Smith Normal Form of a distance matrix inspired by the four-point condition

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**Abstract**

The four point theorem is a condition for distances to arise from trees. Based on this condition, for any tree $T$ on $n$ vertices, we associate an $\binom{n}{2} \times \binom{n}{2}$ matrix $M_T$.

We find the rank and the Smith Normal Form (SNF) of the matrix $M_T$ and show that it only depends on $n$ and is independent of the structure of the tree $T$. Curiously, the non-zero part of the SNF of $M_T$ coincides with the SNF of the distance matrix of $T$. Many such “tree independent” results are known and this result is yet another such result.

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1. Introduction

Let $G$ be a connected graph with edges having nonnegative weights. Let $D$ be its distance matrix whose $(i,j)$-th entry is the shortest path distance between $i$ and $j$ in $G$. Given $G$, finding $D$ is easy with several all-pairs shortest path distance algorithms known. See for example Cormen, Leiserson, Rivest and Stein [7].
The reverse question is this: given a positive integral matrix $D$, is there a connected graph $G$ which induces the distances in $D$? More generally, given a matrix $D$, we want to know if there exists a graph $G$ with weights on its edges and a subset $S$ of the vertices of $G$ such that the restriction of the shortest path distance (with respect to the edge weights) matrix $D_G$ of $G$ to the vertices in $S$ being equal to $D$. This difficult question was raised by Hakimi and Yau [9] and has connections to internet tomography, see Chung, Garrett, Graham and Shallcross [6].

Suppose, we add the restriction that we would like the underlying graph to be a tree. Does some condition on the entries of $D$ ensure that the matrix $D$ arises from tree distances? For trees, Buneman [5] showed that the entries of $D$ need to satisfy the famous four point condition (4PC henceforth). See Baldisseri [1] for more on this work and also see the work of Pachter and Speyer [11].

The 4PC is as follows: consider four vertices $i, j, k$ and $\ell$ and the three terms in $P = \{d_{i,j} + d_{k,\ell}, d_{i,k} + d_{j,\ell}, d_{i,\ell} + d_{j,k}\}$, considered as a multi-set. The 4PC states that for all choices of four distinct vertices, the maximum element in $P$ appears at least twice. Note that if among $i, j, k$ and $\ell$, we only have three distinct vertices (say if $i = j$), then the 4PC gives us the triangle inequality. Thus, the 4PC is stronger than the triangle inequality.

We define a matrix $M_T$ inspired by the 4PC and obtain its rank and its invariant factors. The construction of the matrix $M_T$ does not have any justification other than the rich structure it seems to have.

For the rest of this paper we assume that we have a tree $T$ with distance matrix $D$. For a positive integer $n$, let $[n] = \{1, 2, \ldots, n\}$ and let $T$ have vertex set $V(T) = [n]$. Let $t = \binom{n}{2}$. We assume $n \geq 4$ in what follows as otherwise the trees are isomorphic (hence having the same distance matrix). For vertices $x, y \in V(T)$, let $d(x, y)$ or $d_{xy}$ be the distance between $x$ and $y$.

Form a $t \times t$ matrix $M_T$ as follows. The rows and the columns of $M_T$ are indexed by pairs $e = \{w, x\}$, $f = \{y, z\}$, where $w, x, y, z \in V(T)$ with $w < x, y < z$. We define the $(e, f)$-element of $M$ as

$$m(e, f) = \min(d_{wx} + d_{yz}, d_{wy} + d_{xz}, d_{wz} + d_{xy}).$$

**Example.** Consider the tree $T$

```
   o1
  /   \
 o3 - o4 - o5
  |    |
  2    |
```

Then
Several matrices have been associated with trees $T$ having vertex set $[n]$. Many results show that parameters of the matrices only depend on $n$ and are independent of the structure of $T$. Graham and Pollak in [8] showed that the determinant of $D$ is only a function of $n$. Bapat, Lal and Pati [2] generalized this to the determinant of the $q$-analogue of the distance matrix and the exponential distance matrix. Bapat and Sivasubramanian in [3] later generalized this to the second immanant of the exponential distance matrix. Bapat and Sivasubramanian in [4] also showed that the Smith Normal form of the exponential distance matrix of $T$ is independent of the structure of $T$. In this paper, we present more results of this type involving the matrix $M_T$. Our main results are the following.

