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Signed alternating descent enumeration in classical Weyl groups

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ABSTRACT

The alternating descent statistic on permutations was introduced by Chebikin as a variant of the descent statistic. In this paper, we get a formula for the signed enumeration of alternating descents and in our proof we need a signed convolution type identity involving the Eulerian polynomials. When n is even, we give a more general multivariate version and we also get a formula for the signed enumeration of the alternating major index. We generalize our results to the case when alternating descents are summed up with sign over the elements in classical Weyl groups.

1. Introduction

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and let $\mathfrak{S}_n$ be the group of all permutations of $[n]$. For $\pi \in \mathfrak{S}_n$ with $\pi = \pi_1, \pi_2, \ldots, \pi_n$, define $\mathrm{DES}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$ to be its set of descents and let $\mathrm{des}(\pi) = |\mathrm{DES}(\pi)|$. The classical Eulerian polynomial $A_n(t)$ is the descent enumerating polynomial of $\mathfrak{S}_n$. That is,

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)}. \tag{1}$$

This is a very well-studied polynomial. We refer the reader to the early book by Foata and Schützenberger [9] and the more recent book by Petersen [16] for various properties of Eulerian polynomials. Loday in [13] defined the polynomial $\mathrm{SgnDes}_n(t) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{des}(\pi)}$ and conjectured a recurrence relation satisfied by $\mathrm{SgnDes}_n(t)$. This conjecture was later proved by Désarménien and Foata in [8]. Their result is the following:

Theorem 1 *(Désarménien and Foata). For positive integers n, we have*

$$\mathrm{SgnDes}_n(t) = \begin{cases} (1-t)^k A_k(t) & \text{if } n = 2k, \\ (1-t)^k A_{k+1}(t) & \text{if } n = 2k+1. \end{cases} \tag{2}$$

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Later Wachs in [21] gave a combinatorial proof of Theorem 1. As a variation of descents, *alternating descents* of a permutation are defined as follows. For $\pi \in \mathfrak{S}_n$, define

$$\mathrm{ALTDES}(\pi) = \{2i : \pi_{2i} < \pi_{2i+1}\} \cup \{2i+1 : \pi_{2i+1} > \pi_{2i+2}\},$$

and let $\mathrm{altdes}(\pi) = |\mathrm{ALTDES}(\pi)|$. The notion of alternating descent was introduced by Chebikin in [4], where he showed that the alternating descents on $\mathfrak{S}_n$ are equidistributed with the 3-descent statistic on $\{\pi \in \mathfrak{S}_{n+1} : \pi_1 = 1\}$ where we say that the permutation π has a 3-descent at index i if $\pi_i\pi_{i+1}\pi_{i+2}$ has one of the patterns: 132, 213 or 321. Lin et al. in [12] studied the alternating Eulerian polynomial defined as:

$$\widehat{\mathrm{A}}_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{altdes}(\pi)}.$$

They proved for positive integers n that the polynomial $\widehat{\mathrm{A}}_n(t)$ is palindromic and unimodal.

Recall that for $\pi \in \mathfrak{S}_n$, its *major index* is defined as $\mathrm{maj}(\pi) = \sum_{i \in \mathrm{DES}(\pi)} i$, the sum of π's descent indices. Similarly, define π's *alternate major index* by $\mathrm{altmaj}(\pi) = \sum_{i \in \mathrm{ALTDES}(\pi)} i$. That is, $\mathrm{altmaj}(\pi)$ is the sum of π's alternate descent indices. Remmel in [18] computed the exponential generating function (egf henceforth) of a bivariate polynomial enumerating alternating descents and the alternating major index over $\mathfrak{S}_n$. Remmel's formula is similar to a similar egf for the joint distribution of descents and major index over $\mathfrak{S}_n$ that was proved by Gessel [11] and Gessel and Garsia [10]. Further, in [18], Remmel also computed similar generating functions for the type B and the type D Coxeter groups.

Using derivative polynomials, Ma and Yeh in [14] presented an explicit and complicated formula for the number of permutations in $\mathfrak{S}_n$ with a given number of alternating descents. We are interested in enumerating alternating descents in $\mathfrak{S}_n$ with sign taken into account.

Signed enumeration of several statistics over various Coxeter groups and their subsets are known. Désarménien and Foata in [8] enumerated descents with sign in $\mathfrak{S}_n$ and Reiner in [17] enumerated signed descents for types A, B and D Coxeter groups. Mantaci in [15] enumerated excedances with sign in $\mathfrak{S}_n$, Adin et al. in [1] enumerated signed mahonians in Coxeter groups. Tanimoto in [20] has shown divisibility of coefficients of the signed descent enumerator by prime numbers. Barnabei, Bonetti and Silimbani [2] have enumerated signed ascents over involutions using properties of the RSK correspondence. Motivated by the above signed enumeration results, we define the following:

$$\mathrm{SgnAltDes}_n(t) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{altdes}(\pi)}. \tag{3}$$

Over $\mathfrak{S}_n$, our first result is similar to Theorem 1 and is the following.

Theorem 2. *For positive integers n, we have*

$$\mathrm{SgnAltDes}_n(t) = \begin{cases} (1-t)^{2m} A_{2m}(t) & \text{if } n = 4m, \\ \dfrac{2t}{1+t}(1-t)^{2m} A_{2m}(t) & \text{if } n = 4m+1, \\ (1-t)^{2m+1} A_{2m+1}(t) & \text{if } n = 4m+2, \\ 0 & \text{if } n = 4m+3. \end{cases}$$

The proof of Theorem 2 appears in Section 3. When n is even, our proof gives a more general multivariate version of Theorem 2, see Theorem 8. Over $\mathfrak{S}_n$, we also get a formula for the signed enumeration of the alternating major index as a further artifact of our proof. We generalize our results to the case when alternating descents are summed up with sign over the elements in classical Weyl groups. Let $\mathfrak{B}_n$ denote the group of signed permutations on $[\pm n] = \{\pm 1, \pm 2, \dots, \pm n\}$, that is $\pi \in \mathfrak{B}_n$ consists of all permutations of $[\pm n]$ that satisfy $\pi(-i) = -\pi(i)$ for all $i \in [n]$. Let $\mathfrak{D}_n \subseteq \mathfrak{B}_n$ denote the subset of $\mathfrak{B}_n$ consisting of those elements which have an even number of negative entries. As both $\mathfrak{B}_n$ and $\mathfrak{D}_n$ are Coxeter groups, they have a natural notion of length and hence sign associated to their elements. As mentioned above, Remmel in [18] computed generating functions for the type B and type D alternating Eulerian polynomials. In the type B case, his definition for the alternating descent of a signed permutation is obtained by imposing a different order on the elements of $[\pm n]$. A different Coxeter theoretic definition based on type B descents was given by Ma, Fang, Mansour and Yeh in [7]. We enumerate both versions of alternating-descents with signs. Our type B counterparts are Theorem 13 and Theorem 14 and our type D counterpart is Theorem 21.

2. A signed convolution identity involving the Eulerian polynomials

In our proof of Theorem 2, we need an identity involving binomial coefficients and the Eulerian polynomials with signs for which we could not locate a reference. We thus present a proof in this short section. It is well known that the Eulerian polynomial $A_n(t)$ satisfies the following convolution based quadratic recurrence involving binomial coefficients, see [9, Page 70] or [16, Theorem 1.5].

