A q-analog of the adjacency matrix of the n-cube

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To the memory of Chandan and beloved Lawson

Abstract

Let q be a prime power and define $(n)_q = 1 + q + q^2 + \cdots + q^{n-1}$, for a nonnegative integer n. Let $B_q(n)$ denote the set of all subspaces of $\mathbb{F}_q^n$, the n-dimensional $\mathbb{F}_q$-vector space of all column vectors with n components.

Define a $B_q(n) \times B_q(n)$ complex matrix $M_q(n)$ with entries given by

$$M_q(n)(X,Y) \quad = \quad \begin{cases} 1 & \text{if } Y \subseteq X, \dim(Y) = \dim(X) - 1, \\ q^k & \text{if } X \subseteq Y, \dim(Y) = k+1, \dim(X) = k, \\ 0 & \text{otherwise.} \end{cases}$$

We think of $M_q(n)$ as a q-analog of the adjacency matrix of the n-cube. We show that the eigenvalues of $M_q(n)$ are

$$(n-k)_q - (k)_q \text{ with multiplicity } \binom{n}{k}_q, \quad k = 0, 1, \ldots, n,$$

and we write down an explicit canonical eigenbasis of $M_q(n)$. We give a weighted count of the number of rooted spanning trees in the q-analog of the n-cube.

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1 Introduction

One aspect of algebraic combinatorics is the study of eigenvalues and eigenvectors of certain matrices associated with posets and graphs. Among the most basic such examples is the adjacency matrix of the n-cube, which has an elegant spectral theory and arises in a variety of applications (see **[2, 14]**). This paper defines a q-analog of this matrix, studies its spectral theory, and gives an application to weighted counting of rooted spanning trees in the q-analog of the n-cube.

Let q be a prime power and define $(n)_q = 1 + q + q^2 + \cdots + q^{n-1}$, for a nonnegative integer n. Let $B_q(n)$ denote the set of all subspaces of $\mathbb{F}_q^n$, the n-dimensional $\mathbb{F}_q$-vector space of all column vectors with n components. The set of k-dimensional subspaces in $B_q(n)$ is denoted $B_q(n,k)$ and its cardinality is the *q-binomial coefficient* $\binom{n}{k}_q$. The *Galois number*

$$G_q(n) = \sum_{k=0}^{n} \binom{n}{k}_q$$

is the total number of subspaces in $B_q(n)$. The set $B_q(n)$ has the structure of a graded poset of rank n, under inclusion.

Recall the definition of the adjacency matrix $M(n)$ of the n-cube: let $B(n)$ denote the set of all subsets of $\{1, 2, \ldots, n\}$. The rows and columns of $M(n)$ are indexed by elements of $B(n)$, with the entry in row S, column T equal to 1 if $|(S \setminus T) \cup (T \setminus S)| = 1$ and equal to 0 otherwise. Define a $B_q(n) \times B_q(n)$ complex matrix $M_q(n)$ with entries given by

$$M_q(n)(X,Y) \quad = \quad \begin{cases} 1 & \text{if } Y \subseteq X, \dim(Y) = \dim(X) - 1, \\ q^k & \text{if } X \subseteq Y, \dim(Y) = k+1, \dim(X) = k, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We think of $M_q(n)$ as *a q-analog of the adjacency matrix of the n-cube*. Note that $M(n)$ is symmetric, has entries in $\{0,1\}$, and has all row sums equal to n. The significance of the definition above for the q-analog comes from the fact that though $M_q(n)$ lacks the first two properties it does have all row sums equal. Indeed, let $X \in B_q(n,k)$. Then the number of subspaces covering X is $(n-k)_q$ and the number of subspaces covered by X is $(k)_q$ and so the sum of the entries of row X of $M_q(n)$ is $q^k(n-k)_q + (k)_q = (n)_q$.

A scaling (i.e., a diagonal similarity) of $M_q(n)$ is symmetric. Let $D_q(n)$ be the $B_q(n) \times B_q(n)$ diagonal matrix with diagonal entry in row X, column X given by $\sqrt{q^{\binom{k}{2}}}$, where $k = \dim(X)$. Then for $X \in B_q(n,k)$, $Y \in B_q(n,r)$ the entry in row X, column Y of

