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Smith Normal Form of a distance matrix inspired by the four-point condition

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ABSTRACT

The *four point theorem* is a condition for distances to arise from trees. Based on this condition, for any tree T on n vertices, we associate an $\binom{n}{2} \times \binom{n}{2}$ matrix M_T.
We find the rank and the Smith Normal Form (SNF) of the matrix M_T and show that it only depends on n and is independent of the structure of the tree T. Curiously, the non-zero part of the SNF of M_T coincides with the SNF of the distance matrix of T. Many such "tree independent" results are known and this result is yet another such result.
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1. Introduction

Let G be a connected graph with edges having nonnegative weights. Let D be its distance matrix whose (i, j)-th entry is the shortest path distance between i and j in G. Given G, finding D is easy with several all-pairs shortest path distance algorithms known. See for example Cormen, Leiserson, Rivest and Stein [7].

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The reverse question is this: given a positive integral matrix D, is there a connected graph G which induces the distances in D? More generally, given a matrix D, we want to know if there exists a graph G with weights on its edges and a subset S of the vertices of G such that the restriction of the shortest path distance (with respect to the edge weights) matrix D_G of G to the vertices in S being equal to D. This difficult question was raised by Hakimi and Yau [9] and has connections to internet tomography, see Chung, Garrett, Graham and Shallcross [6].

Suppose, we add the restriction that we would like the underlying graph to be a tree. Does some condition on the entries of D ensure that the matrix D arises from tree distances? For trees, Buneman [5] showed that the entries of D need to satisfy the famous *four point condition (4PC henceforth)*. See Baldisseri [1] for more on this work and also see the work of Pachter and Speyer [11].

The 4PC is as follows: consider four vertices i, j, k and ℓ and the three terms in $P = \{d_{i,j} + d_{k,\ell}, d_{i,k} + d_{j,\ell}, d_{i,\ell} + d_{j,k}\}$, considered as a multi-set. The 4PC states that for all choices of four distinct vertices, the maximum element in P appears at least twice. Note that if among i, j, k and ℓ, we only have three distinct vertices (say if $i = j$), then the 4PC gives us the triangle inequality. Thus, the 4PC is stronger than the triangle inequality.

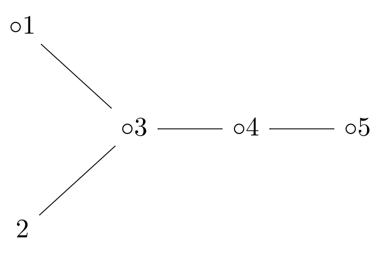
We define a matrix M_T inspired by the 4PC and obtain its rank and its invariant factors. The construction of the matrix M_T does not have any justification other than the rich structure it seems to have.

For the rest of this paper we assume that we have a tree T with distance matrix D. For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and let T have vertex set $V(T) = [n]$. Let $t = \binom{n}{2}$. We assume $n \geq 4$ in what follows as otherwise the trees are isomorphic (hence having the same distance matrix). For vertices $x, y \in V(T)$, let $d(x, y)$ or d_{xy} be the distance between x and y.

Form a $t \times t$ matrix M_T as follows. The rows and the columns of M_T are indexed by pairs $e = \{w, x\}$, $f = \{y, z\}$, where $w, x, y, z \in V(T)$ with $w < x, y < z$. We define the (e, f)-element of M as

$$m(e, f) = \min(d_{wx} + d_{yz}, d_{wy} + d_{xz}, d_{wz} + d_{xy}).$$

Example. Consider the tree T



Then

$$M_T = \begin{array}{c} \\ 13 \\ 23 \\ 34 \\ 45 \\ 12 \\ 14 \\ 15 \\ 24 \\ 25 \\ 35 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 13 & 23 & 34 & 45 & 12 & 14 & 15 & 24 & 25 & 35 \end{array} \\ \left[\begin{array}{cccccccccc} 0 & 2 & 2 & 2 & 1 & 1 & 2 & 3 & 4 & 3 \\ 2 & 0 & 2 & 2 & 1 & 3 & 4 & 1 & 2 & 3 \\ 2 & 2 & 0 & 2 & 3 & 1 & 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 0 & 3 & 3 & 2 & 3 & 2 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 & 3 & 2 & 3 & 4 \\ 1 & 3 & 1 & 3 & 2 & 0 & 1 & 2 & 3 & 2 \\ 2 & 4 & 2 & 2 & 3 & 1 & 0 & 3 & 2 & 1 \\ 3 & 1 & 1 & 3 & 2 & 2 & 3 & 0 & 1 & 2 \\ 4 & 2 & 2 & 2 & 3 & 3 & 2 & 1 & 0 & 1 \\ 3 & 3 & 1 & 1 & 4 & 2 & 1 & 2 & 1 & 0 \end{array}\right] \end{array}.$$

