

Eulerian central limit theorems and Carlitz identities in positive elements of classical Weyl groups

HIRANYA KISHORE DEY AND SIVARAMAKRISHNAN SIVASUBRAMANIAN

The Eulerian numbers counting descents in the symmetric group \mathfrak{S}_n are known to have a Central Limit Theorem. Recently, Fulman, Petersen, Kim and Lee showed similar results over the alternating group \mathcal{A}_n . We show similar Central Limit Theorem like results when one samples permutations uniformly from \mathcal{A}_n and considers excedances. We extend our results to the case when we sample over the positive part of Classical Weyl Groups. For the type D Coxeter groups, Borowiec and Młitkowski recently defined a new type D Eulerian number and we show Central Limit theorems when one samples this new Eulerian statistic over the positive part of \mathfrak{D}_n . Our results give natural refinements of Carlitz type identities as well.

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1. Introduction

For a positive integer n , let $[n] = \{1, 2, \dots, n\}$ and let \mathfrak{S}_n denote the symmetric group on $[n]$. Let $\text{stat} : \mathfrak{S}_n \mapsto \mathbb{Z}_{\geq 0}$ be a statistic on \mathfrak{S}_n . Let $p_{n,k} = |\{\pi \in \mathfrak{S}_n : \text{stat}(\pi) = k\}|$. If we sample permutations uniformly at random from \mathfrak{S}_n , we get the random variable X_{stat} which takes the non negative integral value k with probability $p_{n,k}/|\mathfrak{S}_n|$. Sometimes, we will sample from a subset $S \subseteq \mathfrak{S}_n$, in which case the probability that $X_{\text{stat}} = k$ will have to be suitably modified.

Let $p(t) = \sum_k p_k t^k$ be a polynomial with non negative coefficients and let $p(1) > 0$. We will use the straightforward bijection between such polynomials and non negative random variables X which take the value k with probability $p_k/p(1)$.

Given a statistic stat and subsets $T_n \subseteq \mathfrak{S}_n$, consider the sequence of non negative integers $s_{n,k} = |\{\pi \in T_n : \text{stat}(\pi) = k\}|$. Let $M_n = \max_{\pi \in T_n} \text{stat}(\pi)$ be the maximum value that $\text{stat}(\pi)$ takes over $\pi \in T_n$. Suppose $\text{Sum}_n = \sum_{k=0}^{M_n} s_{n,k}$ is such that $\text{Sum}_n > 0$. The array $\{s_{n,k} : n \geq 1, 0 \leq k \leq M_n\}$, is said to satisfy a *Central Limit Theorem* (CLT henceforth) with mean μ_n and variance σ_n^2 if

$$(1) \quad \lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \sum_{k \leq \lfloor (x)_n \rfloor} \frac{s_{n,k}}{\text{Sum}_n} - \Phi(x) \right| = 0,$$

where $(x)_n = x\sigma_n + \mu_n$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$ is the cumulative distribution function of the standard normal distribution $N(0, 1)$. When this happens, we also say that the statistic stat is *asymptotically normal* over the set S . Canfield’s article [7] is a good reference for background on CLT’s in enumerative combinatorics.

Several *Eulerian statistics* in \mathfrak{S}_n are known. In this paper, we focus on the two most basic such statistics: the number of descents, des and the number of excedances, exc . For $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{S}_n$, define $\text{DES}(\pi) = \{i \in [n - 1] : \pi_i > \pi_{i+1}\}$ and $\text{EXC}(\pi) = \{i \in [n - 1] : \pi_i > i\}$. Let $\text{des}(\pi) = |\text{DES}(\pi)|$ and $\text{exc}(\pi) = |\text{EXC}(\pi)|$. Define $A_{n,k} = |\{\pi \in \mathfrak{S}_n : \text{des}(\pi) = k\}|$ and let $\text{AE}_{n,k} = |\{\pi \in \mathfrak{S}_n : \text{exc}(\pi) = k\}|$. As we will need the generating function later, we define

$$(2) \quad A_n(t) = \sum_{k=0}^{n-1} A_{n,k} t^k \quad \text{and} \quad \text{AE}_n(t) = \sum_{k=0}^{n-1} \text{AE}_{n,k} t^k.$$

David and Barton in [10, Pages 150–154] showed that the random variable X_{des} is asymptotically normal over \mathfrak{S}_n . See the papers [3, Example 3.5] by Bender and [13] by Kahle and Stump as well.

Theorem 1 (David and Barton). *For positive integers n , the random variable X_{des} over \mathfrak{S}_n is asymptotically normal with mean $(n - 1)/2$ and variance $(n + 1)/12$.*

Let $\mathcal{A}_n \subseteq \mathfrak{S}_n$ denote the subset of positive elements in \mathfrak{S}_n . We alternatively denote \mathcal{A}_n as \mathfrak{S}_n^+ and $\mathfrak{S}_n - \mathcal{A}_n$ as \mathfrak{S}_n^- . Let $A_{n,k}^+ = |\{\pi \in \mathfrak{S}_n^+ : \text{des}(\pi) = k\}|$ and $A_{n,k}^- = |\{\pi \in \mathfrak{S}_n^- : \text{des}(\pi) = k\}|$. Define

$$(3) \quad A_n^+(t) = \sum_{k=0}^{n-1} A_{n,k}^+ t^k \quad \text{and} \quad A_n^-(t) = \sum_{k=0}^{n-1} A_{n,k}^- t^k.$$

Fulman, Kim, Lee and Petersen recently in [12, Theorem 1.2] showed that the random variable X_{des} over \mathfrak{S}_n^\pm has a CLT.

Theorem 2 (Fulman, Kim, Lee and Petersen). *The distribution of the coefficients of $A_n^\pm(t)$ is asymptotically normal as $n \rightarrow \infty$. For $n \geq 4$, these numbers have mean $(n - 1)/2$ and for $n \geq 6$, these numbers have variance $(n + 1)/12$.*

It is a well known result of MacMahon [14] that excedances and descents are equidistributed over \mathfrak{S}_n . Thus, Theorem 1 works when the random variable X_{des} is replaced by the random variable X_{exc} . Though excedances and descents are equidistributed over \mathfrak{S}_n , it is easy to check when $n \geq 3$, that they are *not equidistributed* over \mathcal{A}_n . Hence, a version of Theorem 2 for the random variable X_{exc} does not follow immediately. Thus, a natural question is about the existence of a CLT for X_{exc} on the set \mathcal{A}_n . In this work, we give such a CLT. Let $\text{AE}_{n,k}^+ = |\{\pi \in \mathfrak{S}_n^+ : \text{exc}(\pi) = k\}|$ and $\text{AE}_{n,k}^- = |\{\pi \in \mathfrak{S}_n^- : \text{exc}(\pi) = k\}|$. Define

$$(4) \quad \text{AE}_n^+(t) = \sum_{k=0}^{n-1} \text{AE}_{n,k}^+ t^k \quad \text{and} \quad \text{AE}_n^-(t) = \sum_{k=0}^{n-1} \text{AE}_{n,k}^- t^k.$$

One of our main results, proved in subsection 2.1 of this paper is the following.

Theorem 3. *The distribution of the coefficients of $\text{AE}_n^\pm(t)$ is asymptotically normal as $n \rightarrow \infty$. For $n \geq 3$, these numbers have mean $(n - 1)/2$ and for $n \geq 4$, these numbers have variance $(n + 1)/12$.*

\mathfrak{S}_n is a Coxeter group and so are two other families \mathfrak{B}_n and \mathfrak{D}_n . The book [4] by Björner and Brenti is a good reference for the combinatorics of these groups. Hence in both \mathfrak{B}_n and \mathfrak{D}_n , there is a natural notion of descent. We denote the descent statistic in type B and type D Coxeter groups as des_B and des_D (see definitions in Sections 3 and 4) respectively.

Define $B_{n,k} = |\{\pi \in \mathfrak{B}_n : \text{des}_B(\pi) = k\}|$ and $D_{n,k} = |\{\pi \in \mathfrak{D}_n : \text{des}_D(\pi) = k\}|$. Define

$$(5) \quad B_n(t) = \sum_{k=0}^n B_{n,k} t^k \quad \text{and} \quad D_n(t) = \sum_{k=0}^n D_{n,k} t^k.$$

Kahle and Stump in [13], recently computed the first two moments and gave a CLT for the appropriate X_{des} over \mathfrak{B}_n and \mathfrak{D}_n .

