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The 2-Steiner distance matrix of a tree

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Let T be a tree with vertex set $V(T) = \{1, 2, \ldots, n\}$. The Steiner distance of a subset $S \subseteq V(T)$ of vertices of T is defined to be the number of edges in a smallest connected subtree of T that contains all the vertices of S. The k-Steiner distance matrix $\mathfrak{D}_k(T)$ of T is the $\binom{n}{k} \times \binom{n}{k}$ matrix whose rows and columns are indexed by subsets of vertices of size k. The entry in the row indexed by P and column indexed by Q is equal to Steiner distance of $P \cup Q$. We consider the case when $k = 2$ and show that $\text{rank}(\mathfrak{D}_2(T)) = 2n - p - 1$ where p is the number of pendant vertices (or leaves) in T. We construct a basis $\mathfrak{B}$ for the row space of $\mathfrak{D}_2(T)$ and obtain a formula for the inverse of the nonsingular square submatrix $\mathfrak{D} = \mathfrak{D}_2(T)[\mathfrak{B}, \mathfrak{B}]$. We also compute the determinant of $\mathfrak{D}$ and show that its absolute value is independent of the structure of T and apply it to obtain the inertia of $\mathfrak{D}_2(T)$. Lastly, we determine the spectrum of 2-Steiner distance matrix of the star tree.

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1. Introduction

Let $T = (V, E)$ be a tree on n vertices. Associated to T are several matrices whose entries are functions of distance between the vertices. The most well studied of these is the $n \times n$ distance matrix D_T of T whose rows and columns are indexed by vertices of T. The (i, j)-th entry of D_T is $d_{i,j}$, the distance between vertex i and vertex j in T. About fifty years ago, Graham and Pollak in [18] showed that the determinant of D_T is independent of the structure of the tree T and only depends on n, the number of vertices in T. This result has inspired several generalizations (see for example [3–8,16,23]). In this paper, we study the 2-Steiner distance matrix $\mathfrak{D}_2(T)$ of a tree T.

Let $[n] = \{1, 2, \ldots, n\}$ be the vertex set of a tree T and let $P, Q \subseteq [n]$. The Steiner distance $d_{\mathrm{ST}}(P, Q)$ between the subsets P and Q is the number of edges in the smallest connected subtree of T that contains all the vertices of $P \cup Q$. For a connected graph G, when $P = \{x\}$ and $Q = \{y\}$ are singleton sets, then we clearly have

$$d_{\mathrm{ST}}(P, Q) = d_G(x, y) \tag{1}$$

where $d_G(x, y)$ is the usual distance between the vertices x and y in G.

We refer the reader to Mao's paper [21] for a survey of results about Steiner distance in graphs. In related work, Gutman in [19] connected the Weiner index of a tree with a related function of Steiner distances involving three points of the tree T. Dankelmann, Oellermann and Swart in [14] gave bounds for the average Steiner distance in connected graphs. DeVos, McDonald and Pivotto in [15] gave results about packing Steiner trees which span a fixed subset T of the vertex set of a graph G. Thus the Steiner distance has been studied in graph theoretic contexts, but not much from an algebraic perspective.

Azimi, Bapat and Goel in [1] determined the rank, determinant and the inverse of the 2-Steiner distance matrix of caterpillar trees. We take another step in studying algebraic properties of the Steiner distance but consider the case when the graph is an arbitrary tree T and when both P, Q are subsets of $V(T)$ of size two. We term this $\binom{n}{2} \times \binom{n}{2}$ matrix as the **2-Steiner distance matrix of** T and denote it as $\mathfrak{D}_2(T)$. Thus, the rows and columns of $\mathfrak{D}_2(T)$ are indexed by pairs of vertices $P = \{x, y\}$ and $Q = \{w, z\}$ and the entry of $\mathfrak{D}_2(T)$ in the row indexed by P and column indexed by Q equals $d_{\mathrm{ST}}(P, Q)$. The subscript 2 indicates that rows and columns of $\mathfrak{D}_2(T)$ are indexed by subsets of $V(T)$ of size 2.

If we were to use a similar notation, we would denote by $\mathfrak{D}_1(T)$, the $n \times n$ Steiner distance matrix that has both rows and columns indexed by subsets of $V(T)$ of size 1. From (1), it is clear that $\mathfrak{D}_1(T) = D_T$. We thus think of $\mathfrak{D}_2(T)$ as the next higher object in the hierarchy of matrices $\mathfrak{D}_i(T)$ as i increases from 1. This gives us some motivation to study $\mathfrak{D}_2(T)$.

Another reason to study $\mathfrak{D}_2(T)$ comes from the four point condition for distances in trees. Buneman in [10] showed that distances in trees satisfy the *four point condition* (denoted as 4PC henceforth). The 4PC states that for any four vertices w, x, y and z in

T, among the three terms $d_{w,x} + d_{y,z}$, $d_{w,y} + d_{x,z}$ and $d_{w,z} + d_{x,y}$, the maximum value equals the second maximum value. Inspired by this, Bapat and Sivasubramanian in [9] studied the $\binom{n}{2} \times \binom{n}{2}$ matrix M_T whose rows and columns are indexed by pairs of distinct vertices. The entry in the row indexed by $\{w, x\}$ and column $\{y, z\}$ of M_T equals the minimum value among the three terms $d_{w,x} + d_{y,z}$, $d_{w,y} + d_{x,z}$ and $d_{w,z} + d_{x,y}$. They determined the rank and among other results, also gave the Smith Normal Form of M_T. We term the matrix M_T as the *min 4PC matrix*.

Inspired by the definition of the matrix M_T, one can define another $\binom{n}{2} \times \binom{n}{2}$ matrix N_T, again with rows and columns indexed pairs of vertices of T. The entry in the row indexed by $\{w, x\}$ and column $\{y, z\}$ of N_T equals the maximum value among the three terms $d_{w,x} + d_{y,z}$, $d_{w,y} + d_{x,z}$ and $d_{w,z} + d_{x,y}$. We term the matrix N_T as the *max 4PC matrix*.

In Section 2, we show that the 2-Steiner distance matrix $\mathfrak{D}_2(T)$ is the arithmetic mean of the two matrices M_T and N_T. That is, it is the average of the min 4PC and the max 4PC matrices. Thus, we also use the term *average 4PC matrix* for the matrix $\mathfrak{D}_2(T)$. It is interesting to note that among the min 4PC, the max 4PC and the average 4PC matrix, we only know results about the min 4PC matrix. This gives another motivation to study the average 4PC matrix or $\mathfrak{D}_2(T)$.

As we will see, several similarities and differences exist among our results on the 2-Steiner distance matrix $\mathfrak{D}_2(T)$ and results about the distance matrix D_T of T. Our first result is the following.

Theorem 1. *Let T be a tree on n vertices having p pendant vertices (or leaves). Then,*

$$\mathrm{rank}(\mathfrak{D}_2(T)) = 2n - p - 1.$$

We give a proof of Theorem 1 in Section 3. While determining the rank of $\mathfrak{D}_2(T)$, we explicitly give a basis $\mathfrak{B}$ for the row space of $\mathfrak{D}_2(T)$. Graham and Lovasz in [17] gave an explicit formula for the inverse of D_T. They showed that the inverse of D_T is the sum of a scaled Laplacian L_T of T and a rank one matrix. This result has the following implication: for a tree T, a g-inverse of L_T is $-(1/2) \times D_T$ (see [2, Lemma 9.11]). Let $\mathfrak{D}_2(T)[\mathfrak{B}, \mathfrak{B}]$ denote the matrix induced by $\mathfrak{D}_2(T)$ when restricted to the rows and columns in $\mathfrak{B}$. In Section 4 we determine the inverse of $\mathfrak{D}_2(T)[\mathfrak{B}, \mathfrak{B}]$ and find a similar expression as a sum of a Laplacian type matrix M (with zero row and column sums) and a rank-one matrix. Our result is the following.

Theorem 2. *Let T be a tree with n vertices. Let T have an ordered basis $\mathfrak{B}$ for its 2-Steiner distance matrix $\mathfrak{D}_2(T)$. Let $\mathfrak{D} = \mathfrak{D}_2[\mathfrak{B}, \mathfrak{B}]$ and M be the matrix defined in Subsection 4.1. Let $\mathbf{v}$ be the vector defined in (5). Then,*

$$\mathfrak{D}^{-1} = -M + \frac{1}{n-1}\mathbf{v}\mathbf{v}^t$$

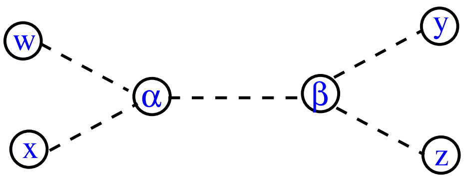

Fig. 1. Dotted lines are paths that may contain several intermediate vertices.

A similar g-inverse interpretation involving the matrices $\mathfrak{D}$ and M is given in Remark 14. Using Theorem 2 in Section 5, we evaluate the determinant of the matrix $\mathfrak{D}_2(T)[\mathfrak{B}, \mathfrak{B}]$. More precisely, we show the following.