**Theorem 1.** Let $T$ be a tree on $n$ vertices. Then, the rank of $M_T$ equals $n$.

We next find the invariant factors of the matrix $M_T$ and show that these are also independent of the structure of the tree $T$. Recalling $t = \binom{n}{2}$, we prove the following.

**Theorem 2.** Let $T$ be a tree on $n$ vertices. The invariant factors of the matrix $M_T$ are

$$0, \ldots, 0, 1, 1, \underbrace{2, \ldots, 2}_{n-3 \text{ times}}, 2(n-1).$$

The invariant factors of the distance matrix $D$ of a tree $T$ on $n$ vertices are known (see, for example, Hou and Woo [10]). It is a curious coincidence that the invariant factors of the distance matrix $D$ are identical to the non-zero terms in Theorem 2. The proof of Theorem 1 appears in Section 2 and the proof of Theorem 2 appears in Section 3.

2. Rank of $M_T$

**Lemma 3.** Let $T$ be a tree with vertex set $V(T) = [n]$. Then, the submatrix of $M$ with rows and columns indexed by $E(T)$ is given by $K = 2(J - I)$.  

\[
M_T = \begin{bmatrix}
13 & 23 & 34 & 45 & 12 & 14 & 15 & 24 & 25 & 35 \\
13 & 0 & 2 & 2 & 1 & 1 & 2 & 3 & 4 & 3 \\
23 & 2 & 0 & 2 & 1 & 3 & 4 & 1 & 2 & 3 \\
34 & 2 & 2 & 0 & 2 & 3 & 1 & 2 & 1 & 2 \!
\]

\[
54 & 2 & 2 & 2 & 0 & 3 & 3 & 2 & 3 & 2 & 1 \\
12 & 1 & 1 & 3 & 3 & 0 & 2 & 3 & 2 & 3 & 4 \\
14 & 1 & 3 & 1 & 3 & 2 & 0 & 1 & 2 & 3 & 2 \\
15 & 2 & 4 & 2 & 2 & 3 & 1 & 0 & 3 & 2 & 1 \\
24 & 3 & 1 & 1 & 3 & 2 & 2 & 3 & 0 & 1 & 2 \\
25 & 4 & 2 & 2 & 2 & 3 & 3 & 2 & 1 & 0 & 1 \\
35 & 3 & 3 & 1 & 1 & 4 & 2 & 1 & 2 & 1 & 0 
\end{bmatrix}.
\]
Proof. Clearly $m(e_i, e_i) = 0$, $i = 1, \ldots, n - 1$, and hence the diagonal elements of $K$ are zero. Let $e_i, e_j \in E(T), i \neq j$, where $e_i = \{w, x\}$ and $e_j = \{y, z\}$. We have $d_{wx} = d_{yz} = 1$ and $d_{wy} + d_{xz} \geq 2, d_{wz} + d_{yx} \geq 2$. Hence
\[
m(e_i, e_j) = \min(d_{wx} + d_{yz}, d_{wy} + d_{xz}, d_{wz} + d_{xy}) = 2.
\]
It follows that the submatrix of $M$ with rows and columns indexed by $E(T)$ is given by $K = 2(J - I)$. □

The proof of the next result is easy and is omitted.

Lemma 4. The $(n - 1) \times (n - 1)$ matrix $K = 2(J - I)$ is nonsingular and $2K^{-1} = -I + \frac{1}{n-2}J$.

