A note on rigidity and triangulability of a derivation

Manoj K. Keshari and Swapnil A. Lokhande

Department of Mathematics, IIT Bombay, Mumbai - 400076, India; (keshari,swapnil)@math.iitb.ac.in

Abstract: Let $A$ be a $\mathbb{Q}$-domain, $K = \text{frac}(A)$, $B = A^{[n]}$ and $D \in \text{LND}_A(B)$. Assume rank $D = \text{rank } D_1$. If $D_1$ is the extension of $D$ to $K^{[n]}$. Then we show that

(i) If $D$ is rigid, then $D$ is rigid.

(ii) Assume $n = 3$, $r = 2$ and $B = A[X, Y, Z]$ with $DX = 0$. Then $D$ is triangulable over $A$ if and only if $D$ is triangulable over $A[X]$. In case $A$ is a field, this result is due to Daigle.


Key words: Locally nilpotent derivation, rigidity, triangulability.

1 Introduction

Throughout this paper, $k$ is a field and all rings are $\mathbb{Q}$-domains. We will begin by setting up some notations from [4]. Let $B = A^{[n]}$ be an $A$-algebra, i.e. $B$ is $A$-isomorphic to the polynomial ring in $n$ variables over $A$. A coordinate system of $B$ over $A$ is an ordered $n$-tuple $(X_1, X_2, ..., X_n)$ of elements of $B$ such that $A[X_1, X_2, ..., X_n] = B$.

An $A$-derivation $D : B \rightarrow B$ is locally nilpotent if for each $x \in B$, there exists an integer $s > 0$ such that $D^s(x) = 0$; $D$ is triangulable over $A$ if there exists a coordinate system $(X_1, ..., X_n)$ of $B$ over $A$ such that $D(X_i) \in A[X_1, ..., X_{i-1}]$ for $1 \leq i \leq n$; rank of $D$ is the least integer $r \geq 0$ for which there exists a coordinate system $(X_1, ..., X_n)$ of $B$ over $A$ satisfying $A[X_1, ..., X_{n-r}] \subset \ker D$; $\text{LND}_A(B)$ is the set of all locally nilpotent $A$-derivations of $B$.

Let $\Gamma(B)$ be the set of coordinate systems of $B$ over $A$. Given $D \in \text{LND}_A(B)$ of rank $r$, let $\Gamma_D(B)$ be the set of $(X_1, ..., X_n) \in \Gamma(B)$ satisfying $A[X_1, ..., X_{n-r}] \subset \ker D$; $D$ is rigid if $A[X_1, ..., X_{n-r}] = A[X_1, ..., X_{n-r}]$ holds whenever $(X_1, ..., X_n)$ and $(X_1, ..., X_n')$ belong to $\Gamma_D(B)$.

For an example, if $D \in \text{LND}_A(B)$ has rank 1, then $D$ is rigid. In this case $\ker(D) = A[X_1, ..., X_{n-1}]$ for some coordinate system $(X_1, ..., X_n)$ and $D = f\partial_{X_n}$ for some $f \in \ker(D)$. If rank $D = n$, then $D$ is obviously rigid, as no variable is in $\ker(D)$. If rank $D \neq 1$, then $\ker(D)$ is not generated by $n - 1$ elements of a coordinate system and is generally difficult to see whether $D$ is rigid. For an example of a non-rigid triangular derivation on $k^{[3]}$, see section 3. We remark that there is also a notion of a ring to be rigid. We say that a ring $A$ is rigid if $\text{LND}(A) = \{0\}$, i.e. there is no non-zero locally nilpotent derivation on $A$. Clearly polynomial rings $k^{[n]}$ are non-rigid rings for $n \geq 1$.

We will state the following result of Daigle ([4], Theorem 2.5) which is used later.

