a generator of $\wedge^n(P)$. Let $e(P, \chi_P) = (J, w_J)$ in $E(A)$, where J is an ideal of height n and w_J is a local orientation of J. Suppose $\widetilde{w}_J$ is another local orientation of J. Then there exists a projective A-module P' of rank n with $[P'] = [P]$ in $K_0(A)$ and a generator $\chi_{P'}$ of $\wedge^n(P')$, such that $e(P', \chi_{P'}) = (J, \widetilde{w}_J)$ in $E(A)$.*

Proof By (5.1), $\widetilde{w}_J = \overline{b}\, w_J$ for some unit $\overline{b} \in A/J$. By (6.4), there exists a projective A-module P' of rank n with $[P'] = [P]$ in $K_0(A)$ and a generator $\chi_{P'}$ of $\wedge^n(P')$, such that $e(P', \chi_{P'}) = (J, \overline{b^{n-1}}\, w_J)$ in $E(A)$. Applying (5.4), we get the result. $\blacksquare$

Proposition 6.7 *Let A be a Noetherian ring of even dimension n. Let J_1 and J_2 be two comaximal ideals of A of heights n and $J_3 = J_1 \cap J_2$. Then*
(i) (Addition Principle) *If J_1 and J_2 are surjective images of stably free A-modules of rank n, then so is the J_3.*
(ii) (Subtraction Principle) *If J_1 and J_3 are surjective images of stably free A-modules of rank n, then so is J_2.*

Proof (i) Suppose that J_1 and J_2 are surjective images of stably free A-modules. Hence, there exist surjections $\psi_1 : P_1 \twoheadrightarrow J_1$ and $\psi_2 : P_2 \twoheadrightarrow J_2$, where P_1 and P_2 are stably free A-modules of rank n. We choose orientations χ_1 and χ_2 of P_1 and P_2 respectively. Then $e(P_1, \chi_1) = (J_1, w_{J_1})$ and $e(P_2, \chi_2) = (J_2, w_{J_2})$ in $E(A)$ for some local orientations w_{J_1} and w_{J_2} of J_1 and J_2 respectively. By (6.5), we can choose two comaximal ideals J_1' and J_2' of height n which are generated by n elements such that

$$(J_1, w_{J_1}) = (J_1', w_{J_1'}) \quad \text{and} \quad (J_2, w_{J_2}) = (J_2', w_{J_2'}) \tag{1}$$

in $E(A)$. Let $J_3' = J_1' \cap J_2'$. Let

$$(J_1, w_{J_1}) + (J_2, w_{J_2}) = (J_3, w_{J_3}) \tag{2}$$

$$(J_1', w_{J_1'}) + (J_2', w_{J_2'}) = (J_3', w_{J_3'}) \tag{3}$$

in $E(A)$. Then we have

$$(J_3, w_{J_3}) = (J_3', w_{J_3'}) \tag{4}$$

in $E(A)$. Since J_1' and J_2' are generated by n elements, by (3.5), J_3' is also generated by n elements. Therefore, applying (6.6) with P free, there exists a stably free A-module P_3 of rank n and an orientation χ_3 of P_3, such that $e(P_3, \chi_3) = (J_3', w_{J_3'})$. Hence, by (4.9), there exists a surjection from P_3 to J_3.

(ii) Assume that J_1 and J_3 are surjective images of stably free A-modules of rank n. Let $\psi_3 : P_3 \twoheadrightarrow J_3$ be a surjection, where P_3 is a stably free A-module of rank n. Let χ_3 be an orientation of P_3. Then

$$e(P_3, \chi_3) = (J_3, w_{J_3}) \tag{5}$$

in $E(A)$ for some local orientation w_{J_3} of J_3. Let w_{J_1} and w_{J_2} be local orientations of J_1 and J_2 respectively, obtained from w_{J_3}. Then

$$(J_1, w_{J_1}) + (J_2, w_{J_2}) = (J_3, w_{J_3}) \tag{6}$$

in $E(A)$. Since J_1 is a surjective image of a stably free A-module of rank n, by (6.6), there exists a stably free A-module P_1 of rank n such that $e(P_1, \chi_1) = (J_1, w_{J_1})$. Now since P_1 is stably free, by (6.5), there exists an ideal J_1' of height n which is generated by n elements and comaximal with J_2, such that