We introduce some notation. Recall that the edges of the tree are denoted $e_1, \ldots, e_{n-1}$. Let $e = \{i, j\}$ be an edge of the complete graph on $V(T)$. We define the column vector $x_e$ as follows. The coordinates of $x_e$ are indexed by $e_1, \ldots, e_{n-1}$. We set the $\ell$-th coordinate of $x_e$ to be $d_{ij} - 1$, if $e_\ell$ is on the $ij$-path and equal to $d_{ij} + 1$, otherwise. It may be observed that
\[
x'_e = [m(e, e_1), \ldots, m(e, e_{n-1})].
\]

Lemma 5. Let $T$ be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $e = \{i, j\}, i, j \in V(T), i < j$. Then $x'_e1 = (n - 3)d_{ij} + n - 1$.

Proof. There are $d_{ij}$ edges on the $ij$-path, and for each such edge, the corresponding coordinate of $x_e$ is $d_{ij} - 1$. Similarly there are $n - 1 - d_{ij}$ edges not on the $ij$-path, and for each such edge, the corresponding coordinate of $x_e$ is $d_{ij} + 1$. Hence the sum of the coordinates of $x_e$ is given by
\[
x'_e1 = d_{ij}(d_{ij} - 1) + (n - 1 - d_{ij})(d_{ij} + 1)
= d_{ij}^2 - d_{ij} + nd_{ij} + n - d_{ij} - 1 - d_{ij} - d_{ij}
= (n - 3)d_{ij} + n - 1,
\]
and the proof is complete. □

Lemma 6. Let $T$ be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $e = \{i, j\}, f = \{u, v\}$, where $i, j, u, v \in V(T), i < j, u < v$. Then
\[
x'_eK^{-1}x_f = -\frac{1}{2}x'_ex_f + \frac{1}{2(n-2)}((n - 3)d_{ij} + n - 1)((n - 3)d_{uv} + n - 1).
\]
Proof. Using Lemmas 4 and 5 we get
\[
x_e'K^{-1}x_f = \frac{1}{2}x_e' \left( -I + \frac{1}{n-2}J \right) x_f
\]
\[
= \frac{1}{2} \left( -x_e'x_f + \frac{1}{n-2}(x_e'1)(x_f'1) \right)
\]
\[
= -\frac{1}{2}x_e'x_f + \frac{1}{2(n-2)}((n-3)d_{ij} + n - 1)((n-3)d_{uv} + n - 1).
\]
This completes the proof. \(\square\)

Corollary 7. Let \(T\) be a tree with vertex set \(V(T) = \{1, \ldots, n\}\) and edge set \(E(T) = \{e_1, \ldots, e_{n-1}\}\). Let \(e = \{i,j\}, i, j \in V(T), i < j, \text{ and let } d = d_{ij}. \text{ Then}
\[
x_e'K^{-1}x_e = \frac{(n-1)(d-1)^2}{2(n-2)}.
\]

Proof. By Lemma 6 we have
\[
x_e'K^{-1}x_e = -\frac{1}{2}x_e'x_e + \frac{1}{2(n-2)}((n-3)d + n - 1)^2.
\]
Note that \(x_e\) has \(d\) coordinates equal to \(d - 1\) and \(n - 1 - d\) coordinates equal to \(d + 1\). Therefore
\[
x_e'x_e = d(d - 1)^2 + (n - 1 - d)(d + 1)^2.
\]
From (3) and (4) we get
\[
x_e'K^{-1}x_e = -\frac{1}{2} \left( d(d - 1)^2 + (n - 1 - d)(d + 1)^2 - \frac{(n-3)d + n - 1)^2}{n-2} \right).
\]
We get the result (2) from (5) after simplification. \(\square\)

Theorem 8. Let \(T\) be a tree with vertex set \(V(T) = \{1, \ldots, n\}\) and edge set \(E(T) = \{e_1, \ldots, e_{n-1}\}\). Let \(e = \{i,j\}, f = \{u,v\}, \text{ where } i, j, u, v \in V(T), i < j, u < v. \text{ Then the (e,f)-element of } M \text{ is given by}
\[
\begin{aligned}
m(e,f) &= x_e'K^{-1}x_f - \sqrt{(x_e'K^{-1}x_e)(x_f'K^{-1}x_f)}.
\end{aligned}
\]