Several matrices have been associated with trees T having vertex set $[n]$. Many results show that parameters of the matrices only depend on n and are independent of the structure of T. Graham and Pollak in [8] showed that the determinant of D is only a function of n. Bapat, Lal and Pati [2] generalized this to the determinant of the q-analogue of the distance matrix and the exponential distance matrix. Bapat and Sivasubramanian in [3] later generalized this to the second immanant of the exponential distance matrix. Bapat and Sivasubramanian in [4] also showed that the Smith Normal form of the exponential distance matrix of T is independent of the structure of T. In this paper, we present more results of this type involving the matrix M_T. Our main results are the following.

Theorem 1. *Let T be a tree on n vertices. Then, the rank of M_T equals n.*

We next find the invariant factors of the matrix M_T and show that these are also independent of the structure of the tree T. Recalling $t = \binom{n}{2}$, we prove the following.

Theorem 2. *Let T be a tree on n vertices. The invariant factors of the matrix M_T are*

$$\overbrace{0,\dots,0}^{t-n \text{ times}}, 1, 1, \overbrace{2,\dots,2}^{n-3 \text{ times}}, 2(n-1).$$

The invariant factors of the distance matrix D of a tree T on n vertices are known (see, for example, Hou and Woo [10]). It is a curious coincidence that the invariant factors of the distance matrix D are identical to the non-zero terms in Theorem 2. The proof of Theorem 1 appears in Section 2 and the proof of Theorem 2 appears in Section 3.

2. Rank of M_T

Lemma 3. *Let T be a tree with vertex set $V(T) = [n]$. Then, the submatrix of M with rows and columns indexed by $E(T)$ is given by $K = 2(J - I)$.*

Proof. Clearly $m(e_i, e_i) = 0, i = 1, \ldots, n-1$, and hence the diagonal elements of K are zero. Let $e_i, e_j \in E(T), i \neq j$, where $e_i = \{w, x\}$ and $e_j = \{y, z\}$. We have $d_{wx} = d_{yz} = 1$ and $d_{wy} + d_{xz} \geq 2, d_{wz} + d_{yx} \geq 2$. Hence

$$m(e_i, e_j) = \min(d_{wx} + d_{yz}, d_{wy} + d_{xz}, d_{wz} + d_{xy}) = 2.$$

It follows that the submatrix of M with rows and columns indexed by $E(T)$ is given by $K = 2(J - I)$. □

The proof of the next result is easy and is omitted.

Lemma 4. *The $(n-1) \times (n-1)$ matrix $K = 2(J - I)$ is nonsingular and $2K^{-1} = -I + \frac{1}{n-2}J$.*

We introduce some notation. Recall that the edges of the tree are denoted $e_1, \ldots, e_{n-1}$. Let $e = \{i, j\}$ be an edge of the complete graph on $V(T)$. We define the column vector x_e as follows. The coordinates of x_e are indexed by $e_1, \ldots, e_{n-1}$. We set the ℓ-th coordinate of x_e to be $d_{ij} - 1$, if e_ℓ is on the ij-path and equal to $d_{ij} + 1$, otherwise. It may be observed that

$$x_e' = [m(e, e_1), \ldots, m(e, e_{n-1})]. \tag{1}$$

Lemma 5. *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $e = \{i, j\}, i, j \in V(T), i < j$. Then $x_e'\mathbf{1} = (n-3)d_{ij} + n - 1$.*