Theorem 3. *For positive integers n, we have*

$$A_n(t) = A_{n-1}(t) + t\sum_{i=0}^{n-2}\binom{n-1}{i}A_i(t)A_{n-1-i}(t)$$

where $A_0(t) = 1$.

Chow in [5, Theorem 2.2] has given a q-analogue of Theorem 3 for the inversion based q-Eulerian polynomial. We need the following similar identity.

Theorem 4. *For non-negative integers k, the following identity is satisfied by the Eulerian polynomials $A_k(t)$:*

$$\sum_{r=0}^{k}(-1)^r\binom{k}{r}A_r(t)A_{k-r}(t) = \begin{cases} 1 & \text{if } k = 0,\\ 2tA_k(t)/(1+t) & \text{if } k > 0 \text{ is even,}\\ 0 & \text{if } k \text{ is odd.}\end{cases} \tag{4}$$

Proof. When $k = 0$, the left hand side is clearly 1 as we have $A_0(t) = 1$. We next consider the case when k is odd. It is easy to see that the left hand side is zero by pairing the term when $r = i$ with its negative when $r = k - i$. We move to the more interesting case when k is even.

Consider $S(t,z) = \sum_{k\geq 0} A_k(t)\frac{z^k}{k!}$, the exponential generating function (egf henceforth) of the Eulerian polynomials $A_k(t)$. We prove (4) by showing the equality of the egf of the left hand side and the right hand side.

Clearly, the coefficient of $\frac{z^k}{k!}$ in $S(t,z)\times S(t,-z)$ equals $\sum_{k=0}^{\infty}(-1)^r\binom{k}{r}A_r(t)A_{k-r}(t)$. Thus, the egf for the left hand side is $S(t,z)\times S(t,-z)$. On the other hand, the egf for the right hand side is

$$1 + \frac{2t}{1+t}\left[\sum_{k=2,\ k\ \text{is even}}^{\infty} A_k(t)\frac{z^k}{k!}\right] = \frac{2t}{1+t}\left[\frac{S(t,z)+S(t,-z)}{2}\right] + \left[1 - \frac{2t}{1+t}\right].$$

Therefore, we need to show the following:

$$S(t,z)\times S(t,-z) = \frac{t}{1+t}\Big(S(t,z)+S(t,-z)\Big) + \left[1-\frac{2t}{1+t}\right]. \tag{5}$$

It is well known (see [16]) that $S(t,z) = \frac{t-1}{t-e^{z(t-1)}}$. Let $p = e^{z(t-1)}$. Plugging this, (5) will follow if we prove the following:

$$\frac{p(t-1)^2}{(t-p)(pt-1)} = \frac{t}{1+t}\left[\frac{t-1}{t-p} + \frac{p(t-1)}{pt-1}\right] + \frac{1-t}{1+t}. \tag{6}$$

As (6) follows by simple algebraic manipulation, our proof is complete. $\square$

3. Proof of Theorem 2

We break the proof in different parts depending on the parity of n in two subsections. We first consider the case when $n = 2k$. Our proof follows along the same lines as the proof of Wachs [21]. Nonetheless, we give a proof as it sets up the stage for enumerating $\text{SgnAltDes}_n(t)$ when $n = 2k+1$.

3.1. Proving Theorem 2 when $n = 2k$

When $n = 2k$, we will show that $\text{SgnAltDes}_{2k}(t) = (1-t)^k A_k(t)$. We first define a map below and prove some of its properties. Consider the set

$$\mathfrak{S}_{2k}^1 = \{\pi \in \mathfrak{S}_{2k} : \text{there exists } i \leq k \text{ such that } 2i-1 \text{ and } 2i \text{ are not adjacent}\}.$$

Define the map $f : \mathfrak{S}_{2k}^1 \mapsto \mathfrak{S}_{2k}^1$ by

$$f(\pi) = (2i-1, 2i)\pi,$$

where $1 \le i \le k$ is the smallest number such that the letters $2i-1$ and $2i$ are not in adjacent positions in π. That is, the map f switches the positions of $2i-1$ and $2i$ where i is the least number such that $2i-1$ and $2i$ are not consecutive in π. For $\pi \in \mathfrak{S}^1_{2k}$, it is easy to note that $\mathrm{sign}(\pi) = -\mathrm{sign}(f(\pi))$ and that $\mathrm{altdes}(\pi) = \mathrm{altdes}(f(\pi))$. Thus, the set $\mathfrak{S}^1_{2k}$ contributes 0 to the polynomial $\mathrm{SgnAltDes}_{2k}(t)$.

Thus consider the set $\mathfrak{S}_{2k} \setminus \mathfrak{S}^1_{2k}$. Note that permutations in $\mathfrak{S}_{2k} \setminus \mathfrak{S}^1_{2k}$ correspond bijectively to signed permutations of the letters $1, 2, \ldots, k$ according to the following rule: each pair of adjacent letters $2i-1, 2i$ is replaced by i if $2i-1$ is to the left of $2i$ in π and is replaced by $-i$ (which we denote as $\overline{i}$ with a bar over i) if $2i$ is to the left of $2i-1$ in π. Let $\mathfrak{B}_k$ be the set of signed permutations on $1, 2, \ldots, k$. Denote this bijection as $h : (\mathfrak{S}_{2k} \setminus \mathfrak{S}^1_{2k}) \mapsto \mathfrak{B}_k$. For example, $h(56214378) = 3\overline{1}\overline{2}4$. For a signed permutation $u \in \mathfrak{B}_k$, let $|u| \in \mathfrak{S}_k$ be the permutation obtained by ignoring the negative signs on u and let $\mathrm{inv}(|u|)$ denote the number of inversions in $|u|$ (computed in $\mathfrak{S}_k$). For $u \in \mathfrak{B}_k$ let $\mathrm{negs}(u)$ denoted the number of negative signs in u. In the above example if $u = 3\overline{1}\overline{2}4$, then we have $\mathrm{inv}(|u|) = \mathrm{inv}(3124) = 2$ and $\mathrm{negs}(u) = 2$. The next claim helps us in moving information across the map h^{-1}.

Claim 5. *For $u \in \mathfrak{B}_k$, with $u = u_1, u_2, \ldots, u_k$, we have $\mathrm{inv}(h^{-1}(u)) = 4 \times \mathrm{inv}(|u|) + \mathrm{negs}(u)$ and $\mathrm{altdes}(h^{-1}(u)) = \mathrm{asc}(|u|) + \mathrm{negs}(u)$, where $\mathrm{asc}(u) = |\{i \in [k-1] : u_i < u_{i+1}\}|$ is the number of ascents of u.*

Proof. For $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{S}_n$, we call the pair (i, j) of indices to be an inversion pair if $i < j$ and $\pi_i > \pi_j$. Clearly, $\mathrm{inv}(\pi)$ is the number of inversion pairs of π. For $|u| = |u|_1 |u|_2 \ldots |u|_n$, if (i, j) is an inversion pair of $|u|$, then $(2i-1, 2j-1), (2i-1, 2j), (2i, 2j-1)$, and $(2i, 2j)$ are inversion pairs of $h^{-1}(u)$. Further, if u_i is negative, then $(2i-1, 2i)$ is a further inversion pair of $h^{-1}(u)$. Thus, $\mathrm{inv}(h^{-1}(u)) = 4 \times \mathrm{inv}(|u|) + \mathrm{negs}(u)$. Moreover, if u_i is negative, then $2i-1$ is a descent and hence an alternating descent of $h^{-1}(u)$. Further, if i is an ascent of $|u|$, then $2i$ is an ascent and hence an alternating descent of $h^{-1}(u)$. Thus, $\mathrm{altdes}(h^{-1}(u)) = \mathrm{asc}(|u|) + \mathrm{negs}(u)$. This completes the proof of the claim. □

Proof of Theorem 2 when $n = 2k$. By Claim 5, when $u \in \mathfrak{B}_k$, we have

$$(-1)^{\mathrm{inv}(h^{-1}(u))} t^{\mathrm{altdes}(h^{-1}(u))} = (-t)^{\mathrm{negs}(u)} t^{\mathrm{asc}(|u|)}. \tag{7}$$

For $w \in \mathfrak{S}_k$, let $\beta(w)$ be the set of 2^k permutations obtained by adding negative signs to elements of w. Thus, if $u \in \beta(w)$ then $u \in \mathfrak{B}_k$ and $|u| = w$.