Proof. Let \(d_{ij} = d, d_{uv} = s. \text{ It follows from Lemma 6 and Corollary 7 that}
\[
x_e'K^{-1}x_f - \sqrt{(x_e'K^{-1}x_e)(x_f'K^{-1}x_f)}
\]
\[
=-\frac{1}{2} x_e' x_f + \frac{1}{2(n-2)}((n-3)d+n-1)((n-3)s+n-1) - \frac{n-1}{2(n-2)}(d-1)(s-1)
= -\frac{1}{2} x_e' x_f + \frac{1}{2}((d+1)(s+1)(n-1) - 4ds). \tag{7}
\]

We introduce some notation. Let \( \mathcal{P}(i,j) \) denote the set of edges on the \( ij \)-path. From the structure of the vectors \( x_e, x_f \) it follows that
\[
x_e' x_f = |\mathcal{P}(i,j) \cap \mathcal{P}(u,v)|(d-1)(s-1) + |\mathcal{P}(i,j) \setminus \mathcal{P}(u,v)|(d-1)(s+1) + |\mathcal{P}(u,v) \setminus \mathcal{P}(i,j)|(d+1)(s-1) + |E(T) \setminus \mathcal{P}(i,j) \setminus \mathcal{P}(u,v)|(d+1)(s+1). \tag{8}
\]

We consider two cases:

**Case (i)** The \( ij \)-path and the \( uv \)-path are edge-disjoint.

Then \( |\mathcal{P}(i,j) \cap \mathcal{P}(u,v)| = 0, |\mathcal{P}(i,j) \setminus \mathcal{P}(u,v)| = d, |\mathcal{P}(u,v) \setminus \mathcal{P}(i,j)| = s \), and \( |E(T) \setminus \mathcal{P}(i,j) \setminus \mathcal{P}(u,v)| = n-1-d-s \). Substituting in (8) we get \( x_e' x_f = ds(d-1) + s(s-1)(d+1) + (n-1-d-s)(d+1)(s+1) \) and then it follows from (7) that
\[
x_e' K^{-1} x_f - \sqrt{(x_e' K^{-1} x_f)(x_f' K^{-1} x_f)} = d + s.
\]
Note that in this case \( m(e, f) = d + s \) as well and hence the result is proved.

**Case (ii)** The \( ij \)-path and the \( uv \)-path are not edge-disjoint.

Let \( |\mathcal{P}(i,j) \cap \mathcal{P}(u,v)| = t \). Then \( |\mathcal{P}(i,j) \setminus \mathcal{P}(u,v)| = d-t, |\mathcal{P}(u,v) \setminus \mathcal{P}(i,j)| = s-t, \) and \( |E(T) \setminus \mathcal{P}(i,j) \setminus \mathcal{P}(u,v)| = n-1-d-s+t \). Substituting in (8) we get \( x_e' x_f = t(d-1)(s-1) + (d-t)(d-1)(s+1) + (s-t)(d+1)(s-1) + (n-1-d-s+t)(d+1)(s+1) \) and then it follows from (7) that
\[
x_e' K^{-1} x_f - \sqrt{(x_e' K^{-1} x_f)(x_f' K^{-1} x_f)} = d + s - 2t.
\]
Note that in this case \( m(e, f) = d + s - 2t \) as well and the result is proved. □

**Theorem 9.** Let \( T \) be a tree with vertex set \( V(T) = \{1, \ldots, n\} \) and edge set \( E(T) = \{e_1, \ldots, e_{n-1}\} \). Then the matrix \( M \) has rank \( n \).