Theorem 3. *Let T be a tree on n vertices having p pendant vertices. Let $\mathfrak{B}$ be our basis and let $\mathfrak{D} = \mathfrak{D}_2[\mathfrak{B}, \mathfrak{B}]$. Then, $\det(\mathfrak{D}) = (-1)^p(n-1)$.*

Theorem 3 shows that the absolute value of the determinant of $\mathfrak{D}_2[\mathfrak{B}, \mathfrak{B}]$ depends only on the number of vertices n in T and is independent of the structure of T. The sign of the determinant, however, depends on the number of leaves in T. In Theorem 18 we show that $\mathfrak{D}_2(T)$ has exactly one positive eigenvalue, $2n-p-2$ negative eigenvalues and nullity $\binom{n}{2} - (2n-p-1)$. The nullity of $\mathfrak{D}_2(T)$ is clearly the multiplicity of 0 as an eigenvalue and Theorem 18 clearly implies that the matrix $\mathfrak{D}_2(T)[\mathfrak{B}, \mathfrak{B}]$ has 1 positive eigenvalue and $2n-p-2$ negative eigenvalues. This result is similar to the result that the distance matrix D_T of a tree T on n vertices has exactly one positive eigenvalue and $n-1$ negative eigenvalues (see for example [2, Lemma 9.15]). Finally, when the tree is the star tree on n vertices Star_n, in Theorem 19 we determine all the eigenvalues of $\mathfrak{D}_2(\mathrm{Star}_n)$.

2. Some preliminaries

We start by showing that the 2-Steiner distance matrix is the arithmetic mean of the min 4PC and the max 4PC matrices.

Lemma 4. *Let T be a tree with min 4PC matrix M_T, max 4PC matrix N_T and 2-Steiner distance matrix $\mathfrak{D}_2(T)$. Then, $\mathfrak{D}_2(T) = \frac{1}{2}\big(M_T + N_T\big)$.*

Proof. Given 4 distinct vertices w, x, y and z in T, they uniquely determine two vertices α and β as follows. Let $P_{w,y}$ be the unique path in T between the vertices w and y. The vertex α is defined to be the unique vertex on $P_{w,y}$ that is closest to vertex x. Likewise, define the vertex β to be the unique vertex on $P_{w,y}$ that is closest to vertex z.

A generic instance of this situation is illustrated in Fig. 1, where the dotted lines are paths which might contain intermediate vertices. It must be noted that instances where x, z and hence α, β are interchanged are possible. It is further possible that $\alpha = x$, that $\beta = z$ or that $\alpha = \beta$. Thus, through six vertices are drawn in the picture, depending on the tree and the four vertices w, x, y and z, there might be fewer vertices. However, our proof will not depend on these cases.

From Fig. 1, it is clear that the entry in the max 4PC matrix corresponding to the row indexed by $\{w, x\}$ and column indexed by $\{y, z\}$ is $s = d_{w,y} + d_{x,z}$ and the corresponding entry in the min 4PC matrix is $t = d_{w,x} + d_{y,z}$. Clearly, $s = t + 2d_{\alpha,\beta}$ where $d_{\alpha,\beta}$ is the distance between vertices α and β in T. The value of t will change if x, z and α, β are interchanged. Nonetheless it is easy to slightly alter the argument and prove our result. We do not write all possible cases in the interest of brevity.

On the other hand, the entry of $\mathfrak{D}_2(T)$ indexed by row $\{w, x\}$ and column $\{y, z\}$ is the least number of edges in a connected subtree containing the four vertices w, x, y and z. This is clearly $t + d_{\alpha,\beta}$ which equals $\frac{1}{2}\Big(s + t\Big)$. This argument is clearly independent of the aforementioned possibilities like $\alpha = x$ and so on. Since this is true for each entry, we get $\mathfrak{D}_2(T) = \frac{1}{2}\big(M_T + N_T\big)$, completing the proof. $\square$

3. Rank of $\mathfrak{D}_2(T)$

For a tree T on n vertices, consider the $\binom{n}{2} \times \binom{n}{2}$ matrix $\mathfrak{D}_2(T)$ with rows and columns indexed by subsets of $V(T)$ of size 2. We will use induction on n, the number of vertices in T. Every tree has at least two leaves, and deletion of a leaf from a tree gives a smaller tree. A leaf vertex is also called a pendant vertex and we use the two terms interchangeably.

Proof of Theorem 1. We will induct on n, the number of vertices with the base case being when $n = 3$. In this case, there is only one tree T with $n = 3$ having $p = 2$ leaves. It is easy to check that $\text{rank}(\mathfrak{D}_2(T)) = 3$ in this case.

Let T be a tree on $n+1$ vertices. We denote degree of a vertex v in T as $\deg_T(v)$. We break the proof into two cases when T has a leaf ℓ whose unique neighbouring vertex v has $\deg_T(v) \geq 3$ and when all leaves ℓ of T are connected to vertices v with $\deg_T(v) = 2$. We write size 2 subsets of $[n]$ as i, j with $i < j$ and denote the row (and column) of $\mathfrak{D}_2(T)$ indexed by i, j as $r_{i,j}$ (and $c_{i,j}$ respectively).

Case 1 (when there exists a leaf vertex ℓ whose neighbour v has $\deg_T(v) \geq 3$): By reordering vertices, let $\ell = 1$ be such a leaf vertex with unique neighbour 2 and let $\deg_T(2) \geq 3$ with vertices $1, \alpha, \beta$ being other neighbours of 2. Let $T' = T - \{1\}$ and let $\mathfrak{D}_2(T)$ be the 2-Steiner distance matrix with rows and columns indexed by subsets of size 2 in lexicographic order. Note that the submatrix of $\mathfrak{D}_2(T)$ induced on size 2 subsets of $\{2, 3, \ldots, n+1\}$ is the same as $\mathfrak{D}_2(T')$.

We claim that $r_{1,2}$, the row of $\mathfrak{D}_2(T)$ indexed by $1, 2$ is the only row linearly independent of the rows of $\mathfrak{D}_2(T')$. For $x \geq 3$, we will make $r_{1,x} = \mathbf{0}^t$ by performing elementary row operations (where $\mathbf{v}^t$ denotes the transpose of vector $\mathbf{v}$). Let $1, x$ be a row of $\mathfrak{D}_2(T)$ with $x \neq 2$. Let $d = d_T(1, x)$ be the classical distance between vertex 1 and vertex x in T. Let the unique path from 1 to x pass via the vertex α (see Fig. 2).

Note that in T, we have $d_T(\beta, x) = d$. That is, the distance between β and x is also d. The argument below does not need $\deg_T(2) = 3$ but only needs $\deg_T(2) \geq 3$. We only

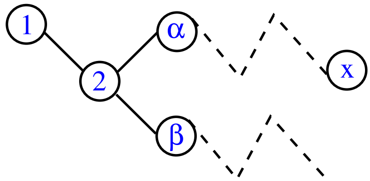

Fig. 2. Case 1 of proof.

need the existence of a neighbour β of 2 which is NOT on the path from 1 to x. We have assumed that $\beta < x$ merely to write the row corresponding to the set $\{\beta, x\}$ as $r_{\beta,x}$. If $\beta > x$, the argument will work with $r_{\beta,x}$ replaced by $r_{x,\beta}$. We claim that

$$r_{1,x} + r_{2,\beta} = r_{1,2} + r_{\beta,x}. \tag{2}$$

We show (2) for each column. For a row indexed by $\{x, y\}$ and column indexed by $\{u, v\}$, let the entry of $\mathfrak{D}_2(T)$ at this row and column be denoted as $a_{x,y|u,v}$. For the column indexed by $\{u, v\}$, the LHS of (2) is number of edges in the disjoint union of the two minimum spanning subtrees of T containing the vertices $1, x, u, v$ and $2, \beta, u, v$. Since this (multi)set of edges also connects the vertices $1, 2, u, v$ and β, x, u, v, we have $a_{1,2|u,v} + a_{\beta,x|u,v} \leq a_{1,x|u,v} + a_{2,\beta|u,v}$. Reversing the argument, we get $a_{1,x|u,v} + a_{2,\beta|u,v} \leq a_{1,2|u,v} + a_{\beta,x|u,v}$. Since we chose arbitrary columns $\{u, v\}$, the proof of (2) is complete.

For $3 \leq x \leq n+1$, perform the elementary row operation $r_{1,x} = r_{1,x} + r_{2,\beta} - r_{1,2} - r_{\beta,x}$. By (2), after this operation we will have $r_{1,x} = \mathbf{0}^t$. Note that in the relations above, $r_{1,x}$ (for $x \geq 3$) only depends on rows $r_{1,2}$ and $r_{x,y}$ where $x, y \in V(T')$. Let $\mathcal{P}_{x,y}$ be the unique path in T from vertex x to y and let $A \uplus B$ denote the multiset union of the sets A and B (where elements can have multiple copies). Underlying the above row operation, we have the following multiset union relation between paths: $\mathcal{P}_{1,2} \uplus \mathcal{P}_{\beta,x} = \mathcal{P}_{1,x} \uplus \mathcal{P}_{2,\beta}$. Since the matrix $\mathfrak{D}_2(T)$ is symmetric, the whole discussion works with rows replaced by columns. Thus, column counterparts of (2) are true. Using these, for $3 \leq x \leq n+1$, we can make columns $c_{1,x}$ zero.

