Spec. Matrices 2016; 4:46–55 **DE GRUYTER** OPEN

**Research Article** **Open Access**

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# The Smith normal form of product distance matrices

DOI 10.1515/spma-2016-0005
Received June 15, 2015; accepted November 10, 2015

**Abstract:** Let $G = (V, E)$ be a connected graph with 2-connected blocks $H_1, H_2, \dots, H_r$. Motivated by the exponential distance matrix, Bapat and Sivasubramanian in [4] defined its product distance matrix $D_G$ and showed that $\det D_G$ only depends on $\det D_{H_i}$ for $1 \le i \le r$ and not on the manner in which its blocks are connected. In this work, when distances are symmetric, we generalize this result to the Smith Normal Form of $D_G$ and give an explicit formula for the invariant factors of $D_G$.

## 1 Introduction

For a positive integer $n$, denote the set $\{1, 2, \dots, n\}$ as $[n]$. Let $G$ be a connected graph with vertex set $V(G) = [n]$. A *block* of $G$ is a maximally connected subgraph without a cut-vertex. In this work, for a graph $G$, we will look at symmetric functions $\eta : V(G) \times V(G) \to R$ where $R$ is a commutative principal ideal domain (PID henceforth). A product distance on $G$ is a function $\eta : V(G) \times V(G) \to R$, that satisfies the following three conditions:

1. $\eta(i, i) = 1$ for all $i \in [n]$.
2. $\eta(i, j) = \eta(j, i)$ for all $i, j \in V$ and
3. if $i, j \in V(G)$ are vertices such that every path from $i$ to $j$ passes through the cut-vertex $k$, then $\eta(i, j) = \eta(i, k)\eta(k, j)$.

Thus, we essentially have the freedom to assign distances within each block subject to symmetry and diagonal entries being 1. Once these distances are fixed, distances across blocks will follow from the third rule above. We sometimes denote $\eta(i, j)$ equivalently as $\eta_{i,j}$, especially when we form a matrix of the distances. Let $G$ have blocks $H_1, H_2, \dots, H_r$. Let $\eta(\cdot, \cdot)$ be a product distance on $G$ and let $D_G = (\eta_{i,j})_{1 \le i,j \le n}$ be the corresponding distance matrix.

The definition of *product distances* is motivated by a concrete example: the exponential distance matrix $\mathsf{ED}_G$ of a connected graph. Given a connected graph $G$ on the vertex set $[n]$, let the distance between two vertices $i, j \in V(G)$ be denoted $d_{i,j}$. That is, $d_{i,j}$ is the length of the minimum length path from $i$ to $j$ in $G$. Define the $n \times n$ matrix $\mathsf{ED}_G = (q^{d_{i,j}})_{1 \le i,j \le n}$ as the exponential distance matrix where $q$ is an indeterminate and $q^0 = 1$. It can be readily checked that $\eta(i, j) = q^{d_{i,j}}$ is a product distance.

A large family of product distances can be obtained from geodetic distances as follows. Let $G = (V, E, w)$ be a graph with weights $w : E \to \mathbb{R}^+$ on its edges. A function $d : V \times V \to \mathbb{R}$ is defined to be *graph geodetic* if for $i, j, k \in V$, the condition $d(i, j) + d(j, k) = d(i, k)$ holds iff every path in $G$ from $i$ to $k$ passes through $j$. It is easy to see that the usual weighted graph distance is graph-geodetic. Klein and Randić in [9] showed that the resistance distance is also graph-geodetic. Chebotarev in [5] has constructed several graph geodetic distances parametrised by a real variable $\alpha$. He showed that at boundary values of $\alpha$, his distance coincides

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with the usual shortest path distance and the resistance distance. In [6], Chebotarev constructed more graph geodetic distances from positive functions $f : V \times V \to \mathbb{R}$ which satisfy the "transition inequality".

Let $d : V \times V \to \mathbb{R}$ be a graph-geodetic distance and consider a new function $e : V \times V \to \mathbb{R}$ defined by $e(x, y) = q^{d(x,y)}$, where $q$ is an indeterminate. Then, $e(x, y)$ becomes a product distance by virtue of the graph-geodetic property. Our results are applicable to this large class of distances.

If $\eta(\cdot,\cdot)$ is a product distance on $G$ and if $G$ has blocks $H_1, H_2, \dots, H_r$, then, each $H_i$ is a graph in its own right and thus has an induced product distance matrix $D_{H_i}$ obtained by restricting $\eta$ to vertices in $H_i$. Indeed, these are the distances within vertices of a block which induce the product-distance on $G$. If the graph $G$ is clear from the context, we abridge $D_G$ to $D$.

If $D$ is a matrix whose entries form a product distance on $G$, Bapat and Sivasubramanian [4] showed that $\det D$ only depends on $\det D_{H_i}$ for individual blocks $H_i$ of $G$ and not on the manner in which the $H_i$'s are connected. Their result is true in a more general asymmetric distance case. We state below, a symmetric version of their result.