Theorem 4 (Kahle and Stump). *For positive integers n , the random variable X_{des_B} over \mathfrak{B}_n is asymptotically normal with mean $n/2$ and variance $(n + 1)/12$. For positive integers n , the random variable X_{des_D} over \mathfrak{D}_n is asymptotically normal with mean $n/2$ and variance $(n + 2)/12$.*

Coxeter groups also have a natural notion of length which in \mathfrak{S}_n , \mathfrak{B}_n and \mathfrak{D}_n will be denoted as ℓ , ℓ_B and ℓ_D respectively. In \mathfrak{S}_n , it is easy to see that $\mathcal{A}_n = \mathfrak{S}_n^+ = \{\pi \in \mathfrak{S}_n : \ell(\pi) \equiv 0 \pmod{2}\}$. Motivated by this, define $\mathfrak{B}_n^+ = \{\pi \in \mathfrak{B}_n : \ell_B(\pi) \equiv 0 \pmod{2}\}$ and $\mathfrak{D}_n^+ = \{\pi \in \mathfrak{D}_n : \ell_D(\pi) \equiv 0 \pmod{2}\}$. Similarly define $\mathfrak{B}_n^- = \{\pi \in \mathfrak{B}_n : \ell_B(\pi) \equiv 1 \pmod{2}\}$ and $\mathfrak{D}_n^- = \{\pi \in \mathfrak{D}_n : \ell_D(\pi) \equiv 1 \pmod{2}\}$. We will define $(\mathfrak{B}_n - \mathfrak{D}_n)^\pm$ as well. For $\pi \in \mathfrak{B}_n - \mathfrak{D}_n$, we choose $\text{inv}_D(\pi)$ as the exponent of -1 to define sign . Had we chosen $\text{inv}_B(\pi)$ for this purpose, Remark 23 makes it clear that the sets $(\mathfrak{B}_n - \mathfrak{D}_n)^\pm$ will be the same, but have their names swapped.

Let $B_{n,k}^+ = |\{\pi \in \mathfrak{B}_n^+ : \text{des}_B(\pi) = k\}|$ and $B_{n,k}^- = |\{\pi \in \mathfrak{B}_n^- : \text{des}_B(\pi) = k\}|$. Define

$$(6) \quad B_n^+(t) = \sum_{k=0}^n B_{n,k}^+ t^k \quad \text{and} \quad B_n^-(t) = \sum_{k=0}^n B_{n,k}^- t^k.$$

For both these groups, CLT results for the random variable X_{des_B} (and X_{des_D}) on the sets \mathfrak{B}_n^\pm (and \mathfrak{D}_n^\pm) are not known to the best of our knowledge. In Theorems 13 and 18, we give such results. Thus, these can be considered as type B and type D counterparts of Theorem 2 and are proved in Sections 3 and 4 respectively.

In [5], Borowiec and Młotkowski enumerated the statistic des_B over \mathfrak{D}_n and called the generating function a *new type D Eulerian polynomial*. See Section 5 for definitions. From this polynomial, one can get CLT results (see Remark 21) for this variant. Our first contribution in this paper on this theme is Theorem 28, a signed enumeration result, where we enumerate des_B over \mathfrak{D}_n (and over $\mathfrak{B}_n - \mathfrak{D}_n$) with sign taken into account. Using this result, in Theorem 32, we give a CLT when one considers the random variable X_{des_B} over \mathfrak{D}_n^\pm (and over $(\mathfrak{B}_n - \mathfrak{D}_n)^\pm$). These results are presented in Section 5. We summarize known CLT results and the new ones in the table below. In the table, we give the statistic rather than the random variable as if one writes the statistic as a subscript, it appears in a smaller font and is hence more difficult to read.

1.1. Carlitz type identities

The following famous powerseries identity involving the Eulerian polynomial is attributed to Carlitz, though MacMahon’s book [14, Vol 2, Chap IV,

Set	Descents	Excedance
\mathfrak{S}_n	For des see David and Barton [10].	For exc, a CLT follows from [10] and MacMahon [14].
\mathfrak{S}_n^\pm	For des see Fulman <i>et al.</i> [12].	For exc see Theorem 3.
\mathfrak{B}_n	For des_B , see Kahle and Stump [13].	For exc_B , result follows from [13] and Brenti [6].
\mathfrak{B}_n^\pm	For des_B see Theorem 13.	For exc_B , see Theorem 13.
\mathfrak{D}_n	For des_D , see Kahle and Stump [13].	For exc_D , see Remark 17.
\mathfrak{D}_n^\pm	For des_D , see Theorem 16.	For exc_D , see Theorem 18.
\mathfrak{D}_n	For des_B , see Remark 21.	For exc_B , see Remark 22.
\mathfrak{D}_n^\pm	For des_B , see Theorem 32.	For exc_B , see Remark 22.

pp 211] contains this. See [17, Corollary 1.1] in the book by Petersen as well.

Theorem 5 (Carlitz). *Let $A_n(t)$ be the Eulerian polynomial as defined in (2). For positive integers n ,*

$$(7) \quad \frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (k+1)^n t^k.$$

Fulman, Kim, Lee and Petersen in [12, Theorem 1.1] gave the following refinement of Theorem 5 involving $A_n^\pm(t)$.

Theorem 6 (Fulman, Kim, Lee and Petersen). *Let $A_n^\pm(t)$ be the restricted versions of the descent based Eulerian polynomial defined in (3). For positive integers n ,*

$$(8) \quad \frac{A_n^\pm(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} \left(\frac{(k+1)^n \pm (k+1)^{\lceil n/2 \rceil}}{2} \right) t^k.$$

In Section 6 of this paper, we prove the following similar but different identity involving the polynomial $\text{AE}_n^\pm(t)$.

Theorem 7. *Let $\text{AE}_n^\pm(t)$ be the restricted versions of the excedance based Eulerian polynomial defined in (4). For positive integers n ,*

$$(9) \quad \frac{\text{AE}_n^\pm(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} \left(\frac{(k+1)^n \pm (k+1)}{2} \right) t^k.$$

Theorem 6 and Theorem 7 are refinements of Theorem 5 as summing up the two equations in (8) and (9) gives us (7). Here too, we consider

type B and D variants. In Theorems 35 and 38 we give type B and type D counterparts of Theorem 6. Brenti proved a type D Carlitz identity by giving a recurrence relation (see [6, Corollary 4.8]) between the polynomials $B_n(t)$, $D_n(t)$ and $A_{n-1}(t)$. This recurrence was also proved by Stembridge in [22, Lemma 9.1]. Our proof of Theorem 38 refines this recurrence by giving two recurrences (see Lemma 39). Borowiec and Młotkowski in [5, Proposition 4.6], also gave a Carlitz type identity involving their new type D Eulerian polynomial. In Theorem 40, we refine their result by giving signed versions. We prove all these results in Section 6.

2. A lemma and the type A result

We start with the following lemma, which is implicit in the second proof of [12, Theorem 1.2]. For positive integers n , let $F_n(t) = \sum_{k=0}^n f_{n,k} t^k$ and $G_n(t) = \sum_{k=0}^n g_{n,k} t^k$ be sequences of polynomials with $f_{n,k}, g_{n,k} \geq 0$ for all k . Further, for all positive integers n , let $F_n(1) > 0$ and $G_n(1) > 0$. We will consider the random variable X_f^n (and X_g^n) which takes the value k with probability $\frac{f_{n,k}}{F_n(1)}$ (and $\frac{g_{n,k}}{G_n(1)}$ respectively). The following lemma gives a condition for moments of X_f^n and X_g^n to be identical.

Lemma 8. *Let $F_n(t)$ and $G_n(t)$ be as described above. Suppose we have a sequence of polynomials $H_n(t)$, a non-zero real number λ and a sequence of positive integers ℓ_n such that*

$$(10) \quad F_n(t) = \lambda G_n(t) \pm (1-t)^{\ell_n} H_n(t).$$

Then, for $r < \ell_n$, the r -th moment of X_f^n is identical to the r -th moment of X_g^n .

Proof. We work with the r -th factorial moment instead. The main idea is to differentiate the relevant generating function r times and then set $t = 1$. We first assume $\ell_n \geq 2$ and show that the first moments of X_f^n and X_g^n are equal. Differentiating (10), we get

$$(11) \quad \frac{dF_n(t)}{dt} = \lambda \frac{dG_n(t)}{dt} \pm \left((1-t)^{\ell_n} \frac{dH_n(t)}{dt} + \ell_n (1-t)^{\ell_n-1} H_n(t) \right).$$

Setting $t = 1$ in (11), we get $\left. \frac{dF_n(t)}{dt} \right|_{t=1} = \lambda \left. \frac{dG_n(t)}{dt} \right|_{t=1}$. That is $F'(1) = \lambda G'(1)$. Here, we have used $\ell_n \geq 2$. From (10), clearly, $F(1) = \lambda G(1)$. Since

the first order factorial moment of X_f^n is $F'(1)/F(1)$, the first order factorial moment of X_g^n is identical to the first order factorial moment of X_f^n . We continue differentiating upto r times and then setting $t = 1$. For $r < \ell_n$, we

$$\text{will have } \left. \frac{d^r F_n(t)}{dt^r} \right|_{t=1} = \left. \frac{d^r G_n(t)}{dt^r} \right|_{t=1}.$$

By a similar argument, when $r < \ell_n$, the r -th factorial moment of X_f^n is identical to the r -th factorial moment of X_g^n . The proof follows from the fact that the moments of two distributions are identical if and only if their factorial moments are identical. □

2.1. Proof of Theorem 3

We will need the following signed excedance enumeration result in \mathfrak{S}_n . Let

$$(12) \quad \text{SgnAE}_n(t) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} t^{\text{exc}(\pi)}.$$

Mantaci (see [15, 16]) showed the following interesting result. Sivasubramanian in [19] gave an alternate proof of Mantaci’s result by evaluating the determinant of appropriately defined $n \times n$ matrices.

