1. INTRODUCTION

Throughout the paper, rings are commutative Noetherian and projective modules are finitely generated and of constant rank.

If $R$ is a ring of dimension $n$, then Serre [Se] proved that projective $R$-modules of rank $> n$ contain a unimodular element. Plumstead [P] generalized this result and proved that projective $R[X] = R[Z_+]$-modules of rank $> n$ contain a unimodular element. Bhatwadekar and Roy [B-R 2] generalized this result and proved that projective $R[X_1, \ldots, X_r] = R[Z_+^r]$-modules of rank $> n$ contain a unimodular element.

In another direction, if $A$ is a ring such that $R[X] \subset A \subset R[X, X^{-1}]$, then Bhatwadekar and Roy [B-R 1] proved that projective $A$-modules of rank $> n$ contain a unimodular element. Rao [Ra] improved this result and proved that if $B$ is a birational overring of $R[X]$, i.e. $R[X] \subset B \subset S^{-1}R[X]$, where $S$ is the set of non-zerodivisors of $R[X]$, then projective $B$-modules of rank $> n$ contain a unimodular element. Bhatwadekar, Lindel and Rao [B-L-R, Theorem 5.1, Remark 5.3] generalized this result and proved that projective $B[Z_+]$-modules of rank $> n$ contain a unimodular element when $B$ is seminormal. Bhatwadekar [Bh, Theorem 3.5] removed the hypothesis of seminormality used in [B-L-R].

All the above results are best possible in the sense that projective modules of rank $n$ over above rings need not have a unimodular element. So it is natural to look for obstructions for a projective module of rank $n$ over above rings to contain a unimodular element. We will prove some results in this direction.

Let $P$ be a projective $R[Z_+][T]$-module of rank $n = \dim R$ such that $P_f$ and $P/TP$ contain unimodular elements for some monic polynomial $f$ in the variable $T$. Then $P$ contains a unimodular element. The proof of this result is implicit in [B-L-R, Theorem 5.1]. We will generalize this result to projective $R[M][T]$-modules of rank $n$, where $M \subset Z_+^r$ is a $\Phi$-simplicial monoid in the class $C(\Phi)$. For this we need the following result whose proof is similar to [B-L-R, Theorem 5.1].
Proposition 1.1. Let $R$ be a ring and $P$ be a projective $R[X]$-module. Let $J \subset R$ be an ideal such that $P_s$ is extended from $R_s$ for every $s \in J$. Suppose that

(a) $P/JP$ contains a unimodular element.

(b) If $I$ is an ideal of $(R/J)[X]$ of height $\text{rank}(P) - 1$, then there exist $\sigma \in \text{Aut}((R/J)[X])$ with $\sigma(X) = X$ and $\sigma \in \text{Aut}(R[X])$ with $\sigma(X) = X$ which is a lift of $\sigma$ such that $\sigma(I)$ contains a monic polynomial in the variable $X$.

(c) $EL(P/(X,J)P)$ acts transitively on $\text{Um}(P/(X,J)P)$.

(d) There exists a monic polynomial $f \in R[X]$ such that $P_f$ contains a unimodular element. Then the natural map $\text{Um}(P) \rightarrow \text{Um}(P/XP)$ is surjective. In particular, if $P/XP$ contains a unimodular element, then $P$ contains a unimodular element.

We prove the following result as an application of (1.1).

Theorem 1.2. Let $R$ be a ring of dimension $n$ and $M \subset \mathbb{Z}_+^n$ a $\Phi$-simplicial monoid in the class $C(\Phi)$. Let $P$ be a projective $R[M][T]$-module of rank $n$ whose determinant is extended from $R$. Assume $P/TP$ and $P_f$ contain unimodular elements for some monic polynomial $f$ in the variable $T$. Then the natural map $\text{Um}(P) \rightarrow \text{Um}(P/TP)$ is surjective. In particular, $P$ contains a unimodular element.

Let $R$ be a ring containing $\mathbb{Q}$ of dimension $n \geq 2$. If $P$ is a projective $R[X]$-module of rank $n$, then Das and Zinna [D-Z] have obtained an obstruction for $P$ to have a unimodular element. Let us fix an isomorphism $\chi : L \sim \wedge^n P$, where $L$ is the determinant of $P$. To the pair $(P, \chi)$, they associated an element $e(P, \chi)$ of the Euler class group $E(R[X], L)$ and proved that $P$ has a unimodular element if and only if $e(P, \chi) = 0$ in $E(R[X], L)$ [D-Z].