After doing these row operations, the matrix $\mathfrak{D}_2(T)$ will be row and column equivalent to a matrix which has the following form. We describe this equivalent matrix in block form with rows and columns partitioned to have size $1, n-1$ and $\binom{n}{2}$ respectively. Further, with respect to this partitioning, the rows and columns are indexed by the subsets $\{1, 2\}$, $\{1, x\}$ for $3 \leq x \leq n+1$ and $\{a, b\} : a, b \in V(T')$. We use $\mathbf{0}$ to denote a zero vector or a zero matrix of appropriate dimensions. After performing these elementary row and column operations $r_{1,2}$ gets transformed to $[1, \mathbf{0}^t, \mathbf{b}^t]$ and likewise $c_{1,2}$ gets transformed to $[1, \mathbf{0}, \mathbf{b}]$. Here, the 1 is a scalar, the $\mathbf{0}$ is an $(n-1)$-dimensional vector and $\mathbf{b}$ is an $\binom{n}{2}$ dimensional vector. Note that the entry corresponding to row $\{1, 2\}$ and column $\{1, 2\}$ was 1 and remains unchanged by these operations. For matrices of identical dimensions, let $A \sim B$ denote that the matrices A and B are row and column equivalent. With this notation, we get that $\mathfrak{D}_2(T) \sim M$ where

$$M = \left(\begin{array}{c|c|c} 1 & \mathbf{0}^t & \mathbf{b}^t \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{b} & \mathbf{0} & \mathfrak{D}_2(T') \end{array} \right). \tag{3}$$

We can delete the zero rows and columns of this equivalent matrix M for rank arguments. For brevity, denote $\mathfrak{D}_2(T')$ as A and recall $\mathbf{b}$ is the $\binom{n}{2}$-dimensional vector appearing in (3). Let $(A|\mathbf{b})$ denote the matrix obtained by augmenting A with the column vector $\mathbf{b}$.

Claim. With the above notation, we have $\text{rank}(A|\mathbf{b}) = \text{rank}(A)$.

Proof of Claim. Consider the matrix $(A|\mathbf{b})$. This is an $\binom{n}{2} \times \left[\binom{n}{2} + 1\right]$ sized matrix with rows indexed by unordered pairs $\{x, y\}$ with $x, y \in V(T')$ where $T' = T - \{1\}$. By induction applied to T', we can get a basis B' comprising of $2n - p' - 1$ pairs, where p' is the number of leaves of T'.

Consider a row of A indexed by $\{x, y\} \notin B'$. We can perform elementary row operations (involving four terms) as in either of the cases. By this, the portion of $r_{x,y}$ in $(A|b)$ corresponding to the columns in A will become zero. We claim that by performing this operation, even the portion of $r_{x,y}$ corresponding to column $\mathbf{b}$ will have zero entries (that is $r_{x,y}$ will become the zero row). To see this, note that row $r_{x,y}$ is made the zero row in A by performing an elementary row operation of the type

$$r_{x,y} = r_{x,y} - r_{u,y} + r_{u,v} - r_{x,v}. \tag{4}$$

Let vertex $2 \in T'$ have neighbour $s \in T'$. In particular, for the column indexed by $\{2, s\}$, we have $a_{x,y|2,s} + a_{u,v|2,s} = a_{u,y|2,s} + a_{x,v|2,s}$. Also recall that we have the smallest (wrt number of edges) subtrees $T_1 \subset T'$ containing $x, y, 2, s$, T_2 containing $u, v, 2, s$, T_3 containing $u, y, 2, s$ and T_4 containing $x, v, 2, s$. The relation (4) means that as multisets $|T_1 \uplus T_2| = |T_3 \uplus T_4|$. Adding the edge $e = \{1, 2\}$ to each T_i for $1 \leq i \leq 4$, we get the same relation in T and hence the entry in $r_{x,y}$ indexed by the column in $\mathbf{b}$ also becomes zero. Hence, after performing these operations, all zero rows in A will also be zero rows in $(A|\mathbf{b})$. Thus, $\text{rank}(A) = \text{rank}(A|\mathbf{b})$, completing the proof of the claim. $\square$

From (3), we have $\mathfrak{D}_2(T) \sim M$. As A is symmetric, rephrasing the claim, $\mathbf{b}^t$ is an element of the rowspace of A (where $A = \mathfrak{D}_2(T')$). Denote the elements of $\mathbf{b}$ as $b_{x,y}$ where $x, y \in T'$. After deleting the zero rows and columns of M, perform the row operations $r_{x,y} = r_{x,y} - b_{x,y} r_{1,2}$. After these operations, we will have the first column of M as e_1 where $e_1 = [1, 0, 0, \ldots, 0]$. Let $r'_{x,y}$ denote the row of A indexed by the pair of vertices $x, y \in T'$. One can check that $r'_{x,y}$ is the row $r_{x,y}$ of M with its first entry omitted. After performing these operations, the matrix A will have corresponding rows $r'_{x,y} - b_{x,y}\mathbf{b}$. Let A' denote this modified matrix A. Since $\mathbf{b}$ is in the row space of A, $\text{rank}(A') = \text{rank}(A)$. Further, we have a 1 entry in $r_{1,2}$ outside A'. Thus, $\text{rank}(\mathfrak{D}_2(T)) = \text{rank}(\mathfrak{D}_2(T')) + 1$, completing the proof.

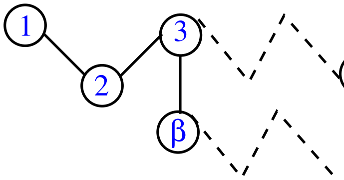

Fig. 3. Illustrating Case 2.

Case 2: (when all leaf vertices have degree 2 neighbours) We can relabel the vertices of T such that vertex 1 is a leaf adjacent to vertex 2. As $\deg_T(2) = 2$, we further assume after relabelling that the neighbours of vertex 2 are vertices $1, 3$. This is illustrated in Fig. 3.

Recall that $\mathcal{P}_{x,y}$ is the unique path in T from vertex x to y. Then, irrespective of the location of the vertex x, it is easy to see that $\mathcal{P}_{1,x} \uplus \mathcal{P}_{2,3} = \mathcal{P}_{2,x} \uplus \mathcal{P}_{1,3}$ where $A \uplus B$ is the disjoint union of the sets A and B. When $x \geq 4$, this relation in linear algebraic terms means that performing the elementary row operation $r_{1,x} = r_{1,x} - r_{2,x} - r_{1,3} + r_{2,3}$ results in $r_{1,x} = \mathbf{0}^t$.

Doing the same operations to columns will give us two non-zero columns (the modified versions of columns $c_{1,2}$ and $c_{1,3}$). If we arrange the rows and columns of $\mathfrak{D}_2(T)$ in lexicographic order, the first two rows and columns will be indexed by $\{1, 2\}$ and $\{1, 3\}$ respectively. Partitioning the rows into the first two rows, the next $(n - 2)$ rows indexed by $\{1, x\}$ for $3 \leq x \leq n+1$ and rows in $T' = T - \{1\}$, we get the following row equivalent matrix $M = \left(\begin{array}{cc|c|c} 1 & 2 & \mathbf{0}^t & \mathbf{u}^t \\ 2 & 2 & \mathbf{0}^t & \mathbf{u}^t \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}^t \\ \hline \mathbf{u} & \mathbf{u} & \mathbf{0} & \mathfrak{D}_2(T') \end{array}\right)$.

Now performing $r_{1,2} = r_{1,2} - r_{1,3}$ and $c_{1,2} = c_{1,2} - c_{1,3}$ gives another equivalent matrix $N = \left(\begin{array}{cc|c|c} -1 & 0 & \mathbf{0}^t & \mathbf{0}^t \\ 0 & 2 & \mathbf{0}^t & \mathbf{u}^t \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}^t \\ \hline \mathbf{0} & \mathbf{u} & \mathbf{0} & \mathfrak{D}_2(T') \end{array}\right)$. The first row is clearly linearly independent of the remaining rows and arguing as in case 1, we get that rank $\left(\begin{array}{c|c|c} 2 & \mathbf{0}^t & \mathbf{u}^t \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0}^t \\ \hline \mathbf{u} & \mathbf{0} & \mathfrak{D}_2(T') \end{array}\right) =$ $\mathrm{rank}(\mathfrak{D}_2(T')) + 1$. Thus, $\mathrm{rank}(\mathfrak{D}_2(T)) = \mathrm{rank}(\mathfrak{D}_2(T')) + 2$. As T' has p leaves, by induction, $\mathrm{rank}(\mathfrak{D}_2(T')) = 2n - p - 1$. As T also has p leaves, we get $\mathrm{rank}(\mathfrak{D}_2(T)) = (2n - p - 1) + 2 = 2(n + 1) - p - 1$, completing the proof in this case. $\square$

Remark 5. The proof of Theorem 1 shows that a basis for the (row space or column space) of $\mathfrak{D}_2(T)$ can be constructed inductively. If T is a tree on $n = 3$ vertices, $\mathfrak{D}_2(T)$ has full row rank and so all rows are needed to form a basis of $\mathrm{RowSpace}(\mathfrak{D}_2(T))$. Let $\mathfrak{B}(T)$ denote a basis of $\mathrm{RowSpace}(\mathfrak{D}_2(T))$.

If $n \geq 4$, check if there is a leaf ℓ in T attached to a vertex with degree strictly more than 2. Let $T' = T - \{\ell\}$. Relabel the vertices of T such that the leaf ℓ gets the label 1 and its neighbour gets the label 2. Extend row vectors in $\mathfrak{B}(T')$ to $\binom{n}{2}$ dimensional space by adding columns corresponding to pairs $\{1, x\}$ for $2 \leq x \leq n$ and computing the Steiner distance of appropriate entries. Then, we have $\mathfrak{B}(T) = \mathfrak{B}(T') \cup \{r_{1,2}\}$.