Proof. There are d_{ij} edges on the ij-path, and for each such edge, the corresponding coordinate of x_e is $d_{ij} - 1$. Similarly there are $n - 1 - d_{ij}$ edges not on the ij-path, and for each such edge, the corresponding coordinate of x_e is $d_{ij} + 1$. Hence the sum of the coordinates of x_e is given by

$$\begin{aligned} x_e'\mathbf{1} &= d_{ij}(d_{ij} - 1) + (n - 1 - d_{ij})(d_{ij} + 1) \\ &= d_{ij}^2 - d_{ij} + nd_{ij} + n - d_{ij} - 1 - d_{ij}^2 - d_{ij} \\ &= (n-3)d_{ij} + n - 1, \end{aligned}$$

and the proof is complete. □

Lemma 6. *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $e = \{i, j\}, f = \{u, v\}$, where $i, j, u, v \in V(T), i < j, u < v$. Then*

$$x_e'K^{-1}x_f = -\frac{1}{2}x_e'x_f + \frac{1}{2(n-2)}((n-3)d_{ij} + n - 1)((n-3)d_{uv} + n - 1).$$

Proof. Using Lemmas 4 and 5 we get

$$\begin{aligned} x_e' K^{-1} x_f &= \frac{1}{2} x_e' \left(-I + \frac{1}{n-2} J \right) x_f \\ &= \frac{1}{2} \left(-x_e' x_f + \frac{1}{n-2} (x_e' \mathbf{1})(x_f' \mathbf{1}) \right) \\ &= -\frac{1}{2} x_e' x_f + \frac{1}{2(n-2)} ((n-3)d_{ij} + n - 1)((n-3)d_{uv} + n - 1). \end{aligned}$$

This completes the proof. □

Corollary 7. *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $e = \{i, j\}, i, j \in V(T), i < j$, and let $d = d_{ij}$. Then*

$$x_e' K^{-1} x_e = \frac{(n-1)(d-1)^2}{2(n-2)}. \tag{2}$$

Proof. By Lemma 6 we have

$$x_e' K^{-1} x_e = -\frac{1}{2} x_e' x_e + \frac{1}{2(n-2)} ((n-3)d + n - 1)^2. \tag{3}$$

Note that x_e has d coordinates equal to $d - 1$ and $n - 1 - d$ coordinates equal to $d + 1$. Therefore

$$x_e' x_e = d(d-1)^2 + (n-1-d)(d+1)^2. \tag{4}$$

From (3) and (4) we get

$$x_e' K^{-1} x_e = -\frac{1}{2} \left(d(d-1)^2 + (n-1-d)(d+1)^2 - \frac{((n-3)d + n - 1)^2}{n-2} \right). \tag{5}$$

We get the result (2) from (5) after simplification. □

Theorem 8. *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $e = \{i, j\}, f = \{u, v\}$, where $i, j, u, v \in V(T), i < j, u < v$. Then the (e, f)-element of M is given by*

$$m(e, f) = x_e' K^{-1} x_f - \sqrt{(x_e' K^{-1} x_e)(x_f' K^{-1} x_f)}. \tag{6}$$

Proof. Let $d_{ij} = d, d_{uv} = s$. It follows from Lemma 6 and Corollary 7 that

$$x_e' K^{-1} x_f - \sqrt{(x_e' K^{-1} x_e)(x_f' K^{-1} x_f)}$$

$$= -\frac{1}{2}x'_e x_f + \frac{1}{2(n-2)}((n-3)d+n-1)((n-3)s+n-1) - \frac{n-1}{2(n-2)}(d-1)(s-1)$$
$$= -\frac{1}{2}x'_e x_f + \frac{1}{2}((d+1)(s+1)(n-1) - 4ds). \tag{7}$$

We introduce some notation. Let $\mathscr{P}(i,j)$ denote the set of edges on the ij-path. From the structure of the vectors x_e, x_f it follows that

$$\begin{aligned} x'_e x_f = {} & |\mathscr{P}(i,j) \cap \mathscr{P}(u,v)|(d-1)(s-1) + |\mathscr{P}(i,j) \setminus \mathscr{P}(u,v)|(d-1)(s+1) \\ & + |\mathscr{P}(u,v) \setminus \mathscr{P}(i,j)|(d+1)(s-1) \\ & + |E(T) \setminus \mathscr{P}(i,j) \setminus \mathscr{P}(u,v)|(d+1)(s+1). \end{aligned} \tag{8}$$

We consider two cases:

Case (i) The ij-path and the uv-path are edge-disjoint.