**Theorem 1.** *([4, Theorem 4]) Let $G$ be a connected graph with blocks $H_i$, $1 \le i \le r$ and product distance matrix $D_G$. For each such $i$, let the distance matrix of each $H_i$ be $D_{H_i}$. Then,*

$$\det D_G = \prod_{i=1}^{r} \det D_{H_i}.$$

In particular, $\det D_G$ is independent of the manner in which the blocks $H_i$ of $G$ are connected. In this paper, we work with matrices $M$ over a PID $R$. In this case, every finite subset $S \subseteq R$ naturally has a greatest common divisor (gcd henceforth). The determinant of an $n \times n$ matrix $M$ with entries from a PID clearly equals the gcd of all $n \times n$ minors of $M$ (as there is only one such minor).

Thus, Theorem 1 can be alternatively stated as "the gcd of $n \times n$ minors of $D_G$ is independent of the manner in which its blocks are connected." Each $k \times k$ minor of $M$ is an element of the PID $R$ and hence, we can talk of the gcd of $k \times k$ minors, with gcd being taken over all the $\binom{n}{k}^2$ choices. If $R$ is a PID, for a multiset $T = \{x_1, x_2, \dots, x_t\} \subseteq R$, the gcd of the elements of $T$ will be denoted as $\gcd(x_1, \dots, x_t)$ or as $\gcd_{x \in T} x$. In the above expression and throughout this paper, when we write $\gcd_{x \in T} x$, we remove those $x \in T$ that are zero and consider the gcd() only over the non-zero elements of $T$.

In this work, we extend the above gcd interpretation of Theorem 1 to all $k \times k$ minors of $D_G$. Our main result is Theorem 3 where we give an explicit formula for the gcd of $k \times k$ minors of $D_G$ as a function of the gcd of smaller minors of the product distance matrix $D_{H_i}$ of blocks $H_i$ of $G$. Since the gcd of $k \times k$ minors occurs in the Smith Normal Form (SNF henceforth) of $D_G$, our results have implications for the SNF of $D_G$ (see Corollary 9).

Shiu [11] has shown some results about the SNF of exponential distance matrices arising from hyperplane arrangements. We are not aware of any other results similar to ours in the literature.

## 2 The main result

Let $D_G$ be the product distance matrix of a graph $G$ and let $H_1, H_2, \dots, H_r$ be the blocks of $G$. Let $G$ have $n$ vertices and similarly, let $H_i$ have $n_i$ vertices for all $i$. Clearly $n = \sum_{i=1}^{r} n_i - (r - 1)$.

Recall that $D_{H_i}$ is the distance matrix of $H_i$. For $1 \le i \le r$ and for $1 \le k \le n_i$, let the gcd of $k \times k$ minors of $D_{H_i}$ be denoted as $g_{i,k-1}$ where the first index is the block number and the second index denotes size minus one. The reason for the second parameter being size minus one will be clear after we see Corollary 7. Thus, for $1 \le i \le r$, we have $g_{i,0}, g_{i,1}, \dots, g_{i,n_i-1}$. For $1 \le i \le r$, and for values $j \ge n_i$ define $g_{i,j} = 0$. It is easy to note for all $i$, that $g_{i,0} = 1$ as each diagonal entry of $D_{H_i}$ is a $1 \times 1$ matrix which equals 1.

Recall $r$ is the number of blocks of $G$ and for positive integers $s$ satisfying $1 \le s \le (\sum_{i=1}^{r} n_i) - r$, define $T_s$ to be the set of ordered integral solutions to the equation $s = s_1 + s_2 + \cdots + s_r$ where $0 \le s_i < n_i$. Here, ordered means that the order of the elements $(s_1, s_2, \dots, s_r)$ is important. We will index elements of $T_s$ by ordered tuples $(s_1, s_2, \dots, s_r)$. Since the graph $G$ is fixed, its number of blocks $r$ is also fixed and hence, the number of summands, $r$ is tacitly obvious. We thus denote the above solution set as $T_s$ instead of the more precise $T_{s,r}$.

**Definition 2.** *With the notation $g_{i,k}$ above, define for $1 \le s \le (\sum_{i=1}^r n_i) - r$,*

$$g_s = \gcd_{(s_1,s_2,\dots,s_r)\in T_s} (\prod_{i=1}^r g_{i,s_i}). \tag{1}$$

With this notation, our main result is the following.

**Theorem 3.** *Let $D_G$ be the product distance matrix of a connected graph $G$ with blocks $H_i$ for $1 \le i \le r$ where $H_i$ has $n_i$ vertices. For $1 \le i \le r$, let the gcd of $k \times k$ minors of $D_{H_i}$ be $g_{i,k-1}$ where $1 \le k \le n_i$. Then, the gcd of $1 \times 1$ minors of $D_G$ is 1 and for $2 \le s \le (\sum_{i=1}^r n_i) - (r-1)$ the gcd of $s \times s$ minors of $D_G$ is $g_{s-1}$, where $g_s$ is defined by (1).*

In order to find the gcd of $k \times k$ minors of a matrix $A$, we look at equivalent matrices $B$ defined as follows. Two $n \times n$ matrices $A, B$ are said to be equivalent, denoted $B \sim A$ if there exist $n \times n$ matrices $U, V$ with both $\det U$ and $\det V$ being units in the ring $R$ and with $B = UAV$. If $R = \mathbb{Z}$, then, we require $\det U = \det V = \pm 1$. We will use elementary row and column operations on matrices. These are the non-multiplicative elementary operations (that is we do not multiply a row or column by a scalar). It is well known that such elementary operations can be accomplished by premultiplying or postmultiplying by matrices whose determinants are $\pm 1$. It follows from the Binet-Cauchy theorem that if $B \sim A$, then, the gcd of $k \times k$ minors of $A$ equals the gcd of $k \times k$ minors of $B$ for all $1 \le k \le n$. (See [10, Theorem II.8].)