From this, we have

$$\begin{aligned} \sum_{\pi \in \mathfrak{S}_{2k} \setminus \mathfrak{S}^1_{2k}} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{altdes}(\pi)} &= \sum_{u \in \mathfrak{B}_k} (-1)^{\mathrm{inv}(h^{-1}(u))} t^{\mathrm{altdes}(h^{-1}(u))} \\ &= \sum_{w \in \mathfrak{S}_k} \sum_{u \in \beta(w)} (-t)^{\mathrm{negs}(u)} t^{\mathrm{asc}(w)} = (1-t)^k \left(\sum_{w \in \mathfrak{S}_k} t^{\mathrm{asc}(w)} \right) \\ &= (1-t)^k A_k(t). \end{aligned}$$

The equality on the first line is due to the bijection between $\mathfrak{S}_{2k} \setminus \mathfrak{S}^1_{2k}$ and $\mathfrak{B}_k$. The second line follows from (7) and the last line from the fact that ascents and descents are equidistributed in $\mathfrak{S}_k$. The proof is complete. □

3.2. *Proving Theorem 2 when n is odd*

When $n = 2k+1$ is odd, consider a similar set

$$\mathfrak{S}^1_{2k+1} = \{\pi \in \mathfrak{S}_{2k+1} : \text{there exists } i \le k \text{ such that } 2i-1 \text{ and } 2i \text{ are not adjacent}\}.$$

Again, we can define the same map $f : \mathfrak{S}^1_{2k+1} \mapsto \mathfrak{S}^1_{2k+1}$ by $f(\pi) = (2i-1, 2i)\pi$ where $1 \le i \le k$ is the smallest number such that the letters $2i-1$ and $2i$ are not in adjacent positions in π. As before, when $\pi \in \mathfrak{S}^1_{2k+1}$, it is easy to note that $\mathrm{sign}(\pi) = -\mathrm{sign}(f(\pi))$ and that $\mathrm{altdes}(\pi) = \mathrm{altdes}(f(\pi))$. Hence the set $\mathfrak{S}^1_{2k+1}$ contributes 0 to the polynomial $\mathrm{SgnAltDes}_{2k+1}(t)$.

Note that for any $\pi \in \mathfrak{S}_{2k+1} \setminus \mathfrak{S}^1_{2k+1}$, the position of $2k+1$ in π must be odd, as otherwise there will be some j such that $2j-1$ and $2j$ will be separated by $2k+1$ and hence non-adjacent. Thus, any $\pi \in \mathfrak{S}_{2k+1} \setminus \mathfrak{S}^1_{2k+1}$ can be written as $\pi = u, 2k+1, v$ where for some $S \subseteq [2k]$ with $|S|$ being even, u, v are permutations of $S, [2k]-S$ respectively with both u, v having pairs $2j-1$ and $2j$ in adjacent positions.

We denote the surviving set $\mathfrak{S}_{2k+1} \setminus \mathfrak{S}^1_{2k+1}$ as Surv_{2k+1} and we break Surv_{2k+1} as the disjoint union of the sets Surv^l_{2k+1} where for $0 \le l \le k$, $\mathrm{Surv}^l_{2k+1} = u, 2k+1, v$ with u being a permutation of $S \subseteq [2k]$ with $|S| = 2l$. Next, we compute the contribution of the sets Surv^l_{2k+1} to $\mathrm{SgnAltDes}_n(t)$. Hence, define $\mathrm{SurvSgnAltDes}^l_{2k+1}(t) = \sum_{\pi \in \mathrm{Surv}^l_{2k+1}} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{altdes}(\pi)}$.

Lemma 6. *For odd positive integers $n = 2k+1$, we have*

$$\text{SurvSgnAltDes}^{l}_{2k+1}(t) = \begin{cases} (-1)^k t(1-t)^k A_k(t) & \text{when } l = 0, \\ (-1)^{k-l}\binom{k}{l} t^2 (1-t)^k A_l(t) A_{k-l}(t) & \text{when } 1 \le l < k, \\ t(1-t)^k A_k(t) & \text{when } l = k. \end{cases}$$

Proof. Let $\pi = u, 2k+1, v \in \text{Surv}^{l}_{2k+1}$. We slightly modify the map h that we used in Claim 5 as follows. We first apply h to u, v. We then convert the letter $2k+1$ to $k+1$ to get $h(\pi)$. Thus if $\pi = 3421765$, then $u = 3421$ and $v = 65$ and so $h(\pi) = 2\overline{1}4\overline{3}$. Thus $h(\pi) \in \mathfrak{B}_{k+1}$, but now the letter $k+1$ will only occur without a negative sign.

We first consider the case when $l = 0$. The set Surv^{0}_{2k+1} has permutations $2k+1, v$ where $v \in \mathfrak{S}_{2k} \setminus \mathfrak{S}^{1}_{2k}$. Let $\mathfrak{B}_k$ be the set of signed permutations on $1, 2, \ldots, k$ and as before, define the map h that bijectively takes the set Surv^{0}_{2k+1} to the set of permutations of the form $k+1, u$ where $u = u_1, u_2, \ldots, u_k \in \mathfrak{B}_k$. We use the same notation as in subsection 3.1 and transfer information about statistics via the map h^{-1}. Let $w = k+1, u \in \mathfrak{B}_{k+1}$ where $u \in \mathfrak{B}_k$. The first position of $h^{-1}(w)$ is clearly an alternating descent. Subsequent odd descents in $h^{-1}(w)$ correspond to descents in u and even ascents in $h^{-1}(u)$ correspond to non-negative (or positive) elements in u. Since $w_1 = k+1$ is positive, this count is $k - \text{negs}(u)$. We thus have $\text{altdes}(h^{-1}(w)) = 1 + \text{des}(|u|) + (k - \text{negs}(u))$. It is easy to see that $\text{inv}(h^{-1}(w)) = 2k + 4\text{inv}(|u|) + \text{negs}(u) = 2k + 4\text{inv}(|w|) - 4k + \text{negs}(u)$. Therefore,

$$(-1)^{\text{inv}(h^{-1}(w))} t^{\text{altdes}(h^{-1}(w))} = (-1)^{\text{negs}(u)} t^{k-\text{negs}(u)+\text{des}(|u|)+1} = (-1)^k (-t)^{k-\text{negs}(u)} t^{\text{des}(|u|)+1}. \tag{8}$$

Arguing as done in subsection 3.1 but using (8) instead of (7) gives an extra term of $(-1)^k t$. This completes the proof when $l = 0$.