Then $|\mathscr{P}(i,j) \cap \mathscr{P}(u,v)| = 0, |\mathscr{P}(i,j) \setminus \mathscr{P}(u,v)| = d, |\mathscr{P}(u,v) \setminus \mathscr{P}(i,j)| = s$, and $|E(T) \setminus \mathscr{P}(i,j) \setminus \mathscr{P}(u,v)| = n-1-d-s$. Substituting in (8) we get $x'_e x_f = ds(d-1) + s(s-1)(d+1) + (n-1-d-s)(d+1)(s+1)$ and then it follows from (7) that

$$x'_e K^{-1} x_f - \sqrt{(x'_e K^{-1} x_e)(x'_f K^{-1} x_f)} = d + s.$$

Note that in this case $m(e,f) = d + s$ as well and hence the result is proved.

Case (ii) The ij-path and the uv-path are not edge-disjoint.

Let $|\mathscr{P}(i,j) \cap \mathscr{P}(u,v)| = t$. Then $|\mathscr{P}(i,j) \setminus \mathscr{P}(u,v)| = d-t, |\mathscr{P}(u,v) \setminus \mathscr{P}(i,j)| = s-t$, and $|E(T) \setminus \mathscr{P}(i,j) \setminus \mathscr{P}(u,v)| = n-1-d-s+t$. Substituting in (8) we get $x'_e x_f = t(d-1)(s-1) + (d-t)(d-1)(s+1) + (s-t)(d+1)(s-1) + (n-1-d-s+t)(d+1)(s+1)$ and then it follows from (7) that

$$x'_e K^{-1} x_f - \sqrt{(x'_e K^{-1} x_e)(x'_f K^{-1} x_f)} = d + s - 2t.$$

Note that in this case $m(e,f) = d + s - 2t$ as well and the result is proved. $\square$

Theorem 9. *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Then the matrix M has rank n.*

Proof. We may write M in partitioned form as

$$M = \begin{bmatrix} K & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

By the Schur complement formula for rank we have

$$\begin{aligned}\operatorname{rank} M &= \operatorname{rank} K + \operatorname{rank}(M_{22} - M_{21}K^{-1}M_{12}) \\ &= n - 1 + \operatorname{rank}(M_{22} - M_{21}K^{-1}M_{12}).\end{aligned} \tag{9}$$

Consider the 2×2 submatrix of $M_{22} - M_{21}K^{-1}M_{12}$ formed by the rows indexed by e, f and the columns indexed by g, h. In view of the observation (1), the matrix is given by

$$\begin{bmatrix} m(e,g) - x'_e K^{-1} x_g & m(e,h) - x'_e K^{-1} x_h \\ m(f,g) - x'_f K^{-1} x_g & m(f,h) - x'_f K^{-1} x_h \end{bmatrix}. \tag{10}$$

It follows by Theorem 8 that the matrix in (10) equals

$$\begin{bmatrix} \sqrt{(x'_e K^{-1} x_e)(x'_g K^{-1} x_g)} & \sqrt{(x'_e K^{-1} x_e)(x'_h K^{-1} x_h)} \\ \sqrt{(x'_f K^{-1} x_f)(x'_g K^{-1} x_g)} & \sqrt{(x'_f K^{-1} x_f)(x'_h K^{-1} x_h)} \end{bmatrix}. \tag{11}$$

Clearly the determinant of the matrix in (11) is zero and hence the rank of $M_{22} - M_{21}K^{-1}M_{12}$ is 1. It follows from (9) that the rank of M is n. □

3. Invariant factors of M_T

Lemma 10. *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $i, j, u, v \in V(T)$. Let $\alpha = d_{ij} + d_{uv}, \beta = d_{iu} + d_{jv}$ and $\gamma = d_{iv} + d_{ju}$. Then $\alpha - \beta, \beta - \gamma$ and $\alpha - \gamma$ are even integers.*