## 2.1 Proof of Theorem 3

We will calculate the gcd of $s \times s$ minors of $D_G$ by getting an equivalent matrix $M_r$ which is a direct sum of several diagonal blocks $K_i$. We will know the gcd of $k \times k$ minors of each direct summand $K_i$ for all $1 \le k \le |K_i|$. From this, we will get the gcd of $s \times s$ minors of $D_G$.

We need two results, one on getting the gcd of $s \times s$ minors of a direct-sum matrix when we know the gcds of minors of its direct-sum constituents. Secondly, we need to get the gcd of $k \times k$ minors of each direct summmand $K_i$ - this will be done inductively. The first point is addressed by the following. We now change our notation slightly and denote gcds of $k \times k$ minors by $a_k$, $b_k$ and so on. See Remark 10 later for an explanation for this change.

**Lemma 4.** *Let $M$ be an $n \times n$ square matrix over a PID $R$ and let $M = A \oplus B$ be a direct sum of two square matrices $A, B$, where $A$ is an $s \times s$ matrix and $B$ is an $(n-s) \times (n-s)$ matrix. That is, $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Let the gcd of $k \times k$ minors of $A$ be denoted $a_k$ for $1 \le k \le s$ and likewise, let $b_k$ denote the gcd of $k \times k$ minors of $B$ for $1 \le k \le n-s$. Define $a_0 = b_0 = 1$. Let the gcd of $k \times k$ minors of $M$ be denoted $m_k$ for $1 \le k \le n$. For an integer $1 \le k \le n$, let $T_k$ be the set of ordered integral solutions $(x, y)$ to the equation $k = x + y$, where $0 \le x \le s$ and $0 \le y \le n-s$. Then, $m_k = gcd_{(x,y)\in T_k} a_x b_y$.*

*Proof.* Let $Y$ be a $k \times k$ submatrix of $M$. Then, $Y$ is obtained by choosing $P, Q \subset [s]$, where $P$ is a set of chosen rows and $Q$ is a set of chosen columns, and choosing $L, N \subset [n] - [s]$, where $L$ is a set of chosen rows and $N$ is a set of chosen columns. Clearly, $|P| + |L| = k$ and $|Q| + |N| = k$. That is, the submatrices $A[P, Q]$ and $B[L, N]$ are chosen and so $Y = A[P, Q] \oplus B[L, N]$ can be written as a direct sum. If $|L| + |Q| < k$, then $|P| + |N| > k$ and so there will exist a zero submatrix (induced on the rows indexed by $P$ and columns indexed by $N$) of order $|P| \times |N|$ where $|P| + |N| > k$. For any $k \times k$ matrix $Y = (y_{i,j})_{1 \le i,j \le k}$, if there exists $P, N \subseteq [k]$

with $|P| + |N| > k$ and with $Y[P, N] = 0$, then we claim that $\det Y = 0$. We will show a stronger statement that all terms in the Laplace expansion of $\det Y$ will be zero. Without loss of generality, assume that the zero submatrix of $Y$ is formed on the rows $P = \{1, 2, \ldots, |P|\}$ and the columns $N = \{1, 2, \ldots, |N|\}$. We claim that all permutations $\pi \in \mathfrak{S}_k$ satisfy $y_{i,\pi_i} = 0$ for some $i \in [k]$. To see this, suppose this does not happen. Then, we have $\{\pi_1, \pi_2, \ldots, \pi_{|P|}\} \cap \{1, 2, \ldots, |N|\} = \emptyset$, which contradicts our assumption that $|P| + |N| > k$. (This proof is very similar to the proof of the Frobenius-König theorem [3, Theorem 2.1.4].)

Thus, if we want $\det Y \neq 0$, we must have $|L| + |Q| \geq k$. A similar argument shows that we must have $|P| + |N| \geq k$ if we want $\det Y \neq 0$. Hence, if $\det Y \neq 0$, we must have $|L| + |Q| = k = |P| + |N|$. We already know $|L| + |P| = k = |Q| + |N|$. Thus, we infer that $|P| = |Q|$ and $|L| = |N|$. That is, only square submatrices of the direct summands can be chosen if we want $\det Y \neq 0$. Thus the determinant of any non-singular $k \times k$ submatrix of $M$ equals the product of determinants of non-singular submatrices $M_1$ of $A$ and $M_2$ of $B$. Hence the gcd $m_k$ of $k \times k$ minors of $M$ equals $\gcd_{(x,y)\in T_k} a_x b_y$. The proof is complete. □

More generally, if $M$ is the direct sum of $r$ diagonal blocks $D_1, D_2, \ldots, D_r$, then it is clear that we only need to modify the definition of $T_k$ to contain $r$ ordered summands adding upto $k$. We state this generalisation without proof as a corollary. We just remark that a proof by induction on the number of summands is straightforward.

For $1 \leq i \leq r$, let $D_i$ be an $n_i \times n_i$ matrix. and for $1 \leq k \leq n_i$, let the gcd of $k \times k$ minors of $D_i$ be denoted as $g_{i,k}$. For $1 \leq i \leq r$, define $g_{i,0} = 1$. Define $T_s$ to be the set of ordered integral solutions to the equation $s = s_1 + s_2 + \cdots + s_r$ where $0 \leq s_i \leq n_i$. This definition is similar, but slightly different from Definition 2. See Remark 10 later for an explanation.