If no such leaf vertex exists in T, then all leaf vertices are adjacent to vertices with degree 2. Relabel the vertices such that 1 is a leaf vertex adjacent to vertex 2 and let the neighbours of vertex 2 be vertices $1, 3$. Again let $T' = T - \{1\}$. As done above, extend row vectors in $\mathfrak{B}(T')$ to $\binom{n}{2}$ dimensional space. Then, we have $\mathfrak{B}(T) = \mathfrak{B}(T') \cup \{r_{1,2}, r_{1,3}\}$.

Remark 6. From the inductive construction of our basis given in Remark 5, it follows that all two subsets of vertices corresponding to edges of T will be in $\mathfrak{B}(T)$. Thus, if $\{x, y\} \in E(T)$, then $\{x, y\} \in \mathfrak{B}(T)$. For each non-pendant vertex w, we need to choose a pair $\{u, v\}$ with $\{u, w\}, \{v, w\} \in E(T)$. Since the number of blocks (maximally 2-connected subgraph) in the line graph $\mathrm{LG}(T)$ equals $n-p$, choosing such a pair $f = \{u, v\}$ is equivalent to choosing $e_1 = \{u, w\}, e_2 = \{v, w\}$ where $e_1, e_2 \in E(T)$ and $f = e_1 \ominus e_2$. This can be considered as choosing an edge $f' = \{e_1, e_2\}$ where $f' \in E(\mathrm{LG}(T))$. Thus all elements in our basis $\mathfrak{B}(T)$ have pairs of vertices which are at distance at most two in T. We will use the notation f and f' that are given in Definition 7.

Definition 7. We will use this notation throughout this work. If $f \in \mathfrak{B}$ and $f \notin E(T)$, then there clearly exists $e_r, e_s \in E(T)$ such that $f = e_r \ominus e_s$. Here, we think of e_r and e_s as sets of vertices and $\ominus$ is the symmetric difference of these two sets. We denote the corresponding edge in $E(\mathrm{LG}(T))$ as $f' = \{e_r, e_s\}$.

Remark 8. A poset, called the *generalized tree shift poset* denoted GTS_n on the set of trees on n vertices was defined by Csikvari (see [12] and [13]) where several properties were shown to be monotonic as one goes up on GTS_n. It is known that GTS_n is a graded poset with number of leaves (or pendant vertices) as its height parameter. Theorem 1 shows that the rank of $\mathfrak{D}_2(T)$ decreases as one goes up GTS_n and hence adds another property that is monotonic on elements of the poset GTS_n.

4. Inverse of $\mathfrak{D}_2(T)[\mathfrak{B}, \mathfrak{B}]$

Let $\mathfrak{B} = \mathfrak{B}(T)$ be a basis for the row space of $\mathfrak{D}_2(T)$. Thus $\mathfrak{B}$ is a set of 2-subsets of $V(T)$. In this section, we obtain a formula for the inverse of the square matrix $\mathfrak{D} = \mathfrak{D}_2(T)[\mathfrak{B}, \mathfrak{B}]$. We need a few preliminaries before moving on to our proof. Let T be a tree on n vertices with p pendant vertices and let $\mathrm{LG}(T)$ be its line graph. Let $B_1, \ldots, B_{n-p}$ be the blocks of $\mathrm{LG}(T)$ with $|B_i| = b_i$ for $1 \leq i \leq n-p$. Indeed each B_i is a clique with b_i vertices. As mentioned in Remark 6, we can choose a basis $\mathfrak{B} = \{e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-p}\}$ with $e_i \in V(\mathrm{LG}(T))$ and f_i is the symmetric difference

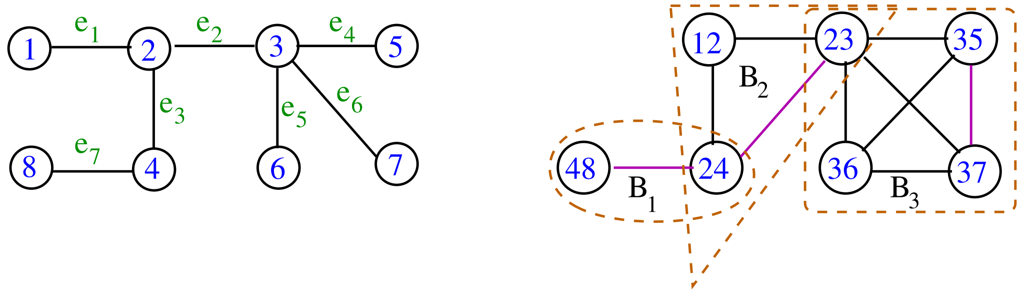

Fig. 4. A tree T and its line graph $\mathrm{LG}(T)$.

of endpoints of edge $f'_i \in B_i$ in $\mathrm{LG}(T)$. Define the $2n-p-1$ dimensional column vector $\mathbf{v}$ (that depends on the basis $\mathfrak{B}$) as follows

$$v_{f_i} = b_i - 1, \text{ where } f'_i \in B_i, \quad \text{and} \quad v_{e_i} = 1 - \sum_{b_j : e_i \in f'_j} (b_j - 1). \tag{5}$$

Recall that $\mathbf{1}$ is the column vector of appropriate size all of whose entries are 1s. We give some properties of the vector $\mathbf{v}$.

Lemma 9. *Let T be a tree on n vertices having p pendant vertices. Let $\mathfrak{B}$ be a basis of $\mathfrak{D}_2(T)$ with $\mathfrak{B} = \{e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-p}\}$ and let $\mathbf{v}$ be the column vector described above. Then, $\mathbf{1}^t\mathbf{v} = 1$.*

Proof.

$$\begin{aligned}
\mathbf{1^t v} &= \sum_{i=1}^{n-1} v_{e_i} + \sum_{i=1}^{n-p} v_{f_i} = \sum_{i=1}^{n-1} \Big(1 - \sum_{b_j : e_i \in f'_j} (b_j - 1)\Big) + \sum_{i=1}^{n-p} (b_i - 1) \\
&= \sum_{i=1}^{n-1} 1 - \sum_{i=1}^{n-1} \sum_{b_j : e_i \in f'_j} (b_j - 1) + \sum_{i=1}^{n-p} b_i - \sum_{i=1}^{n-p} 1 \\
&= \sum_{i=1}^{n-1} 1 - 2\sum_{i=1}^{n-p} b_i + 2\sum_{i=1}^{n-p} 1 + \sum_{i=1}^{n-p} b_i - \sum_{i=1}^{n-p} 1 = \sum_{i=1}^{n-1} 1 - \sum_{i=1}^{n-p} b_i + \sum_{i=1}^{n-p} 1 \\
&= (n-1) - \Big(2(n-1) - p\Big) + (n-p) = 1, \text{ completing the proof.} \qquad \square
\end{aligned}$$

Example 10. Consider the tree T and its line graph $\mathrm{LG}(T)$ illustrated in Fig. 4. The vector $\mathbf{v}$ depends on the ordered basis $\mathfrak{B} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, f_1, f_2, f_3\}$ where $f_1 = \{2, 8\}$, $f_2 = \{3, 4\}$ and $f_3 = \{5, 7\}$. It can be checked that $\mathbf{v}^t = [1, -1, -2, -2, 1, -2, 0,\ 1, 2, 3]$. With this, it is easy to see that $\mathbf{1}^t\mathbf{v} = \sum v_i = 1$.

We move on to our next property of the vector $\mathbf{v}$.

Lemma 11. *Let T be a tree on n vertices having p pendant vertices. Let $\mathfrak{B}$ be an ordered basis of $\mathfrak{D}_2(T)$ with $\mathfrak{B} = \{e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-p}\}$. Then,*

$$\mathfrak{D}_2(T)[\mathfrak{B}, \mathfrak{B}]\mathbf{v} = (n-1)\mathbf{1}.$$

Proof. We prove the result by induction on n. We check the result when $n = 3$. In this case, there is a unique tree T with $\mathfrak{D} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$. The matrix has full rank and we clearly have $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. It is easy to verify that $\mathfrak{D}\mathbf{v} = 2 \cdot \mathbf{1}$. Let T be a tree on $n > 3$ vertices and let the hypothesis be true for any tree with less than n vertices. We will remove a leaf ℓ of T to get $T' = T - \{\ell\}$. We break our proof into two cases, based on whether there exists a leaf a in T whose unique neighbour $N(a)$ has degree at least three or not.