It is desirable to have such an obstruction for projective $R[X,Y]$-module $P$ of rank $n$. As an application of (1.2), we obtain such a result. Recall that $R(X)$ denotes the ring obtained from $R[X]$ by inverting all monic polynomials in $X$. Let $L$ be the determinant of $P$ and $\chi : L \sim \wedge^n(P)$ be an isomorphism. We define the Euler class group $E(R[X,Y], L)$ of $R[X,Y]$ as the product of Euler class groups $E(R(X)[Y], L \otimes R(X)[Y])$ of $R(X)[Y]$ and $E(R[Y], L \otimes R[Y])$ of $R[Y]$ defined by Das and Zinna [D-Z]. To the pair $(P, \chi)$, we associate an element $e(P, \chi)$ in $E(R[X,Y], L)$ and prove the following result (3.4).

Theorem 1.3. Let the notations be as above. Then $e(P, \chi) = 0$ in $E(R[X,Y], L)$ if and only if $P$ has a unimodular element.

Let $R$ be a local ring and $P$ be a projective $R[T]$-module. Roitman [Ro, Lemma 10] proved that if the projective $R[T]_f$-module $P_f$ contains a unimodular element for some monic polynomial $f \in R[T]$, then $P$ contains a unimodular element. Roy [Ry,
Theorem 1.1] generalized this result and proved that if $P$ and $Q$ are projective $R[T]$-modules with rank($Q$) < rank($P$) such that $Q_f$ is a direct summand of $P_f$ for some monic polynomial $f \in R[T]$, then $Q$ is a direct summand of $P$. Mandal [M, Theorem 2.1] extended Roy’s result to Laurent polynomial rings.

We prove the following result (4.4) which gives Mandal’s [M] in case $A = R[X, X^{-1}]$. Recall that a monic polynomial $f \in R[X]$ is called special monic if $f(0) = 1$.

**Theorem 1.4.** Let $R$ be a local ring and $R[X] \subset A \subset R[X, X^{-1}]$. Let $P$ and $Q$ be two projective $A$-modules with rank($Q$) < rank($P$). If $Q_f$ is a direct summand of $P_f$ for some special monic polynomial $f \in R[X]$, then $Q$ is also a direct summand of $P$.

## 2. Preliminaries

**Definition 2.1.** Let $R$ be a ring and $P$ be a projective $R$-module. An element $p \in P$ is called unimodular if there is a surjective $R$-linear map $\varphi : P \to R$ such that $\varphi(p) = 1$. Note that $P$ has a unimodular element if and only if $P \cong Q \oplus R$ for some $R$-module $Q$. The set of all unimodular elements of $P$ is denoted by $\text{Um}(P)$.

**Definition 2.2.** Let $M$ be a finitely generated submonoid of $\mathbb{Z}_+^r$ of rank $r$ such that $M \subset \mathbb{Z}_+^r$ is an integral extension, i.e. for any $x \in \mathbb{Z}_+^r$, $nx \in M$ for some integer $n > 0$. Such a monoid $M$ is called a $\Phi$-simplicial monoid of rank $r$ [G2].

**Definition 2.3.** Let $M \subset \mathbb{Z}_+^r$ be a $\Phi$-simplicial monoid of rank $r$. We say that $M$ belongs to the class $\mathcal{C}(\Phi)$ if $M$ is seminormal (i.e. if $x \in gp(M)$ and $x^2, x^3 \in M$, then $x \in M$) and if we write $\mathbb{Z}_+^r = \{t_1^{s_1} \cdots t_r^{s_r} | s_i \geq 0\}$, then for $1 \leq m \leq r$, $M_m = M \cap \{t_1^{s_1} \cdots t_m^{s_m} | s_i \geq 0\}$ satisfies the following properties: Given a positive integer $c$, there exist integers $c_i > c$ for $i = 1, \ldots, m - 1$ such that for any ring $R$, the automorphism $\eta \in \text{Aut}_{R[t_1]}(R[t_1, \ldots, t_m])$ defined by $\eta(t_i) = t_i + t_m^c$ for $i = 1, \ldots, m - 1$, restricts to an $R$-automorphism of $R[M_m]$. It is easy to see that $M_m \subset C(\Phi)$ and rank $M_m = m$ for $1 \leq m \leq r$.

**Example 2.4.** The following monoids belong to $C(\Phi)$ [K-S, Example 3.5, 3.9, 3.10].

(i) If $M \subset \mathbb{Z}_+^2$ is a finitely generated and normal monoid (i.e. $x \in gp(M)$ and $x^n \in M$ for some $n > 1$, then $x \in M$) of rank 2, then $M \subset C(\Phi)$.

(ii) For a fixed integer $n > 0$, if $M \subset \mathbb{Z}_+^r$ is the monoid generated by all monomials in $t_1, \ldots, t_r$ of total degree $n$, then $M$ is a normal monoid of rank $r$ and $M \subset C(\Phi)$. In particular, $\mathbb{Z}_+^r \subset C(\Phi)$ and $< t_1^2, t_2^2, t_3 t_2, t_1 t_3, t_2 t_3 > \in C(\Phi)$.

