Inverse of the distance matrix of a block graph

R.B. Bapat\textsuperscript{a} and Sivaramakrishnan Sivasubramanian\textsuperscript{b,*}

\textsuperscript{a}Stat-Math Unit, Indian Statistical Institute, Delhi, 7-SJSS Marg, New Delhi 110 016, India; \textsuperscript{b}Department of Mathematics, Indian Institute of Technology, Mumbai 400 076, India

Communicated by S. Kirkland

(Received 28 May 2009; final version received 19 January 2011)

A connected graph $G$, whose 2-connected blocks are all cliques (of possibly varying sizes) is called a block graph. Let $D$ be its distance matrix. By a theorem of Graham, Hoffman and Hosoya, we have $\det(D) \neq 0$. We give a formula for both the determinant and the inverse, $D^{-1}$ of $D$.

\textbf{Keywords:} distance matrix; determinant; Laplacian matrix

\textbf{AMS Subject Classifications:} 15A09; 15A15; 15A24

1. Introduction

Graham et al. [3] proved a very attractive theorem about the determinant of the distance matrix $D_G$ of a connected graph $G$ as a function of the distance matrix of its 2-connected blocks. In a connected graph, the distance between two vertices $d(u,v)$ is the length of the shortest path between them. Let $A$ be an $n \times n$ matrix. Recall that for $1 \leq i, j \leq n$, the cofactor $c_{i,j}$ is defined as $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting row $i$ and column $j$ of $A$. For a matrix $A$, let $\#(A) = \sum_{i,j} c_{i,j}$ be the sum of its cofactors. Graham et al. [3] showed the following theorem.

\textbf{Theorem 1} If $G$ is a connected graph with 2-connected blocks $G_1, G_2, \ldots, G_r$, then $\#(D_G) = \prod_{i=1}^{r} \#(D_{G_i})$ and $\det(D_G) = \sum_{i=1}^{r} \det(D_{G_i}) \prod_{i \neq j} \#(D_{G_j})$.

Let $D$ be the distance matrix of a connected graph, all of whose blocks are cliques. Such graphs are called block graphs in [2] and let $G_i$ denote the blocks of $G$ (for $1 \leq i \leq r$). See Figure 1 for an example.

Further, we give a formula for $\det(D)$ for the distance matrix $D$ of a block graph $G$ in terms of its block sizes and $n$, its number of vertices.

From the formula it will be clear that $\det(D) \neq 0$. Hence, we are interested in finding $D^{-1}$. For the case when all blocks are $K_2$'s (i.e. the graph $G$ is a tree) it is known [1,4] that $D^{-1} = -\frac{1}{2} + \frac{1}{\tau_{n-1}} \tau^t$, where $L$ is the Laplacian matrix of $G$ and $\tau$ is the $n \times 1$ column vector with $\tau_i = 2 - \deg_i$. Similarly, it is known that when all blocks are $K_3$'s [5], we have $D^{-1} = -\frac{1}{2} + \frac{1}{\tau_{n-1}} \mu^t$ where $L$ is again the Laplacian of $G$ and
is the column vector with \( \mu_i = 3 - \deg_i \). Thus, \( D^{-1} \) is a constant times \( L \) plus a multiple of a rank one matrix. We show a similar statement for block graphs.

2. Determinant and inverse of \( D \)

Let \( G \) be a block graph on \( n \) vertices with blocks \( G_i, 1 \leq i \leq r \), where each \( G_i \) is a \( p_i \)-clique. Denote by \( \lambda_G \) the non-zero constant

\[
\lambda_G = \sum_{i=1}^{r} \frac{p_i - 1}{p_i}.
\]  

(1)

The following theorem is easily derived from Theorem 1.

**Theorem 2** Let \( G \) be a block graph on \( n \) vertices with blocks \( G_i, 1 \leq i \leq r \), where each \( G_i \) is a \( p_i \)-clique. Let \( D \) be its distance matrix. Then,

\[ \det(D) = (-1)^{r-1} \lambda_G \prod_{j=1}^{r} p_j. \]

**Proof** As each \( D_{G_i} \) is the matrix \( J - I \) where \( J \) is the all ones matrix and \( I \) is the identity matrix of dimension \( p_i \times p_i \), it is easy to see that \( \det(D_{G_i}) = (-1)^{p_i-1}(p_i - 1) \) and \( \#(D_{G_i}) = (-1)^{p_i-1}p_i \) (the \( \#(D_{G_i}) \) calculation is immediate if we use [3, Lemma 1]). Since \( \sum_{i=1}^{r} p_i = n + r - 1 \), the equality of the theorem follows from Theorem 1.