$D_q(n)M_q(n)D_q(n)^{-1}$ is given by

$$
\begin{aligned}
&\sqrt{q^{\binom{k}{2}}}\, M_q(X,Y)\, \sqrt{q^{-\binom{r}{2}}} \\
&= \begin{cases} \sqrt{q^{\binom{k}{2}}}\, q^k \sqrt{q^{-\binom{k+1}{2}}} = \sqrt{q^k} & \text{if } X \subseteq Y \text{ and } r = k+1, \\ \sqrt{q^{\binom{k}{2}}}\sqrt{q^{-\binom{k-1}{2}}} = \sqrt{q^r} & \text{if } Y \subseteq X \text{ and } r = k-1, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \sqrt{q^{\min\{\dim(X),\dim(Y)\}}} & \text{if } X \subseteq Y \text{ or } Y \subseteq X, \text{ and } |\dim(X) - \dim(Y)| = 1, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
$$

yielding a symmetric matrix. It follows that $M_q(n)$ is diagonalizable and that its eigenvalues are real. In fact they are integral and the (eigenvalue, multiplicity) pairs of $M_q(n)$ are a q-analog of those for $M(n)$; the eigenvalues of $M(n)$ are $n - 2k = (n-k) - (k)$ with multiplicity $\binom{n}{k}$, $k = 0, 1, \ldots, n$ (see**[2, 14]**).

Theorem 1.1. *The eigenvalues of the matrix $M_q(n)$ are*

$$(n-k)_q - (k)_q \text{ with multiplicity } \binom{n}{k}_q, \quad k = 0, 1, \ldots, n.$$

We give two proofs of Theorem 1.1 in this paper. The first proof, given in Section 2, is based on two basic results of Terwilliger **[16, 17, 19]**. A result from **[16]** on the existence of a symmetric Jordan basis with respect to the up operator, together with formulas for the action of the up and down operators on this basis, reduces Theorem 1.1 to showing that the eigenvalues of $K_q(n)$, a certain $(n+1) \times (n+1)$ tridiagonal matrix, are $(n-k)_q - (k)_q,\ 0 \le k \le n$. The matrix $K_q(n)$ is a q-analog of the famous tridiagonal matrix $K(n)$ of Mark Kac **[1, 8, 15]** (both these matrices are defined in Section 2). Now $K_q(n)$ occurs in Terwilliger's classification of Leonard pairs (**[17, 19]**) and as such its eigenvalues/eigenvectors were known (see, for example, Lemma 4.20 in **[18]**), completing the first proof of Theorem 1.1.

Another natural problem is to write down eigenvectors of $M_q(n)$. For $X \in B_q(n)$ with $\dim(X) = k$ define

$$\pi(X) = \frac{q^{\binom{k}{2}}}{P_q(n)},$$

where $P_q(n) = \prod_{k=0}^{n-1}(1 + q^k)$. We have

$$\sum_{X \in B_q(n)} \pi(X) = \frac{\sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q}{P_q(n)} = 1$$

where the second equality follows by the q-binomial theorem (so π is a probability vector on $B_q(n)$).

Define an inner product on the (complex) vector space of column vectors with components indexed by $B_q(n)$ as follows: given vectors u, v define

$$\langle u,\, v\rangle_\pi = \sum_{X\in B_q(n)} \overline{u(X)}v(X)\pi(X). \tag{2}$$

Since $P_q(n)$ is independent of k, the argument showing that $D_q(n)M_q(n)D_q(n)^{-1}$ is symmetric shows that $M_q(n)$ is self-adjoint with respect to the inner product (2).

Recall that a classical result exhibits an explicit orthogonal eigenbasis of $M(n)$ (under the standard inner product), see **[2, 14]**. Up to scalars, this basis is canonical in the sense that no choices are involved in writing it down. Moreover, the product $M(n)v$, for v in the eigenbasis, can be evaluated easily, determining the eigenvalues of $M(n)$. In Section 4 we extend this method to $M_q(n)$.

Theorem 1.2. *There is an inductive procedure to write down a canonical eigenbasis of $M_q(n)$, orthogonal with respect to the inner product (2).*

In the course of proving Theorem 1.2 we also evaluate the products $M_q(n)v$, for v in the eigenbasis, thereby giving an alternate proof of Theorem 1.1.