Theorem 9 (Mantaci). *For positive integers $n \geq 1$, $\text{SgnAE}_n(t) = (1-t)^{n-1}$.*

Proof of Theorem 3. For $n \geq 1$, we clearly have

$$(13) \quad \text{AE}_n^\pm(t) = \frac{1}{2} \left(\text{AE}_n(t) \pm (1-t)^{n-1} \right).$$

Thus, by Lemma 8, the first $n-2$ moments of $\text{AE}_n^\pm(t)$ are identical to the first $n-2$ moments of $\text{AE}_n(t)$. Since $\text{AE}_n(t) = A_n(t)$ and $A_n(t)$ is asymptotically normal, by the method of moments, $\text{AE}_n^\pm(t)$ is also asymptotically normal. Further, they have the same expected value as $A_n(t)$ when $n \geq 3$ and the same variance as $A_n(t)$ when $n \geq 4$. The proof is complete. □

3. Type B Coxeter groups

It is known that \mathfrak{B}_n can be thought as the group of permutations π of the set $[\pm n] = \{-n, -(n-1), \dots, -1, 1, 2, \dots, n\}$ which satisfy $\pi(-i) = -\pi(i)$ for $1 \leq i \leq n$. See the book by Björner and Brenti [4, Chapter 8]. Clearly, we only need $\pi(i)$ for $1 \leq i \leq n$ to know $\pi \in \mathfrak{B}_n$. We denote $-i$ alternatively as \bar{i} as well. For $\pi \in \mathfrak{B}_n$, let $\text{negs}(\pi) = |\{i \in [n] : \pi(i) < 0\}|$ be the number of negative elements in the image of $\pi(i)$ for $i \in [n]$.

For a positive integer n , define $[n]_0 = \{0, 1, \dots, n\}$. For a permutation $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{B}_n$, let $\pi_0 = 0$. Define $\text{DES}_B(\pi) = \{i \in [n-1]_0 : \pi_i > \pi_{i+1}\}$ and let $\text{des}_B(\pi) = |\text{DES}_B(\pi)|$. Define $B_{n,k} = |\{\pi \in \mathfrak{B}_n : \text{des}_B(\pi) = k\}|$. Following Brenti's definition of excedance from [6], let $\text{EXC}_B(\pi) = \{i \in [n] : \pi_{|\pi(i)|} > \pi_i\} \cup \{i \in [n] : \pi_i = -i\}$ and let $\text{exc}_B(\pi) = |\text{EXC}_B(\pi)|$. Define $\text{BExc}_{n,k} = |\{\pi \in \mathfrak{B}_n : \text{exc}_B(\pi) = k\}|$. Let

$$(14) \quad B_n(t) = \sum_{k=0}^n B_{n,k} t^k \quad \text{and} \quad \text{BExc}_n(t) = \sum_{k=0}^n \text{BExc}_{n,k} t^k.$$

Let $B_{n,k}^+ = |\{\pi \in \mathfrak{B}_n^+ : \text{des}_B(\pi) = k\}|$ and $B_{n,k}^- = |\{\pi \in \mathfrak{B}_n^- : \text{des}_B(\pi) = k\}|$. Define $B_n^+(t) = \sum_{k=0}^n B_{n,k}^+ t^k$ and $B_n^-(t) = \sum_{k=0}^n B_{n,k}^- t^k$. Let $\text{BExc}_{n,k}^+ = |\{\pi \in \mathfrak{B}_n^+ : \text{exc}_B(\pi) = k\}|$ and $\text{BExc}_{n,k}^- = |\{\pi \in \mathfrak{B}_n^- : \text{exc}_B(\pi) = k\}|$. Define $\text{BExc}_n^+(t) = \sum_{k=0}^n \text{BExc}_{n,k}^+ t^k$ and $\text{BExc}_n^-(t) = \sum_{k=0}^n \text{BExc}_{n,k}^- t^k$. Lastly define $\text{Sgn}B_n(t) = \sum_{\pi \in \mathfrak{B}_n} (-1)^{\text{inv}_B(\pi)} t^{\text{des}_B(\pi)}$ and $\text{SgnBExc}_n(t) = \sum_{\pi \in \mathfrak{B}_n} (-1)^{\text{inv}_B(\pi)} t^{\text{exc}_B(\pi)}$. Reiner in [18] and Sivasubramanian in [20] showed the following.

Theorem 10 (Reiner). *For positive integers n , $\text{Sgn}B_n(t) = (1 - t)^n$.*

Theorem 11 (Sivasubramanian). *For positive integers n , $\text{SgnBExc}_n(t) = (1 - t)^n$.*

Remark 12. *Brenti in [6, Theorem 3.15] showed that for positive integers n , we have $B_n(t) = \text{BExc}_n(t)$. From Theorems 10 and 11, we infer that $\text{Sgn}B_n(t) = \text{SgnBExc}_n(t)$ for all positive integers n . Thus, we further have $B_n^+(t) = \text{BExc}_n^+(t)$ and $B_n^-(t) = \text{BExc}_n^-(t)$. This is a noteworthy difference between the type A and the type B Coxeter groups.*

The main result of this Section is the following type B counterpart of Theorem 3.

Theorem 13. *The distribution of the coefficients of $\text{BExc}_n^\pm(t)$ (and hence of $B_n^\pm(t)$) is asymptotically normal as $n \rightarrow \infty$. The random variables X_{exc_B} and X_{des_B} over \mathfrak{B}_n^\pm have mean $n/2$ when $n \geq 2$, and have variance $(n+1)/12$ when $n \geq 3$.*

Proof. For $n \geq 1$, we clearly have

$$(15) \quad \text{BExc}_n^\pm(t) = \frac{1}{2} \left(\text{BExc}_n(t) \pm \text{SgnBExc}_n(t) \right) = \frac{1}{2} \left(B_n(t) \pm (1 - t)^n \right).$$

We have used $\text{BExc}_n(t) = B_n(t)$ above. Thus, by Lemma 8, the first $n - 1$ moments of $\text{BExc}_n^\pm(t)$ are identical to the first $n - 1$ moments of $B_n(t)$. By Theorem 4, $B_n(t)$ is asymptotically normal. Hence, by the method of moments, $\text{BExc}_n^\pm(t)$ is also asymptotically normal. Further, they have the same expected value as $B_n(t)$ when $n \geq 2$ and the same variance as $B_n(t)$ when $n \geq 3$. The proof is complete. \square

4. Type D Coxeter groups

The type D Coxeter group $\mathfrak{D}_n \subseteq \mathfrak{B}_n$ can be combinatorially defined as $\mathfrak{D}_n = \{\pi \in \mathfrak{B}_n : \text{negs}(\pi) \text{ is even}\}$. That is, \mathfrak{D}_n consists of those signed permutations in \mathfrak{B}_n with an even number of negative elements. See the book by Björner and Brenti [4, Chapter 8].