(iii) The submonoid $M$ of $\mathbb{Z}_+^3$ generated by $< t_1^2, t_2^2, t_3^2, t_1 t_3, t_2 t_3 > \in C(\Phi)$.

**Remark 2.5.** Let $R$ be a ring and $M \subset \mathbb{Z}_+^r = \{t_1^{m_1} \cdots t_r^{m_r} | m_i \geq 0\}$ be a monoid of rank $r$ in the class $C(\Phi)$. Let $I$ be an ideal of $R[M]$ of height $> \dim R$. Then by [G2, Lemma 6.5]
and [K-S, Lemma 3.1], there exists an $R$-automorphism $\sigma$ of $R[M]$ such that $\sigma(t_r) = t_r$ and $\sigma(I)$ contains a monic polynomial in $t_r$ with coefficients in $R[M] \cap R[t_1, \ldots, t_{r-1}]$.

We will state some results for later use.

**Theorem 2.6.** [K-S, Theorem 3.4] Let $R$ be a ring and $M$ be a $\Phi$-simplicial monoid such that $M \in C(\Phi)$. Let $P$ be a projective $R[M]$-module of rank $> \dim R$. Then $P$ has a unimodular element.

**Theorem 2.7.** [D-K, Theorem 4.5] Let $R$ be a ring and $M$ be a $\Phi$-simplicial monoid. Let $P$ be a projective $R[M]$-module of rank $\geq \max\{\dim R + 1, 2\}$. Then $EL(P \oplus R[M])$ acts transitively on $Um(P \oplus R[M])$.

The following result is proved in [B-L-R, Criterion-1 and Remark] in case $J = Q(P, R_0)$ is the Quillen ideal of $P$ in $R_0$. The same proof works in our case.

**Theorem 2.8.** Let $R = \bigoplus_{i \geq 0} R_i$ be a graded ring and $P$ be a projective $R$-module. Let $J$ be an ideal of $R_0$ such that $J$ is contained in the Quillen ideal $Q(P, R_0)$. Let $p \in P$ be such that $p_{1+R^+} \in Um(P_{1+R^+})$ and $p_{1+J} \in Um(P_{1+J})$, where $R^+ = \bigoplus_{i \geq 1} R_i$. Then $P$ contains a unimodular element $p_1$ such that $p = p_1$ modulo $R^+ P$.

The following result is a consequence of Eisenbud-Evans [E-E], as stated in [P, p. 1420].

**Lemma 2.9.** Let $A$ be a ring and $P$ be a projective $A$-module of rank $n$. Let $(\alpha, a) \in (P^* \oplus A)$. Then there exists an element $\beta \in P^*$ such that $\ht(I_0) \geq n$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq n$, then $\ht I \geq n$. Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and $I$ is a proper ideal of $A$, then $\ht I = n$.

3. **Proofs of (1.1), (1.2) and (1.3)**

3.1. **Proof of Proposition 1.1.** Let $p_0 \in Um(P/JP)$ and $p_1 \in Um(P/XP)$. Let $\tilde{p}_0$ and $\tilde{p}_1$ be the images of $p_0$ and $p_1$ in $P/(X, J)P$. By hypothesis (c), there exist $\tilde{\delta} \in EL(P/(X, J)P)$ such that $\tilde{\delta}(\tilde{p}_0) = \tilde{p}_1$. By [B-R 2, Proposition 4.1], $\tilde{\delta}$ can be lifted to an automorphism $\delta$ of $P/JP$. Consider the fiber product diagram for rings and modules

\[
\begin{array}{ccc}
R[X]_{(X)} & \rightarrow & R[X]_{(X)} \\
\downarrow & & \downarrow \\
R[X]_{(X)} & \rightarrow & R[X]_{(X, J)}.
\end{array}
\]

\[
\begin{array}{ccc}
P_{(X)P} & \rightarrow & P_{(X)P} \\
\downarrow & & \downarrow \\
P_{(X)P} & \rightarrow & P_{(X, J)P}.
\end{array}
\]

We can patch $\delta(p_0)$ and $p_1$ to get a unimodular element $p \in Um(P/XJP)$ such that $p = \delta(p_0)$ modulo $JP$ and $p = p_1$ modulo $XP$. Writing $\delta(p_0)$ by $p_0$, we assume that $p = p_0$ modulo $JP$ and $p = p_1$ modulo $XP$. 


Using hypothesis (d), we get an element \( q \in P \) such that the order ideal \( O_P(q) = \{ \phi(q) | \phi \in \text{Hom}_{\mathbb{R}[X]}(P, \mathbb{R}[X]) \} \) contains a power of \( f \). We may assume that \( f \in O_P(q) \).