Proof. We assume, without loss of generality, that the ij-path and the uv-path are edge-disjoint. Recall that $\mathscr{P}(i,j)$ denotes the set of edges on the ij-path. Then $\alpha = d_{ij} + d_{uv} = |\mathscr{P}(i,j)| + |\mathscr{P}(u,v)|, \beta = d_{iu} + d_{jv} = |\mathscr{P}(i,j)| + |\mathscr{P}(u,v)| + 2|\mathscr{P}(i,u) \cap \mathscr{P}(j,v)|$ and $\gamma = d_{iv} + d_{ju} = |\mathscr{P}(i,j)| + |\mathscr{P}(u,v)| + 2|\mathscr{P}(i,v) \cap \mathscr{P}(j,u)|$. Note that $|\mathscr{P}(i,u) \cap \mathscr{P}(j,v)| = |\mathscr{P}(i,v) \cap \mathscr{P}(j,u)|$. Hence $\beta = \gamma$. Since $\alpha - \beta = -2|\mathscr{P}(i,u) \cap \mathscr{P}(j,v)|$, it follows that $\alpha - \beta, \beta - \gamma$ and $\alpha - \gamma$ are even integers. □

Lemma 11. *The invariant factors of the $n \times n$ matrix ($n \geq 3$)*

$$\begin{bmatrix} -2 & \cdots & 0 & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -2 & 0 & 1 \\ 0 & \cdots & 0 & -2 & 1 \\ 1 & \cdots & 1 & 1 & 0 \end{bmatrix}$$

are given by

$$1, 1, \overbrace{2, \ldots, 2}^{n-3}, 2(n-1).$$

Proof. The result is a special case of [10, Corollary 2]. □

Lemma 12. *The invariant factors of the $n \times n$ matrix ($n \geq 3$)*

$$\begin{bmatrix} 0 & 2 & \cdots & 2 & 2 & 2 & 3 \\ 2 & 0 & \cdots & 2 & 2 & 2 & 3 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & \cdots & \cdots & 0 & 2 & 2 & 1 \\ 2 & \cdots & \cdots & 2 & 0 & 2 & 1 \\ 2 & \cdots & \cdots & 2 & 2 & 0 & 1 \\ 3 & \cdots & \cdots & 3 & 1 & 1 & 0 \end{bmatrix} \tag{12}$$

are given by

$$1, 1, \overbrace{2, \ldots, 2}^{n-3}, 2(n-1).$$

Proof. Perform the following elementary operations on the matrix given in the Lemma: From column j, subtract column $n-1$, column $n-2$, and add column n; from row j, subtract row $n-2$, row $n-1$, and add row n, $j = 1, 2, \ldots, n-3$. Then from column $n-2$, subtract column n, from row $n-2$, subtract row n, from column $n-1$, subtract column n, from row $n-1$ subtract row n. Then we get the matrix in Lemma 11. The two matrices thus have the same invariant factors and the proof is complete by Lemma 11. □

Lemma 13. *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $f_i = \{u_i, v_i\}, i = 1, \ldots, n$ be edges of the complete graph on $V(T)$. Let S be the $n \times (n-1)$ matrix with its j-th row equal to $x'_{f_j}, j = 1, \ldots, n$. Let h be the $n \times 1$ column vector with its j-th element equal to $d(u_j, v_j) - 1$. Then*

$$\det[S, h] = (n-2)2^{n-1}\theta,$$

for some integer θ.

Proof. Perform the following column operations on $[S, h]$. For $k = 1, \ldots, n-1$, subtract column n from column k. Since each element of x_{f_j} is either $d(u_j, v_j) - 1$ or $d(u_j, v_j) + 1$, the resulting matrix has each element equal to 0 or 2 in its first $n-1$ columns. Divide columns $1, \ldots, n-1$ by 2 and let the resulting matrix be W. Then we can write

$$\det[S, h] = 2^{n-1} \det W. \tag{13}$$

In view of the definition of x_{f_j}, we observe that the k-th row of W has its last element equal to $d(u_k, v_k) - 1$, while among its first $n-1$ elements, there are precisely $d(u_k, v_k)$ zeros and $n - 1 - d(u_k, v_k)$ ones. To the last column of W, add columns $1, \ldots, n-1$.