For a block graph \( G \), consider the \( |V(G)| \)-dimensional column vector \( \beta \) defined as follows. Let a vertex \( v \in V \) be in \( k \geq 1 \) cliques of sizes \( p_1, p_2, \ldots, p_k \) (where each \( p_i > 1 \)). Let

\[
\beta_v = \left( \sum_{i=1}^{k} \frac{1}{p_i} \right) - (k - 1).
\]  

(2)

For the block graph given in Figure 1, we have \( \lambda_G = \frac{163}{60} \), and \( \beta' = (-1/4, 1/4, 1/4, 1/4, -3/10, 1/5, 1/5, 1/5, -7/15, 1/3, 1/3) \).

**Lemma 1** Let \( G \) be a block graph and let \( \beta \) be the vector defined above. Then, \( \sum_{v \in V(G)} \beta_v = 1 \).

**Proof** By induction on \( b \), we have the number of blocks of \( G \), with the case \( b = 1 \) being clear. When \( G \) has more than one block, let \( H \) be any leaf block (i.e. a block whose deletion does not disconnect \( G \)) connected through cut-vertex \( c \). Clearly, a leaf block \( H \) exists and let \( F = G - \{H - c\} \) be the smaller graph obtained by deleting \( H - c \) from \( G \). Let \( H \) be a \( p \)-clique (i.e. \( H = K_p \)). By induction, for the graph \( F \), we know \( S = \sum_{v \in V(F)} \beta_v = 1 \). It is simple to note that when we move to \( G \) from \( F \), the

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{block_graph.png}
\caption{An example of a block graph.}
\end{figure}
\]
vector $\beta$ is different from that for $F$ only for the vertices of $H$. The change in $\beta$ for $G$ is easily seen to be $(1/p-1)$ for $c$ and $1/p$ for the other $p-1$ vertices of $H$. Thus the sum of the changes is zero, completing the proof.

**Lemma 2** Let $D$ be the distance matrix of a block graph $G$. Let $|V|=n$ and $\beta$ be the vector defined by Equation (2). Let $1$ be the $n$-dimensional vector with all components equal to $1$. Then $D\beta = \lambda_G 1$, where $\lambda_G$ is as given in Equation (1).

**Proof** We again induct on $b$, the number of blocks of $G$ with the case $b=1$ being simple. Delete a leaf block $H$ connected to $G$ through $c$ and let $F=G\setminus \{H-c\}$. Let $H$ be a $p$-clique and let $D_F$ be the distance matrix of $F$. Let $1_F$ be the vector $1$ restricted to vertices of $F$. Let $\alpha$ be the column of $D_F$ corresponding to the vertex $c$. The $v$-th component of $\alpha$ is $\alpha_v=d_{v,c}$ where $d_{u,v}$ is the distance between vertices $u,v \in F$. It is simple to note that

$$D = \begin{pmatrix}
D_F & \alpha \cdot 1_F & \cdots & \alpha \cdot 1_F \\
(\alpha \cdot 1_F)^t & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
(\alpha \cdot 1_F)^t & 1 & \cdots & 0
\end{pmatrix}.$$ 

If $\beta_F$ is the restriction of $\beta$ to $F$, then by induction we have $D_F \beta_F = \lambda_F 1_F$. Here $\lambda_F$ is the vector $\lambda$ as in Equation (1) for the graph $F$. Let $t=D\beta$ and for $v \in F$ and let $R_v(D_F)$ be the $v$-th row of $D_F$. For vectors $a, b$ with identical dimension, $(a, b)$ denotes the usual (real) inner product of two vectors. For a vertex $v \in F$, the $v$-th component of $t$ is $t_v = (R_v(D_F), \beta_F) + (\frac{1}{p}-1)\alpha_v + (\alpha_v + 1) \cdot \frac{p-1}{p}$. Hence, $t_v = \lambda_F + \frac{p-1}{p}$. Thus for all vertices in $F$, we have $\lambda_G = \lambda_F + \frac{p-1}{p}$. For vertices $u \in H\setminus \{c\}$, we have

$$t_u = ((\alpha \cdot 1_F)^t, \beta_F) + \frac{1}{p} - 1 + \frac{p-2}{p}$$

$$= \lambda_F + \sum_{v \in F} (\beta_F)_v - 1 + \frac{p-1}{p} = \lambda_F + \frac{p-1}{p},$$

where in the first line we have used the fact that $\alpha_c = 0$ and in the second line we have used Lemma 1. Thus, for all vertices $u \in V(G)$, $t_u = \lambda_F + \frac{p-1}{p}$. Since $\lambda_G = \lambda_F + \frac{p-1}{p}$, the proof is complete.