Let “bar” denote reduction modulo the ideal (J). Write \( \overline{P} = \overline{\mathbb{R}[X]p_0} \oplus Q \) for some projective \( \overline{\mathbb{R}[X]} \)-module \( Q \) and \( \overline{q} = (\pi p_0, q') \) for some \( q' \in Q \). By Eisenbud-Evans (2.9), there exist \( \tau \in EL(\overline{P}) \) such that \( \tau(q) = (\pi p_0, q') \) and \( \text{ht}(\overline{O}_q(q'))\overline{\mathbb{R}[X]}_{\tau} \geq \text{rank}(P) - 1 \). Since \( \tau \) can be lifted to \( \tau \in \text{Aut}(P) \), replacing \( P \) by \( \tau(P) \), we may assume that \( \text{ht}(\overline{O}_q(q')) \geq \text{rank}(P) - 1 \) on the Zariski-open set \( D(\pi) \) of \( \text{Spec}(\overline{\mathbb{R}[X]}) \).

Let \( p_1, \ldots, p_r \) be minimal prime ideals of \( \overline{O}_q(q') \) in \( \overline{\mathbb{R}[X]} \) not containing \( \pi \). Then \( \text{ht}(\cap_i p_i) \geq \text{rank}(P) - 1 \). By hypothesis (b), we can find \( \sigma \in \text{Aut}(\overline{\mathbb{R}[X]}) \) with \( \sigma(X) = X \) and \( \sigma \in \text{Aut}(\mathbb{R}[X]) \) with \( \sigma(X) = X \) which is a lift of \( \sigma \) such that \( \sigma(\cap_i p_i) \) contains a monic polynomial in \( \mathbb{R}[X] = \overline{\mathbb{R}[X]} \). Note that \( \sigma(f) \) is a monic polynomial. Replacing \( \mathbb{R}[X] \) by \( \sigma(\mathbb{R}[X]) \), we may assume that \( \cap_i p_i \) contains a monic polynomial in \( \overline{\mathbb{R}[X]} \), and \( f \in O_P(q) \) is a monic polynomial.

If \( p \) is a minimal prime ideals of \( O_q(q') \) in \( \overline{\mathbb{R}[X]} \) containing \( \pi \), then \( p \) contains \( \overline{O}_q(q) \). Since \( f \in O_P(q) \), \( p \) contains the monic polynomial \( \overline{f} \). Therefore, all minimal primes of \( O_q(q') \) contains a monic polynomial, hence \( O_q(q') \) contains a monic polynomial, say \( \overline{g} \in \overline{\mathbb{R}[X]} \). Let \( g \in \mathbb{R}[X] \) be a monic polynomial which is a lift of \( \overline{g} \).

**Claim:** For large \( N > 0 \), \( p_2 = p + X^ng^Nq \in \text{Um}(P_1 + J_R) \).

Choose \( \phi \in P^* \) such that \( \phi(q) = f \). Then \( \phi(p_2) = \phi(p) + X^Ng^Nf \) is a monic polynomial for large \( N \). Since \( p = p_0 \) module \( JP, \overline{p} = p_0 \) and \( \overline{q} = (\overline{p}, q') \). Therefore,

\[
\overline{p}_2 = \overline{p} + X^Ng^N(\overline{mp}, q') = ((1 + T^Ng^N\overline{g})\overline{p}, X^Ng^Nq').
\]

Since \( \overline{g} \in O_q(q') \subset O_{\overline{p}}(\overline{p}_2) \), we get \( O_{\overline{p}}(\overline{g}) \subset O_\tau(\overline{p}_2) \). Since \( \overline{g} \in \text{Um}(\overline{P}) \), we get \( \overline{p}_2 \in \text{Um}(\overline{P}) \) and hence \( p_2 \in \text{Um}(P_1 + J_{R[X]}) \). Since \( O_P(p_2) \) contains a monic polynomial, by [La, Lemma 1.1, p. 79], \( p_2 \in \text{Um}(P_1 + J_R) \).

Now \( p_2 = p_1 \) modulo \( XP \), we get \( p_2 \in \text{Um}(P/XP) \). By (2.8), there exist \( p_3 \in \text{Um}(P) \) such that \( p_3 = p_2 = p_1 \) modulo \( XP \). This completes the proof.

**3.2. Proof of Theorem 1.2.** Without loss of generality, we may assume that \( R \) is reduced. When \( n = 1 \), the result follows from well known Quillen [Q] and Suslin [Su]. When \( n = 2 \), the result follows from Bhatwadekar [Bh, Proposition 3.3] where he proves that if \( P \) is a projective \( R[T] \)-module of rank 2 such that \( P \) contains a unimodular element for some monic polynomial \( f \in R[T] \), then \( P \) contains a unimodular element. So now we assume \( n \geq 3 \).