**Proof.** We may write \( M \) in partitioned form as
\[
M = \begin{bmatrix}
K & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}.
\]
By the Schur complement formula for rank we have
rank $M = \text{rank } K + \text{rank}(M_{22} - M_{21}K^{-1}M_{12})$
\[= n - 1 + \text{rank}(M_{22} - M_{21}K^{-1}M_{12}). \] (9)

Consider the $2 \times 2$ submatrix of $M_{22} - M_{21}K^{-1}M_{12}$ formed by the rows indexed by $e, f$ and the columns indexed by $g, h$. In view of the observation (1), the matrix is given by
\[
\begin{bmatrix}
m(e, g) - x_e'K^{-1}x_g & m(e, h) - x_e'K^{-1}x_h \\
m(f, g) - x_f'K^{-1}x_g & m(f, h) - x_f'K^{-1}x_h
\end{bmatrix}.
\] (10)

It follows by Theorem 8 that the matrix in (10) equals
\[
\begin{bmatrix}
\sqrt{(x_e'K^{-1}x_e)(x_g'K^{-1}x_g)} & \sqrt{(x_e'K^{-1}x_e)(x_h'K^{-1}x_h)} \\
\sqrt{(x_f'K^{-1}x_f)(x_g'K^{-1}x_g)} & \sqrt{(x_f'K^{-1}x_f)(x_h'K^{-1}x_h)}
\end{bmatrix}.
\] (11)

Clearly the determinant of the matrix in (11) is zero and hence the rank of $M_{22} - M_{21}K^{-1}M_{12}$ is 1. It follows from (9) that the rank of $M$ is $n$. \(\Box\)

3. Invariant factors of $M_T$

**Lemma 10.** Let $T$ be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $i, j, u, v \in V(T)$. Let $\alpha = d_{ij} + d_{uv}, \beta = d_{iu} + d_{jv}$ and $\gamma = d_{iv} + d_{ju}$. Then $\alpha - \beta, \beta - \gamma$ and $\alpha - \gamma$ are even integers.

**Proof.** We assume, without loss of generality, that the $ij$-path and the $uv$-path are edge-disjoint. Recall that $P(i, j)$ denotes the set of edges on the $ij$-path. Then $\alpha = d_{ij} + d_{uv} = |P(i, j)| + |P(u, v)|$, $\beta = d_{iu} + d_{jv} = |P(i, j)| + |P(u, v)| + 2|P(i, u) \cap P(j, v)|$ and $\gamma = d_{iv} + d_{ju} = |P(i, j)| + |P(u, v)| + 2|P(i, v) \cap P(j, u)|$. Note that $|P(i, u) \cap P(j, v)| = |P(i, v) \cap P(j, u)|$. Hence $\beta = \gamma$. Since $\alpha - \beta = -2|P(i, u) \cap P(j, v)|$, it follows that $\alpha - \beta, \beta - \gamma$ and $\alpha - \gamma$ are even integers. \(\Box\)

**Lemma 11.** The invariant factors of the $n \times n$ matrix ($n \geq 3$)
\[
\begin{bmatrix}
-2 & \cdots & 0 & 0 & 1 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & -2 & 0 & 1 \\
0 & \cdots & 0 & -2 & 1 \\
1 & \cdots & 1 & 1 & 0
\end{bmatrix}
\]
are given by
\[
\begin{bmatrix}
-2 & \cdots & 0 & 0 & 1 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & -2 & 0 & 1 \\
0 & \cdots & 0 & -2 & 1 \\
1 & \cdots & 1 & 1 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
1, 1, 2, \ldots, 2, 2(n - 1).
\end{bmatrix}
\]
Proof. The result is a special case of [10, Corollary 2]. □

Lemma 12. The invariant factors of the $n \times n$ matrix ($n \geq 3$)

$$
\begin{bmatrix}
0 & 2 & \cdots & 2 & 2 & 2 & 3 \\
2 & 0 & \cdots & 2 & 2 & 2 & 3 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2 & \cdots & \cdots & 0 & 2 & 2 & 1 \\
2 & \cdots & \cdots & 2 & 0 & 2 & 1 \\
2 & \cdots & \cdots & 2 & 2 & 0 & 1 \\
3 & \cdots & \cdots & 3 & 1 & 1 & 0
\end{bmatrix}
$$