Let $G$ have vertex set $[n]$ and blocks $G_i$ where $1 \leq i \leq r$. Each $G_i$ is also considered as a graph on $[n]$ with perhaps isolated vertices and let its edge set be $E_i$ (i.e. $G_i$ is a clique on say $p_i$ vertices, but consider it as a graph on $[n]$). Let $L_i$ be the Laplacian of $G_i=([n], E_i)$. Let $I$ be the $|V| \times |V|$ identity matrix. Define

$$\hat{L} = \sum_{i=1}^r \frac{1}{p_i} L_i.$$ 

**Lemma 3** With the above notation, $\hat{L} D + I = \beta 1$.

**Proof** We again induct on $b$, the number of blocks of $G$ with the base case $b=1$ being simple. Let $H, F, c$ be as in the proof of Lemma 1 and let $H$ be a $p$-clique. Let $\hat{L}_F$ be the combination of the Laplacian as before, but only for the blocks of $F$ and let $D_F$ be the distance matrix of $F$. Similarly, let $I_F$ be the identity matrix of order $|F| \times |F|$.
Let $e_c$ be the $|F|$-dimensional column vector with a 1 in position $c$ and zero elsewhere and let $\alpha = D_p e_c$. Let $R_{H-c}$ be a $(|H|-1)$-dimensional all ones column vector. $H$ is a leaf-block, but considering it as a graph on $[n]$, let its Laplacian be denoted by $L_H$. Let $L_{H-c}$ be $L_H$ restricted to the set of vertices $V(H) - \{c\}$ and $D_{H-c}$ the distance matrix of $G$, restricted to the set of vertices $V(H) - \{c\}$. We clearly have

$$\hat{L} = \begin{pmatrix} L_F + \frac{p-1}{p} (e_c \times e_c') - \frac{1}{p} (e_c \times R_{H-c}) \\ -\frac{1}{p} (R_{H-c} \times e_c') - \frac{1}{p} L_{H-c} \end{pmatrix}. $$

$$D = \begin{pmatrix} D_F (\alpha + \Pi_F) (\alpha + \Pi_F)' \\ R_H \times (\alpha + \Pi_F)' \end{pmatrix}. $$

We need to show that for all $i, j$, $(\hat{L}D + I)_{i,j} = \beta_i$.

For rows $i \in F - \{c\}$: For such a row $i$ and for columns $j \in F$, we have by induction, $\hat{L}_F D_F + I_F = (\beta_F)_i$. We denote the $i$-th row (and $j$-th column) of matrix $M$ as $C_j(M)$ (and $C_j(M)$, respectively). Since $\beta_i = (\beta_F)_i$ for $i \in F - c$, we are done for all columns in $F$. For columns $j \in H - \{c\}$, we note that $(\hat{L}D + I)_{i,j} = (R_j(\hat{L}_F), \alpha + \Pi_F)$. Since the row-sum of a linear combination of Laplacians is zero, $(R_j(\hat{L}_F), \Pi_F) = 0$. Thus $(\hat{L}D + I)_{i,j} = (\hat{L}D + I)_{i,c} = (\beta_F)_i = \beta_i$.

For rows $i \in H - \{c\}$: For such rows $i$, it is easy to see that $\beta_i = \frac{1}{\rho}$. For all columns $j \in H$, since the entries $L_{i,j} \neq 0$ only if $x \in H$, it is simple to see that $(\hat{L}D + I)_{i,j} = \frac{1}{\rho}$. In the above result, in case $i = j$, since the diagonal entry $D_{i,i} = 0$, we get a 1 from the identity matrix to get $(\hat{L}D + I)_{i,i} = 1 + (p-1)\frac{1}{\rho} = \frac{1}{\rho}$. For columns $j \in H$, using the matrices $L_{H-c}$ and $D_{H-c}$, we see that $(\hat{L}D + I)_{i,j} = (R_j(L_{H-c}), C_j(D_{H-c}))$. Since $C_j(D_{H-c}) = C_j(D_{H-c}) + d_{i,c} R_H$ and since the column sum of a Laplacian is zero, we get $(\hat{L}D + I)_{i,j} = (\hat{L}D + I)_{i,c} = \frac{1}{\rho} = \beta_i$.