Next, we consider the case when $1 \le l \le k-1$. The set Surv^{l}_{2k+1} has permutations of the form $\pi = u, 2k+1, v$ where u has l consecutive pairs $2j-1, 2j$ and v has the remaining $(k-l)$ consecutive pairs. In this case, the function h maps the set Surv^{l}_{2k+1} bijectively to the set of permutations of the form $u, k+1, v$ where $u = u_1, u_2, \ldots, u_l \in \mathfrak{B}_l$ and $v = v_1, v_2, \ldots, v_{k-l} \in \mathfrak{B}_{k-l}$. Consider $\pi \in \text{Surv}^{l}_{2k+1}$ and let $h(\pi) = u, k+1, v$ where $u = u_1, u_2, \ldots, u_l \in \mathfrak{B}_l$ and $v = v_1, v_2, \ldots, v_{k-l} \in \mathfrak{B}_{k-l}$. We again transfer information about statistics via the map h^{-1} and use the same notation from earlier. Let $w = u, k+1, v \in \mathfrak{B}_{k+1}$. We can choose a set of l pairs from k in $\binom{k}{l}$ ways and permute them in $\mathfrak{B}_l$ ways to get u. As $k+1$ appears in an odd position, $h^{-1}(w)$ has 2 alternating descents, at the position before and after $2k+1$. Arguing as before, one can check that $\text{altdes}(h^{-1}(w)) = 2 + \text{asc}(|u|) + \text{des}(|v|) + \text{negs}(u) + (k - l - \text{negs}(v))$ and $(-1)^{\text{inv}(h^{-1}(w))} = (-1)^{\text{negs}(u)+\text{negs}(v)}$. Thus, for $w \in \mathfrak{B}_{k+1}$ with $w = u, k+1, v$ where u, v have lengths $l, k-l$ respectively, we have

$$(-1)^{\text{inv}(h^{-1}(w))} t^{\text{altdes}(h^{-1}(w))} = t^2 (-1)^{k-l} (-t)^{\text{negs}(u)+k-l+\text{negs}(v)} t^{\text{asc}(|u|)+\text{des}(|v|)}. \tag{9}$$

From this, we have

$$\begin{aligned}
&\sum_{\pi \in \text{Surv}^{l}_{2k+1}} (-1)^{\text{inv}(\pi)} t^{\text{altdes}(\pi)} \\
&= \binom{k}{l} \sum_{u \in \mathfrak{B}_l} \sum_{v \in \mathfrak{B}_{k-l}} (-1)^{\text{inv}(h^{-1}(u,k+1,v))} t^{\text{altdes}(h^{-1}(u,k+1,v))} \\
&= (-1)^{k-l}\binom{k}{l} t^2 \sum_{x \in \mathfrak{S}_l} \sum_{u \in \beta(x)} \sum_{y \in \mathfrak{S}_{k-l}} \sum_{v \in \beta(y)} (-t)^{\text{negs}(u)} (-t)^{k-l-\text{negs}(v)} t^{\text{asc}(x)+\text{des}(y)} \\
&= \binom{k}{l} t^2 (-1)^{k-l} (1-t)^k \left(\sum_{w \in \mathfrak{S}_l} t^{\text{asc}(w)} \right) \left(\sum_{x \in \mathfrak{S}_{k-l}} t^{\text{des}(x)} \right) \\
&= (-1)^{k-l}\binom{k}{l} t^2 (1-t)^k A_l(t) A_{k-l}(t).
\end{aligned}$$

The second line above follows from the bijection h The third line follows from (9) and the last line from the fact that ascents and descents are equidistributed in $\mathfrak{S}_k$. This completes the proof when $1 \le l < k$.

We remark that the proof when $l = k$ is very similar to the proof when k is even as we just add $2k+1$ at the end and this addition has no change on both the number of inversions and the number of alternating descents. As the proof of this case follows in the same manner, we omit it. The proof is now complete. $\square$

We are now ready to prove Theorem 2 for n odd.

Proof of Theorem 2. Let $n=4m+1$, and let $k=2m$ so that $n=2k+1$. As k is even, note that $(-1)^k=1$. By Lemma 6 and Theorem 4 we have

$$\begin{aligned}\mathrm{SgnAltDes}_n(t) &= (1-t)^k\left[\sum_{l=0}^{k}\sum_{\pi\in\mathrm{Surv}^l_{2k+1}}(-1)^{\mathrm{inv}(\pi)}t^{\mathrm{altdes}(\pi)}\right]\\ &= (1-t)^k\left[(-1)^k tA_k(t)+\left(\sum_{l=1}^{k-1}(-1)^{k-l}t^2\binom{k}{l}A_{k-l}(t)A_l(t)\right)+tA_k(t)\right]\\ &= (1-t)^k\left[t^2\left(\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}A_{k-l}(t)A_l(t)\right)+(2t-2t^2)A_k(t)\right]\\ &= (1-t)^k\left[t^2\frac{2tA_k(t)}{(1+t)}+(2t-2t^2)A_k(t)\right]=\frac{2t(1-t)^k}{1+t}A_k(t).\end{aligned}$$

Thus when $n=4m+1$, the proof is complete. We move to the case when $n=4m+3$. Consider the bijection $\mathrm{Rev}:\mathfrak{S}_{4m+3}\mapsto\mathfrak{S}_{4m+3}$, defined by reversing the permutation. Formally,

$$\mathrm{Rev}(\pi_1,\pi_2,\ldots,\pi_{4m+2},\pi_{4m+3})=\pi_{4m+3},\pi_{4m+2},\ldots,\pi_2,\pi_1.$$

One can check that $\mathrm{sign}(\pi)=-\mathrm{sign}(\mathrm{Rev}(\pi))$ and that $\mathrm{altdes}(\pi)=\mathrm{altdes}(\mathrm{Rev}(\pi))$. Summing, we get zero, completing the proof. □

3.3. A multivariate refinement when n is even

In this subsection, we show that when n is even, our proof from subsection 3.1 gives us a multivariate refinement. It further refines Theorem 1 of Désarménien and Foata. We will see that Wachs proof of Theorem 1 goes through for this multivariate refinement. We start with the following bivariate refinement.

For $\pi\in\mathfrak{S}_n$ define $\mathrm{OddDES}(\pi)=\{2i+1\in[n]:\pi_{2i+1}>\pi_{2i+2}\}$ to be the set of odd positions where a descent occurs in π. Similarly, define $\mathrm{EvenDES}(\pi)=\{2i\in[n]:\pi_{2i}>\pi_{2i+1}\}$ to be the set of even positions where a descent occurs in π and define $\mathrm{EvenASC}(\pi)=\{2i\in[n]:\pi_{2i}<\pi_{2i+1}\}$ to be the set of even positions where an ascent occurs in π. Let $\mathrm{odes}(\pi)=|\mathrm{OddDES}(\pi)|$, $\mathrm{edes}(\pi)=|\mathrm{EvenDES}(\pi)|$ and $\mathrm{easc}(\pi)=|\mathrm{EvenASC}(\pi)|$. As can be seen from the definitions, we have $\mathrm{ALTDES}(\pi)=\mathrm{OddDES}(\pi)\cup\mathrm{EvenASC}(\pi)$ and $\mathrm{DES}(\pi)=\mathrm{OddDES}(\pi)\cup\mathrm{EvenDES}(\pi)$. Define the signed bivariate alternating descent enumerator to be

$$\mathrm{SgnBivAltDes}_n(p,q)=\sum_{\pi\in\mathfrak{S}_n}(-1)^{\mathrm{inv}(\pi)}p^{\mathrm{odes}(\pi)}q^{\mathrm{easc}(\pi)}.$$

Similarly define the signed bivariate descent enumerator to be

$$\mathrm{SgnBivDes}_n(p,q)=\sum_{\pi\in\mathfrak{S}_n}(-1)^{\mathrm{inv}(\pi)}p^{\mathrm{odes}(\pi)}q^{\mathrm{edes}(\pi)}.$$

As $\mathrm{SgnAltDes}_n(t)=\mathrm{SgnBivAltDes}_n(t,t)$, this bivariate polynomial refines $\mathrm{SgnAltDes}_n(t)$. We similarly have $\mathrm{SgnDes}_n(t)=\mathrm{SgnBivDes}_n(t,t)$. The following refinement of Theorem 2 when n is even is clear from the same proof given in subsection 3.1. Wachs proof of Désarménien and Foata's result also gives the following refinement of Theorem 1 when n is even. As the proof is identical, we just mention the result.