For a positive integer $n \geq 2$, define $[n]_0 = \{0, 1, \dots, n\}$. For $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{D}_n$, let $\pi_0 = -\pi_2$. Define $\text{DES}_D(\pi) = \{i \in [n - 1]_0 : \pi_i > \pi_{i+1}\}$ and let $\text{des}_D(\pi) = |\text{DES}_D(\pi)|$. Define $D_{n,k} = |\{\pi \in \mathfrak{D}_n : \text{des}_D(\pi) = k\}|$. In \mathfrak{D}_n^\pm , define $D_{n,k}^+ = |\{\pi \in \mathfrak{D}_n^+ : \text{des}_D(\pi) = k\}|$ and $D_{n,k}^- = |\{\pi \in \mathfrak{D}_n^- : \text{des}_D(\pi) = k\}|$. Define

$$(16) \quad D_n^+(t) = \sum_{k=0}^n D_{n,k}^+ t^k \quad \text{and} \quad D_n^-(t) = \sum_{k=0}^n D_{n,k}^- t^k.$$

As $\mathfrak{D}_n \subseteq \mathfrak{B}_n$, we use the same definition of excedance in \mathfrak{D}_n . Thus, for $\pi \in \mathfrak{D}_n$, we have $\text{exc}_D(\pi) = \text{exc}_B(\pi)$. Let $\text{DExc}_{n,k}^+ = |\{\pi \in \mathfrak{D}_n^+ : \text{exc}_B(\pi) = k\}|$ and $\text{DExc}_{n,k}^- = |\{\pi \in \mathfrak{D}_n^- : \text{exc}_B(\pi) = k\}|$. Define the generating function $\text{DExc}_n^+(t) = \sum_{k=0}^n \text{DExc}_{n,k}^+ t^k$ and $\text{DExc}_n^-(t) = \sum_{k=0}^n \text{DExc}_{n,k}^- t^k$. Lastly, let $\text{SgnD}_n(t) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\text{inv}_D(\pi)} t^{\text{des}_D(\pi)}$ and define $\text{SgnDExc}_n(t) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\text{inv}_D(\pi)} t^{\text{exc}_D(\pi)}$. Reiner in [18] and Sivasubramanian in [21] showed the following.

Theorem 14 (Reiner). *For positive integers n ,*

$$\text{SgnD}_n(t) = \begin{cases} (1 - t)^n & \text{if } n \text{ is even,} \\ (1 + t)(1 - t)^{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 15 (Sivasubramanian). *For positive integers n ,*

$$\text{SgnDExc}_n(t) = \begin{cases} (1 - t)^n & \text{if } n \text{ is even,} \\ (1 - t)^{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

The first main result of this Section is the following type D counterpart of Theorem 3.

Theorem 16. *The distribution of the coefficients of $D_n^\pm(t)$ is asymptotically normal as $n \rightarrow \infty$. Over \mathfrak{D}_n^\pm , the random variable X_{des_D} has mean $n/2$ when $n \geq 3$, and has variance $(n + 2)/12$ when $n \geq 4$.*

Proof. For $n \geq 1$, we clearly have

$$D_n^\pm(t) = \frac{1}{2} \left(D_n(t) \pm \text{Sgn} D_n(t) \right) = \begin{cases} \frac{D_n(t) \pm (1-t)^n}{2} & \text{if } n \text{ is even,} \\ \frac{D_n(t) \pm (1+t)(1-t)^{n-1}}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Thus, irrespective of the parity of n , by Lemma 8, the first $n - 2$ moments of $D_n^\pm(t)$ are identical to the first $n - 2$ moments of $D_n(t)$. Since the coefficients of $D_n(t)$ are asymptotically normal, by the method of moments, so are the coefficients of $D_n^\pm(t)$. Further, they have the same expected value as $D_n(t)$ when $n \geq 3$ and the same variance as $D_n(t)$ when $n \geq 4$. The proof is complete. \square

Remark 17. *In [11, Theorem 16], Dey and Sivasubramanian show that the sum of exc_B over \mathfrak{B}_n^+ equals the sum of exc_D over \mathfrak{D}_n . Hence by Theorem 13, a CLT follows for X_{exc_D} over \mathfrak{D}_n .*

Another result of this Section is the following type D counterpart of Theorem 3.

Theorem 18. *The distribution of the coefficients of $\text{DExc}_n^\pm(t)$ is asymptotically normal as $n \rightarrow \infty$. Over \mathfrak{D}_n^\pm , the random variable X_{exc_D} has mean $n/2$ when $n \geq 3$, and has variance $(n + 1)/12$ when $n \geq 4$.*

Proof. For $n \geq 1$, as $\text{DExc}_n^\pm(t) = \frac{1}{2} \left(\text{DExc}_n(t) \pm \text{Sgn} \text{DExc}_n(t) \right)$, we have

$$\text{DExc}_n^\pm(t) = \begin{cases} \frac{\text{DExc}_n(t) \pm (1-t)^n}{2} & \text{if } n \text{ is even,} \\ \frac{\text{DExc}_n(t) \pm (1-t)^{n-1}}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Thus, irrespective of the parity of n , by Lemma 8, the first $n - 2$ moments of $\text{DExc}_n^\pm(t)$ are identical to the first $n - 2$ moments of $\text{DExc}_n(t)$. By Remark 17 the coefficients of $\text{DExc}_n(t)$ are asymptotically normal. Thus, by the method of moments, the coefficients of $\text{DExc}_n^\pm(t)$ are also asymptotically normal. Further, they have the same expected value as $\text{DExc}_n(t)$ when $n \geq 3$ and the same variance as $\text{DExc}_n(t)$ when $n \geq 4$. The proof is complete. \square

Note that Theorem 18 refines Theorem 13.

5. Results on Borowiec and Młotkowski’s variant

In [5], Borowiec and Młotkowski enumerated type B descents des_B over \mathfrak{D}_n and $\mathfrak{B}_n - \mathfrak{D}_n$. Let $\text{BDes}_{n,k}^D = |\{\pi \in \mathfrak{D}_n : \text{des}_B(\pi) = k\}|$ and $\text{BDes}_{n,k}^{B-D} = |\{\pi \in \mathfrak{B}_n - \mathfrak{D}_n : \text{des}_B(\pi) = k\}|$. They considered the polynomials

$$(17) \quad \text{BDes}_n^D(t) = \sum_{k=0}^n \text{BDes}_{n,k}^D t^k \quad \text{and} \quad \text{BDes}_n^{B-D}(t) = \sum_{k=0}^n \text{BDes}_{n,k}^{B-D} t^k.$$

In [5, Equations (24), (25)], they showed the following.

Theorem 19 (Borowiec and Młotkowski). *For positive integers n , the two polynomials $\text{BDes}_n^D(t)$ and $\text{BDes}_n^{B-D}(t)$ satisfy the following:*

$$(18) \quad \begin{aligned} \text{BDes}_n^D(t) &= \frac{1}{2} \left(B_n(t) + (1-t)^n \right) \quad \text{and} \\ \text{BDes}_n^{B-D}(t) &= \frac{1}{2} \left(B_n(t) - (1-t)^n \right). \end{aligned}$$

Using Theorem 10, we restate Theorem 19 as follows.

Corollary 20. *For positive integers n ,*

$$(19) \quad \text{BDes}_n^D(t) = B_n^+(t) \quad \text{and} \quad \text{BDes}_n^{B-D}(t) = B_n^-(t).$$

Remark 21. *From Theorem 13 and Remark 12, it follows that the distribution of the coefficients of $\text{BDes}_n^D(t)$ and $\text{BDes}_n^{B-D}(t)$ are asymptotically normal as $n \rightarrow \infty$.*

Mimicking the approach of Borowiec and Młotkowski, suppose instead of type B descents, we wish to sum type B excedances over \mathfrak{D}_n . Similar to (17) we would need to define $\text{BExc}_{n,k}^D = |\{\pi \in \mathfrak{D}_n : \text{exc}_B(\pi) = k\}|$ and $\text{BExc}_{n,k}^{B-D} = |\{\pi \in \mathfrak{B}_n - \mathfrak{D}_n : \text{exc}_B(\pi) = k\}|$. Define

$$(20) \quad \text{BExc}_n^D(t) = \sum_{k=0}^n \text{BExc}_{n,k}^D t^k \quad \text{and} \quad \text{BExc}_n^{B-D}(t) = \sum_{k=0}^n \text{BExc}_{n,k}^{B-D} t^k.$$

Remark 22. *It is simple to see that $\text{BExc}_n^D(t) = \text{DExc}_n(t)$ and hence we have no new results on this “variant”.*

We start work in the next subsection towards proving a CLT for the random variable X_{des_B} over \mathfrak{D}_n^\pm .

5.1. Enumerating type B descents over \mathfrak{D}_n^+ and \mathfrak{D}_n^-

Let $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{D}_n$. The following combinatorial definition of type D inversions is known (see Petersen’s book [17, Page 302]): $\text{inv}_D(\pi) = \text{inv}_A(\pi) + |\{1 \leq i < j \leq n : -\pi_i > \pi_j\}|$. Here $\text{inv}_A(\pi)$ is computed with respect to the usual order on \mathbb{Z} . Let $\pi \in \mathfrak{B}_n$. We will need the following alternate definition of $\text{inv}_B(\pi)$ (see Petersen’s book, [17, Page 294]): $\text{inv}_B(\pi) = \text{inv}_A(\pi) + |\{1 \leq i < j \leq n : -\pi_i > \pi_j\}| + |\text{negs}(\pi)|$.