**Theorem 1.1** All locally nilpotent derivations of $k^{[3]}$ are rigid.
Our first result extends this as follows:

**Theorem 1.2** Let \( A \) be a ring, \( B = A^{[n]} \), \( K = \text{frac}(A) \) and \( D \in \text{LND}_A(B) \). Assume that rank \( D = \text{rank} \ D_K \), where \( D_K \) is the extension of \( D \) to \( K^{[n]} \). If \( D_K \) is rigid, then \( D \) is rigid.

In ([4], Corollary 3.4), Daigle obtained the following triangulability criteria: Let \( D \) be an irreducible, locally nilpotent derivation of \( R = k^{[3]} \) of rank at most 2. Let \( (X,Y,Z) \in \Gamma(R) \) be such that \( DX = 0 \). Then \( D \) is triangulable over \( k \) if and only if \( D \) is triangulable over \( k[X] \). Our second result extends this result as follows:

**Theorem 1.3** Let \( A \) be a ring, \( B = A^{[3]} \), \( K = \text{frac}(A) \) and \( D \in \text{LND}_A(B) \). Assume that rank \( D = \text{rank} \ D_K = 2 \). Then \( D \) is triangulable over \( A \) if and only if \( D \) is triangulable over \( A[X] \).

## 2 Preliminaries

Recall that a ring is called a \( HCF \)-ring if intersection of two principal ideal is again a principal ideal. We state some results for later use.

**Lemma 2.1** (Daigle [4], 1.2) Let \( D \) be a \( k \)-derivation of \( R = k^{[n]} \) of rank 1 and let \( (X_1,X_2,...,X_n) \in \Gamma(R) \) be such that \( k[X_1,X_2,...,X_{n-1}] \subset \ker D \). Then

\( i \) ker \( D = k[X_1,X_2,...,X_{n-1}] \);

\( ii \) \( D \) is locally nilpotent if and only if \( D(X_n) \in \ker D \).

**Proposition 2.2** (Abhyankar, Eakin and Heinzer [1], Proposition 4.8) Let \( R \) be a \( HCF \)-ring, \( A \) a ring of transcendence degree one over \( R \) and \( R \subset A \subset R^{[n]} \) for some \( n \geq 1 \). If \( A \) is a factorially closed subring of \( R^{[n]} \), then \( A = R^{[1]} \).

**Lemma 2.3** (Abhyankar, Eakin and Heinzer [1], 1.7) Suppose \( A^{[n]} = R = B^{[n]} \). If \( b \in B \) is such that \( bR \cap A \neq 0 \), then \( b \in A \).

**Theorem 2.4** ([6], Theorem 4.11) Let \( R \) be a \( HCF \)-ring and \( 0 \neq D \in \text{LND}_R(R[X,Y]) \). Then there exists \( P \in R[X,Y] \) such that ker \( D = R[P] \).

**Theorem 2.5** (Bhatwadekar and Dutta [3]) Let \( A \) be a ring and \( B = A^{[2]} \). Then \( b \in B \) is a variable of \( B \) over \( A \) if and only if for every prime ideal \( p \) of \( A \), \( \overline{b} \in \overline{B} := B_p/pB_p \) is a variable of \( \overline{B} \) over \( A_p/pA_p \).

## 3 Rigidity

**Theorem 3.1** Let \( A \) be a ring, \( B = A^{[n]} \), \( K = \text{frac}(A) \) and \( D \in \text{LND}_A(B) \). Assume that rank \( D = \text{rank} \ D_K \), where \( D_K \) is the extension of \( D \) to \( K^{[n]} \). If \( D_K \) is rigid, then \( D \) is rigid.
Proof Assume rank $D = \text{rank } D_K = r$ and $D_K$ is rigid. We need to show that $D$ is rigid, i.e. if $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are two coordinate systems of $B$ satisfying $A[x_1, \ldots, x_{n-r}] \subset \ker D$ and $A[y_1, \ldots, y_{n-r}] \subset \ker (D)$, then we have to show that $A[x_1, \ldots, x_{n-r}] = A[y_1, \ldots, y_{n-r}]$. By symmetry, it is enough to show that $A[x_1, \ldots, x_{n-r}] \subset A[y_1, \ldots, y_{n-r}]$.