Case 1: When there exists a leaf vertex a such that $\deg_T(N(a)) \geq 3$: Let a be a pendant vertex of T with neighbour $N(a) = b$ and let $\deg_T(b) \geq 3$. Label the edges of T such that $e_1 = \{a, b\}$, $e_2 \cap e_3 = b$ and $f_1 = e_2 \ominus e_3$ the symmetric difference of e_2 and e_3. The choice of the edges e_2 and e_3 is arbitrary. Let $T' = T - \{a\}$ and let $\mathfrak{B}' = \{e_2, e_3, \ldots, e_{n-1}, f_1, \ldots, f_{n-p}\}$ be a basis of $\mathfrak{D}_2(T')$. For brevity, denote $\mathfrak{D}_2(T)[\mathfrak{B}, \mathfrak{B}]$ as $\mathfrak{D}$ and denote $\mathfrak{D}_2(T')[\mathfrak{B}', \mathfrak{B}']$ as $\mathfrak{D}'$. Therefore,

$$\mathfrak{D} = \left(\begin{array}{c|c} 1 & \mathbf{x}^t \\ \hline \mathbf{x} & \mathfrak{D}' \end{array}\right).$$

By induction, there exists a vector $\mathbf{v}'$ (that depends on the basis $\mathfrak{B}'$) such that $\mathfrak{D}'\mathbf{v}' = (n-2)\mathbf{1}$. Define $\mathbf{y} = \begin{pmatrix} 0 \\ \mathbf{v}' \end{pmatrix}$ and define $\mathbf{z}$ by $\mathbf{v} = \mathbf{y} + \mathbf{z}$. We have the following alternate definition of $\mathbf{z}$ as follows

$$z_{S_i} = \begin{cases} 1 & \text{if } S_i \in \{e_1, f_1\} \\ -1 & \text{if } S_i \in \{e_2, e_3\} \\ 0 & \text{otherwise} \end{cases}.$$

As $\mathbf{v} = \mathbf{y} + \mathbf{z}$, we have $\mathfrak{D}\mathbf{v} = \mathfrak{D}(\mathbf{y} + \mathbf{z})$, On the other hand, for every $S_i \in \mathfrak{B}$ we have

$$d_{\mathrm{ST}}(S_i, e_1) - d_{\mathrm{ST}}(S_i, e_2) - d_{\mathrm{ST}}(S_i, e_3) + d_{\mathrm{ST}}(S_i, f_1) = \begin{cases} 1 & \text{if } S_i \neq e_1 \\ 0 & \text{if } S_i = e_1 \end{cases}.$$

Hence $\mathfrak{D}\mathbf{y} = \begin{pmatrix} \mathbf{x}^t\mathbf{v}' \\ (n-2)\mathbf{1} \end{pmatrix}$ and $\mathfrak{D}\mathbf{z} = \begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix}$. Thus $\mathfrak{D}\mathbf{v} = \begin{pmatrix} \mathbf{x}^t\mathbf{v}' \\ (n-1)\mathbf{1} \end{pmatrix}$. We show that $\mathbf{x}^t\mathbf{v}' = n-1$. Let $e_2 = \{b, c\}$ and T_c be the connected component of $T - e_2$ that contains c.

$$\mathbf{x}^t\mathbf{v}' = \sum_{S_i \in \mathfrak{B}'} d_{\mathrm{ST}}(e_1, S_i)v'_{S_i} = \sum_{\substack{S_i \subseteq V(T_c)\\ S_i = e_2, f_1}} \Big(d_{\mathrm{ST}}(e_2, S_i) + 1\Big)v'_{S_i} + \sum_{\substack{S_i \not\subseteq V(T_c)\\ S_i \neq e_2, f_1}} d_{\mathrm{ST}}(e_2, S_i)v'_{S_i}$$

$$= \sum_{S_i \in \mathfrak{B}'} d_{\mathrm{ST}}(e_2, S_i)v'_{S_i} + \sum_{\substack{S_i \subseteq V(T_c)\\ S_i = e_2, f_1}} v'_{S_i}.$$

Since $\mathfrak{D}(T')\mathbf{v}' = (n-2)\mathbf{1}$, we have

$$\sum_{S_i \in \mathfrak{B}'} d_{\mathrm{ST}}(e_2, S_i)v'_{S_i} = n - 2. \tag{6}$$

Now, suppose that $e_2 \in B_1 \cap B_2$ and $f'_2 \in B_2$, where B_1 and B_2 are blocks of $\mathrm{LG}(T')$. Consider the tree $T'' = T_c \cup \{b\}$ and the vector $\mathbf{v}''$ depending on the basis $\mathfrak{B}'' = \mathfrak{B}''(T'') \subseteq \mathfrak{B}'$. Clearly, $v''_{S_i} = v'_{S_i}$ for every $S_i \neq e_2$, and thus

$$\sum_{\substack{S_i \subseteq V(T_c)\\ S_i = e_2, f_1}} v'_{S_i} = v'_{e_2} + v'_{f_1} + \sum_{S_i \in \mathfrak{B}'' \setminus \{e_2\}} v'_{S_i} = v'_{e_2} + v'_{f_1} + \sum_{S_i \in \mathfrak{B}'' \setminus \{e_2\}} v''_{S_i}.$$

By Lemma 9, $\sum_{S_i \in \mathfrak{B}''} v''_{S_i} = 1$. Further, $v'_{f_1} = b_1 - 1$ then

$$\sum_{\substack{S_i \subseteq V(T_c)\\ S_i = e_2, f_1}} v'_{S_i} = v'_{e_2} + (b_1 - 1) + (1 - v''_{e_2}) = v'_{e_2} - v''_{e_2} + b_1.$$

However,

$$v'_{e_2} = \begin{cases} 1 - (b_1 - 1) & \text{if } e_2 \notin f'_2 \\ 1 - (b_1 - 1) - (b_2 - 1) & \text{if } e_2 \in f'_2 \end{cases},$$

and

$$v''_{e_2} = \begin{cases} 1 & \text{if } e_2 \notin f'_2 \\ 1 - (b_2 - 1) & \text{if } e_2 \in f'_2 \end{cases}.$$

Thus in both these cases we have

$$\sum_{\substack{S_i \subseteq V(T_c)\\ S_i = e_2, f_1}} v'_{S_i} = 1. \tag{7}$$

From (6) and (7) it follows that $\mathbf{x}^t\mathbf{v}' = n - 1$. Hence $\mathfrak{D}\mathbf{v} = (n-1)\mathbf{1}$.

Case 2: When all leaf vertices a are such that $\deg_T(N(a)) = 2$: Suppose a is a pendant vertex of T with $N(a) = b$ and $\deg_T(b) = 2$. Let $e_1 = \{a, b\}$, $e_1 \cap e_2 = b$

and $f_1 = e_1 \ominus e_2$ be the symmetric difference of e_1 and e_2. Let $T' = T - \{a\}$ and $\mathfrak{B}' = \{e_2, e_3, \ldots, e_{n-1}, f_2, \ldots, f_{n-p}\}$. By interchanging rows and columns so that we have the first and second rows (and columns) indexed by e_1 and f_1, we get

$$\mathfrak{D} = \left(\begin{array}{cc|c} 1 & 2 & \mathbf{x}^t \\ 2 & 2 & \mathbf{x}^t \\ \hline \mathbf{x} & \mathbf{x} & \mathfrak{D}(T') \end{array}\right),$$

where $\mathbf{x}^t = (x_1, x_2, \ldots, x_{2n-p-3})$. Let $\mathbf{v}'$ be the vector depending on the basis $\mathfrak{B}'$ such that $\mathfrak{D}(T')\mathbf{v}' = (n-2)\mathbf{1}$. Define $\mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{v}' \end{pmatrix}$ and $\mathbf{z}$ by $\mathbf{v} = \mathbf{y} + \mathbf{z}$. It can be seen that $\mathbf{z}$ can be alternatively defined as follows

$$z_{S_i} = \begin{cases} 1 & \text{if } S_i = f_1 \\ -1 & \text{if } S_i = e_2 \\ 0 & \text{otherwise.} \end{cases}.$$

Clearly $\mathfrak{D}\mathbf{v} = \mathfrak{D}(\mathbf{y} + \mathbf{z})$, On the other hand, for every $S_i \in \mathfrak{B}$ we have

$$d_{\mathrm{ST}}(S_i, f_1) - d_{\mathrm{ST}}(S_i, e_2) = \begin{cases} 1 & \text{if } S_i \notin \{e_1, f_1\} \\ 0 & \text{if } S_i \in \{e_1, f_1\} \end{cases}.$$

Hence, $\mathfrak{D}\mathbf{y} = \begin{pmatrix} \mathbf{x}^t\mathbf{v}' \\ \mathbf{x}^t\mathbf{v}' \\ (n-2)\mathbf{1} \end{pmatrix}$ and $\mathfrak{D}\mathbf{z} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1} \end{pmatrix}$. Thus, $\mathfrak{D}\mathbf{v} = \begin{pmatrix} \mathbf{x}^t\mathbf{v}' \\ \mathbf{x}^t\mathbf{v}' \\ (n-1)\mathbf{1} \end{pmatrix}$. Clearly

$$\begin{aligned} \mathbf{x}^t\mathbf{v}' &= \sum_{S_i \in \mathfrak{B}'} d_{\mathrm{ST}}(e_1, S_i) v'_{S_i} = \sum_{S_i \in \mathfrak{B}'} \Big(d_{\mathrm{ST}}(e_2, S_i) + 1 \Big) v'_{S_i} \\ &= \sum_{S_i \in \mathfrak{B}'} d_{\mathrm{ST}}(e_2, S_i) v'_{S_i} + \sum_{S_i \in \mathfrak{B}'} v'_{S_i}. \end{aligned}$$

Using the fact that $\mathfrak{D}(T')\mathbf{v}' = (n-2)\mathbf{1}$ and Lemma 9 we get $\sum_{S_i \in \mathfrak{B}'} d_{\mathrm{ST}}(e_2, S_i) v'_{S_i} = n-2$ and $\sum_{S_i \in \mathfrak{B}'} v'_{S_i} = 1$. Thus $\mathbf{x}^t\mathbf{v}' = n-1$ and hence $\mathfrak{D}\mathbf{v} = (n-1)\mathbf{1}$, completing the proof. $\square$