Let $\pi \in \mathfrak{D}_n$. We can also think of π as an element of \mathfrak{B}_n . Since we have a combinatorial definition of inv_B , even though $\pi \in \mathfrak{D}_n$, $\text{inv}_B(\pi)$ is defined. Similarly, we also have $\text{inv}_D(\pi)$ when $\pi \in \mathfrak{B}_n$, especially when $\pi \in \mathfrak{B}_n - \mathfrak{D}_n$. From the above definitions of $\text{inv}_B(\pi)$ and $\text{inv}_D(\pi)$, the following remark follows. This appears in [6, Equation (45)] of Brenti and we will need this later.

Remark 23. *Let $\pi \in \mathfrak{B}_n$. Then, $\text{inv}_B(\pi) = \text{inv}_D(\pi) + |\text{negs}(\pi)|$.*

Let $\pi \in \mathfrak{B}_n$. Define its number of type B ascents to be $\text{asc}_B(\pi) = n - \text{des}_B(\pi)$. We are interested in the following two quantities:

$$(21) \quad \text{SgnBDes}_n^D(s, t) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\text{inv}_D \pi} s^{\text{asc}_B \pi} t^{\text{des}_B \pi}$$

$$(22) \quad \text{SgnBDes}_n^{B-D}(s, t) = \sum_{\pi \in \mathfrak{B}_n - \mathfrak{D}_n} (-1)^{\text{inv}_D \pi} s^{\text{asc}_B \pi} t^{\text{des}_B \pi}.$$

Our proof involves an elaborate sign reversing involution to describe which we need a few preliminaries. Let $\pi \in \mathfrak{D}_n$. For an index $k \in [n]$, and positive value $r \in [n]$, define $\text{pos}_r(\pi) = k$ if $\pi_k = r$ and define $\text{pos}_{-r}(\pi) = k$ if $\pi_k = \bar{r}$. Moreover, let $\text{pos}_{\pm r}(\pi) = k$ if $\pi_k = r$ or $\pi_k = \bar{r}$.

Let $n \geq 3$. For $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{D}_n$, let π'' be obtained from π by deleting the letters n and $n - 1$. Suppose $\pi \in \mathfrak{D}_n$. Then, it is easy to see that both $\pi'' \in \mathfrak{D}_{n-2}$ and $\pi'' \in \mathfrak{B}_{n-2} - \mathfrak{D}_{n-2}$ are possible. From a permutation $\pi'' \in \mathfrak{D}_{n-2}$, to get a permutation $\pi \in \mathfrak{D}_n$, we have to insert either n and $n - 1$ or insert \bar{n} and $\overline{n - 1}$. Similarly, from $\pi'' \in \mathfrak{B}_{n-2} - \mathfrak{D}_{n-2}$, to get $\pi \in \mathfrak{D}_n$, we have to insert either n and $\overline{n - 1}$ or insert \bar{n} and $n - 1$. A similar statement is true when we want to get $\pi \in \mathfrak{B}_n - \mathfrak{D}_n$ from $\pi'' \in \mathfrak{B}_{n-2} - \mathfrak{D}_{n-2}$ or from $\pi'' \in \mathfrak{D}_{n-2}$.

We partition \mathfrak{D}_n into the following 6 disjoint subsets and will consider the contribution of each set to $\text{SgnBDes}_n(s, t)$.

1. $\mathfrak{D}_n^1 = \{\pi \in \mathfrak{D}_n : \pi'' \in \mathfrak{D}_{n-2}, |\text{pos}_{\pm n}(\pi) - \text{pos}_{\pm(n-1)}(\pi)| = 1, \pi_n \in \{\pm(n - 1), \pm n\}\}$,

2. $\mathfrak{D}_n^2 = \{\pi \in \mathfrak{D}_n : \pi'' \in \mathfrak{D}_{n-2}, |\text{pos}_n(\pi) - \text{pos}_{n-1}(\pi)| = 1, \pi_n \notin \{n-1, n\}\},$
3. $\mathfrak{D}_n^3 = \{\pi \in \mathfrak{D}_n : \pi'' \in \mathfrak{D}_{n-2}, |\text{pos}_{-n}(\pi) - \text{pos}_{-(n-1)}(\pi)| = 1, \pi_n \notin \{\overline{n-1}, \overline{n}\}\},$
4. $\mathfrak{D}_n^4 = \{\pi \in \mathfrak{D}_n : \pi'' \in \mathfrak{D}_{n-2}, |\text{pos}_n(\pi) - \text{pos}_{n-1}(\pi)| > 1\},$
5. $\mathfrak{D}_n^5 = \{\pi \in \mathfrak{D}_n : \pi'' \in \mathfrak{D}_{n-2}, |\text{pos}_{-n}(\pi) - \text{pos}_{-(n-1)}(\pi)| > 1\},$
6. $\mathfrak{D}_n^6 = \{\pi \in \mathfrak{D}_n : \pi'' \in \mathfrak{B}_{n-2} - \mathfrak{D}_{n-2}\}.$

We prove some preliminary lemmas which we will use later.

Lemma 24. *For positive integers $n \geq 3$, the contribution of $\mathfrak{D}_n^2 \cup \mathfrak{D}_n^3$ to $\text{SgnBDes}_n^D(s, t)$ is 0. That is,*

$$\sum_{\pi \in \mathfrak{D}_n^2 \cup \mathfrak{D}_n^3} (-1)^{\text{inv}_D(\pi)} s^{\text{asc}_B(\pi)} t^{\text{des}_B(\pi)} = 0$$

Proof. Let $\pi \in \mathfrak{D}_n^2$. Suppose $\text{pos}_n(\pi) - \text{pos}_{n-1}(\pi) = -1$. Then, π has the following form: $\pi = \pi_1, \dots, \pi_{i-1}, \pi_i = n, \pi_{i+1} = n-1, \pi_{i+2}, \dots, \pi_n$. Define $f : \mathfrak{D}_n^2 \mapsto \mathfrak{D}_n^3$ by $f(\pi) = \pi_1, \dots, \pi_{i-1}, \pi_i = -(n-1), \pi_{i+1} = -n, \pi_{i+2}, \dots, \pi_n$. We have

$$\begin{aligned} \text{inv}_D(f(\pi)) &= \text{inv}_B(f(\pi)) - |\text{negs}(f(\pi))| \\ (23) \quad &\equiv \text{inv}_B(\pi_1, \dots, \pi_{i-1}, \pi_i = n-1, \pi_{i+1} = n, \pi_{i+2}, \dots, \pi_n) \\ &\quad - |\text{negs}(\pi)| \pmod{2} \end{aligned}$$

$$(24) \quad \equiv \text{inv}_B(\pi) - 1 \pmod{2} \equiv \text{inv}_D(\pi) - 1 \pmod{2}.$$

The second line above uses the fact that flipping the sign of a single π_i changes the parity of the number of type B inversions (see [20, Lemma 3]) The last line uses the fact that swapping two letters of π changes the parity of type B inversions. Moreover, it is easy to check that $\text{des}_B(f(\pi)) = \text{des}_B(\pi)$.

When $\text{pos}_n(\pi) - \text{pos}_{n-1}(\pi) = 1$, π has the form $\pi = \pi_1, \dots, \pi_{i-1}, \pi_i = n-1, \pi_{i+1} = n, \pi_{i+2}, \dots, \pi_n$. We define $f(\pi) = \pi_1, \dots, \pi_{i-1}, \pi_i = \overline{n}, \pi_{i+1} = \overline{n-1}, \pi_{i+2}, \dots, \pi_n$. As done before, one can check that $\text{des}_B(f(\pi)) = \text{des}_B(\pi)$ and $\text{inv}_D(f(\pi)) - \text{inv}_D(\pi) \equiv 1 \pmod{2}$. Moreover, f is invertible. The proof is complete. \square

Lemma 25. *For positive integers $n \geq 3$, the contribution to $\text{SgnBDes}_n^D(s, t)$ from \mathfrak{D}_n^4 is 0. That is,*

$$\sum_{\pi \in \mathfrak{D}_n^4} (-1)^{\text{inv}_D(\pi)} s^{\text{asc}_B(\pi)} t^{\text{des}_B(\pi)} = 0$$

Proof. Let $\pi = \pi_1, \dots, \pi_{i-1}, \pi_i = x, \pi_{i+1}, \dots, \pi_j = y, \dots, \pi_n \in \mathfrak{D}_n^4$ where $\{x, y\} = \{n-1, n\}$. Define $g : \mathfrak{D}_n^4 \mapsto \mathfrak{D}_n^4$ by $g(\pi) = \pi_1, \dots, \pi_{i-1}, \pi_i = y, \pi_{i+1}, \dots, \pi_j = x, \dots, \pi_n$. The map g clearly satisfies $\text{des}_B(\pi) = \text{des}_B(g(\pi))$ but changes the parity of inv_D as the pair (i, j) flips being an inversion. The proof is complete. \square

Lemma 26. *For positive integers $n \geq 3$, the contribution to $\text{SgnBDes}_n^D(s, t)$ from \mathfrak{D}_n^5 is 0. That is,*

$$\sum_{\pi \in \mathfrak{D}_n^5} (-1)^{\text{inv}_D(\pi)} s^{\text{asc}_B(\pi)} t^{\text{des}_B(\pi)} = 0.$$

The proof of Lemma 26 is very similar to the proof of Lemma 25 and so we omit it.