**Definition 5.** *With the notation $g_{i,k}$ above, define for $1 \leq s \leq \sum_{i=1}^{r} n_i$,*

$$g_s = \gcd_{(s_1,s_2,\ldots,s_r)\in T_s} \left(\prod_{i=1}^{r} g_{i,s_i}\right). \tag{2}$$

**Corollary 6.** *Let $M = \oplus_{i=1}^{r} D_i$ be an $n \times n$ matrix where $D_i$ has dimension $n_i \times n_i$. Then, for $1 \leq s \leq n$, the gcd of $s \times s$ minors of $M$ is $g_s$.*

We now move on to a useful corollary of Lemma 4, where we find the gcd of $k \times k$ minors when one direct summand is 1. This corollary will be useful in the proof of Theorem 3.

**Corollary 7.** *Let $A$ be an $n \times n$ matrix and for $1 \leq k \leq n$, let the gcd of $k \times k$ minors of $A$ be denoted $a_k$. Let $M$ be a matrix obtained by performing elementary row and column operations on $A$ such that $M$ is the direct sum of a $1 \times 1$ matrix 1 and an $(n-1) \times (n-1)$ matrix $B$. That is, $M = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$. For $1 \leq k \leq n-1$, let $b_k$ be the gcd of $k \times k$ minors of $B$. Then, $b_k = a_{k+1}$ for $1 \leq k \leq n-1$.*

*Proof.* As $M$ is obtained by performing row/column operations on $A$, $M \sim A$. Thus, the gcd of $(k+1) \times (k+1)$ minors of $A$ equals the gcd of $(k+1) \times (k+1)$ minors of $M$. For $1 \leq k \leq n$, denote the gcd of $k \times k$ minors of $M$ as $m_k$. Then, for $0 \leq k \leq n-1$, we have $a_{k+1} = m_{k+1}$. By Lemma 4, since $M$ is a direct-sum of a $1 \times 1$ matrix and an $(n-1) \times (n-1)$ matrix, $m_{k+1} = \gcd(b_k, b_{k+1}) = b_k$, where the last equality follows as each $(k+1) \times (k+1)$ minor of $B$ is a linear combination of $k \times k$ minors of $B$. The proof is complete. □

The above lemma says that the gcd of 1x1 minors of $B$ equals the gcd of 2x2 minors of $A$; the gcd of 2x2 minors of $B$ equals the gcd of 3x3 minors of $A$ and so on. We are now in a position to prove Theorem 3. As the proof is algorithmic, we illustrate our proof later in Example 11 for clarity.

**Remark 8.** *Corollary 7 will be repeatedly used in this work and thus at several places, we will denote the gcds of $k \times k$ minors of matrices with index $k-1$. (That is, as $a_{k-1}$ and so on).*

*Proof.* (Of Theorem 3) Among the blocks $H_j$ of $G$, define a leaf-block as one with exactly one cut-vertex. Clearly, leaf-blocks exist and let $H_1$ be a leaf-block of $G$ connected via cut-vertex $\mathrm{cv}_1$. Let $L_0 = G$ and let $L_1 = L_0 -$

$(H_1 - \{\mathsf{cv}_1\})$. Let the vertices of $L_1$ be $\{u_1, u_2, \dots, u_p\}$ where $u_p = \mathsf{cv}_1$. Let the rows and columns of $D_G$ be written in the order $u_1, u_2, \dots, u_p$, followed by the vertices of of $H_1 - \{\mathsf{cv}_1\}$ in some order $w_2, w_3, \dots, w_{n_1}$.

For an $n \times n$ matrix $M$, denote its $i$-th row for $1 \le i \le n$ as $\mathsf{Row}_i$ and its $j$-th column for $1 \le j \le n$ as $\mathsf{Col}_j$. For vertices $u, v \in H_1$, recall the distance between them is $\eta(u, v)$. Obtain a matrix $M_1$ equivalent to $D_G$ as follows: perform the elementary row operations $\mathsf{Row}_{w_i} = \mathsf{Row}_{w_i} - \eta(\mathsf{cv}_1, w_i)\mathsf{Row}_{\mathsf{cv}_1}$ for $2 \le i \le n_1$ and then perform the elementary column operations $\mathsf{Col}_{w_i} = \mathsf{Col}_{w_i} - \eta(\mathsf{cv}_1, w_i)\mathsf{Col}_{\mathsf{cv}_1}$ for $2 \le i \le n_1$. The resulting matrix $M_1 = ((M_1)_{u,v})$ will have $(M_1)_{u,w} = 0$ for $u \in L_1, w \in H_1 - \{\mathsf{cv}_1\}$ (likewise for $(M_1)_{w,u}$). Thus, $M_1$ can be written as the following direct sum. $M_1 = \begin{pmatrix} D_{L_1} & 0 \\ 0 & K_1 \end{pmatrix}$, where $K_1$ is an $(n_1-1)\times(n_1-1)$ sized matrix with rows and columns indexed by $w_2, \dots, w_{n_1}$ with the $(w_r, w_s)$-th entry being $\eta(w_r, w_s) - \eta(w_r, \mathsf{cv}_1)\eta(\mathsf{cv}_1, w_s)$.