4.1. A Laplacian type matrix M

For a tree T with ordered basis $\mathfrak{B} = \mathfrak{B}(T)$, we define a symmetric matrix M, with rows and columns indexed by elements of $\mathfrak{B}$ as follows. Recall that $e_i, f'_i \in B_i$ and similarly $e_j, f'_j \in B_j$. The entries $M(e_i, e_j)$, $M(e_i, f_j)$ and $M(f_i, f_j)$ are zero if $i \neq j$. On the other hand, if $e_i, e_j, f'_k \in B_k$ then define

$$M(e_i,e_j)=\begin{cases} 0 & \text{if } e_i,e_j\notin f'_k \\ b_k-2 & \text{if } e_i,e_j\in f'_k \\ -1 & \text{if } e_i\in f'_k, e_j\notin f'_k \end{cases} \quad \text{and} \quad M(e_i,f_k)=\begin{cases} 1 & \text{if } e_i\notin f'_k \\ 1-b_k & \text{if } e_i\in f'_k \end{cases}. \tag{8}$$

Define its diagonal entries so that M has zero row and column sums. Therefore

$$M(e_i,e_i)=-\sum_{S_i\in\mathfrak{B}\setminus\{e_i\}} M(e_i,S_i)$$

and

$$\begin{aligned} M(f_k,f_k) &= -\sum_{S_i\in\mathfrak{B}\setminus\{f_k\}} M(f_k,S_i) = -\sum_{e_k\in B_k} M(f_k,e_k) \\ &= -[(b_k-2)+2(1-b_k)] = b_k, \end{aligned}$$

where $f_k\in B_k$. As we will need the following later, we record it below. From the definition of M, it is clear that

$$M\mathbf{1}=\mathbf{0}. \tag{9}$$

Example 12. Let T be the tree in Fig. 4. We illustrate the definition of the entries in rows r_{e_1} and r_{f_2}. Let B_1, B_2 and B_3 be the blocks of $\mathrm{LG}(T)$ induced by $\{e_3,e_7\}, \{e_1,e_2,e_3\}$ and $\{e_2,e_4,e_5,e_6\}$ respectively. Since $e_1,e_2,e_3,f'_2\in B_2$ and $f_2=e_2\ominus e_3$ we have $M(e_1,e_2)=M(e_1,e_3)=-1$ and $M(e_1,f_2)=1$. Moreover, $M(e_1,S)=0$ for every $S\in\mathfrak{B}\setminus B_2$. Thus $r_{e_1}=[1,-1,-1,0,0,0,0,0,1,0]$. Since $f_2=e_2\ominus e_3$ we have $M(f_2,e_2)=M(f_2,e_3)=1-b_2=-2$ and $M(f_2,e_1)=1$. Thus $r_{f_2}=[1,-2,-2,0,0,0,0,0,3,0]$. One can check that the matrix M of the tree T (with a separator drawn in the rows and columns between the e_i's and the f_j's) is

$$M=\left(\begin{array}{rrrrrrr|rrr} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 3 & 1 & -1 & 0 & -1 & 0 & 0 & -2 & 1 \\ -1 & 1 & 3 & 0 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & -1 & 0 & 3 & -1 & 2 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 2 & -1 & 3 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 1 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & -3 & 1 & -3 & 0 & 0 & 0 & 4 \end{array}\right).$$

Lemma 13. *Let T be a tree on n vertices having p pendant vertices. Let $\mathfrak{B}$ be an ordered basis of T with respect to which we have $\mathfrak{D}=\mathfrak{D}_2[\mathfrak{B},\mathfrak{B}]$ and let M be the matrix defined in Subsection 4.1. Let $\mathbf{v}$ be the vector defined in (5). Then,*

$$\mathfrak{D}M + I = \mathbf{1}\mathbf{v}^t. \tag{10}$$

Proof. Let $e_i \in \{e_1, \ldots, e_{n-1}\}$. We consider two cases.

Case 1. Let e_i be solely in block B_k. Let $f_k = e_r \ominus e_s$ and $\{e_r, e_s\} = f'_k \in B_k$. If $e_i = e_r$, then for any $S_i \in \mathfrak{B}$, we have

$$\begin{aligned}(\mathfrak{D}M)_{S_i,e_i} &= (b_k - 1)d_{\mathrm{ST}}(S_i, e_i) + (1 - b_k)d_{\mathrm{ST}}(S_i, f_k) \\ &\quad + (b_k - 2)d_{\mathrm{ST}}(S_i, e_s) - \sum_{e_k \in B_k \setminus f'_k} d_{\mathrm{ST}}(S_i, e_k) \\ &= \begin{cases} 1 - (b_k - 1) & \text{if } S_i \neq e_i \\ -(b_k - 1) & \text{if } S_i = e_i \end{cases}.\end{aligned} \tag{11}$$

If $e_i \notin f'_k$, then for any $S_i \in \mathfrak{B}$ we have

$$\begin{aligned}(\mathfrak{D}M)_{S_i,e_i} &= d_{\mathrm{ST}}(S_i, e_i) + d_{\mathrm{ST}}(S_i, f_k) - d_{\mathrm{ST}}(S_i, e_r) - d_{\mathrm{ST}}(S_i, e_s) \\ &= \begin{cases} 1 & \text{if } S_i \neq e_i \\ 0 & \text{if } S_i = e_i \end{cases}.\end{aligned} \tag{12}$$

Case 2. Let $e_i \in B_k \cap B_l$. Let $f_k = e_r \ominus e_s$, $\{e_r, e_s\} = f'_k \in B_k$ and $f_l = e_t \ominus e_u$, $\{e_t, e_u\} = f'_l \in B_l$. If $e_i \notin f'_k \cup f'_l$, then for any $S_i \in \mathfrak{B}$ we have

$$(\mathfrak{D}M)_{S_i,e_i} = 2d_{\mathrm{ST}}(S_i, e_i) + d_{\mathrm{ST}}(S_i, f_k) + d_{\mathrm{ST}}(S_i, f_l) - \sum_{e_x \in f'_k \cup f'_l} d_{\mathrm{ST}}(S_i, e_x).$$

If $S_i = e_i$, then $d_{\mathrm{ST}}(S_i, e_j) = 2$ for $e_j \in \{e_r, e_s, e_t, e_u\}$ and $d_{\mathrm{ST}}(S_i, f_j) = 3$ for $f_j \in \{f_k, f_l\}$. Therefore

$$(\mathfrak{D}M)_{e_i,e_i} = 2 + 3 + 3 - 4 \times 2 = 0.$$

Now, suppose that $S_i \neq e_i$. Let T_k and T_l be connected components of $T - e_i$ contain f_k and f_l respectively. Without lost of generality let $S_i \subseteq V(T_k)$. If $S_i = f_k$, then

$$(\mathfrak{D}M)_{f_k,e_i} = 2 \times 3 + 2 + 5 - 12 = 1.$$

If $S_i \neq f_k$, we have three following cases where S_i is a subset of vertices in branch contain e_r or e_s or any other branches of T_k respectively.

$$\begin{aligned}d_{\mathrm{ST}}(S_i, e_i) &= d_{\mathrm{ST}}(S_i, e_r) + 1 = d_{\mathrm{ST}}(S_i, e_s) = d_{\mathrm{ST}}(S_i, f_k) \\ &= d_{\mathrm{ST}}(S_i, e_t) - 1 = d_{\mathrm{ST}}(S_i, e_u) - 1 = d_{\mathrm{ST}}(S_i, f_l) - 2,\end{aligned}$$

or

$$\begin{aligned} d_{\mathrm{ST}}(S_i, e_i) &= d_{\mathrm{ST}}(S_i, e_r) = d_{\mathrm{ST}}(S_i, e_s) + 1 = d_{\mathrm{ST}}(S_i, f_k) \\ &= d_{\mathrm{ST}}(S_i, e_t) - 1 = d_{\mathrm{ST}}(S_i, e_u) - 1 = d_{\mathrm{ST}}(S_i, f_l) - 2. \end{aligned}$$

or

$$\begin{aligned} d_{\mathrm{ST}}(S_i, e_i) &= d_{\mathrm{ST}}(S_i, e_r) = d_{\mathrm{ST}}(S_i, e_s) = d_{\mathrm{ST}}(S_i, f_k) - 1 \\ &= d_{\mathrm{ST}}(S_i, e_t) - 1 = d_{\mathrm{ST}}(S_i, e_u) - 1 = d_{\mathrm{ST}}(S_i, f_l) - 2, \end{aligned}$$

Therefore

$$(\mathfrak{D}M)_{S_i,e_i} = \begin{cases} 1 & \text{if } S_i \neq e_i \\ 0 & \text{if } S_i = e_i \end{cases}. \tag{13}$$

If $e_i = e_r = e_t \in f'_k \cap f'_l$.

$$\begin{aligned} (\mathfrak{D}M)_{S_i,e_i} &= (b_k + b_l - 2)d_{\mathrm{ST}}(S_i, e_i) + (b_k - 2)d_{\mathrm{ST}}(S_i, e_s) \\ &\quad + (b_l - 2)d_{\mathrm{ST}}(S_i, e_u) + (1 - b_k)d_{\mathrm{ST}}(S_i, f_k) + (1 - b_l)d_{\mathrm{ST}}(S_i, f_l) \\ &\quad - \sum_{e_k \in B_k \setminus f'_k} d_{\mathrm{ST}}(S_i, e_k) - \sum_{e_k \in B_l \setminus f'_l} d_{\mathrm{ST}}(S_i, e_k) \\ &= \begin{cases} 1 - (b_k - 1) - (b_l - 1) & \text{if } S_i \neq e_i \\ -(b_k - 1) - (b_l - 1) & \text{if } S_i = e_i \end{cases}. \end{aligned} \tag{14}$$