Theorem 7. *Let $n=2k$ be an even positive integer. Then,*

$$\mathrm{SgnBivAltDes}_n(p,q)=(1-p)^kA_k(q), \tag{10}$$

$$\mathrm{SgnBivDes}_n(p,q)=(1-p)^kA_k(q). \tag{11}$$

Moreover, it is easy to see that the following multivariate generalization of Theorem 7 also holds. For a permutation $\pi\in\mathfrak{S}_n$, define the monomial $m_\pi=\prod_{i\in\mathrm{OddDES}(\pi)}p_i$ where p_is are commuting variables. When $n=2k$, define the following multivariate polynomials

$$\mathrm{SgnMultDes}_n(p_1,p_3,\ldots,p_{2k-1},q)=\sum_{\pi\in\mathfrak{S}_n}(-1)^{\mathrm{inv}(\pi)}m_\pi q^{\mathrm{edes}(\pi)}$$

and

$$\mathrm{SgnMultAltDes}_n(p_1, p_3, \ldots, p_{2k-1}, q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\pi)} m_\pi q^{\mathrm{easc}(\pi)}.$$

Wachs proof of Désarménien and Foata's result also gives the following multivariate refinement of Theorem 7.

Theorem 8. *Let $n = 2k$ be an even positive integer. Then,*

$$\mathrm{SgnMultAltDes}_n(p_1, p_3, \ldots, p_{2k-1}, q) = \left(\prod_{r=1}^{k}(1 - p_{2r-1})\right) A_k(q), \tag{12}$$

$$\mathrm{SgnMultDes}_n(p_1, p_3, \ldots, p_{2k-1}, q) = \left(\prod_{r=1}^{k}(1 - p_{2r-1})\right) A_k(q). \tag{13}$$

It is easy to see that setting $p_1 = p_3 = \cdots = p_{2k-1} = p$, we immediately get Theorem 7.

For $\pi \in \mathfrak{S}_n$, recall that $\mathrm{DES}(\pi)$ is its set of descents. Define the major index of π as $\mathrm{maj}(\pi) = \sum_{i \in \mathrm{DES}(\pi)} i$. Let $\mathrm{SgnMaj}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{des}(\pi)}$ denote the signed major index enumerator over $\mathfrak{S}_n$. Gessel and Simion (see [21]) showed that enumerating the major index with signs gives interesting results. For a positive integer n, let $[n]_q = \dfrac{1-q^n}{1-q}$. In particular, they showed the following.

Theorem 9 *(Gessel and Simion). For positive integers n, we have*

$$\mathrm{SgnMaj}_n(q) = [1]_q[2]_{-q}[3]_q[4]_{-q} \cdots [n]_{(-1)^{n-1}q}.$$

For $\pi \in \mathfrak{S}_n$, recall that $\mathrm{altmaj}(\pi) = \sum_{i \in \mathrm{ALTDES}} i$ is the sum of the alternate descent indices of π. For even positive integers n, the following result follows from Wachs' involution.

Remark 10. For even positive integers n, Wachs' involution also gives the following:

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{altmaj}(\pi)} = [1]_q[2]_{-q}[3]_q[4]_{-q} \cdots [n]_{(-1)^{n-1}q}.$$

4. Type B Coxeter groups

Let $\mathfrak{B}_n$ be the set of permutations of $[\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$ satisfying $\pi(-i) = -\pi(i)$. The group $\mathfrak{B}_n$ is referred to as the hyperoctahedral group or the group of signed permutations on $[n]$. For $\pi \in \mathfrak{B}_n$ and for $1 \leq i \leq n$, we alternately denote $\pi(i)$ as π_i and for $1 \leq k \leq n$, we denote $-k$ by $\overline{k}$. For $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{B}_n$, define $\mathrm{DES}_B(\pi) = \{i : \pi_i > \pi_{i+1}, i \geq 0\}$ to be its set of descents and let $\mathrm{des}_B(\pi) = |\mathrm{DES}_B(\pi)|$. Define $\mathrm{Negs}(\pi) = \{\pi_i : i > 0, \pi_i < 0\}$ as the set of elements of π which occur with a negative sign. The following definition of type B inversions is known (see Petersen's book [16, Page 294]):

$$\mathrm{inv}_B(\pi) = |\{1 \leq i < j \leq n : \pi_i > \pi_j\}| + |\{1 \leq i < j \leq n : -\pi_i > \pi_j\}| + |\mathrm{Negs}(\pi)|. \tag{14}$$

We refer to $\mathrm{inv}_B(\pi)$ alternatively as the *length of* $\pi \in \mathfrak{B}_n$. For $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{B}_n$, let $\pi_0 = 0$, and define

$$\mathrm{ALTDES}_B(\pi) = \{2i : \pi_{2i} < \pi_{2i+1}, i \geq 0\} \cup \{2i+1 : \pi_{2i+1} > \pi_{2i+2}, i \geq 0\},$$

$$\mathrm{ALTDESRM}_B(\pi) = \{2i : \pi_{2i} < \pi_{2i+1}, i \geq 1\} \cup \{2i+1 : \pi_{2i+1} > \pi_{2i+2}, i \geq 0\}.$$

Let $\mathrm{altdes}_B(\pi) = |\mathrm{ALTDES}_B(\pi)|$, $\mathrm{altdesrm}_B(\pi) = |\mathrm{ALTDESRM}_B(\pi)|$ and define the type B alternating Eulerian polynomial by

$$\hat{A}_n^B(t) = \sum_{\pi \in \mathfrak{B}_n} t^{\mathrm{altdes}_B(\pi)}.$$

The polynomial $\hat{A}_n^B(t)$ was studied by Ma, Fang, Mansour and Yeh in [7] where a recurrence relation satisfied by these polynomials was given. As $\pi_0 = 0$, position 0 could contribute to even ascents. We are again interested in enumerating alternating descents with (the type B) sign taken into account. Thus, define

$$\mathrm{SgnBAltDes}_n(t) = \sum_{\pi\in\mathfrak{B}_n} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdes}_B(\pi)}, \ \mathrm{SgnBAltDesRm}_n(t) = \sum_{\pi\in\mathfrak{B}_n} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdesrm}_B(\pi)}. \tag{15}$$

Consider the identity permutation $\mathrm{id} = 1, 2, \ldots, n-1, n \in \mathfrak{B}_n$ and the set $\mathfrak{B}_n(\mathrm{id}) = \{\pi \in \mathfrak{B}_n : |\pi| = \mathrm{id}\}$. We clearly have $|\mathfrak{B}_n(\mathrm{id})| = 2^n$. We first show that the signed enumeration of alternating descents outside the set $\mathfrak{B}_n(\mathrm{id})$ is 0.