Recall that $g_{1,k-1}$ is the gcd of $k \times k$ minors of $D_{H_1}$ for $1 \le k \le n_1 - 1$ and let $b_k$ be the gcd of $k \times k$ minors of $K_1$ for $1 \le k \le n_1 - 1$. By performing the same row/column operations as before, we get that $D_{H_1}$ is equivalent to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & K_1 \end{pmatrix}$ and hence by Corollary 7, it follows that

$$\text{for } 1 \le k < n_1, \text{ we have } b_k = g_{1,k+1}. \tag{3}$$

Upto equivalence, we have decomposed $D_G$ into a direct sum of $K_1$ and $D_{L_1}$ and we know the gcd of $k \times k$ minors of $K_1$ for all $k$. If we also know the gcd of $t \times t$ minors of $D_{L_1}$, then we can get the gcd of all minors of the direct sum of $D_{L_1}$ and $K_1$ using Lemma 4.

Towards doing this, we iterate this process, just that we now work with the graph $L_1$. Pick a leaf block $H_2$ of $L_1$ ($H_2$ need not necessarily be a leaf-block of $G$). Let $H_2$ be connected via cut-vertex $\mathsf{cv}_2$ to $L_1$. Define $L_2 = L_1 - (H_2 - \{\mathsf{cv}_2\})$. We may assume (by multiplying by a permutation matrix if necessary) that the rows and columns of $D_{L_1}$ are listed with vertices of $L_2 - \{\mathsf{cv}_2\}$ in some order, followed by $\mathsf{cv}_2$ and then the vertices of $H_2 - \{\mathsf{cv}_2\}$ in some order. Denote the vertices of $H_2 - \{\mathsf{cv}_2\}$ as $z_2, z_3, \dots, z_{n_2}$. Obtain a matrix equivalent to $D_{H_2}$ as follows: perform the elementary row operations $\mathsf{Row}_{z_i} = \mathsf{Row}_{z_i} - \eta(\mathsf{cv}_2, z_i)\mathsf{Row}_{\mathsf{cv}_2}$ for $2 \le i \le n_2$ and then perform the elementary column operations $\mathsf{Col}_{z_i} = \mathsf{Col}_{z_i} - \eta(\mathsf{cv}_2, z_i)\mathsf{Col}_{\mathsf{cv}_2}$ for $2 \le i \le n_2$. We note that though it appears that we perform row and column operations on $D_{L_1}$, we can actually perform operations on $D_G$ to get an equivalent matrix $M_2$ to $M_1$ (and hence equivalent to $D_G$) where $D_{L_1}$ is broken into a direct sum of $K_2$, a square matrix with dimension $n_2 - 1$ and $D_{L_2}$. ie we get $M_2 = \begin{pmatrix} D_{L_2} & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_1 \end{pmatrix}$. Thus, we get an equivalent matrix with one more direct summand. Iterate this procedure till we have a graph with only one block $H_r$. From our process, it is clear that we get one direct summand for each 2-connected block of $G$. We stop iterating when we have one block $H_r$ of $G$. In this case, the leaf-block is $H_r$ itself and denote its vertices as $1, 2, \dots, n_r$. Treat 1 as a cut-vertex connecting $H_r$ to the empty graph and perform the same row and column operations for all vertices of $H_r - \{1\}$. This will result in a decomposition of $D_{H_r}$ as a direct sum of $K_r$ of dimension $(n_r - 1) \times (n_r - 1)$ and a $1 \times 1$ matrix consisting of the entry 1. Thus, we get a matrix $M_r$, equivalent to $D_G$, of the form $M_r = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & K_r & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & K_2 & 0 \\ 0 & 0 & \cdots & 0 & K_1 \end{pmatrix}$.

In this direct-sum decomposition, each square block $K_i$ has size $n_i - 1$ for $1 \le i \le r$ and we have a $1 \times 1$ block $K_0$ with entry 1. We apply Corollary 6 to get the gcd of $k \times k$ minors of the direct sum $\oplus_{i=1}^r K_i$ and combine with Corollary 7 to complete the proof of the theorem. □

A simple corollary of Theorem 3 is the following.

**Corollary 9.** *The SNF of $D_G$ is independent of the manner in which its blocks $H_i$ are connected.*

*Proof.* By Theorem 3, the gcd $h_k$ of $k \times k$ minors for $1 \le k \le n$ of $D_G$ is independent of the manner in which the blocks $H_i$ are connected. Set $h_0 = 1$. The invariant factors $s_k$ for $1 \le k \le n$ of $D_G$ are $s_k = \frac{h_k}{h_{k-1}}$. Clearly the $h_k$'s only depend on the blocks of $G$ and as taking gcds is also independent of the ordering of the blocks, the $h_k$'s are independent of the manner in which the blocks $H_i$ are connected. Thus, so are the $s_k$'s, completing the proof. □

**Remark 10.** *The reason for denoting the gcd of $k \times k$ minors of $D_{H_i}$ as $g_{i,k-1}$ stems from the proof of Theorem 3 and Corollary 7.*

**Example 11.** *We illustrate the inductive argument occurring in the proof of Theorem 3 on an example graph, shown in Figure 1) where we use the exponential distance matrix. Let $D_G$ be the exponential distance matrix $\mathsf{ED}_G$ of $L_0 = G$.*