$$(J_1, w_{J_1}) = e(P_1, \chi_1) = (J_1', w_{J_1'}). \tag{7}$$

Let $J_1' \cap J_2 = J_4$. Then $w_{J_1'}$ and w_{J_2} induce a local orientation w_{J_4} of J_4 such that

$$(J_1', w_{J_1'}) + (J_2, w_{J_2}) = (J_4, w_{J_4}). \tag{8}$$

By (6,7), we have $(J_3, w_{J_3}) = (J_4, w_{J_4})$ in $E(A)$. By (5), we have $e(P_3, \chi_3) = (J_4, w_{J_4})$. Let $J_1' = (a_1, \dots, a_n)$ and let

$$(J_1', [\overline{a}_1, \dots, \overline{a}_n]) + (J_2, w_{J_2}) = (J_4, \widetilde{w}_{J_4}).$$

Then since $(J_1', [\overline{a}_1, \dots, \overline{a}_n]) = 0$ in $E(A)$, we have $(J_2, w_{J_2}) = (J_4, \widetilde{w}_{J_4})$ in $E(A)$. By (6.6), there exists a projective A-module Q of rank n which is stably isomorphic to P_3 and hence stably free, and an orientation χ_Q of Q such that

$$e(Q, \chi_Q) = (J_4, \widetilde{w}_{J_4}).$$

Therefore $e(Q, \chi_Q) = (J_2, w_{J_2})$. By (4.9), there exists a surjection from Q to J_2 and hence the proposition is proved. ■

Theorem 6.8 *Let A be a Noetherian ring of even dimension n. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Then $(J) = 0$ in $E_0(A)$ if and only if J is a surjective image of a stably free A-module of rank n.*

Proof We will apply lemma (4.1) to prove this theorem.

Let F be the free abelian group on the set $(\mathcal{N})$ where $\mathcal{N}$ is $\mathcal{M}$-primary ideal of height n such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements. Define an equivalence relation on the set $(\mathcal{N})$ by $(\mathcal{N}) \sim (\mathcal{N}_1)$ if $\sqrt{\mathcal{N}} = \sqrt{\mathcal{N}_1}$, i.e. $\mathcal{N}$ and $\mathcal{N}_1$ both are $\mathcal{M}$-primary ideals of A. If $J \subset A$ is an ideal of height n and $J = \cap \mathcal{N}_i$ is a reduced primary decomposition of J, then denote (J) the element $\sum (\mathcal{N}_i)$ of F. Let S be the set

$\{ (J) \in F \mid J$ is surjective image of a stably free A-module of rank $n\}$. Then

(1) Every element of S is nicely reduced.

(2) Let x, $y \in F$ be nicely reduced such that $x + y$ is also nicely reduced. Then if any two of x, y and $x + y$ belongs to S then so does the third. This follows from (6.7).

(3) Let $x \in F$ be nicely reduced and $x \notin S$ and let $(\mathcal{N}_i)$, for $i = 1, \ldots, r$ be finitely many elements of F. Since x is nicely reduced, we have $x = (J)$ for some height n ideal J. Applying (2.14), there exists an ideal J' of height n which is comaximal with $J, \mathcal{N}_i$, $i = 1, \ldots, r$, such that $J \cap J'$ is generated by n elements. Let $y = (J')$. Then $x + y \in S$.

Let H' be the subgroup of F generated by S. Then by (4.1), if $x \in H'$ is nicely reduced then $x \in S$. $(*)$

Let H be the subgroup of F generated by $(J) \in F$ where J is generated by n elements. We claim that $H = H'$.

Clearly $H \subset H'$. For other inclusion, it is enough to show that $S \subset H$. For this, let $(J) \in S$. Then J is surjective image of a stably free A-module P of rank n. Applying (6.5), there exists an ideal J' of height n which is generated by n elements such that J' is surjective image of P. By (4.7), there exists an ideal K of height n comaximal with J and J' such that $J \cap K$ and $J' \cap K$ are generated by n elements. Hence $(J) \in H$ and $H = H'$.

Suppose $(J) = 0$ in $E_0(A)$. Then $(J) \in H = H'$ is a nicely reduced element. Hence by $(*)$, we get that J is surjective image of a stably free A-module of rank n.