Lemma 11. *For positive integers n, we have $\sum_{\pi\in\mathfrak{B}_n\setminus\mathfrak{B}_n(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdes}_B(\pi)} = 0$.*

Proof. Let $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{B}_n \setminus \mathfrak{B}_n(\mathrm{id}))$. As $\pi \notin \mathfrak{B}_n(\mathrm{id})$, there exists $1 \le j < n$ such that $|\pi_j| \ne j$. Let $1 \le k \le n$ be the smallest index such that $|\pi_k| \ne k$. Let $|\pi_r| = k$. Thus $0 \le r-1$. Further, as $\pi \in \mathfrak{B}_n \setminus \mathfrak{B}_n(\mathrm{id})$, we have $r \le n$. Define f by flipping the sign of k or $\overline{k}$ in π. Thus, define $f : (\mathfrak{B}_n \setminus \mathfrak{B}_n(\mathrm{id})) \mapsto (\mathfrak{B}_n \setminus \mathfrak{B}_n(\mathrm{id}))$ by

$$f(\pi_1, \ldots, \pi_{r-1}, k, \pi_{r+1}, \ldots, \pi_n) = \pi_1, \ldots, \pi_{r-1}, \overline{k}, \pi_{r+1}, \ldots, \pi_n.$$

For example, if $\pi = 1, \overline{2}, 3, 5, \overline{4}$, then $\overline{4}$ is the smallest k with $|\pi_4| \ne 4$. Thus, $f(\pi) = 1, \overline{2}, 3, 5, 4$. For $\pi \in \mathfrak{B}_n$, it is easy to check that $\mathrm{inv}_B(\pi) \not\equiv \mathrm{inv}_B(f(\pi)) \pmod 2$. That is, the map f flips the parity of the type B length (see [19, Lemma 3]). Recall r is the index with $|\pi_r| = |k|$. Recall that we have $r-1 \ge 0$ and $r \le n$. Moreover, as $|\pi_{r-1}| > k$ we have $\pi_{r-1} > k$ if and only if $\pi_{r-1} > \overline{k}$. That is, $r-1$ is an alternating descent of π if and only if $r-1$ is an alternating descent of $f(\pi)$. When $r \ne n$, as $|\pi_{r+1}| > k$, we have $\pi_{r+1} > k$ if and only if $\pi_{r+1} > \overline{k}$. A similar case when $r = n$ is also easy to see. Thus, the map f preserves the number of alternating descents and is a sign-reversing involution. This completes the proof. □

Remark 12. The map f constructed in Lemma 11 not only satisfies $\mathrm{altdes}_B(\pi) = \mathrm{altdes}_B(f(\pi))$, but it also satisfies equality of the appropriate sets. That is, for all $\pi \in \mathfrak{B}_n$,

$$\mathrm{ALTDES}_B(\pi) = \mathrm{ALTDES}_B(f(\pi)) \qquad \text{and} \qquad \mathrm{ALTDESRM}_B(\pi) = \mathrm{ALTDESRM}_B(f(\pi)).$$

Theorem 13. *For positive integers n, with $m = \lfloor (n+1)/2 \rfloor$, we have*

$$\mathrm{SgnBAltDes}_n(t) = (-1)^m (1-t)^n.$$

Proof. By Lemma 11 for positive integers n, we have

$$\sum_{\pi\in\mathfrak{B}_n} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdes}_B(\pi)} = \sum_{\pi\in\mathfrak{B}_n(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdes}_B(\pi)}. \tag{16}$$

To compute $\sum_{\pi\in\mathfrak{B}_n(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdes}_B(\pi)}$, note that one can inductively get all permutations in $\mathfrak{B}_n(\mathrm{id})$ by either inserting 'n' or $\overline{n}$ at the end of permutations of $\mathfrak{B}_{n-1}(\mathrm{id})$. Let $\pi \in \mathfrak{B}_{n-1}(\mathrm{id})$. We will consider $\pi, n \in \mathfrak{B}_n(\mathrm{id})$ or $\pi, \overline{n} \in \mathfrak{B}_n(\mathrm{id})$.

When $n-1$ is even, for $\pi \in \mathfrak{B}_{n-1}(\mathrm{id})$, it is easy to see the following: that $\mathrm{altdes}_B(\pi, n) = \mathrm{altdes}_B(\pi) + 1$ and $\mathrm{inv}_B(\pi, n) = \mathrm{inv}_B(\pi)$; and that $\mathrm{altdes}_B(\pi, \overline{n}) = \mathrm{altdes}_B(\pi)$ and $\mathrm{inv}_B(\pi, \overline{n}) = \mathrm{inv}_B(\pi) + 2n - 1$. Therefore, we have

$$\sum_{\pi\in\mathfrak{B}_n(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdes}_B(\pi)} = (t-1) \sum_{\pi\in\mathfrak{B}_{n-1}(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdes}_B(\pi)}. \tag{17}$$

When $n-1$ is odd, for $\pi \in \mathfrak{B}_{n-1}(\mathrm{id})$, it is easy to see the following: that $\mathrm{altdes}_B(\pi, n) = \mathrm{altdes}_B(\pi)$ and $\mathrm{inv}_B(\pi, n) = \mathrm{inv}_B(\pi)$; and that $\mathrm{altdes}_B(\pi, \overline{n}) = \mathrm{altdes}_B(\pi) + 1$ and $\mathrm{inv}_B(\pi, \overline{n}) = \mathrm{inv}_B(\pi) + 2n - 1$. Therefore, we have

$$\sum_{\pi\in\mathfrak{B}_n(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdes}_B(\pi)} = (1-t) \sum_{\pi\in\mathfrak{B}_{n-1}(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdes}_B(\pi)}. \tag{18}$$

The following base cases are easy to check:

$$\sum_{\pi\in\mathfrak{B}_1(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdes}_B(\pi)} = -1(1-t), \quad \text{and} \quad \sum_{\pi\in\mathfrak{B}_2(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdes}_B(\pi)} = -(1-t)^2.$$

Combining (16), (17) and (18) completes the proof. □

Theorem 14. *For positive integers n, we have*

$$\mathrm{SgnBAltDesRm}_n(t) = 0.$$

Proof. By Remark 12, we have

$$\sum_{\pi \in \mathfrak{B}_n \setminus \mathfrak{B}_n(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdesrm}_B(\pi)} = 0.$$

We now consider $\sum_{\pi \in \mathfrak{B}_n(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdesrm}_B(\pi)}$. For this, consider the map $g : \mathfrak{B}_n(\mathrm{id}) \mapsto \mathfrak{B}_n(\mathrm{id})$ defined by $g(\pi_1, \ldots, \pi_n) = \overline{\pi_1}, \ldots, \pi_n$. The map g clearly flips the parity of type B inversions and preserves $\mathrm{ALTDESRM}_B(\pi)$. Thus, we have $\sum\limits_{\pi \in \mathfrak{B}_n(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altdesrm}_B(\pi)} = 0$, completing the proof. □

4.1. A multivariate version

We next give a multivariate version of Theorem 13 and Theorem 14. As in subsection 3.3, for $\pi \in \mathfrak{B}_n$, define the following monomial $m_\pi^B = \prod_{i \in \mathrm{ALTDES}_B(\pi)} t_i$ and $m_\pi^{RM,B} = \prod_{i \in \mathrm{ALTDESRM}_B(\pi)} t_i$, where the t_i's are commuting variables. Define the following multivariate polynomials

$$\mathrm{SgnBMultAltDes}_n(t_0, t_1, \ldots, t_{n-1}) = \sum_{\pi \in \mathfrak{B}_n} (-1)^{\mathrm{inv}_B(\pi)} m_\pi^B,$$

and

$$\mathrm{SgnBMultAltDesRm}_n(t_0, t_1, \ldots, t_{n-1}) = \sum_{\pi \in \mathfrak{B}_n} (-1)^{\mathrm{inv}_B(\pi)} m_\pi^{RM,B}.$$

We have the following multivariate version of Theorem 13.