(12)

are given by

$$
1, 1, 2, \ldots, 2, 2(n-1).
$$

Proof. Perform the following elementary operations on the matrix given in the Lemma: From column $j$, subtract column $n - 1$, column $n - 2$, and add column $n$; from row $j$, subtract row $n - 2$, row $n - 1$, and add row $n, j = 1, 2, \ldots, n - 3$. Then from column $n - 2$, subtract column $n$, from row $n - 2$, subtract row $n$, from column $n - 1$, subtract column $n$, from row $n - 1$ subtract row $n$. Then we get the matrix in Lemma 11. The two matrices thus have the same invariant factors and the proof is complete by Lemma 11. □

Lemma 13. Let $T$ be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $f_i = \{u_i, v_i\}, i = 1, \ldots, n$ be edges of the complete graph on $V(T)$. Let $S$ be the $n \times (n-1)$ matrix with its $j$-th row equal to $x'_{f_j}, j = 1, \ldots, n$. Let $h$ be the $n \times 1$ column vector with its $j$-th element equal to $d(u_j, v_j) - 1$. Then

$$
\det[S, h] = (n - 2)2^{n-1}\theta,
$$

for some integer $\theta$.

Proof. Perform the following column operations on $[S, h]$. For $k = 1, \ldots, n - 1$, subtract column $n$ from column $k$. Since each element of $x_{f_j}$ is either $d(u_j, v_j) - 1$ or $d(u_j, v_j) + 1$, the resulting matrix has each element equal to 0 or 2 in its first $n - 1$ columns. Divide columns $1, \ldots, n - 1$ by 2 and let the resulting matrix be $W$. Then we can write

$$
\det[S, h] = 2^{n-1} \det W.
$$

(13)

In view of the definition of $x_{f_j}$, we observe that the $k$-th row of $W$ has its last element equal to $d(u_k, v_k) - 1$, while among its first $n - 1$ elements, there are precisely $d(u_k, v_k)$ zeros and $n - 1 - d(u_k, v_k)$ ones. To the last column of $W$, add columns $1, \ldots, n - 1$. 
Then the last column becomes \((n - 2)\mathbf{1}\). Thus \(\det W = (n - 2)\theta\) for some integer \(\theta\). The proof is complete in view of (13) \(\square\)

**Lemma 14.** Let \(T\) be a tree with vertex set \(V(T) = \{1, \ldots, n\}\) and edge set \(E(T) = \{e_1, \ldots, e_{n-1}\}\). Let \(f_i = \{u_i, v_i\}, i = 1, \ldots, n,\) and \(g = \{w, z\}\) be edges of the complete graph on \(V(T)\). Let \(M_1\) be the \(n \times n\) submatrix of \(M\) with rows indexed by \(f_1, \ldots, f_n\) and columns indexed by \(e_1, \ldots, e_{n-1}, g\). Then

\[
\det M_1 = (n - 1)2^{n-2}\theta',
\]

for some integer \(\theta'\).

**Proof.** Let \(S\) be the matrix defined in Lemma 13. Then note that \(M_1 = [S, \hat{h}]\) for some vector \(\hat{h}\).

By Theorem 8, the \(i\)-th element of \(\hat{h}\) is given by

\[
m(f_i, g) = x'_f K^{-1}x_g - \sqrt{(x'_f K^{-1}x_f)(x'_g K^{-1}x_g)}
= x'_f K^{-1}x_g - \frac{n - 1}{2(n - 2)}(d(u_i, v_i) - 1)(d(w, z) - 1), \tag{14}
\]

in view of Corollary 7.

Let \(h^1\) be the \(n \times 1\) vector with its \(i\)-th element \(x'_f K^{-1}x_g\), and let \(h^2\) be the \(n \times 1\) vector with its \(i\)-th element \(\frac{n - 1}{2(n - 2)}(d(u_i, v_i) - 1)(d(w, z) - 1), i = 1, \ldots, n\). By (14),

\[
M_1 = [S, \hat{h}] = [S, h^1 - h^2],
\]

and by Laplace expansion along the last column,

\[
\det M_1 = \det[S, h^1] - \det[S, h^2]. \tag{15}
\]

Note that

\[
[S, h^1] = [x_{f_1}, \ldots, x_{f_n}]' K^{-1} [x_{e_1}, \ldots, x_{e_{n-1}}, x_g].
\]

Thus the rank of \([S, h^1]\) is at most \(n - 1\), and hence \(\det[S, h^1] = 0\). Furthermore,

\[
[S, h^2] = [S, \frac{n - 1}{2(n - 2)}(d(w, z) - 1)h],
\]

where \(h\) is defined in Lemma 13.