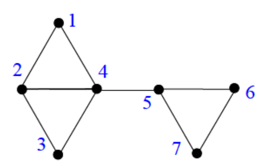


**Figure 1:** An example of a graph.

*Let $H_1$ be the subgraph on the vertices $\{5, 6, 7\}$ with cut vertex $\mathsf{cv}_1 = 5$ and let $L_1 = G - (H_1 - \{\mathsf{cv}_1\})$.*

*Clearly,* $$\mathsf{ED}_G = \begin{pmatrix} 1 & q & q^2 & q & q^2 & q^3 & q^3 \\ q & 1 & q & q & q^2 & q^3 & q^3 \\ q^2 & q & 1 & q & q^2 & q^3 & q^3 \\ q & q & q & 1 & q & q^2 & q^2 \\ q^2 & q^2 & q^2 & q & 1 & q & q \\ q^3 & q^3 & q^3 & q^2 & q & 1 & q \\ q^3 & q^3 & q^3 & q^2 & q & q & 1 \end{pmatrix}.$$

*Let the i-th row of $\mathsf{ED}_G$ for $1 \le i \le n$ be denoted $\mathsf{Row}_i$ and the i-th column of $\mathsf{ED}_G$ be denoted $\mathsf{Col}_i$ respectively.*

*After performing the row operations $\mathsf{Row}_i = \mathsf{Row}_i - q \cdot \mathsf{Row}_5$ for $i = 6, 7$ and then performing the column operations $\mathsf{Col}_i = \mathsf{Col}_i - q \cdot \mathsf{Col}_5$ for $i = 6, 7$, we get an equivalent matrix $M_1 = \begin{pmatrix} \mathsf{ED}_{L_1} & 0 \\ 0 & K_1 \end{pmatrix}$, where we have $K_1 = \begin{pmatrix} 1-q^2 & q-q^2 \\ q-q^2 & 1-q^2 \end{pmatrix}$. Note that $K_1$ is a $2 \times 2$ matrix as $H_1$ has 3 vertices. Next, if we perform the row operation $\mathsf{Row}_5 = \mathsf{Row}_5 - q \cdot \mathsf{Row}_4$ and the column operation $\mathsf{Col}_5 = \mathsf{Col}_5 - q \cdot \mathsf{Col}_4$ on $H_2$ (the subgraph induced on $\{4, 5\}$) with $\mathsf{cv}_2 = 4$, we get an equivalent matrix $M_2 = \begin{pmatrix} \mathsf{ED}_{L_2} & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_1 \end{pmatrix}$, where $K_2 = \begin{pmatrix} 1-q^2 \end{pmatrix}$ (i.e. $K_2$ is a $1 \times 1$ matrix). We are now left with $H_3$, the induced subgraph on $\{1, 2, 3, 4\}$. Choosing $\mathsf{cv}_3 = 1$ and performing the row operations $\mathsf{Row}_2 = \mathsf{Row}_2 - q \cdot \mathsf{Row}_1$; $\mathsf{Row}_3 = \mathsf{Row}_3 - q^2 \cdot \mathsf{Row}_1$ and $\mathsf{Row}_4 = \mathsf{Row}_4 - q \cdot \mathsf{Row}_1$ and then the column operations $\mathsf{Col}_2 = \mathsf{Col}_2 - q \cdot \mathsf{Col}_1$; $\mathsf{Col}_3 = \mathsf{Col}_3 - q^2 \cdot \mathsf{Col}_1$ and $\mathsf{Col}_4 = \mathsf{Col}_4 - q \cdot \mathsf{Col}_1$, we get the following equivalent matrix*

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & K_3 & 0 & 0 \\ 0 & 0 & K_2 & 0 \\ 0 & 0 & 0 & K_1 \end{pmatrix}, \text{ where } K_3 = \begin{pmatrix} 1-q^2 & q-q^3 & q-q^2 \\ q-q^3 & 1-q^4 & q-q^3 \\ q-q^2 & q-q^3 & 1-q^2 \end{pmatrix}.$$

*Thus, we get $\mathsf{ED}_G \sim M_3$. We know the gcd of $k \times k$ minors in each $K_i$ by Corollary 7 and using Corollary 6, we can get the gcd of $s \times s$ minors of $\mathsf{ED}_G$.*

# 3 Special Cases

In this section, we consider some instances of product distance matrices and their inverses and draw corollaries on their SNFs. We first consider exponential distance matrices $\mathsf{ED}_G$ of connected graphs $G$ defined in Section 1. Clearly, they are examples of a product distance and so Theorem 3 is applicable to them.

## 3.1 Exponential distance matrix of Trees

Firstly consider the case when $G$ is a tree $T$. In this case, each block of $G$ is the complete graph on two vertices. Hence, in $\mathsf{ED}_H$, for each block $H$ of $G$, the gcd of $1 \times 1$ minors is 1 and the gcd of $2 \times 2$ minors is $(q^2 - 1)$. In this case, as $n_i = 2$, we will get an *equivalent diagonal* matrix.