Theorem 15. *For positive integers n, we have*

$$\mathrm{SgnBMultAltDes}_n(t_0, t_1, \ldots, t_{n-1}) = (-1)^m \prod_{r=0}^{n-1} \left(1 - t_r\right),$$

where $m = \lfloor (n+1)/2 \rfloor$.

Proof. By Remark 12, we have $\mathrm{ALTDES}_B(\pi) = \mathrm{ALTDES}_B(f(\pi))$. Thus,

$$\sum_{\pi \in \mathfrak{B}_n \setminus \mathfrak{B}_n(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} m_\pi^B = 0.$$

Hence it suffices to compute $\sum_{\pi \in \mathfrak{B}_n(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} m_\pi^B$. Proceeding as in the proof of Theorem 13, we note the following analogues of (17) and (18). When $n-1$ is even, we have

$$\sum_{\pi \in \mathfrak{B}_n(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} m_\pi^B = (t_{n-1} - 1) \sum_{\pi \in \mathfrak{B}_{n-1}(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} m_\pi^B, \tag{19}$$

and when $n-1$ is odd, we have

$$\sum_{\pi \in \mathfrak{B}_n(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} m_\pi^B = (1 - t_{n-1}) \sum_{\pi \in \mathfrak{B}_{n-1}(\mathrm{id})} (-1)^{\mathrm{inv}_B(\pi)} m_\pi^B. \tag{20}$$

It is easy to see that the base cases when $n = 1, 2$ are $\mathrm{SgnBMultAltDes}_1(t_0, t_1) = (-1)(1 - t_0)$ and $\mathrm{SgnBMultAltDes}_2(t_0, t_1) = (-1)(1 - t_0)(1 - t_1)$. Combining these with (19) and (20) completes the proof. □

The proof of Theorem 14 also gives us the following refinement whose proof we omit.

Theorem 16. *For positive integers n, we have*

$$\mathrm{SgnBMultAltDesRm}_n(t_0, t_1, \ldots, t_{n-1}) = 0.$$

Corollary 17. *For* $\pi \in \mathfrak{B}_n$, *define* $\mathrm{altmaj}_B(\pi) = \sum_{i \in \mathrm{ALTDES}_B(\pi)}$. *In a similar manner, define* $\mathrm{altmajrm}_B(\pi) = \sum_{i \in \mathrm{ALTDESRM}_B(\pi)} i$. *We have*

$$\sum_{\pi \in \mathfrak{B}_n} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altmaj}_B(\pi)} = 0 \text{ and } \sum_{\pi \in \mathfrak{B}_n} (-1)^{\mathrm{inv}_B(\pi)} t^{\mathrm{altmajrm}_B(\pi)} = 0.$$

Proof. Set $t_0 = 1$ and $t_i = t^i$ for $1 \le i \le n-1$ in Theorem 15 and in Theorem 16. □

4.2. A type B counterpart of Theorem 4

In this subsection, we give a type B counterpart of Theorem 4. Though we do not need it for any proof in this work, we present it as it may find applications elsewhere. Define the following polynomial

$$B_n(t) = \sum_{\pi \in \mathfrak{B}_n} t^{\mathrm{des}_B(\pi)}.$$

Theorem 18. *For even positive integers k, we have the following identity for the type B Eulerian polynomials $B_k(t)$:*

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} B_r(t) B_{k-r}(t) = \frac{2^{k+1} t A_k(t)}{(1+t)}. \tag{21}$$

Proof. Our proof will be similar to the proof of Theorem 4. Consider the exponential generating functions for

$$B(t,z) = \sum_{k \geq 0} B_k(t) \frac{z^k}{k!},$$

and recall

$$S(t,z) = \sum_{k \geq 0} A_k(t) \frac{z^k}{k!}.$$

We proceed by using egfs in a similar way to that of Theorem 3. Subsequently, to show (21), we need to prove the following:

$$B(t,z)B(t,-z) = \frac{t}{1+t}\left[S(t,2z) + S(t,-2z)\right] + \left[1 - \frac{2t}{1+t}\right]. \tag{22}$$

Recall that $S(t,z) = \dfrac{t-1}{t-e^{z(t-1)}}$ and it is known that $B(t,z) = \dfrac{(t-1)e^{z(t-1)}}{t-e^{-2z(t-1)}}$ (see [16, Theorem 13.3]). The proof of (22) now follows by simple algebraic manipulation and so we skip the details. □

5. Type D Coxeter groups

Let $\mathfrak{D}_n \subseteq \mathfrak{B}_n$ be the subset of type B permutations that have an even number of negative signs. For $\pi = \pi_1, \pi_2, \ldots, \pi_n \in \mathfrak{D}_n$, the following combinatorial definition of type D inversions is well known (see, for example, Petersen's book [16, Page 302]):

$$\mathrm{inv}_D(\pi) = \mathrm{inv}_A(\pi) + |\{1 \leq i < j \leq n : -\pi_i > \pi_j\}|.$$

Here $\mathrm{inv}_A(\pi)$ is computed with respect to the usual order on $\mathbb{Z}$. Let $\pi \in \mathfrak{D}_n$. We can also think of π as an element of $\mathfrak{B}_n$. Similarly, if $\pi \in \mathfrak{B}_n$, as the above definition of $\mathrm{inv}_D(\pi)$ is combinatorial, we can use it to define $\mathrm{inv}_D(\pi)$. From the above definition of $\mathrm{inv}_D(\pi)$ and the definition of $\mathrm{inv}_B(\pi)$ that we saw in Section 4, we get the following simple corollary which we will need later.

Corollary 19. *Let $\pi \in \mathfrak{B}_n$. Then, $\mathrm{inv}_B(\pi) = \mathrm{inv}_D(\pi) + |\mathrm{Negs}(\pi)|$.*

In this section, we are interested in enumerating Type D alternating descents with sign taken into account. For $\pi \in \mathfrak{D}_n$, we define its type D alternating descent set of π to be identical to the type B alternating descent set of π. That is, $\mathrm{ALTDES}_D(\pi) = \mathrm{ALTDES}_B(\pi)$ and let $\mathrm{altdes}_D(\pi) = |\mathrm{ALTDES}_D(\pi)|$. Consider the following polynomial:

$$\mathrm{SgnDAltDes}_n(t) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}_D(\pi)} t^{\mathrm{altdes}_D(\pi)}. \tag{23}$$

Remark 20. In the above definition, the exponent of -1 could have been $\mathrm{inv}_B(\pi)$ as well.