Lemma 27. *For positive integers $n \geq 3$, the contribution to $\text{SgnBDes}_n^D(s, t)$ from \mathfrak{D}_n^6 is 0. That is,*

$$\sum_{\pi \in \mathfrak{D}_n^6} (-1)^{\text{inv}_D(\pi)} s^{\text{asc}_B(\pi)} t^{\text{des}_B(\pi)} = 0$$

Proof. Let $\pi \in \mathfrak{D}_n^6$ and $\pi'' \in \mathfrak{B}_{n-2} - \mathfrak{D}_{n-2}$. Thus, the one line notation of π either contains n and $\overline{n-1}$ or contains \overline{n} and $n-1$. Suppose

$$\pi = \pi_1, \dots, \pi_{i-1}, \pi_i = x, \pi_{i+1}, \dots, \pi_j = y, \dots, \pi_n$$

where $\{|x|, |y|\} = \{n-1, n\}$ and $\text{sign}(x) \neq \text{sign}(y)$. Define $h : \mathfrak{D}_n^6 \mapsto \mathfrak{D}_n^6$ by $h(\pi) = \pi_1, \dots, \pi_{i-1}, \pi_i = \overline{y}, \pi_{i+1}, \dots, \pi_j = \overline{x}, \dots, \pi_n$. It is easy to check that h preserves des_B but changes the parity of inv_D , completing the proof. \square

Thus, the total contribution of the sets \mathfrak{D}_n^k for $k \geq 2$, to $\text{SgnBDes}_n^D(s, t)$, equals 0. Hence

$$(25) \quad \text{SgnBDes}_n^D(s, t) = \sum_{\pi \in \mathfrak{D}_n^1} (-1)^{\text{inv}_D(\pi)} s^{\text{asc}_B(\pi)} t^{\text{des}_B(\pi)}.$$

Theorem 28. *For positive integers $n \geq 2$, the following recurrence relations hold:*

$$(26) \quad \text{SgnBDes}_n^D(s, t) = (s-t)^2 \text{SgnBDes}_{n-2}^D(s, t),$$

$$(27) \quad \text{SgnBDes}_n^{B-D}(s, t) = (s-t)^2 \text{SgnBDes}_{n-2}^{B-D}(s, t).$$

We thus have

$$(28) \quad \text{SgnBDes}_n^D(s, t) = \begin{cases} (s-t)^n & \text{when } n \text{ is even,} \\ s(s-t)^{n-1} & \text{when } n \text{ is odd.} \end{cases}$$

$$(29) \quad \text{SgnBDes}_n^{B-D}(s, t) = \begin{cases} 0 & \text{when } n \text{ is even,} \\ t(s-t)^{n-1} & \text{when } n \text{ is odd.} \end{cases}$$

Proof. We consider (26) first. Each $\pi = \pi_1, \pi_2, \dots, \pi_{n-2} \in \mathfrak{D}_{n-2}$ gives rise to the following four permutations ψ_1, ψ_2, ψ_3 and ψ_4 in \mathfrak{D}_n^1 :

$$\begin{aligned} \psi_1 &= \pi_1, \pi_2, \dots, \pi_{n-2}, n-1, n, & \psi_2 &= \pi_1, \pi_2, \dots, \pi_{n-2}, n, n-1, \\ \psi_3 &= \pi_1, \pi_2, \dots, \pi_{n-2}, \overline{n}, \overline{n-1}, & \psi_4 &= \pi_1, \pi_2, \dots, \pi_{n-2}, \overline{n-1}, \overline{n}. \end{aligned}$$

It is simple to note that

1. $\text{des}_B(\psi_1) = \text{des}_B(\pi)$ and $\text{inv}_D(\psi_1) - \text{inv}_D(\pi) \equiv 0 \pmod 2$.
2. $\text{des}_B(\psi_i) = \text{des}_B(\pi) + 1$ and $\text{inv}_D(\psi_i) - \text{inv}_D(\pi) \equiv 1 \pmod 2$ when $i = 2, 3$.
3. $\text{des}_B(\psi_4) = \text{des}_B(\pi) + 2$ and $\text{inv}_D(\psi_4) - \text{inv}_D(\pi) \equiv 0 \pmod 2$.

Thus, we get

$$\begin{aligned} \text{SgnBDes}_n^D(s, t) &= \sum_{\pi \in \mathfrak{D}_n^1} (-1)^{\text{inv}_D(\pi)} s^{\text{asc}_B(\pi)} t^{\text{des}_B(\pi)} \\ &= (s^2 - 2st + t^2) \sum_{\pi \in \mathfrak{D}_{n-2}} (-1)^{\text{inv}_D(\pi)} s^{\text{asc}_B(\pi)} t^{\text{des}_B(\pi)} \\ &= (s-t)^2 \text{SgnBDes}_{n-2}^D(s, t) \end{aligned}$$

Variants of Lemmas 24, 25, 26 and 27 can be proved for $\mathfrak{B}_n - \mathfrak{D}_n$ by defining $(\mathfrak{B}_n - \mathfrak{D}_n)^k$ for $1 \leq k \leq 6$. Using these, in a similar manner, one can prove (27). One can check the following base cases: $\text{SgnBDes}_1^D(s, t) = s$, $\text{SgnBDes}_2^D(s, t) = (s-t)^2$ and $\text{SgnBDes}_1^{B-D}(s, t) = t$, $\text{SgnBDes}_2^{B-D}(s, t) = 0$. Using these with (26) and (27) completes the proof. \square

We mention two uses of Theorem 28 before moving on to the proof of CLTs. Firstly, subtracting (29) from (28), we get an alternative proof of a bivariate version of Theorem 10.

Theorem 29 (Reiner). *For positive integers n , we have*

$$\sum_{\pi \in \mathfrak{B}_n} (-1)^{\text{inv}_B(\pi)} s^{\text{asc}_B(\pi)} t^{\text{des}_B(\pi)} = (s-t)^n.$$

Next, as mentioned in Remark 12, we have $B_n^+(t) = \text{BExc}_n^+(t)$ and $B_n^-(t) = \text{BExc}_n^-(t)$. Using Theorem 28, we get the following further refinement which we record below.

Remark 30. *From [11, Remark 26] of Dey and Sivasubramanian, we see that des_B and exc_B are equidistributed over \mathfrak{D}_n and hence over $\mathfrak{B}_n - \mathfrak{D}_n$. From Theorem 15 and (28) of Theorem 28, we get that $\text{SgnDExc}_n(t) = \text{SgnBDes}_n^D(1, t)$. Thus, we get $\text{DExc}_n^+(t) = \text{SgnBDes}_n^{D,+}(1, t)$, $\text{DExc}_n^-(t) = \text{SgnBDes}_n^{D,-}(1, t)$. That is, des_B and exc_B are equidistributed over \mathfrak{D}_n^\pm . One can show in a similar manner that des_B and exc_B are equidistributed over $(\mathfrak{B}_n - \mathfrak{D}_n)^\pm$.*

Remark 31. *Borowiec and Młotkowski in [5, Proposition 4.7] show another signed enumeration result which is similar to Theorem 28, but is different from it.*

5.2. Fixed points of the involution in the proof of Theorem 28

It is clear that our proof of Theorem 28 is a sign reversing involution, though it is described in several parts. The set of fixed points of this involution is thus a natural question which we consider next.