Since $D_K$ is rigid and rank $D_K = r$, we get $K[x_1, \ldots, x_{n-r}] = K[y_1, \ldots, y_{n-r}]$. If $f \in A[x_1, \ldots, x_{n-r}]$, then $f \in K[y_1, \ldots, y_{n-r}]$. We can choose $a \in A$ such that $af \in A[y_1, \ldots, y_{n-r}]$ and hence $fB \cap A[y_1, \ldots, y_{n-r}] \neq 0$. Applying (2.3) to $A[x_1, \ldots, x_{n-r}] = B = A[y_1, \ldots, y_{n-r}]$, we get $f \in A[y_1, y_2, \ldots, y_{n-r}]$. Therefore $A[x_1, \ldots, x_{n-r}] \subset A[y_1, \ldots, y_{n-r}]$. This completes the proof.

The following result is immediate from (3.1) and (1.1).

Corollary 3.2 Let $A$ be a ring, $B = A[3]$, $D \in \text{LND}_A(B)$. If rank $D = \text{rank } D_K$, then $D$ is rigid.

Remark 3.3 (1) If $D \in \text{LND}_A(B)$, then rank $D$ and rank $D_K$ need not be same. For an example, consider $A = \mathbb{Q}[X]$ and $B = A[T, Y, Z]$. Define $D \in \text{LND}_A(B)$ as $DT = 0$, $D(Y) = X$ and $D(Z) = Y$. Then rank $D = 2$ and rank $D_K = 1$. Further, $(T' = T - Y^2 + 2XZ, Y, Z) \in \Gamma_D(B)$ and $A[T] \neq A[T']$. Therefore, $D$ is not rigid, whereas $D_K$ is rigid, by (1.1).

Above example gives a $D \in \text{LND}(k[4])$ which is not rigid. Hence Daigle’s result (1.1) is best possible. Note that $D$ is a triangular derivation and by [2], $\ker(D)$ is a finitely generated $k$-algebra.

(2) The condition in (3.1) is sufficient but not necessary, i.e. $D \in \text{LND}_A(B)$ may be rigid even if rank $D \neq \text{rank } D_K$. For an example consider $A = \mathbb{Q}[X]$ and $B = A[Y, Z]$. Define $D \in \text{LND}_A(B)$ as $D(Y) = X$ and $D(Z) = Y$. Then rank $D = 2$ and hence $D$ is rigid. Further, rank $D_K = 1$ and $D_K$ is also rigid, by (1.1).

(3) It will be interesting to know if $D \in \text{LND}(k[n])$ being rigid implies that $\ker(D)$ is a finitely generated $k$-algebra. The following example could provide an answer.

Let $D = X^3\partial_5 + S\partial_T + T\partial_U + X^2\partial_V \in \text{LND}(B)$, where $B = k[3] = k[X, S, T, U, V]$. Daigle and Freudenberg [5] have shown that $\ker(D)$ is not a finitely generated $k$-algebra. We do not know if $D$ is rigid. We will show that rank $D = 3$. Clearly $X, S - XV \in \ker(D)$ is part of a coordinate system. Hence rank $D \leq 3$. If rank $D = 1$, then there exists a coordinate system $(X_1, \ldots, X_4, Y)$ of $B$ over $k$ such that $X_1, \ldots, X_4 \in \ker(D)$. Hence $D = f\partial_Y$ for some $f \in k[X_1, \ldots, X_4]$ and $\ker(D) = k[X_1, \ldots, X_4]$ is a finitely generated $k$-algebra, a contradiction. If rank $D = 2$, then there exists a coordinate system $(X_1, X_2, X_3, Y, Z)$ of $B$ over $k$ such that $X_1, X_2, X_3 \in \ker(D)$. If we write $A = k[X_1, X_2, X_3]$, then $D \in \text{LND}_A(A[Y, Z])$. Since $A$ is UFD, by ([6], Theorem 4.11), $\ker(D) = A[1]$, hence $\ker(D)$ is a finitely generated $k$-algebra, a contradiction. Therefore, rank of $D$ is 3.