Write \( A = R[M] \). Let \( J(A, P) = \{ s \in A | p_4 \text{ is extended from } A \} \) be the Quillen ideal of \( P \) in \( A \). Let \( J = J(A, P) \cap R \) be the ideal of \( R \) and \( J = J_{R[M]} \). We will show that \( J \)

satisfies the properties of (1.1).

Let \( p \in \text{Spec}(R) \) with \( \text{ht}(p) = 1 \) and \( S = R - p \). Then \( S^{-1}P \) is a projective module over \( S^{-1}A[T] = R_p[M][T] \). Since \( \text{dim}(R_p) = 1 \), by (2.6), \( S^{-1}P = A[n]P \oplus S^{-1}A[T]^{-1} \).
Since determinant of \( P \) is extended from \( R, \wedge^n P_S = A[T]_S \) and hence \( S^{-1}P \) is free. Therefore there exists \( s \in R - p \) such that \( P_s \) is free. Hence \( s \in \tilde{J} \) and so \( \text{ht}(\tilde{J}) \geq 2 \).

Since \( \dim(R/\tilde{J}) \leq n - 2 \) and \( A[T]/(J) = (R/\tilde{J})[M][T] \), by (2.6), \( P/JP \) contains a unimodular element.

If \( I \) is an ideal of \( (A/J)[T] = (R/\tilde{J})[M][T] \) of height \( \geq n - 1 \), then by (2.5), there exists an \( R[T]-\text{automorphism} \ \sigma \in \text{Aut}_{R[T]}(A[T]) \) such that if \( \pi \) denotes the induced automorphism of \( (A/J)[T] \), then \( \pi(I) \) contains a monic polynomial in \( T \).

By (2.7), \( E \hat{L}(P/(T,J)P) \) acts transitively on \( \text{Um}(P/(J,T)P) \).

Therefore, the result now follows from (1.1).

\[ \square \]

**Corollary 3.1.** Let \( R \) be a ring of dimension \( n \), \( A = R[X_1, \cdots , X_m] \) a polynomial ring over \( R \) and \( P \) be a projective \( A[T] \)-module of rank \( n \). Assume that \( P/TP \) and \( P_f \) both contain a unimodular element for some monic polynomial \( f(T) \in A[T] \). Then \( P \) has a unimodular element.

Proof. If \( n = 1 \), the result follows from well known Quillen [Q] and Suslin [Su] Theorem. When \( n = 2 \), the result follows from Bhatwadekar [Bh, Proposition 3.3]. Assume \( n \geq 3 \). Let \( L \) be the determinant of \( P \). If \( \tilde{R} \) is the seminormalization of \( R \), then by Swan [Sw], \( \tilde{L} \otimes \tilde{R}[X_1, \cdots , X_m] \) is extended from \( \tilde{R} \). By (1.2), \( \tilde{P} \otimes \tilde{R}[X_1, \cdots , X_m] \) has a unimodular element. Since \( \tilde{R}[X_1, \cdots , X_n] \) is the seminormalization of \( A \), by Bhatwadekar [Bh, Lemma 3.1], \( P \) has a unimodular element.

\[ \square \]

**3.3. Obstruction for Projective Modules to have a Unimodular Element.** Let \( R \) be a ring of dimension \( n \geq 2 \) containing \( \mathbb{Q} \) and \( P \) be a projective \( R[X,Y] \)-module of rank \( n \) with determinant \( L \). Let \( \chi : L \overset{\sim}{\twoheadrightarrow} \wedge^n(P) \) be an isomorphism. We call \( \chi \) an *orientation* of \( P \). In general, we shall use ‘hat’ when we move to \( R(X)[Y] \) and ‘bar’ when we move modulo the ideal \( (X) \). For instance, we have:

1. \( L \otimes R(X)[Y] = \hat{L} \) and \( L/XYL = \hat{L} \),
2. \( P \otimes R(X)[Y] = \hat{P} \) and \( P/XP = \hat{P} \).

Similarly, \( \bar{\chi} \) denotes the induced isomorphism \( \hat{L} \overset{\sim}{\twoheadrightarrow} \wedge^n \hat{P} \) and \( \bar{\chi} \) denotes the induced isomorphism \( \bar{\wedge}^n \bar{P} \).