We claim that the following two families of sets $L_n \subseteq \mathfrak{D}_n$ and $M_n \subseteq \mathfrak{B}_n - \mathfrak{D}_n$ are those which survives the cancellations in our proof. We first define these sets inductively. Let $L_1 = \mathfrak{D}_1$, $L_2 = \mathfrak{D}_2$, $M_1 = \mathfrak{B}_1 - \mathfrak{D}_1$, and for even positive integers n , let $M_n = \emptyset$. We define the sets L_n for all positive integers $n \geq 3$ and M_n for odd integers $n \geq 3$. Consider $\pi = \pi_1, \pi_2, \dots, \pi_n \in L_{n-2}$. Using π , we form four signed permutations τ_1, τ_2, τ_3 and $\tau_4 \in L_n$ as follows:

$$\begin{aligned} \tau_1 &= \pi_1, \pi_2, \dots, \pi_{n-2}, n-1, n, & \tau_2 &= \pi_1, \pi_2, \dots, \pi_{n-2}, n, n-1, \\ \tau_3 &= \pi_1, \pi_2, \dots, \pi_{n-2}, \overline{n}, \overline{n-1}, & \tau_4 &= \pi_1, \pi_2, \dots, \pi_{n-2}, \overline{n-1}, \overline{n}. \end{aligned}$$

We do the same construction to get M_n from M_{n-2} for odd n . Clearly, $L_n \subseteq \mathfrak{D}_n^1$ for positive integers n and $M_n \subseteq (\mathfrak{B}_n - \mathfrak{D}_n)^1$ for odd positive integers n . Further, it is clear that $|L_n| = 2^n$ when n is even and $|L_n| = 2^{n-1}$ when n is odd. Further, it is simple to see that the elements of L_n have the following property. Recall that for $\pi \in \mathfrak{D}_n$, π'' is obtained by deleting the two highest elements in absolute value. For $\pi \in L_n$, let $\pi_1 = \pi''$. Then $\pi_1 \in L_{n-2}$. Next, let $\pi_2 = \pi_1''$. Then $\pi_2 \in L_{n-4}$ and so on till $\pi_{\lfloor n/2 \rfloor}$.

We want to show that L_n is the set of permutations that survive cancellations. We do this when n is even. The argument is very similar when n is odd. For even n , let $\pi \in \mathfrak{D}_n^1$. Denote $\pi_{n/2} = \pi$. Let $\pi_{n/2-1} = \pi''_{n/2}$,

$\pi_{n/2-2} = \pi''_{n/2-1}$ and so on. Consider the smallest positive integer r such that either $\pi_{n/2-r} \notin L_{n-2r}$.

As mentioned above, it is clear that if there is no such r , then $\pi \in L_n$. Suppose such an r exists. Suppose $\pi_{n/2-r} \notin L_{n-2r}$. The contribution of $\pi_{n/2-r}$ will cancel with the contribution of some other $\psi \in \mathfrak{D}_{n/2-r}^p$ for some p . Padding ψ up with the same sequence of $2r$ elements that we removed from $\pi_{n/2}$ gives an element $\sigma \in \mathfrak{D}_n^1$ which cancels with π . Two points need to be checked and both are easy. The first point is to see why $\sigma \in \mathfrak{D}_n^1$. This is straightforward from the definition. The second point is to see that π and σ have opposite signs but have the same des_B value. This is also easy to see.

Finally, it is straightforward to check when n is even that

$$\sum_{\pi \in L_n} (-1)^{\text{inv}_D(\pi)} s^{\text{asc}_B(\pi)} t^{\text{des}_B(\pi)} = (s - t)^n$$

and that $\sum_{\pi \in L_n} (-1)^{\text{inv}_D(\pi)} s^{\text{asc}_B(\pi)} t^{\text{des}_B(\pi)} = s(s - t)^{n-1}$ when n is odd. Similarly, it is not hard to show when n is odd, that,

$$\sum_{\pi \in M_n} (-1)^{\text{inv}_D(\pi)} s^{\text{asc}_B(\pi)} t^{\text{des}_B(\pi)} = t(s - t)^{n-1}.$$

When n is even, clearly $\sum_{\pi \in M_n} (-1)^{\text{inv}_D(\pi)} s^{\text{asc}_B(\pi)} t^{\text{des}_B(\pi)} = 0$.

5.3. CLT results over \mathfrak{D}_n^\pm and $(\mathfrak{B}_n - \mathfrak{D}_n)^\pm$

In this subsection, we give our CLT results for the random variable X_{des_B} over \mathfrak{D}_n^\pm and $(\mathfrak{B}_n - \mathfrak{D}_n)^\pm$. Let $\text{BDes}_{n,k}^{D,+} = |\{\pi \in \mathfrak{D}_n^+ : \text{des}_B(\pi) = k\}|$ and $\text{BDes}_{n,k}^{D,-} = |\{\pi \in \mathfrak{D}_n^- : \text{des}_B(\pi) = k\}|$. Further, let $\text{BDes}_{n,k}^{B-D,+} = |\{\pi \in (\mathfrak{B}_n - \mathfrak{D}_n)^+ : \text{des}_B(\pi) = k\}|$ and $\text{BDes}_{n,k}^{B-D,-} = |\{\pi \in (\mathfrak{B}_n - \mathfrak{D}_n)^- : \text{des}_B(\pi) = k\}|$. Define

$$(30) \quad \text{BDes}_n^{D,+}(t) = \sum_{k=0}^n \text{BDes}_{n,k}^{D,+} t^k$$

$$(31) \quad \text{BDes}_n^{D,-}(t) = \sum_{k=0}^n \text{BDes}_{n,k}^{D,-} t^k,$$

$$(32) \quad \text{BDes}_n^{B-D,+}(t) = \sum_{k=0}^n \text{BDes}_{n,k}^{B-D,+} t^k$$

$$(33) \quad \text{BDes}_n^{B-D,-}(t) = \sum_{k=0}^n \text{BDes}_{n,k}^{B-D,-} t^k.$$

We are now ready to prove our CLT results.

Theorem 32. *The distribution of the coefficients of the generating functions $\text{BDes}_n^{D,\pm}(t)$ and $\text{BDes}_n^{B-D,\pm}(t)$ are asymptotically normal as $n \rightarrow \infty$. Over \mathfrak{D}_n^\pm , the random variable X_{des_B} has mean $n/2$ when $n \geq 3$, and has variance $(n+1)/12$ when $n \geq 4$. Over $(\mathfrak{B}_n - \mathfrak{D}_n)^\pm$, the random variable X_{des_B} has mean $n/2$ when $n \geq 3$ and variance $(n+1)/12$ when $n \geq 4$.*

Proof. We first consider X_{des_B} over \mathfrak{D}_n^\pm . Let $n \geq 1$. Using Theorem 19 and Theorem 28, we get

$$(34) \quad \begin{aligned} \text{BDes}_n^{D,\pm}(t) &= \frac{1}{2} \left(\text{BDes}_n^D(t) \pm \text{SgnBDes}_n^D(1, t) \right) \\ &= \frac{1}{2} \left(B_n^+(t) \pm \text{SgnBDes}_n^D(1, t) \right) \end{aligned}$$

$$(35) \quad = \begin{cases} \frac{1}{2} \left(B_n^+(t) \pm (1-t)^n \right) & \text{if } n \text{ is even,} \\ \frac{1}{2} \left(B_n^+(t) \pm (1-t)^{n-1} \right) & \text{if } n \text{ is odd.} \end{cases}$$

Similarly, we have

$$(36) \quad \begin{aligned} \text{BDes}_n^{B-D,\pm}(t) &= \frac{1}{2} \left(\text{BDes}_n^{B-D}(t) \pm \text{SgnBDes}_n^{B-D}(1, t) \right) \\ &= \frac{1}{2} \left(B_n^-(t) \pm \text{SgnBDes}_n^{B-D}(1, t) \right) \end{aligned}$$

$$(37) \quad = \begin{cases} \frac{1}{2} B_n^-(t) & \text{if } n \text{ is even,} \\ \frac{1}{2} \left(B_n^-(t) \pm t(1-t)^{n-1} \right) & \text{if } n \text{ is odd.} \end{cases}$$

Thus, irrespective of the parity of n , by Lemma 8, the first $n-2$ moments of $\text{BDes}_n^{D,\pm}(t)$ are identical to the first $n-2$ moments of $B_n^+(t)$. Since the coefficients of $B_n^+(t)$ are asymptotically normal, by the method of moments, so are the coefficients of $\text{BDes}_n^{D,\pm}(t)$. Further, they have the same expected value as $B_n^+(t)$ when $n \geq 3$ and the same variance as $B_n^+(t)$ when $n \geq 4$. The proof for a CLT for X_{des_B} over $(\mathfrak{B}_n - \mathfrak{D}_n)^\pm$ is very similar and is hence omitted. \square

Remark 33. By Remark 30, similar to Theorem 32, one can get CLTs for X_{exc_B} over \mathfrak{D}_n^\pm and $(\mathfrak{B}_n - \mathfrak{D}_n)^\pm$.

6. Carlitz identities involving even and odd excedance based Eulerian polynomials

In this Section, we prove Carlitz type identities which refine various known identities associated with Eulerian polynomials arising from Coxeter groups. Our identities are obtained using signed enumeration results that either exists in the literature or those proved in Section 5. We begin with our proof of Theorem 7.