Then the last column becomes $(n-2)\mathbf{1}$. Thus $\det W = (n-2)\theta$ for some integer θ. The proof is complete in view of (13) $\square$

Lemma 14. *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $f_i = \{u_i, v_i\}, i = 1, \ldots, n$, and $g = \{w, z\}$ be edges of the complete graph on $V(T)$. Let M_1 be the $n \times n$ submatrix of M with rows indexed by $f_1, \ldots, f_n$ and columns indexed by $e_1, \ldots, e_{n-1}, g$. Then*

$$\det M_1 = (n-1)2^{n-2}\theta',$$

for some integer θ'.

Proof. Let S be the matrix defined in Lemma 13. Then note that $M_1 = [S, \hat{h}]$ for some vector $\hat{h}$.

By Theorem 8, the i-th element of $\hat{h}$ is given by

$$\begin{aligned} m(f_i, g) &= x'_{f_i} K^{-1} x_g - \sqrt{(x'_{f_i} K^{-1} x_{f_i})(x'_g K^{-1} x_g)} \\ &= x'_{f_i} K^{-1} x_g - \frac{n-1}{2(n-2)}(d(u_i, v_i) - 1)(d(w, z) - 1), \end{aligned} \tag{14}$$

in view of Corollary 7.

Let h^1 be the $n \times 1$ vector with its i-th element $x'_{f_i} K^{-1} x_g$, and let h^2 be the $n \times 1$ vector with its i-th element $\frac{n-1}{2(n-2)}(d(u_i, v_i) - 1)(d(w, z) - 1), i = 1, \ldots, n$. By (14),

$$M_1 = [S, \hat{h}] = [S, h^1 - h^2],$$

and by Laplace expansion along the last column,

$$\det M_1 = \det[S, h^1] - \det[S, h^2]. \tag{15}$$

Note that

$$[S, h^1] = [x_{f_1}, \ldots, x_{f_n}]' K^{-1} [x_{e_1}, \ldots, x_{e_{n-1}}, x_g].$$

Thus the rank of $[S, h^1]$ is at most $n-1$, and hence $\det[S, h^1] = 0$. Furthermore,

$$[S, h^2] = [S, \frac{n-1}{2(n-2)}(d(w, z) - 1)h],$$

where h is defined in Lemma 13.

It follows from Lemma 13, that

$$\det[S, h^2] = \frac{n-1}{2(n-2)}(d(w, z) - 1) \det[S, h]$$

$$= \frac{n-1}{2(n-2)}(d(w,z)-1)(n-2)2^{n-1}\theta$$
$$= (n-1)2^{n-2}\theta', \tag{16}$$

for some integer θ'. The proof is complete by substituting (16) in (15). □

Theorem 15. *Let T be a tree with vertex set $V(T) = \{1,\dots,n\}$ and edge set $E(T) = \{e_1,\dots,e_{n-1}\}$. Then the invariant factors of M are given by*

$$\overbrace{0,\dots,0}^{t-n},1,1,\overbrace{2,\dots,2}^{n-3},2(n-1).$$

Proof. By Theorem 9, M has rank n and hence M has n nonzero invariant factors. As observed in the proof of Lemma 13, (12) is a submatrix of M. Thus M has an element equal to 1 and hence the g.c.d of the elements of M is 1. Consider the $k \times k$ submatrix X of $M, 2 \le k \le n$. We will show that $\det X$ is divisible by 2^{k-2}. Let the rows of X be indexed by $s_1 = \{i_1, j_1\}, \dots, s_k = \{i_k, j_k\}$, and the columns of X be indexed by $t_1 = \{u_1, v_1\}, \dots, t_k = \{u_k, v_k\}$. By Lemma 10, for $i = 1,\dots,k$,

$$\begin{aligned} m(s_i,t_i) &= \min\{d(i_1,u_1)+d(j_1,v_1), d(i_1,v_1)+d(j_1,u_1), d(i_1,j_1)+d(u_1,v_1)\} \\ &= d(i_1,j_1)+d(u_1,v_1)+2\theta_k, \end{aligned}$$

for some integer θ_k. Thus we may write

$$X = X_1 + X_2 + X_3,$$

where

$$X_1 = \begin{bmatrix} d(i_1,j_1) \\ \vdots \\ d(i_k,j_k) \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} d(u_1,v_1) & \cdots & d(u_k,v_k) \end{bmatrix},$$