It follows from Lemma 13, that

\[
\det[S, h^2] = \frac{n - 1}{2(n - 2)}(d(w, z) - 1) \det[S, \hat{h}]
\]
\[
\begin{align*}
&\quad = \frac{n-1}{2(n-2)}(d(w,z) - 1)(n-2)2^{n-1}\theta \\
&= (n-1)2^{n-2}\theta', \\
&\quad \text{(16)}
\end{align*}
\]
for some integer \(\theta'\). The proof is complete by substituting (16) in (15). \(\square\)

**Theorem 15.** Let \(T\) be a tree with vertex set \(V(T) = \{1, \ldots, n\}\) and edge set \(E(T) = \{e_1, \ldots, e_{n-1}\}\). Then the invariant factors of \(M\) are given by

\[
\begin{bmatrix}
1 & \cdots & 1 & 0, \ldots, 0, 1, 1, 2, \ldots, 2, 2(n-1) \\
\end{bmatrix}
\]

**Proof.** By Theorem 9, \(M\) has rank \(n\) and hence \(M\) has \(n\) nonzero invariant factors. As observed in the proof of Lemma 13, (12) is a submatrix of \(M\). Thus \(M\) has an element equal to 1 and hence the g.c.d. of the elements of \(M\) is 1. Consider the \(k \times k\) submatrix \(X\) of \(M\), \(2 \leq k \leq n\). We will show that \(\det X\) is divisible by \(2^{k-2}\). Let the rows of \(X\) be indexed by \(s_1 = \{i_1, j_1\}, \ldots, s_k = \{i_k, j_k\}\), and the columns of \(X\) be indexed by \(t_1 = \{u_1, v_1\}, \ldots, t_k = \{u_k, v_k\}\). By Lemma 10, for \(i = 1, \ldots, k\),

\[
m(s_i, t_i) = \min\{d(i_1, u_1) + d(j_1, v_1), d(i_1, v_1) + d(j_1, u_1), d(i_1, j_1) + d(u_1, v_1)\}
\]

\[
= d(i_1, j_1) + d(u_1, v_1) + 2\theta_k,
\]

for some integer \(\theta_k\). Thus we may write

\[
X = X_1 + X_2 + X_3,
\]

where

\[
X_1 = \begin{bmatrix}
d(i_1, j_1) \\
\vdots \\
d(i_k, j_k)
\end{bmatrix} \begin{bmatrix}
1 & \cdots & 1 \\
\end{bmatrix},
X_2 = \begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix} \begin{bmatrix}
d(u_1, v_1) & \cdots & d(u_k, v_k)
\end{bmatrix},
\]

and \(X_3\) has all entries even. By the multilinearity of the determinant, \(\det X\) can be evaluated as \(\sum S \det S\), where the summation is over all \(k \times k\) matrices \(S\) such that each column of \(S\) is chosen to be a column of one of the matrices \(X_1, X_2, X_3\). If \(S\) contains at least \(k-2\) columns of \(X_3\), then since all elements of \(X_3\) are even integers, \(\det S\) is divisible by \(2^{k-2}\). If \(S\) contains at least 3 columns which do not come from \(X_3\), then \(S\) must have either 2 columns of \(X_1\) or 2 columns of \(X_2\). Then, since \(\text{rank } X_1 = \text{rank } X_2 = 1\), by Laplace expansion we see that \(\det S = 0\). Therefore we conclude that \(\det X\) is divisible by \(2^{k-2}\). Thus the g.c.d. of the \(k \times k\) minors of \(M\) is divisible by \(2^{k-2}\).