Proof of Theorem 7. By Theorem 5 and Theorem 9, we have

$$\begin{aligned} \frac{AE_n(t)}{(1-t)^{n+1}} &= \sum_{k \geq 0} (k+1)^n t^k \text{ and } \frac{\text{Sgn}AE_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (k+1)t^k. \text{ Thus,} \\ \frac{AE_n^\pm(t)}{(1-t)^{n+1}} &= \sum_{k \geq 0} \frac{1}{2} \left((k+1)^n \pm (k+1) \right) t^k. \end{aligned}$$

The proof is complete. □

Theorem 6 and Theorem 7 are two different refinements of Theorem 5. This stems from the fact that when $n \geq 3$, $A_n^+(t) \neq AE_n^+(t)$.

6.1. Type B Carlitz identities involving $BExc_n^\pm(t)$

Type B analogues of the Carlitz identity are also known. We start with the following result of Brenti [6, Theorem 3.4].

Theorem 34 (Brenti). *Let $B_n(t)$ be the type B Eulerian polynomial defined in (5). Then,*

$$(38) \quad \frac{B_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (2k+1)^n t^k.$$

By Remark 12, for positive integers n , we have $B_n^+(t) = BExc_n^+(t)$. Thus, we have only one refinement of Theorem 34.

Theorem 35. *Let $B_n^\pm(t)$ be the signed type B Eulerian polynomial defined in (6). Then,*

$$(39) \quad \frac{B_n^\pm(t)}{(1-t)^{n+1}} = \frac{BExc_n^\pm(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} \frac{(2k+1)^n \pm 1}{2} t^k.$$

Proof. By Theorem 34 and Theorem 11, we have

$$\frac{\text{BExc}_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (2k+1)^n t^k \quad \text{and} \quad \frac{\text{SgnBExc}_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} t^k. \quad \text{Thus,}$$

$$\frac{\text{BExc}_n^\pm(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} \frac{1}{2} \left((2k+1)^n \pm 1 \right) t^k.$$

The proof is complete. □

6.2. Type D Carlitz identities involving $D_n^\pm(t)$

Brenti in [6, Corollary 4.8] gave the following recurrence involving the polynomials $B_n(t)$, $D_n(t)$ and $A_{n-1}(t)$. He also mentions that Stembridge proves it in [22, Lemma 9.1].

Lemma 36 (Brenti). *For positive integers n , the numbers $A_{n-1,k}$, $B_{n,k}$ and $D_{n,k}$ satisfy the following relation:*

$$(40) \quad B_{n,k} = D_{n,k} + n2^{n-1}A_{n-1,k-1}.$$

Equivalently, the Eulerian polynomials of types A, B and D are related as follows:

$$(41) \quad B_n(t) = D_n(t) + n2^{n-1}tA_{n-1}(t).$$

Using Lemma 36, Brenti in [6, Theorem 4.10] gave the following type D analogue of the Carlitz identity. Let $\mathcal{B}_n(x)$ be the n -th Bernoulli polynomial.

Theorem 37 (Brenti). *For positive integers $n \geq 2$,*

$$(42) \quad \frac{D_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} \left((2k+1)^n - 2^{n-1}(\mathcal{B}_n(k+1) - \mathcal{B}_n(k)) \right) t^k.$$

Our refinement involving $D_n^\pm(t)$ is the following. The proof uses Theorem 37 and Theorem 14 in a manner very similar to the proof of Theorem 35. Thus, we omit its proof and merely state our result.

Theorem 38. *Let $D_n^\pm(t)$ be the signed type D Eulerian polynomial defined in (16). Then, for positive integers $n \geq 2$, when n is even, we have*

$$(43) \quad \frac{D_n^\pm(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} \frac{1}{2} \left((2k+1)^n - 2^{n-1}(\mathcal{B}_n(k+1) - \mathcal{B}_n(k)) \pm 1 \right) t^k$$

and when n is odd, we have

$$(44) \quad \frac{D_n^\pm(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} \frac{1}{2} \left((2k+1)^n - 2^{n-1} (\mathcal{B}_n(k+1) - \mathcal{B}_n(k)) \pm (2k+1) \right) t^k$$

Lemma 36 and Theorem 37 are intimately connected. Similar to their relation, we get the following Lemma from Theorem 38. It is clear Lemma 39 refines Lemma 36.

Lemma 39. *For positive integers n , the Eulerian numbers $A_{n-1,k}$, $B_{n,k}^\pm$ and $D_{n,k}^\pm$ satisfy the following relation:*

$$(45) \quad B_{n,k}^\pm = \begin{cases} D_{n,k}^\pm + \frac{1}{2}n2^{n-1}A_{n-1,k-1} & \text{if } n \text{ is even,} \\ D_{n,k}^\pm + \frac{1}{2}n2^{n-1}A_{n-1,k-1} \mp (-1)^{k-1} \binom{n-1}{k-1} & \text{if } n \text{ is odd.} \end{cases}$$

Equivalently, the signed Eulerian polynomials of types A , B and D are related as follows:

$$(46) \quad B_n^\pm(t) = \begin{cases} D_n^\pm(t) + \frac{1}{2}n2^{n-1}tA_{n-1}(t), & \text{if } n \text{ is even,} \\ D_n^\pm(t) + \frac{1}{2}n2^{n-1}tA_{n-1}(t) \mp t(1-t)^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. (Sketch) Since (45) and (46) are equivalent, we only sketch the proof of (45). Clearly, $B_n^\pm(t) = \frac{1}{2} \left(B_n(t) \pm \text{Sgn}B_n(t) \right)$ and similarly $D_n^\pm(t) = \frac{1}{2} \left(D_n(t) \pm \text{Sgn}D_n(t) \right)$. The proof follows by combining the above two with Theorem 10, Theorem 14 and (41). □

6.3. Identities involving $\text{BDes}_n^{D,\pm}(t)$ and $\text{BDes}_n^{B-D,\pm}(t)$

We wish to prove similar identities involving $\text{BDes}_n^{D,\pm}(t)$. Combining Theorem 19 with Theorem 28, we get the following Carlitz type result.

Theorem 40. *For positive integers n ,*

$$(47) \quad \frac{\text{BDes}_n^{D,\pm}(t)}{(1-t)^{n+1}} = \begin{cases} \sum_{k \geq 0} \frac{1}{2} \left(\frac{1}{2} \left((2k+1)^n + 1 \right) \pm 1 \right) t^k & \text{if } n \text{ is even,} \\ \sum_{k \geq 0} \frac{1}{2} \left(\frac{1}{2} \left((2k+1)^n + 1 \right) \pm (k+1) \right) t^k & \text{if } n \text{ is odd.} \end{cases}$$

$$(48) \quad \frac{\text{BDes}_n^{B-D, \pm}(t)}{(1-t)^{n+1}} = \begin{cases} \sum_{k \geq 0} \frac{1}{2} \left(\frac{1}{2} ((2k+1)^n - 1) \right) t^k & \text{if } n \text{ is even,} \\ \sum_{k \geq 0} \frac{1}{2} \left(\frac{1}{2} ((2k+1)^n - 1) \pm k \right) t^k & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We clearly have

$$\begin{aligned} \frac{B_n^\pm(t)}{(1-t)^{n+1}} &= \sum_{k \geq 0} \frac{1}{2} \left((2k+1)^n \pm 1 \right) t^k \\ \frac{(1-t)^{n-1}}{(1-t)^{n+1}} &= \sum_{k \geq 0} (k+1) t^k \quad \text{and} \quad \frac{(1-t)^n}{(1-t)^{n+1}} = \sum_{k \geq 0} t^k \end{aligned}$$

Combining the above three with (35) and (37) completes the proof. \square

Theorem 40 clearly refines the Worpitzky type identity [5, Proposition 4.6] proved by Borowiec and Młotkowski.

In \mathfrak{S}_n and \mathfrak{B}_n , bivariate versions of Carlitz’s identity are known with respect to the major index and descent statistics see [8, 1, 9]. It would be interesting to see if any counterparts can be given for \mathfrak{S}_n^\pm and \mathfrak{B}_n^\pm .

The preprint [2] by Bagno, Garber and Novick gives combinatorial proofs of Worpitzky’s identities, which are a rephrasement of Carlitz’s identities. It is easy to recast all Carlitz type results of this Section as Worpitzky identities. On the lines of [2], it would be interesting to see if combinatorial proofs of our Worpitzky’s identities can be given.

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HIRANYA KISHORE DEY
DEPARTMENT OF MATHEMATICS
IIT BOMBAY
POWAI, MUMBAI 400076
INDIA
E-mail address: hkdey@math.iitb.ac.in
URL: sites.google.com/view/hiranyadey/home

SIVARAMAKRISHNAN SIVASUBRAMANIAN
DEPARTMENT OF MATHEMATICS
IIT BOMBAY
POWAI, MUMBAI 400076
INDIA
E-mail address: krishnan@math.iitb.ac.in
URL: <http://www.math.iitb.ac.in/~krishnan/>

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