4 Triangulability

We begin with the following result which is of independent interest.
Lemma 4.1  Let \( A \) be a UFD, \( K = \text{frac}(A), \) \( B = A^{[n]} \) and \( D \in \text{LND}_A(B). \) Let \( D_K \) be the extension of \( D \) on \( K^{[n]} \). If \( D \) is irreducible, then \( D_K \) is irreducible.

Proof  We prove that if \( D_K \) is reducible, then so is \( D \). Let \( D_K(K^{[n]}) \subset fK^{[n]} \) for some \( f \in B \). If \( B = A[x_1, \ldots, x_n] \), then we can write \( \text{dx}_i = \frac{g_i}{c_i} \) for some \( g_i \in B \) and \( c_i \in A \) with \( \text{gcd}_B(g_i, c_i) = 1 \). Since \( \text{dx}_i \in B \), we get \( c_i \) divides \( f \) in \( B \). If \( c \) is lcm of \( c_i \)'s, then \( c \) divides \( f \). If we take \( f' = f/c \in B \), then \( \text{dx}_i \in f'B \) and hence \( D \) is reducible. \( \square \)

Proposition 4.2  Let \( A \) be a ring, \( B = A^{[3]} \), and \( D \in \text{LND}_A(B) \) be of rank one. Let \( (X, Y, Z) \in \Gamma(B) \) be such that \( DX = 0 \). Assume that either \( A \) is a UFD or \( D \) is irreducible. Then \( D \) is triangulable over \( A[X] \).

Proof  As rank \( D = 1 \), there exists \( (X', Y', Z') \in \Gamma(B) \) such that \( DX' = DY' = 0 \). By (2.1), ker \( D = A[X', Y'] \) and \( DZ' \in \text{ker} \; D \).


(ii) Assume \( D \) is irreducible. Then \( DZ' \) must be a unit. To show that \( X \) is a variable of \( A[X', Y'] \) over \( A \). By (2.5), it is enough to prove that for every prime ideal \( p \) of \( A \), if \( \kappa(p) = A_p/pA_p \) then \( X \) is a variable of \( \kappa(p)[X', Y'] \) over \( \kappa(p) \). Extend \( D \) on \( A_p[X, Y, Z] \) and let \( D' \) be \( D \) modulo \( pA_p \). Then \( \text{ker} \; D' = \kappa(p)[X', Y'] \). By (2.2), \( \text{ker} \; D' = \kappa(p)[X]^{[1]} \). Therefore \( X \) is a variable of \( A[X', Y'] \), i.e. \( A[X', Y'] = A[X, P] \) for some \( P \in B \). Hence \( B = A[X, P, Z'] \). Thus \( D \) is triangulable over \( A[X] \). \( \square \)

Proposition 4.3  Let \( A \) be a ring, \( K = \text{frac}(A), \) \( B = A^{[3]} \) and \( D \in \text{LND}_A(B) \). Let \( (X, Y, Z) \in \Gamma(B) \) be such that \( DX = 0 \). Assume rank \( D = \text{rank} \; D_K = 2 \). Then \( D \) is triangulable over \( A \) if and only if \( D \) is triangulable over \( A[X] \).

Proof  We need to show only \( \Rightarrow \). Suppose that \( D \) is triangulable over \( A \). Then there exists \( (X', Y', Z') \in \Gamma(B) \) such that \( DX' \in A, DY' \in A[X'] \) and \( DZ' \in A[X', Y'] \). If \( a = DX' \neq 0 \), then \( D_K(X'/a) = 1 \); which implies that rank \( D_K = 1 \), a contradiction. Hence \( DX' = 0 \).

Since \( D_K \) is rigid, by (3.1), \( D \) is rigid of rank 2. Therefore \( A[X] = A[X'] \) and \( D \) is triangulable over \( A[X] \). \( \square \)

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References