$\blacksquare$

Proposition 6.9 *Let A be a Noetherian ring of dimension n. Let P and P_1 be two projective A-modules of rank n such that $[P] = [P_1]$ in $K_0(A)$. Then, there exists an ideal $J \subset A$ of height $\geq n$ such that J is a surjective image of both P and P_1.*

Proof Since dim $(A) = n$ and $[P] = [P_1]$ in $K_0(A)$, we have $P \oplus Q \simeq P_1 \oplus Q$ for some projective A-module Q. We may assume that Q is free by replacing Q by $Q \oplus Q' \simeq A^t$. Now it follows that $P \oplus A \stackrel{\sim}{\to} P_1 \oplus A$ by the Bass Cancellation theorem (1.27). Therefore, there exists a short exact sequence

$$0 \to P_1 \to A \oplus P \stackrel{(b,-\alpha)}{\longrightarrow} A \to 0.$$

Further, without loss of generality, we may replace α by $\alpha + b\gamma$ by a transvection, where $\gamma \in P^*$, because this will not change the isomorphism class of ker $((b, -\alpha)) = P_1$, i.e. if ker $((b, -(\alpha + b\gamma))) = P_2$ then $P_1 \stackrel{\sim}{\to} P_2$. Therefore, using (2.12), we may assume that the ideal $\alpha(P) = J$ is such that height $(J) \geq n$. By $(6.3(i))$, J is also a surjective image of P_1. This proves the proposition. $\blacksquare$

Corollary 6.10 *Let A be a Noetherian ring of even dimension n. Let P be a projective A-module of rank n with trivial determinant. Then $e(P) = 0$ in $E_0(A)$ if and only if $[P] = [Q \oplus A]$ in $K_0(A)$ for some projective A-module Q of rank $n-1$.*

Proof First assume that $[P] = [Q \oplus A]$ in $K_0(A)$ for some projective A-module Q of rank $n-1$. Then by (6.9), $e(P) = e(Q \oplus A)$. By (6.8), $e(Q \oplus A) = 0$ in $E_0(A)$. Hence $e(P) = 0$.

Now we assume that $e(P) = 0$ in $E_0(A)$. Let $\psi : P \twoheadrightarrow J$ be a surjection, where J is an ideal of height n. Let $e(P, \chi) = (J, w_J)$, where χ is a generator of $\wedge^n(P)$ and w_J is a local orientation of J. Since $e(P) = (J) = 0$ in $E_0(A)$, it follows from (6.8) that J is a surjective image of a stably free A-module of rank n. It follows now from (6.6), that there exists a stably free A-module $\widetilde{P}$ of rank n and a generator $\widetilde{\chi}$ of $\wedge^n(\widetilde{P})$ such that $e(\widetilde{P}, \widetilde{\chi}) = (J, w_J)$. Since $\widetilde{P}$ is a stably free A-module of rank n, by (6.5), there exists an ideal J_1 of height n which is generated by n elements and a local orientation w_{J_1} of J_1 such that $(J, w_J) = (J_1, w_{J_1})$. Hence we have $e(P, \chi) = (J, w_J) = (J_1, w_{J_1})$. Let $J_1 = (b_1, \ldots, b_n)$. Then by (6.6), there exists a projective A-module P' of rank n with $[P'] = [P]$ in $K_0(A)$ and a generator $\chi_{P'}$ of $\wedge^n(P')$, such that $e(P', \chi_{P'}) = (J_1, [\overline{b}_1, \ldots, \overline{b}_n]) = 0$ in $E(A)$. But then by (4.10), P' has a unimodular element. Hence $P' = Q \oplus A$. But $[P] = [P']$ in $K_0(A)$. This proves the corollary. ■

Corollary 6.11 *Let A be a Noetherian ring of even dimension n. Let P be a projective A-module of rank n with trivial determinant. Suppose that $e(P) = (J)$ in $E_0(A)$, where $J \subset A$ is an ideal of height n. Then, there exists a projective A-module Q of rank n, such that $[Q] = [P]$ in $K_0(A)$ and J is a surjective image of Q.*