We now define the *Euler class of \( (P, \chi) \).*

**Definition 3.2.** First we consider the case \( n \geq 2 \) and \( n \neq 3 \). Let \( E(R(X)[Y], \hat{L}) \) be the \( n \)-th Euler class group of \( R(X)[Y] \) with respect to the line bundle \( \hat{L} \) over \( R(X)[Y] \) and \( E(R[Y], \bar{L}) \) be the \( n \)-th Euler class group of \( R[Y] \) with respect to the line bundle \( \bar{L} \) over \( R[Y] \) (see [D-Z, Section 6] for definition). We define the \( n \)-th *Euler class group of \( R[X,Y] \),* denoted by \( E(R[X,Y], L) \), as the product \( E(R(X)[Y], \hat{L}) \times E(R[Y], \bar{L}) \).
To the pair \((P, \chi)\), we associate an element \(e(P, \chi)\) of \(E(R[X, Y], L)\), called the Euler class of \((P, \chi)\), as follows:

\[ e(P, \chi) = (e(\hat{P}, \tilde{\chi}), e(\overline{P}, \overline{\chi})) \]

where \(e(\hat{P}, \tilde{\chi}) \in E(R(X)[Y], \hat{L})\) is the Euler class of \((\hat{P}, \tilde{\chi})\) and \(e(\overline{P}, \overline{\chi}) \in E(R[Y], \overline{L})\) is the Euler class of \((\overline{P}, \overline{\chi})\), defined in [D-Z, Section 6].

Now we treat the case when \(n = 3\). Let \(\tilde{E}(R(X)[Y], \hat{L})\) be the \(n\)th restricted Euler class group of \(R(X)[Y]\) with respect to the line bundle \(\hat{L}\) over \(R(X)[Y]\) and \(\tilde{E}(R[Y], \overline{L})\) be the \(n\)th restricted Euler class group of \(R[Y]\) with respect to the line bundle \(\overline{L}\) over \(R[Y]\) (see [D-Z, Section 7] for definition). We define the Euler class group of \(R[X, Y]\), again denoted by \(E(R[X, Y], L)\), as the product \(\tilde{E}(R(X)[Y], \hat{L}) \times \tilde{E}(R[Y], \overline{L})\).

To the pair \((P, \chi)\), we associate an element \(e(P, \chi)\) of \(E(R[X, Y], L)\), called the Euler class of \((P, \chi)\), as follows:

\[ e(P, \chi) = (e(\hat{P}, \tilde{\chi}), e(\overline{P}, \overline{\chi})) \]

where \(e(\hat{P}, \tilde{\chi}) \in \tilde{E}(R(X)[Y], \hat{L})\) is the Euler class of \((\hat{P}, \tilde{\chi})\) and \(e(\overline{P}, \overline{\chi}) \in \tilde{E}(R[Y], \overline{L})\) is the Euler class of \((\overline{P}, \overline{\chi})\), defined in [D-Z, Section 7].

Remark 3.3. Note that when \(n = 2\), the definition of the Euler class group \(E(R[T], L)\) is slightly different from the case \(n \geq 4\). See [D-Z, Remark 7.8] for details.

**Theorem 3.4.** Let \(R\) be a ring containing \(\mathbb{Q}\) of dimension \(n \geq 2\) and \(P\) be a projective \(R[X, Y]\)-module of rank \(n\) with determinant \(L\). Let \(\chi : L \sim \wedge^n(P)\) be an isomorphism. Then \(e(P, \chi) = 0\) in \(E(R[X, Y], L)\) if and only if \(P\) has a unimodular element.

Proof. First we assume that \(P\) has a unimodular element. Therefore, \(\hat{P}\) and \(\overline{P}\) also have unimodular elements. If \(n \geq 4\), by [D-Z, Theorem 6.12], we have \(e(\hat{P}, \tilde{\chi}) = 0\) in \(E(R(X)[Y], \hat{L})\) and \(e(\overline{P}, \overline{\chi}) = 0\) in \(E(R[Y], \overline{L})\). The case \(n = 2\) is taken care by [D-Z, Remark 7.8]. Now if \(n = 3\), it follows from [D-Z, Theorem 7.4] that \(e(\hat{P}, \tilde{\chi}) = 0\) in \(E(R(X)[Y], \hat{L})\) and \(e(\overline{P}, \overline{\chi}) = 0\) in \(\tilde{E}(R[Y], \overline{L})\). Consequently, \(e(P, \chi) = 0\).

Conversely, assume that \(e(P, \chi) = 0\). Then \(e(\hat{P}, \tilde{\chi}) = 0\) in \(E(R(X)[Y], \hat{L})\) and \(e(\overline{P}, \overline{\chi}) = 0\) in \(E(R[Y], \overline{L})\). If \(n \neq 3\), by [D-Z, Theorem 6.12] and [D-Z, Remark 7.8], \(\hat{P}\) and \(\overline{P}\) have unimodular elements. If \(n = 3\), by [D-Z, Theorem 7.4], \(\hat{P}\) and \(\overline{P}\) have unimodular elements. Since \(\hat{P}\) has a unimodular element, we can find a monic polynomial \(f \in R[X]\) such that \(P_f\) contains a unimodular element. But then by Theorem 3.1, \(P\) has a unimodular element. \(\square\)