We prefer to give a proof of the multivariate version this time and infer the univariate variant by setting values to variables. Thus, for a permutation $\pi \in \mathfrak{D}_n$, define the monomial $m_\pi^D = \prod_{\imath \in \mathrm{ALTDES}_D(\pi)} t_i$ where the t_is are commuting variables. Define the following multivariate polynomial

$$\text{SgnDMultAltDes}_n(t_0,t_1,\ldots,t_{n-1}) = \sum_{\pi\in\mathfrak{D}_n} (-1)^{\text{inv}_D(\pi)} m_\pi^D.$$

Theorem 21. *For even positive integers $n = 2k$, we have*

$$\text{SgnDMultAltDes}_n(t_0,t_1,\ldots,t_{2k-1}) = (-1)^k \prod_{r=0}^{2k-1}\Big(1-t_r\Big). \tag{24}$$

For odd positive integers $n = 2k+1$, we have

$$\text{SgnDMultAltDes}_n(t_0,t_1,\ldots,t_{2k}) = (-1)^k t_{2k} \prod_{r=0}^{2k-1}\Big(1-t_r\Big). \tag{25}$$

Proof. We first consider the case when $n = 2k$. Let $\pi = \pi_1,\ldots,\pi_n \in \mathfrak{B}_n \setminus \mathfrak{D}_n$. As n is even, there exists $i \in [n]$ such that the letters $2i-1$ and $2i$ are of opposite signs in π. Thus, π is of the form $\pi_1,\ldots,\pi_i = x,\ldots,\pi_j = y,\ldots,\pi_n$ where the indices i, j may be consecutive, where $\{|x|,|y|\} = \{2i-1, 2i\}$ and x, y are of different signs. Define

$$f(\pi) = \pi_1,\ldots,\pi_i = \overline{y},\ldots,\pi_j = \overline{x},\ldots,\pi_n.$$

It is easy to verify that the map f preserves the indices where alternating descents occur. Moreover, this map clearly changes the parity of inv_D. Thus, we have

$$\sum_{\pi\in\mathfrak{B}_n\setminus\mathfrak{D}_n} (-1)^{\text{inv}_D(\pi)} m_\pi^D = 0.$$

By Theorem 15, the proof of (24) is now complete. We now consider the case when n is odd, say $n = 2k+1$. Let $\mathfrak{D}_n^1 = \{\pi \in \mathfrak{D}_n : \pi_n = n\}$. We show that $\sum_{\pi\in\mathfrak{D}_n\setminus\mathfrak{D}_n^1} (-1)^{\text{inv}_D(\pi)} m_\pi^D = 0$. For this, we break $\mathfrak{D}_n \setminus \mathfrak{D}_n^1$ as the disjoint union of following three subsets:

1. $\mathfrak{D}_n^2 = \{\pi \in \mathfrak{D}_n : \pi_n = \overline{n}\}$,
2. $\mathfrak{D}_n^3 = \{\pi \in \mathfrak{D}_n : |\pi_n| < n$, there exists $i \le k$ such that $\pm(2i-1)$ and $\pm(2i)$ are not adjacent$\}$,
3. $\mathfrak{D}_n^4 = \{\pi \in \mathfrak{D}_n : |\pi_n| < n$, for all $i \le k$, $\pm(2i-1)$ and $\pm(2i)$ are adjacent$\}$.

We first note that

$$\sum_{\pi\in\mathfrak{D}_n^2} (-1)^{\text{inv}_D(\pi)} m_\pi^D = \sum_{\pi\in\mathfrak{B}_{n-1}\setminus\mathfrak{D}_{n-1}} (-1)^{\text{inv}_D(\pi)} m_\pi^D = 0.$$

This follows by observing that inserting $\overline{n}$ in the last place of any permutation $\pi \in \mathfrak{B}_{n-1} \setminus \mathfrak{D}_{n-1}$ does not change ALTDES and the parity of inv_D.

We also clearly have $\sum_{\pi\in\mathfrak{D}_n^3} (-1)^{\text{inv}_D(\pi)} m_\pi^D = 0$ as we can swap the letters x and y where $\{|x|,|y|\} = \{2i-1, 2i\}$ and this swapping does not change the set of alternating descents but change the parity of inv_D.

We now consider the set $\mathfrak{D}_n^4$. For any $\pi \in \mathfrak{D}_n^4$, we must have $i \ge 1$ such that $\{|\pi_{2i-1}|, |\pi_{2i}|\} \ne \{2i-1, 2i\}$. We take the smallest such i. So, π is of the form $\pi_1,\ldots,\pi_j = x,\ \pi_{j+1} = y,\ldots,\pi_n$ where $j > 2i-1$ and $\{|x|,|y|\} = \{2i-1, 2i\}$. Define

$$f(\pi) = \pi_1,\ldots,\pi_j = \overline{y}, \pi_{j+1} = \overline{x},\ldots,\pi_n.$$

It is easy to verify that the map f preserves the indices where alternating descents occur. Moreover, this map clearly changes the parity of inv_D. Therefore, here also we have $\sum_{\pi\in\mathfrak{D}_n^4} (-1)^{\text{inv}_D(\pi)} m_\pi^D = 0$.

Thus, we are only left with enumerating $\sum_{\pi\in\mathfrak{D}_n^1} (-1)^{\text{inv}_D(\pi)} m_\pi^D$. As $n = 2k+1$ and $\pi_n = n$ the $2k$-th position is always an alternating descent. Thus,

$$\sum_{\pi\in\mathfrak{D}_n^1} (-1)^{\text{inv}_D(\pi)} m_\pi^D = t_{2k}\text{SgnDMultAltDes}_n(t_0,t_1,t_2,t_3,\ldots,t_{2k-1}).$$

By using (24), the proof of (25) is now complete. □

Enumerating m_π^D over $\mathfrak{B}_n \setminus \mathfrak{D}_n$ gives us nice results and so we define

$$\text{Sgn(B}-\text{D)MultAltDes}_n(t_0,t_1,\ldots,t_{n-1}) = \sum_{\pi\in\mathfrak{B}_n\setminus\mathfrak{D}_n} (-1)^{\text{inv}_D(\pi)} m_\pi^D.$$

Corollary 22. *When $n \geq 1$, we have*

$$\mathrm{Sgn(B-D)MultAltDes}_n(t_0, t_1, \ldots, t_{n-1}) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ (-1)^{(n-1)/2} \prod_{r=0}^{n-1}(1-t_r) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. When n is even, the result follows from the proof of Theorem 21. When n is odd, the result follows by using Theorem 15 and Theorem 21. □

We end with the following corollary obtained by setting $t_i = t$ for all i in Theorem 21.

Corollary 23. *When $n \geq 1$, we have*

$$\mathrm{SgnDAltDes}_n(t) = \begin{cases} (-1)^k(1-t)^{2k} & \text{when } n = 2k, \\ (-1)^k t(1-t)^{2k} & \text{when } n = 2k+1. \end{cases} \tag{26}$$

We end this work with two remarks: the first on a type D version of Theorem 4 and another on Central Limit Theorems for alternating descents.

Remark 24. Using Brenti's result (see [3, Corollary 4.8]) that connects the Eulerian polynomials of types A, B and D, and both Theorem 4 and Theorem 18, one can get a type D counterpart of Theorem 4. We do not do so as it gets somewhat complicated to state. It would be interesting to see if a cleaner statement similar to Theorem 4 or Theorem 18 can be made.

Remark 25 *(Central Limit Theorems).* It is not clear if there is a Central Limit Theorem (CLT henceforth) for alternating descents over $\mathfrak{S}_n$. However, if there is a CLT over $\mathfrak{S}_n$, then using Theorem 2 and results from [6], there is a CLT for alternating descents over $\mathcal{A}_n$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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