For the row $c$: We need to show that for any column $v \neq c$, $(R_j(\hat{L}_c), C_j(D_c)) = \beta_c$ and that for column $c$, $(R_j(\hat{L}_c), C_j(D_c)) + 1 = \beta_c$. We first show that $(R_j(\hat{L}_c), C_j(D_c)) + 1 = \beta_c$. By induction, we know that $(R_j(\hat{L}_F)), C_j(D_F)) + 1 = (\beta_F)_c$. Since $d_{i,c} = 0$, the required proof is easily seen to be $(R_j(\hat{L}_F)), C_j(D_F)) + \frac{1}{\rho \rho^{-1}}$, which is $\beta_c$. We now show for $v \neq c$, $(R_j(\hat{L}_c), C_j(D_c)) = \beta_c$. First, consider columns $v \in F - \{c\}$. By induction, we know that $(R_j(\hat{L}_F)), C_j(D_F)) = (\beta_F)_c$. Since $L_{i,e} = (\hat{L}_F)_{i,e} + \frac{e-1}{\rho}$ and for all $u \in H$, $L_{u,e} = \frac{1}{\rho}$, we get $(R_j(\hat{L}_c), C_j(D_c)) = (\beta_F)_c + d_{i,c} \cdot 0 - \frac{e-1}{\rho}$ Since $\beta_c = (\beta_F)_c + \frac{1}{\rho}$, we are done. Next consider columns $v \in H - \{c\}$. We have just shown that $(R_j(\hat{L}_c), C_j(D_c)) = \beta_c - 1$. The column vectors $C_c(D)$ and $C_c(D)$ only differ in the entries corresponding to row $c$ and $v$, when restricted to rows in $H$ and differ for all entries in $F$: each entry of $C_c(D)$ is larger than the corresponding entry of $C_c(D)$ by 1. Since a linear combination of the Laplacian has zero row-sum, we have $(R_j(\hat{L}_c), C_j(D_c)) = (R_j(\hat{L}_c), C_j(D_c)) + \frac{e-1}{\rho} + \frac{1}{\rho}$, where the term $\frac{e-1}{\rho}$ arises as $L_{i,e} = (\hat{L}_F)_{i,e} + \frac{e-1}{\rho}$ and $d_{i,e} = 1$ and the term $\frac{1}{\rho}$ arises as $d_{c,e} = -\frac{1}{\rho}$ is to be subtracted from $(R_j(\hat{L}_c), C_j(D_c))$. Thus, we have $(R_j(\hat{L}_c), C_j(D_c)) = \beta_c - 1 + \frac{e-1}{\rho} + \frac{1}{\rho} = \beta_c$, completing the proof.

**Theorem 3** Let $\hat{L}$, $D$, $\lambda_G$ and $\beta$ be as above. Then $D^{-1} = -\hat{L} + \frac{1}{\lambda_G} \beta \beta'$.

**Proof** By Lemma 3, we see that $\hat{L}D + I = \beta \Pi'$. By Lemma 2, we get $\beta D = \lambda_G \Pi'$ or $\beta' D = \lambda_G \beta' \Pi'$ where clearly $\lambda_G \neq 0$. Thus, $\hat{L}D + I = \frac{1}{\lambda_G} \beta \beta' D$, i.e. $I = (-\hat{L} + \frac{1}{\lambda_G} \beta \beta') D$, completing the proof.
Theorem 3 says that even if all the blocks of $G$ are arbitrary sized cliques, $D^{-1}$ is a scalar multiple of a kind of Laplacian matrix plus a constant multiple of a rank one matrix. The following known corollaries are easily derived from Theorem 3.

**Corollary 1** [4] Let $D$ be the distance matrix of a tree $T$ on $n$ vertices and let $L$ be its Laplacian matrix. Let $\tau$ be the $n$-dimensional column vector with $\tau_u = 2 - \deg_u$, where $\deg_u$ is the degree of vertex $u$ in $T$. Then $D^{-1} = \frac{2}{\tau} + \frac{1}{\pi(n-1)} \tau \tau'$.

**Corollary 2** [5] Let $D$ be the distance matrix of a graph $G$ on $n$ vertices, all of whose blocks are $K_3$'s and let $L$ be its Laplacian matrix. Let $\mu$ be the $n$-dimensional column vector with $\mu_u = 3 - \deg_u$, where $\deg_u$ is the degree of vertex $u$ in $G$. Then $D^{-1} = \frac{L}{\mu} + \frac{1}{\pi(n-1)} \tau \mu'$.

Acknowledgements

Some results in this work were in their conjecture form, tested using the computer package ‘Sage’. We thank the authors for generously releasing their software as an open-source package. We sincerely thank the referees for bringing several inaccuracies to our notice, resulting in a considerable improvement in the presentation. R.B. Bapat gratefully acknowledges the support of the JC Bose Fellowship, Department of Science and Technology, Government of India. S. Sivasubramanian thanks Professor Murali K. Srinivasan for his support in making a trip to ISI Delhi possible and the Stat-Math unit of ISI Delhi for their hospitality.

References