Remark 3.5. Let \(R\) be a ring containing \(\mathbb{Q}\) of dimension \(n \geq 2\) and \(P\) be a projective \(R[X_1, \ldots, X_r]\)-module \((r \geq 3)\) of rank \(n\) with determinant \(L\). Let \(\chi : L \sim \wedge^r(P)\) be an isomorphism. By induction on \(r\), we can define the Euler class group of \(R[X_1, \ldots, X_r]\) with respect to the line bundle \(L\), denoted by \(E(R[X_1, \ldots, X_r], L)\), as the product of \(E(R(X_r)[X_1, \ldots, X_{r-1}], \hat{L})\) and \(E(R[X_1, \ldots, X_{r-1}], \overline{L})\).
To the pair \((P, \chi)\), we can associate an invariant \(e(P, \chi)\) in \(E(R[X_1, \ldots, X_r], L)\) as follows:

\[
e(P, \chi) = (e(\hat{P}, \hat{\chi}), e(\overline{P}, \overline{\chi}))
\]

where \(e(\hat{P}, \hat{\chi}) \in E(R(X_1)[X_1, \ldots, X_{r-1}], \hat{L})\) is the Euler class of \((\hat{P}, \hat{\chi})\) and \(e(\overline{P}, \overline{\chi}) \in E(R[X_1, \ldots, X_{r-1}], \overline{L})\) is the Euler class of \((\overline{P}, \overline{\chi})\). Finally we have the following result.

**Theorem 3.6.** Let \(R\) be a ring containing \(Q\) of dimension \(n \geq 2\) and \(P\) be a projective \(R[X_1, \ldots, X_r]\)-module of rank \(n\) with determinant \(L\). Let \(\chi : L \sim \wedge^n(P)\) be an isomorphism. Then \(e(P, \chi) = 0\) in \(E(R[X_1, \ldots, X_r], L)\) if and only if \(P\) has a unimodular element.

4. **ANALOGUE OF ROY AND MANDAL**

In this section we will prove (1.4). We begin with the following result from [Ry, Lemma 2.1].

**Lemma 4.1.** Let \(R\) be a ring and \(P, Q\) be two projective \(R\)-modules. Suppose that \(\phi : Q \rightarrow P\) is an \(R\)-linear map. For an ideal \(I\) of \(R\), if \(\phi\) is a split monomorphism modulo \(I\), then \(\phi_{1+I} : Q_{1+I} \rightarrow P_{1+I}\) is also a split monomorphism.

**Lemma 4.2.** Let \((R, M)\) be a local ring and \(A\) be a ring such that \(R[X] \hookrightarrow A \hookrightarrow R[X, X^{-1}]\). Let \(P\) and \(Q\) be two projective \(A\)-modules and \(\phi : Q \rightarrow P\) be an \(R\)-linear map. If \(\phi\) is a split monomorphism modulo \(M\) and if \(\phi_f\) is a split monomorphism for some special monic polynomial \(f \in R[X]\), then \(\phi\) is also a split monomorphism.

Proof. By Lemma 4.1 \(\phi_{1+MA}\) is a split monomorphism. So, there is an element \(h\) in \(1 + MA\) such that \(\phi_h\) is a split monomorphism. Since \(f\) is a special monic polynomial, \(R \rightarrow A/f\) is an integral extension and hence, \(h\) and \(f\) are comaximal. As \(\phi_f\) is also a split monomorphism, it follows that \(\phi\) is a split monomorphism. \(\square\)

**Lemma 4.3.** Let \(R\) be a local ring and \(A\) be a ring such that \(R[X] \hookrightarrow A \hookrightarrow R[X, X^{-1}]\). Let \(P\) and \(Q\) be two projective \(A\)-modules and \(\phi, \psi : Q \rightarrow P\) be \(R\)-linear maps. Further assume that \(\gamma : P \rightarrow Q\) is an \(A\)-linear map such that \(\gamma \psi = f1_Q\) for some special monic polynomial \(f \in R[X]\). For large \(m\), there exists a special monic polynomial \(g_m \in A\) such that \(X\phi + (1 + X^m)\psi\) becomes a split monomorphism after inverting \(g_m\).

Proof. As in [Ry, M], first we assume that \(Q\) is free. We have \(\gamma(X\phi + (1 + X^m)\psi) = X\gamma\phi + (1 + X^m)f1_Q\). Since \(Q\) is free, \(X\gamma\phi + (1 + X^m)f1_Q\) is a matrix. Clearly for large integer \(m\), \(det(X\gamma\phi + (1 + X^m)f1_Q)\) is a special monic polynomial which can be taken for \(g_m\).