If $e_i \in f'_k$ or $e_i \in f'_l$. Without loss of generality, let $e_i \in f'_k$, $e_i = e_r$ and $e_i \notin f'_l$. Then, for any $S_i \in \mathfrak{B}$ we have

$$\begin{aligned} (\mathfrak{D}M)_{S_i,e_i} &= b_k d_{\mathrm{ST}}(S_i, e_i) + (b_k - 2)d_{\mathrm{ST}}(S_i, e_s) + (1 - b_k)d_{\mathrm{ST}}(S_i, f_k) \\ &\quad - d_{\mathrm{ST}}(S_i, e_t) - d_{\mathrm{ST}}(S_i, e_u) + d_{\mathrm{ST}}(S_i, f_l) - \sum_{e_k \in B_k \setminus f'_k} d_{\mathrm{ST}}(S_i, e_k) \\ &= \begin{cases} 1 - (b_k - 1) & \text{if } S_i \neq e_i \\ -(b_k - 1) & \text{if } S_i = e_i \end{cases}. \end{aligned} \tag{15}$$

Now, suppose that $f_i \in \{f_1, \ldots, f_{n-p}\}$, $f_i = e_r \ominus e_s$ and $f'_i \in B_i$. For any $S_i \in \mathfrak{B}$,

$$\begin{aligned} (\mathfrak{D}M)_{S_i,f_i} &= b_i d_{\mathrm{ST}}(S_i, f_i) + (1 - b_i)d_{\mathrm{ST}}(S_i, e_r) + (1 - b_i)d_{\mathrm{ST}}(S_i, e_s) \\ &\quad + \sum_{e_k \in B_i \setminus f'_i} d_{\mathrm{ST}}(S_i, e_k) \\ &= \begin{cases} b_i - 1 & \text{if } S_i \neq f_i \\ b_i - 2 & \text{if } S_i = f_i \end{cases}. \end{aligned} \tag{16}$$

It follows from (11), (12), (13), (14), (15) and (16) that $(\mathfrak{D}M + I)_{S_i,S_j} = v_{S_j}$ for all $S_i, S_j \in \mathfrak{B}$. Hence $\mathfrak{D}M + I = \mathbf{1}\mathbf{v}^t$, completing the proof. $\square$

Remark 14. Lemma 13 implies that $(-\mathfrak{D})$ is a g-inverse of the matrix M defined in Subsection 4.1.

Proof. Multiply (10) on the left by M and use (9) to get $M\mathfrak{D}M = -M$ to complete the proof. $\square$

We are now in a position to prove our formula for the inverse of $\mathfrak{D} = \mathfrak{D}_2(T)[\mathfrak{B}, \mathfrak{B}]$.

Proof of Theorem 2. We have

$$\mathfrak{D}\left(-M + \frac{1}{n-1}\mathbf{v}\mathbf{v}^t\right) = -\mathfrak{D}M + \frac{1}{n-1}\mathfrak{D}\mathbf{v}\mathbf{v}^t = I - \mathbf{1}\mathbf{v}^t + \frac{1}{n-1}(n-1)\mathbf{1}\mathbf{v}^t = I.$$

This completes the proof. $\square$

5. Determinant of $\mathfrak{D}_2(T)[\mathfrak{B}, \mathfrak{B}]$ and inertia of $\mathfrak{D}_2(T)$

In this section, we compute the determinant of $\mathfrak{D} = \mathfrak{D}_2(T)[\mathfrak{B}, \mathfrak{B}]$. Using this result, we also compute the inertia of $\mathfrak{D}_2(T)$. Lastly, we determine all eigenvalues of the 2-Steiner distance matrix of the star tree.

Let A be an $n \times n$ matrix partitioned as $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{11} and A_{22} are square matrices. If A_{11} is nonsingular then the Schur complement of A_{11} is defined to be the matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$. Similarly, if A_{22} is nonsingular then the Schur complement of A_{22} is defined to be $A_{11} - A_{12}A_{22}^{-1}A_{21}$.

Lemma 15. *Let A be an $n \times n$ matrix partitioned as $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. If A_{11} is square and nonsingular then $\det(A) = \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12})$. Similarly, if A_{22} is square and nonsingular then det$(A) = \det(A_{22})\det(A_{11} - A_{12}A_{22}^{-1}A_{21})$.*

With these preliminaries, we are now ready to prove Theorem 3.

Proof of Theorem 3. We use induction on n. When $n = 2$ it is easy to see that $\det(\mathfrak{D}) = 1$. Assume that the result is true for any tree with $n - 1$ vertices. Let T be a tree on n vertices and $\mathfrak{B}$ be an ordered basis for the row space of $\mathfrak{D}_2(T)$.

By relabelling, let 1 be a pendant vertex in T with neighbour vertex 2. Suppose $\deg_T(2) \geq 3$. Fig. 2 gives a picture of this scenario. We assume that $\{1, 2\} = e_1$, that $e_2 = \{2, \alpha\}$, $e_3 = \{2, \beta\}$ and $f_1 = e_2 \ominus e_3$. Let $T' = T - \{1\}$ and $\mathfrak{B}' = \mathfrak{B} \setminus \{e_1\}$.

If $\mathfrak{D}(e_1|e_1)$ denotes the submatrix of T obtained by deleting its first row and column, then clearly $\mathfrak{D}(e_1|e_1) = \mathfrak{D}(T')$ and $\mathfrak{D} = \left(\begin{array}{c|c} 1 & \mathbf{y}' \\ \hline \mathbf{y} & \mathfrak{D}(T') \end{array}\right)$. Let T_α be the connected component of $T - e_2$ that contains α. Let $\mathfrak{C}$ be the set of all 2-subsets of $V(T_\alpha \cup e_2)$. So for $S_i \in \mathfrak{B}'$ we have

$$d_{\mathrm{ST}}(S_i, e_1) = \begin{cases} d_{\mathrm{ST}}(S_i, e_2) + 1 & S_i \in \mathfrak{C} \cap \mathfrak{B}' \text{ or } S_i = f_1 \\ d_{\mathrm{ST}}(S_i, e_2) & \text{otherwise.} \end{cases}$$

Performing the row and column operations $r_{e_1} = r_{e_1} - r_{e_2}$ and $c_{e_1} = c_{e_1} - c_{e_2}$ changes matrix $\mathfrak{D}$ to

$$A = \left(\begin{array}{c|c} -2 & \mathbf{x}^t \\ \hline \mathbf{x} & \mathfrak{D}(T') \end{array}\right), \text{ where } x_{S_i} = \begin{cases} 1 & S_i \in \mathfrak{C} \cap \mathfrak{B}' \text{ or } S_i = f_1, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 15 and Theorem 2 we get

$$\begin{aligned} \det(\mathfrak{D}) = \det(A) &= \det(\mathfrak{D}(T')) \det\Big(-2 - \mathbf{x}^t \mathfrak{D}(T')^{-1} \mathbf{x} \Big) \\ &= \det(\mathfrak{D}(T')) \det\Big(-2 + \mathbf{x}^t M' \mathbf{x} - \frac{1}{n-2} \mathbf{x}^t \mathbf{v}' \mathbf{v}'^t \mathbf{x} \Big). \end{aligned}$$

Since $M'\mathbf{1} = \mathbf{0}$ it is easy to see that $M'_{S_i}\mathbf{x} = 0$ for all $S_i \subseteq V(\mathfrak{C})$ and that $M'_{f_1}\mathbf{x} = 1$. Thus $\mathbf{x}^t M' \mathbf{x} = 1$. Further, from (7) we have $\mathbf{x}^t \mathbf{v}' = 1$. By the induction assumption, $\det(\mathfrak{D}(T')) = (-1)^{p-1}(n-2)$. Hence

$$\det(\mathfrak{D}) = (-1)^{p-1}(n-2)\left(-2 + 1 - \frac{1}{n-2} \right) = (-1)^p (n-1).$$

We can thus assume that every pendant vertex u in T has $\deg_T(N(u)) = 2$. By relabelling, let 1 be a pendant vertex with neighbour 2. As $\deg_T(2) = 2$, let $1, 3$ be the neighbours of 2. Let $e_1 = \{1,2\}$, $e_2 = \{2,3\}$ and $f_1 = \{1,3\}$. Let $T' = T - \{1\}$. Let $\mathfrak{D}(e_1, f_1|e_1, f_1)$ denote the submatrix of $\mathfrak{D}$ obtained by deleting the rows and columns indexed by e_1 and f_1. Thus $\mathfrak{D}(e_1, f_1|e_1, f_1) = \mathfrak{D}(T')$. Clearly, $d_{\mathrm{ST}}(e_1, S_i) = d_{\mathrm{ST}}(f_1, S_i)$ for every $S_i \in \mathfrak{B} \setminus \{e_1\}$ and $d_{\mathrm{ST}}(e_1, S_i) = d_{\mathrm{ST}}(e_2, S_i) + 1$ for every $S_i \in \mathfrak{B} \setminus \{f_1, e_1\}$. Perform the following row and column operations $r_{f_1} = r_{f_1} - r_{e_1}$, $r_{e_1} = r_{e_1} - r_{e_2}$ and $c_{f_1} = c_{f_1} - c_{e_1}$, $c_{e_1} = c_{e_1} - c_{e_2}$. After these are done, the matrix $\mathfrak{D}$ clearly changes to

$$B = \left(\begin{array}{cc|ccc} -2 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ \hline 1 & 0 & & & \\ \vdots & \vdots & & \mathfrak{D}(T') & \\ 1 & 0 & & & \end{array}\right).$$

Denote $X^t = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix}$. By Lemma 15 and Theorem 2, we have

$$\begin{aligned} \det(\mathfrak{D}) = \det(B) &= \det(\mathfrak{D}(T'))\det\left(\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} - X^t\mathfrak{D}(T')^{-1}X\right) \\ &= \det(\mathfrak{D}(T'))\det\left(\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + X^tM'X - \frac{1}{n-2}X^t\mathbf{v}'(\mathbf{v}')^tX\right). \end{aligned}$$

By equation (9), Lemma 9 and the induction hypothesis we have

$$\begin{aligned} \det(\mathfrak{D}) &= \det(\mathfrak{D}(T'))\det\left(\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{n-2} & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= (-1)^p(n-2)\det\left(\begin{pmatrix} \frac{3-2n}{n-2} & 1 \\ 1 & -1 \end{pmatrix}\right) = (-1)^p(n-1). \end{aligned}$$

The proof is complete. $\square$

We move on to our results about Inertia. Let A be an $n \times n$ real symmetric matrix. Let n_+, n_- and n_0 be the number of positive, negative and zero eigenvalues of A respectively. Recall that the inertia of A, denoted $\mathrm{Inertia}(A)$ is the triple (n_+, n_-, n_0). As all eigenvalues of A are real, we have $n_+ + n_- + n_0 = n$ (see [11,24,20]). We will need the following famous result of Sylvester (see [22, Theorem 10.43].)