Proof By (2.14), there exists a surjection $\psi : P \twoheadrightarrow J \cap J_1$, where J_1 is a height n ideal such that $J + J_1 = A$. Since $e(P) = (J) = (J \cap J_1)$ in $E_0(A)$, it follows that $(J_1) = 0$ in $E_0(A)$. Using ψ we have

$$e(P, \chi) = (J, w_J) + (J_1, w_{J_1})$$

in $E(A)$, where χ is a generator of $\wedge^n(P)$. Since $(J_1) = 0$ in $E_0(A)$, it follows from (6.8) that J_1 is the surjective image of a stably free A-module of rank n. Therefore by (6.6), there exists a stably free A-module P_1 of rank n such that $e(P_1, \chi_1) = (J_1, w_{J_1})$, where χ_1 is a generator of $\wedge^n(P_1)$ and w_{J_1} is a local orientation of J_1. By (6.5), we can choose an ideal J_2 of height n which is generated by n elements and is comaximal with J, such that $(J_1, w_{J_1}) = (J_2, w_{J_2})$ for some local orientation w_{J_2} of J_2. Therefore

$$e(P, \chi) = (J, w_J) + (J_2, w_{J_2}) = (J \cap J_2, w_{J \cap J_2}),$$

where $w_{J \cap J_2}$ is a local orientation of $J \cap J_2$ induced from w_J and w_{J_2}. Therefore by (4.9), there exists a surjection from P to $J \cap J_2$. Since J_2 is generated by n elements, we can choose a set of generators $b_1, \ldots, b_n$ of J_2. Let

$$(J, w_J) + (J_2, [\,\overline{b}_1, \ldots, \overline{b}_n]) = (J \cap J_2, \widetilde{w}_{J \cap J_2}).$$

By (6.6), there exists a projective A-module Q with $[Q] = [P]$ in $K_0(A)$ such that

$$e(Q, w_Q) = (J \cap J_2, \widetilde{w}_{J \cap J_2}) = (J, w_J).$$

Hence, by (4.9), there exists a surjection from Q to J. This proves the corollary. ■

Proposition 6.12 *Let A be a Noetherian ring of even dimension n and let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let $w_J : (A/J)^n \twoheadrightarrow J/J^2$ be a surjection. Suppose that the element (J, w_J) of $E(A)$ belongs to the kernel of the canonical homomorphism $E(A) \twoheadrightarrow E_0(A)$. Then, there exists a stably free A-module P_1 of rank n and a generator χ_1 of $\wedge^n(P_1)$ such that $e(P_1, \chi_1) = (J, w_J)$ in $E(A)$.*

Proof Since $(J, w_J) \in E(A)$ belongs to the kernel of the canonical homomorphism $E(A) \to E_0(A)$, it follows that $(J) = 0$ in $E_0(A)$. Hence, by (6.8), there exists a surjection $\alpha : P \twoheadrightarrow J$, where P is stably free A-module of rank n. Let χ be a generator of $\wedge^n(P)$. Suppose that $(J, \widetilde{w}_J)$ is obtained from (α, χ). By (5.1), there exists $a \in A$ such that $\overline{a} \in A/J$ is a unit and $w_J = \overline{a}\,\widetilde{w}_J$. By (6.4), there exists a projective A-module P_1 of rank n with $[P_1] = [P]$ in $K_0(A)$ and a generator χ_1 of $\wedge^n P_1$, such that $e(P_1, \chi_1) = (J, \overline{a^{n-1}}\,\widetilde{w}_J)$ in $E(A)$. Since n is even, by (5.4), we have $(J, \overline{a^{n-1}}\,\widetilde{w}_J) = (J, \overline{a}\,\widetilde{w}_J)$ in $E(A)$. Hence, $e(P_1, \chi_1) = (J, w_J)$ in $E(A)$. ■

Corollary 6.13 *Let A be a Noetherian ring of even dimension n. Let P be a projective A-module of rank n with trivial determinant. Let $\alpha : P \twoheadrightarrow J$ be a surjection, where $J \subset A$ is an ideal of height n. Then J is a surjective image of a stably free A-module of rank n if and only if $[P] = [Q \oplus A]$ in $K_0(A)$ for some projective A-module Q of rank $n-1$.*

Proof Let J be a surjective image of a stably free A-module of rank n. Then by (6.8), $(J) = 0$ in $E_0(A)$. Hence $e(P) = (J) = 0$ in $E_0(A)$. Applying (6.10), the result follows.

The converse also follows from (6.8) and (6.10). ■

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