and X_3 has all entries even. By the multilinearity of the determinant, $\det X$ can be evaluated as $\sum_S \det S$, where the summation is over all $k \times k$ matrices S such that each column of S is chosen to be a column of one of the matrices X_1, X_2, X_3. If S contains at least $k-2$ columns of X_3, then since all elements of X_3 are even integers, $\det S$ is divisible by 2^{k-2}. If S contains at least 3 columns which do not come from X_3, then S must have either 2 columns of X_1 or 2 columns of X_2. Then, since $\operatorname{rank} X_1 = \operatorname{rank} X_2 = 1$, by Laplace expansion we see that $\det S = 0$. Therefore we conclude that $\det X$ is divisible by 2^{k-2}. Thus the g.c.d. of the $k \times k$ minors of M is divisible by 2^{k-2}.

We assume, without loss of generality, that the edges e_{n-2} and e_{n-1} have a vertex in common. Let $e_{n-2} = \{s,t\}, e_{n-1} = \{t,w\}$ and let $e = \{s,w\}$. Then note that

$m(e, e_{n-1}) = m(e, e_{n-2}) = 1$ and $m(e, e_j) = 3, j = 1, \ldots, n-3$. Thus the principal submatrix of M with rows and columns indexed by $e_1, \ldots, e_{n-1}, e$ is given by (12).

By Lemma 12, the invariant factors of (12) are given by

$$1, 1, \overbrace{2, \ldots, 2}^{n-3}, 2(n-1).$$

Thus the g.c.d. of the $k \times k$ minors of (12), and hence of M, is $2^{k-2}, 2 \leq k < n-1$.

Our next objective is to show that the g.c.d. of the $n \times n$ minors of M is $(n-1)2^{n-2}$. Let $g = \{w, z\}$ be an edge of the complete graph on $V(T)$, distinct from $e_1, \ldots, e_{n-1}$. If M_1 is an $n \times n$ submatrix of M whose columns are indexed by $e_1, \ldots, e_{n-1}, g$, then by Lemma 14, $\det M_1$ is divisible by $(n-1)2^{n-2}$. By symmetry, if M_2 is an $n \times n$ submatrix of M whose rows are indexed by $e_1, \ldots, e_{n-1}, g$, then $\det M_2$ is divisible by $(n-1)2^{n-2}$.

Let $\mathscr{C}_n(M)$ be the n-th compound of M. The rows and the columns of $\mathscr{C}_n(M)$ are indexed by the n-subsets of $\{1, 2, \ldots, \binom{t}{2}\}$. We assume that the first subset is $\{e_1, \ldots, e_{n-1}, e\}$. It follows from the preceding discussion that any element in the first row and the first column of $\mathscr{C}_n$ is divisible by $(n-1)2^{n-2}$.

The $(1, 1)$-element of $\mathscr{C}_n(M)$ is the determinant of the matrix (7), which equals $\pm(n-1)2^{n-2}$ by Lemma 12.

Let $\begin{bmatrix} c_{11} & c_{1q} \\ c_{p1} & c_{pq} \end{bmatrix}$ be the 2×2 submatrix of $\mathscr{C}_n(M)$ formed by rows $1, p$ and columns $1, q$. By Theorem 9, M has rank n and hence $\mathscr{C}_n(M)$ has rank 1. Thus $c_{11}c_{pq} - c_{1q}c_{p1} = 0$ and hence

$$c_{pq} = \frac{c_{1q}c_{p1}}{c_{11}}. \tag{17}$$

As noted earlier in this proof, c_{1q} and c_{p1} are divisible by $(n-1)2^{n-2}$, while $c_{11} = \pm(n-1)2^{n-2}$. it follows from (17) that c_{pq} is divisible by $(n-1)2^{n-2}$. Furthermore, since $c_{11} = \pm(n-1)2^{n-2}$, it follows that the g.c.d. of the $n \times n$ minors of M is $(n-1)2^{n-2}$. We conclude that the invariant factors of M are

$$\overbrace{0, \ldots, 0}^{t-n}, 1, 1, \overbrace{2, \ldots, 2}^{n-3}, 2(n-1),$$

and the proof is complete. □

Declaration of competing interest

None declared.

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