In the general case, find projective \(A\)-module \(Q'\) such that \(Q \oplus Q'\) is free. Define maps \(\phi', \psi' : Q \oplus Q' \rightarrow P \oplus Q'\) and \(\gamma' : P \oplus Q' \rightarrow Q \oplus Q'\) as \(\phi' = \phi \oplus 0\), \(\psi' = \psi \oplus f1_{Q'}\).
and $\gamma' = \gamma \oplus 1_Q$. By the previous case, we can find a special monic polynomial $g_m$ for some large $m$ such that $(X \phi' + (1 + X^m)\psi')g_m$ becomes a split monomorphism. Hence $X \phi + (1 + X^m)\psi$ becomes a split monomorphism after inverting $g_m$. □

The following result generalizes Mandal’s [M].

**Theorem 4.4.** Let $(R, M)$ be a local ring and $R[X] \subset A \subset R[X, X^{-1}]$. Let $P$ and $Q$ be two projective $A$-modules with rank($Q$) < rank($P$). If $Q_f$ is a direct summand of $P_f$ for some special monic polynomial $f \in R[X]$, then $Q$ is also a direct summand of $P$.

Proof. The method of proof is similar to [Ry, Theorem 1.1], hence we give an outline of the proof.

Since $Q_f$ is a direct summand of $P_f$, we can find $A$-linear maps $\psi : Q \to P$ and $\gamma : P \to Q$ such that $\gamma \psi = f 1_Q$ (possibly after replacing $f$ by a power of $f$).

Let 'bar' denote reduction modulo $M$. Then we have $\bar{\gamma} \bar{\psi} = \bar{f} 1_{\bar{Q}}$. As $f$ is special monic, $\bar{\psi}$ is a monomorphism.

We may assume that $A = R[X, f_1/X^t, \ldots, f_n/X^t]$ with $f_i \in R[X]$. If $f_i \in MR[X]$, then $R[X, f_i/X^t] = R[X, Y]/(X^t Y)$. If $f_i \in R[X] - MR[X]$, then $R[X, f_i/X^t]$ is either $R[X]$ or $R[X, X^{-1}]$ depending on whether $f_i$ is a polynomial in $R[X]$ or $F_i/X^s$ with $F_i(0) \neq 0$ and $s > 0$.

In general, $\bar{A}$ is one of $\bar{R[X]}$, $\bar{R[X, X^{-1}]}$ or $\bar{R[X, Y_1, \ldots, Y_m]}/(X^t Y_1, \ldots, Y_m)$ for some $m > 0$. By [V, Theorem 3.2], any projective $\bar{R[X, Y_1, \ldots, Y_m]}/(X^t Y_1, \ldots, Y_m)$-module is free. Therefore, in all cases, projective $\bar{A}$-modules are free and hence extended from $\bar{R[X]}$. In particular, $\bar{P}$ and $\bar{Q}$ are extended from $\bar{R[X]}$, which is a PID.

Let rank($P$) = $r$ and rank($Q$) = $s$. Therefore, using elementary divisors theorem, we can find bases $\{\bar{p}_1, \ldots, \bar{p}_r\}$ and $\{\bar{q}_1, \ldots, \bar{q}_s\}$ for $\bar{P}$ and $\bar{Q}$, respectively, such that $\bar{\psi}(\bar{q}_i) = \bar{f}_i \bar{p}_i$ for some $f_i \in R[X]$ and $1 \leq i \leq s$.

For the rest of the proof, we can follow the proof of [Ry, Theorem 1.1]. □

Now we have the following consequence of (4.4).

**Corollary 4.5.** Let $R$ be a local ring and $R[X] \subset A \subset R[X, X^{-1}]$. Let $P, Q$ be two projective $A$-modules such that $P_f$ is isomorphic to $Q_f$ for some special monic polynomial $f \in R[X]$. Then,

1. $Q$ is a direct summand of $P \oplus L$ for any projective $A$-module $L$.
2. $P$ is isomorphic to $Q$ if $P$ or $Q$ has a direct summand of rank one.
3. $P \oplus L$ is isomorphic to $Q \oplus L$ for all rank one projective $A$-modules $L$.
4. $P$ and $Q$ have same number of generators.

Proof. (1) trivially follows from Theorem 4.4 and (3) follows from (2).

The proof of (4) is same as [Ry, Proposition 3.1 (4)].
For (2), we can follow the proof of [M, Theorem 2.2 (ii)] by replacing doubly monic polynomial by special monic polynomial in his arguments. □

**Corollary 4.6.** Let $R$ be a local ring and $R[X] \subset A \subset R[X, X^{-1}]$. Let $P$ be a projective $A$-module such that $P_f$ is free for some special monic polynomial $f \in R[X]$. Then $P$ is free.

Proof. Follows from second part of (4.5). □

**References**


UNIMODULAR ELEMENTS IN PROJECTIVE MODULES AND AN ANALOGUE OF A RESULT OF MANDAL

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY, POWAI, MUMBAI 400076, INDIA.
E-mail address: keshari@math.iitb.ac.in

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY, POWAI, MUMBAI, INDIA - 400 076
E-mail address: zinna@math.iitb.ac.in