Thus if $T$ is a tree on $n$ vertices, by Theorem 3, we infer that the gcd of $k \times k$ minors of $\mathsf{ED}_G$ is $g_{k-1} = (q^2 - 1)^{k-1}$ Hence, we get the invariant factors $s_k$ of $\mathsf{ED}_T$ to be $s_1 = 1$, $s_k = (q^2 - 1)$ for $1 < k \le n$. This is a reproof of the following fact: the invariant factors of $\mathsf{ED}_T$ only depend on $n$, the number of vertices of $T$ and is independent of the structure of the tree $T$. That is, the SNF of $\mathsf{ED}_T$ is the equivalent diagonal matrix $M = 1 \oplus \underbrace{(q^2 - 1) \oplus \cdots \oplus (q^2 - 1)}_{(n-1)\text{ times}}$.

By using Jacobi's Theorem we can get the gcds of $k \times k$ minors of inverses of the exponential distance matrices of trees. We cover this in the next few lines. Bapat, Lal and Pati [2, Proposition 3.3] showed that the inverse of the exponential distance matrix $\mathsf{ED}_T$ of a tree $T$ is upto scalar, the $q$-analogue of $T$'s laplacian $\mathcal{L}_q$ which is defined as $\mathcal{L}_q = I - qA + (D - I)q^2$. Here, $q$ is a variable, $A$ is the adjacency matrix of $T$ and $D$ a diagonal matrix with the $(i, i)$-th entry being the degree of vertex $i$.

**Theorem 12** (Bapat, Lal and Pati). *Let $T$ be a tree with exponential distance matrix $ED_T$ and let $\mathcal{L}_q$ be the $q$-analogue of $T$'s laplacian matrix. Then* $\mathsf{ED}_T^{-1} = \dfrac{1}{1 - q^2}\mathcal{L}_q$.

By Jacobi's Theorem [7, Section 4.2], we know that for invertible matrices $M$, its minors are related to complementary minors of $M^{-1}$. From this, we see that the gcd $b_k$ of $k \times k$ minors of $\mathcal{L}_q$ is 1 for $1 \le k < n$ and that $b_n = q^2 - 1$.

## 3.2 Exponential distance matrices when all blocks are $K_t$'s

We consider exponential distance matrices of graphs $G$, in the case when all blocks of $G$ are the complete graph $K_t$ on $t$ vertices. For concreteness, we assume $t = 3$, though our results are applicable for larger $t$ as well (see the paragraph before Corollary 13, below).

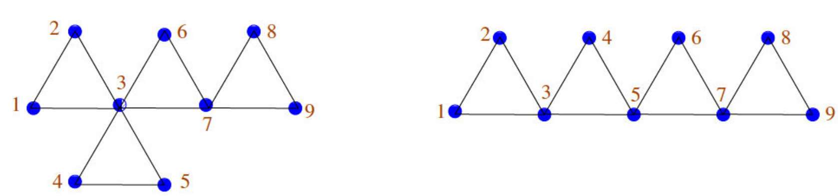


**Figure 2:** Two examples of graphs with only $K_3$ as blocks.

Let $G$ be a graph with blocks being $K_3$'s. Two examples are given in Figure 2. It can be checked that the gcd of $k \times k$ minors of $\mathsf{ED}(K_3)$ is as given in the following table.

| $k$ | gcd of $k \times k$ minors |
|---|---|
| 1 | 1 |
| 2 | $q-1$ |
| 3 | $(2q+1)(q-1)^2$ |

In general when we consider $\mathsf{ED}_{K_t}$, it can be shown that the gcd of $k \times k$ blocks is $(q-1)^{k-1}$ for $1 \le k < t$ and that the gcd of $t \times t$ blocks is $(1+(t-1)q)(q-1)^{t-1}$. We can derive the following corollary of Theorem 3.

**Corollary 13.** *Let $G$ be a graph with $r$ blocks, each of which is a $K_3$ and let $\mathsf{ED}_G$ be its exponential distance matrix. Let the gcd of $s \times s$ minors of $\mathsf{ED}_G$ be denoted $g_{s-1}$. Then, for $1 \le s \le r+1$, we have $g_s = (q-1)^{s-1}$ and for $r+2 \le s \le 2r+1$, we have $g_s = (2q+1)^{s-r-1}(q-1)^{s-1}$.*

*Proof.* Since each block is a $K_3$, we have in the notation of Theorem 3 that $g_{i,0} = 1$, $g_{i,1} = q-1$ and $g_{i,2} = (2q+1)(q-1)^2$ for each $1 \le i \le r$. It is easy to see that the number of vertices in $G$ is $n = 2r+1$. By Theorem 3, when $2 \le s \le n$, the gcd of $s \times s$ minors of $\mathsf{ED}_G$ is $g_{s-1}$. To calculate $g_{s-1}$, when $2 \le s \le r$, note that $\gcd(\prod_{i=1}^r g_{i,s_i})$ over choices $(s_1, s_2, \ldots, s_r) \in T_{s-1}$ is attained when each $s_i = 0$ or 1 (ie the choice $s_i = 2$ for some $i$ will result in a larger product and hence not be chosen). Thus in this case, the gcd of $s \times s$ minors of $\mathsf{ED}_G$ is $(q-1)^{s-1}$. When $r+1 \le s \le n$, we will have all $s_i \ge 1$ and some $s-r-1$ indices $j$ with $s_j = 2$. In this case, the gcd will be $(2q+1)^{s-r-1}(q-1)^{s-1}$, completing the proof. □

The above corollary states that if $G$ has $r$ blocks, then the topmost $r$ largest-sized minors have one form as their gcd and the remaining smaller sized minors all have another form for their gcd. This is true when all blocks of $G$ are $K_p$'s as well (i.e. Corollary 13 can be generalised to this case). A similar proof can be given, though we omit it. We note that all blocks are required to be $K_t$ for the above corollary. It $G$ has two blocks, a $K_3$ and a $K_4$, then it can be checked that only the topmost gcd is different (as opposed to the two topmost gcds as asserted by Corollary 13.)