We assume, without loss of generality, that the edges \(e_{n-2}\) and \(e_{n-1}\) have a vertex in common. Let \(e_{n-2} = \{s, t\}\), \(e_{n-1} = \{t, w\}\) and let \(e = \{s, w\}\). Then note that
\[m(e, e_{n-1}) = m(e, e_{n-2}) = 1 \text{ and } m(e, e_j) = 3, j = 1, \ldots, n - 3. \] Thus the principal submatrix of \(M\) with rows and columns indexed by \(e_1, \ldots, e_{n-1}, e\) is given by (12).

By Lemma 12, the invariant factors of (12) are given by

\[
\begin{array}{c}
1, 1, 2, \ldots, 2, 2(n-1) \\
\end{array}
\]

Thus the g.c.d. of the \(k \times k\) minors of (12), and hence of \(M\), is \(2^{k-2}, 2 \leq k < n - 1\).

Our next objective is to show that the g.c.d. of the \(n \times n\) minors of \(M\) is \((n - 1)2^{n-2}\).

Let \(g = \{w, z\}\) be an edge of the complete graph on \(V(T)\), distinct from \(e_1, \ldots, e_{n-1}\). If \(M_1\) is an \(n \times n\) submatrix of \(M\) whose columns are indexed by \(e_1, \ldots, e_{n-1}, g\), then by Lemma 14, \(\det M_1\) is divisible by \((n - 1)2^{n-2}\). By symmetry, if \(M_2\) is an \(n \times n\) submatrix of \(M\) whose rows are indexed by \(e_1, \ldots, e_{n-1}, g\), then \(\det M_2\) is divisible by \((n - 1)2^{n-2}\).

Let \(\mathcal{C}_n(M)\) be the \(n\)-th compound of \(M\). The rows and the columns of \(\mathcal{C}_n(M)\) are indexed by the \(n\)-subsets of \(\{1, 2, \ldots, (\frac{n}{2})\}\). We assume that the first subset is \(\{e_1, \ldots, e_{n-1}, e\}\). It follows from the preceding discussion that any element in the first row and the first column of \(\mathcal{C}_n\) is divisible by \((n - 1)2^{n-2}\).

The (1,1)-element of \(\mathcal{C}_n(M)\) is the determinant of the matrix (7), which equals \(\pm(n - 1)2^{n-2}\) by Lemma 12.

Let

\[
\begin{bmatrix}
c_{11} & c_{1q} \\
c_{p1} & c_{pq}
\end{bmatrix}
\]

be the \(2 \times 2\) submatrix of \(\mathcal{C}_n(M)\) formed by rows 1, \(p\) and columns 1, \(q\). By Theorem 9, \(M\) has rank \(n\) and hence \(\mathcal{C}_n(M)\) has rank 1. Thus \(c_{11}c_{pq} - c_{1q}c_{pq} = 0\) and hence

\[
c_{pq} = \frac{c_{1q}c_{p1}}{c_{11}}. \tag{17}
\]

As noted earlier in this proof, \(c_{1q}\) and \(c_{p1}\) are divisible by \((n - 1)2^{n-2}\), while \(c_{11} = \pm(n - 1)2^{n-2}\). It follows from (17) that \(c_{pq}\) is divisible by \((n - 1)2^{n-2}\).

Furthermore, since \(c_{11} = \pm(n - 1)2^{n-2}\), it follows that the g.c.d. of the \(n \times n\) minors of \(M\) is \((n - 1)2^{n-2}\). We conclude that the invariant factors of \(M\) are

\[
\begin{array}{c}
0, 1, 2, \ldots, 2(n-1), \\
\end{array}
\]

and the proof is complete. \(\square\)

**Declaration of competing interest**

None declared.
Acknowledgements

The first author acknowledges the support of the Indian National Science Academy under the INSA Senior Scientist scheme.

The second author acknowledges support from project grant P07 IR052, given by IRCC, IIT Bombay and from project SERB/F/252/2019-2020 given by the Science and Engineering Research Board (SERB), India.

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