Lemma 16 *(Sylvester's law of inertia). Let A be a real symmetric $n \times n$ matrix and let S be a real $n \times n$ nonsingular matrix. Then, as a triple,* $\mathrm{Inertia}(A) = \mathrm{Inertia}(S^tAS)$.

For matrices M, N we treat $\mathrm{Inertia}(M), \mathrm{Inertia}(N)$ as vectors and do component-wise addition when we write $\mathrm{Inertia}(M) + \mathrm{Inertia}(N)$.

Lemma 17. *Let A be an $n \times n$ matrix partitioned as $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. If A_{11} is square and nonsingular, then*

$$\mathrm{Inertia}(A) = \mathrm{Inertia}(A_{11}) + \mathrm{Inertia}(A_{22} - A_{21}A_{11}^{-1}A_{12}).$$

Similarly, if A_{22} is square and nonsingular, then

$$\mathrm{Inertia}(A) = \mathrm{Inertia}(A_{22}) + \mathrm{Inertia}(A_{11} - A_{12}A_{22}^{-1}A_{21}).$$

Theorem 18. *Let T be a tree of order n with p pendant vertices. Then*

$$\Big(n_+(\mathfrak{D}_2(T)), n_-(\mathfrak{D}_2(T)), n_0(\mathfrak{D}_2(T))\Big) = \left(1, 2n-p-2, \binom{n}{2} - 2n + p + 1\right).$$

Proof. First we compute $\mathrm{Inertia}(\mathfrak{D}_2[\mathfrak{B},\mathfrak{B}])$. Our proof is identical to the proof of Theorem 3. In spirit, we use the Schur complement inertia version instead of the determinant version. Similar to the two cases that appear in the proof of Theorem 3 we either have

$$\mathrm{Inertia}(\mathfrak{D}_2[\mathfrak{B},\mathfrak{B}]) = \mathrm{Inertia}(\mathfrak{D}(T')) + \mathrm{Inertia}\left(-2+1-\frac{1}{n-2}\right),$$

or

$$\mathrm{Inertia}(\mathfrak{D}_2[\mathfrak{B},\mathfrak{B}]) = \mathrm{Inertia}(\mathfrak{D}(T')) + \mathrm{Inertia}\left(\begin{pmatrix} \frac{3-2n}{n-2} & 1 \\ 1 & -1 \end{pmatrix}\right).$$

In both cases, it can be seen that $\mathrm{Inertia}(\mathfrak{D}_2[\mathfrak{B},\mathfrak{B}]) = (1, 2n-p-2, 0)$. As $\mathfrak{D}_2(T)$ is a real symmetric matrix, there exists an orthogonal matrix S such that $S\mathfrak{D}_2(T)S^t = \begin{pmatrix} \mathfrak{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. By Lemma 16, we get

$$\mathrm{Inertia}(\mathfrak{D}_2(T)) = \mathrm{Inertia}\left(\begin{pmatrix} \mathfrak{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right) = \left(1, 2n-p-2, \binom{n}{2} - 2n+p+1\right).$$

The proof is complete. □

Theorem 19. *Let* Star_n *be the star tree on n vertices. Then, the eigenvalues of* $\mathfrak{D}_2(\mathrm{Star}_n)$ *are:* 0 *with multiplicity* $\binom{n}{2} - n$, $2-n$ *with multiplicity* $n-2$ *and the following two eigenvalues.*

$$\frac{(n-1)^2 + (n-2)^2 \pm \sqrt{(n-1)^4 + (n-2)^4 + 2(n-1)^3(n-2)}}{2}, \tag{17}$$

each with multiplicity 1.

Proof. Let e_r, e_s be two distinct edges of Star_n. Let $\mathbf{v}$ be the vector indexed by 2-subsets of $V(\mathrm{Star}_n)$ defined as follows.

$$v_S = \begin{cases} 1 & \text{if } S = e_r \text{ or } S = e_r \ominus e_{j\notin\{r,s\}}, \\ -1 & \text{if } S = e_s \text{ or } S = e_s \ominus e_{j\notin\{r,s\}}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that if $S \notin \{e_r, e_s, e_r \ominus e_{j\notin\{r,s\}}, e_s \ominus e_{j\notin\{r,s\}}\}$ then both $d_{\mathrm{ST}}(S, e_r) = d_{\mathrm{ST}}(S, e_s)$ and $d_{\mathrm{ST}}(S, e_r \ominus e_{j\notin\{r,s\}}) = d_{\mathrm{ST}}(S, e_s \ominus e_{j\notin\{r,s\}})$. Therefore, $(\mathfrak{D}_2(\mathrm{Star}_n))_S\mathbf{v} = 0$ where $(\mathfrak{D}_2(\mathrm{Star}_n))_S$ denotes the S-th row of the matrix $\mathfrak{D}_2(\mathrm{Star}_n)$. Now, suppose that $S \in \{e_r, e_s, e_r \ominus e_{j\neq\{r,s\}}, e_s \ominus e_{j\notin\{r,s\}}\}$, then

$$
\begin{aligned}
(\mathfrak{D}_2(\mathrm{Star}_n))_S \mathbf{v} &= d_{\mathrm{ST}}(S, e_r) - d_{\mathrm{ST}}(S, e_s) \\
&\quad + \sum_{e_i \notin \{e_r, e_s\}} d_{\mathrm{ST}}(S_i, e_r \ominus e_i) - \sum_{e_i \notin \{e_r, e_s\}} d_{\mathrm{ST}}(S_i, e_s \ominus e_i) \\
&= \begin{cases} 1 - 2 + 2(n-3) - 3(n-3) = 2 - n & \text{if } S = e_r \\ 2 - 1 + 3(n-3) - 2(n-3) = n - 2 & \text{if } S = e_s \\ 2 - 3 + (3(n-4) + 2) - (3 + 4(n-4)) = 2 - n & \text{if } S = e_r \ominus e_i \\ 3 - 2 + (3 + 4(n-4)) - (2 + 3(n-4)) = n - 2 & \text{if } S = e_s \ominus e_i \end{cases}.
\end{aligned}
$$

Hence $\mathfrak{D}_2(\mathrm{Star}_n)\mathbf{v} = (2-n)\mathbf{v}$. We can choose $n-2$ distinct edges apart from e_r and define eigenvectors depending on the above. It can be seen that these vectors are linearly independent so the multiplicity of $2-n$ is at least $n-2$. Let λ and μ be two non zero eigenvalues of $\mathfrak{D}_2(\mathrm{Star}_n)$. Then

$$
(n-1) + 2\binom{n-1}{2} = \mathrm{Trace}(\mathfrak{D}_2(\mathrm{Star}_n)) = (n-2)(2-n) + \lambda + \mu,
$$

and consequently

$$
\lambda + \mu = (n-1)^2 + (n-2)^2. \tag{18}
$$

Sum of the 2×2 minors of $\mathfrak{D}_2(\mathrm{Star}_n)$ equals

$$
-3\binom{n-1}{2} - 2 \times 2\binom{n-1}{2} - 7\binom{n-1}{1}\binom{n-2}{2} - 5\binom{n-1}{2}\binom{n-3}{1} - 12\frac{\binom{n-1}{2}\binom{n-3}{2}}{2},
$$

which is equal to $-\binom{n-1}{2}(3n^2 - 9n + 7)$. Also, sum of the 2×2 minors of $\mathfrak{D}_2(\mathrm{Star}_n)$ is equal to

$$
\begin{aligned}
\sum_{i<j} \lambda_i \lambda_j &= \lambda(2-n)(n-2) + \mu(2-n)(n-2) + \binom{n-2}{2}(n-2)^2 + \lambda\mu \\
&= -(n-2)^2(\lambda + \mu) + \binom{n-2}{2}(n-2)^2 + \lambda\mu \\
&= -\binom{n-1}{2}(3n^2 - 10n + 8) + \lambda\mu,
\end{aligned}
$$

and thus

$$
\lambda\mu = -\frac{1}{2}(n-1)^2(n-2). \tag{19}
$$

The two (18) and (19) give (17). Since $\mathrm{rank}(\mathfrak{D}_2(\mathrm{Star}_n)) = n$, the proof is complete. □

Declaration of competing interest

There are no competing interests.

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