For graphs that are not necessarily trees by Theorem 1, $\mathsf{ED}_G$ is invertible provided each of its blocks are. In this case, Bapat and Sivasubramanian [4, Theorem 6] give an explicit inverse of $\mathsf{ED}_G$. It would be interesting to get a list of 2-connected graphs for which the SNF of its exponential distance matrix is known.

Similarly, the SNF of exponential distance matrices of block graphs (defined as graphs all of whose blocks are cliques) can be determined. The formulae for these are not as attractive as in the case when $G$ has all blocks being $K_t$ with the same $t$ and so we leave things here.

## 3.3 Exponential version of scaled resistance matrix

As mentioned in Section 1, exponential versions of resistance distances are also product distances. Let $G$ have two blocks $B$ and $F$ where $B = C_4$ is the four-cycle and $F = C_5$ is the five-cycle (see Figure 3). Bapat in [1, Equation 4] has shown that the resistance distance between vertices $i, j$ in the $n$-cycle is given by $r_{i,j} = (j-i)(n-j+i)/n$.

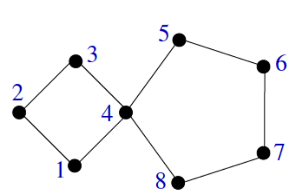


**Figure 3:** Graphs with distances given by scaled resistance distance.

Among vertices $i, j \in V(B)$, let $r_{i,j}$ be its resistance distance. Let $\kappa_B$ denote the number of spanning trees of $B$. For vertices $i, j \in V(B)$, define its distance by $d_{i,j} = q^{\kappa_B r_{i,j}}$, where $r_{i,j}$ is the resistance distance between $i$ and $j$ in $B$, and $q$ is an indeterminate. Similarly, let $\kappa_F$ denote the number of spanning trees of $F$ and define the distance between two vertices $u, v$ of $F$ by $d_{u,v} = q^{\kappa_F r_{u,v}}$ where $r_{u,v}$ is the resistance distance in $F$. For concreteness, we give the distance matrices of $B$ and $F$.

$$D_B = \begin{pmatrix} 1 & q^3 & q^4 & q^3 \\ q^3 & 1 & q^3 & q^4 \\ q^4 & q^3 & 1 & q^3 \\ q^3 & q^4 & q^3 & 1 \end{pmatrix} \qquad D_F = \begin{pmatrix} 1 & q^4 & q^6 & q^6 & q^4 \\ q^4 & 1 & q^4 & q^6 & q^6 \\ q^6 & q^4 & 1 & q^4 & q^6 \\ q^6 & q^6 & q^4 & 1 & q^4 \\ q^4 & q^6 & q^6 & q^4 & 1 \end{pmatrix}$$

Let $B$ and $F$ have a common cut vertex $c$. For vertices $i \in V(B)$, $u \in V(F)$, define $d_{i,u} = d_{i,c} \times d_{c,u}$. Let $D_G$ be the matrix of $d_{i,j}$ for $i, j \in G$. Let $b_k, f_k$ and $g_k$ be the gcd of $k \times k$ minors of $D_B$, $D_F$ and $D_G$ respectively. A table of $b_k, f_k$ and $g_k$ for $1 \le k \le 8$ is as follows. Here, $p = q^3 - q^2 - q - 1$, $r = q^3 + q^2 - q + 1$, $u = q^8 - q^6 - 2q^4 - 2q^2 - 1$ and $w = 2q^6 + 2q^4 + 1$. It is easy to check that Theorem 3 agrees with computed gcd $g_k$.

| $k$ | $b_k$ | $f_k$ | $g_k$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | $q^2 - 1$ | $q^2 - 1$ | $q^2 - 1$ |
| 3 | $(q^2 - 1)^2(1 + q^2)$ | $(q^2 - 1)^2$ | $(q^2 - 1)^2$ |
| 4 | $(q^2 - 1)^3(1 + q^2)^2 pr$ | $(q^2 - 1)^3 u$ | $(q^2 - 1)^3$ |
| 5 | 0 | $(q^2 - 1)^4 wu^2$ | $(q^2 - 1)^4$ |
| 6 | 0 | 0 | $(q^2 - 1)^5$ |
| 7 | 0 | 0 | $(q^2 - 1)^6(1 + q^2)u$ |
| 8 | 0 | 0 | $(q^2 - 1)^7(1 + q^2)^2 prwu^2$ |

**Acknowledgement:** We sincerely thank the referees for several invaluable suggestions that have resulted in a vastly improved exposition of this work. Some Theorems in this work were as conjectures, tested using the computer package "Sage" [8]. We thank the authors of Sage for generously releasing their software as an open-source package.

The first author gratefully acknowledges the support of the JC Bose Fellowship, Department of Science and Technology, Government of India. The second author thanks the Stat-Math unit of ISI-Delhi for their hospitality during several trips. He also acknowledges support from project grant Spons/MA/KS-1/2015 given by IIT Bombay.

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