Let us make a few informal remarks about the proof of Theorem 1.2. For the matrix $M(n)$ one writes down an orthogonal eigenbasis, directly and explicitly for each n. It is not clear how to extend this direct approach to $M_q(n)$. On the other hand, we can inductively understand the eigenbasis of $M(n)$ and then try to extend that approach to $M_q(n)$. Roughly speaking, $M(n+1)$ can be built using two copies of $M(n)$ and this allows us to write down an eigenbasis of $M(n+1)$ given an eigenbasis of $M(n)$. The Goldman-Rota recurrence for the Galois numbers (**[5, 9, 10]**)

$$G_q(n+1) \;=\; 2G_q(n) + (q^n-1)G_q(n-1),\; n\geq 1,\;\; G_q(0)=1,\; G_q(1)=2, \tag{3}$$

suggests the possibility of building $M_q(n+1)$ from two copies of $M_q(n)$ and (q^n-1) copies of $M_q(n-1)$ and using this to write down an eigenbasis of $M_q(n+1)$ given eigenbases of $M_q(n)$ and $M_q(n-1)$. This is implemented in Section 4 using a linear algebraic interpretation of the Goldman-Rota recurrence that was worked out in **[13]** (and summarized in Section 3 of the present paper). The two copies of $M_q(n)$ occurring in $M_q(n+1)$ is along the same lines as the $q=1$ case (although we need some powers of q not visible in the $q=1$ case, see (24) and cases (a), (b) in the proof of Theorem 4.4). The extra feature here is the q^n-1 copies of $M_q(n-1)$ in $M_q(n+1)$. Here a central role is played by the characters of the additive abelian group $\mathbb{F}_q^n$. There is a copy of $M_q(n-1)$ inside $M_q(n+1)$ for every nontrivial irreducible character of $\mathbb{F}_q^n$, so q^n-1 copies in all. These introductory remarks on Theorem 1.2 are continued at the beginning of Section 4.

We originally arrived at the matrix $M_q(n)$ through a reversible Markov chain with state space $B_q(n)$, transition matrix $\frac{1}{(n)_q}M_q(n)$, and stationary distribution π (see **[6]**). Since the spectral theory of $M_q(n)$, as a q-analog of the spectral theory of $M(n)$, is of independent interest we are presenting it separately in this paper. Another application concerns weighted enumeration of rooted spanning trees in the q-analog of the n-cube.

Let us first recall the remarkable product formula (see Example 9.12 in **[14]**) for the number of rooted spanning trees in the n-cube. Let $C(n)$ denote the n-cube (i.e., the Hasse diagram of the poset $B(n)$ treated as a graph) and let $\mathcal{F}(n)$ denote the set of all rooted spanning trees of $C(n)$. The Laplacian eigenvalues of $C(n)$ are well known to be $2k,\ k = 0, 1, \ldots, n$ with multiplicity $\binom{n}{k}$. It thus follows from the Matrix-Tree theorem that the number of rooted spanning trees of $C(n)$ is given by

$$\sum_{F \in \mathcal{F}(n)} 1 \;=\; \prod_{k=1}^{n} (2k)^{\binom{n}{k}}. \tag{4}$$

The *q-analog* $C_q(n)$ of $C(n)$ is defined to be the Hasse diagram of $B_q(n)$ treated as a graph (note that $M_q(n)$ is not the adjacency matrix of the graph $C_q(n)$). The eigenvalues of the Laplacian of $C_q(n)$ are not known (note that $C_q(n)$ is not regular) but it was shown in **[12]** that the product of the nonzero eigenvalues of the Laplacian is more tractable and led to a product formula for the number of spanning trees of $C_q(n)$, although the individual terms in this product are not explicitly given but only as a positive combinatorial sum. Here we obtain an explicit q-analog of (4) by a weighted count of the rooted spanning trees of $C_q(n)$.

Let $F \in \mathcal{F}_q(n)$, the set of all rooted spanning trees of $C_q(n)$. Orient every edge of F by pointing it towards the root. Let $e = (X, Y)$ be an oriented edge of F. We say e is *spin up* if $\dim(Y) = \dim(X) + 1$ and is *spin down* if $\dim(Y) = \dim(X) - 1$. The *weight* of F is defined by

$$w(F) \;=\; \sum_{(X,Y)} \dim(X),$$

where the sum is over all spin up oriented edges of F. In Section 5 we prove the following generalization of (4) (the proof can be read at this point, assuming Theorem 1.1).

Theorem 1.3. *We have*

$$\sum_{F \in \mathcal{F}_q(n)} q^{w(F)} = \prod_{k=1}^{n} \left((1 + q^{n-k})(k)_q\right)^{\binom{n}{k}_q}.$$

2 Eigenvalues of $M_q(n)$

The spectral theory of $M_q(n)$ goes hand in hand with that of a q-analog of the Kac matrix. Recall that the Kac matrix is a $(n + 1) \times (n + 1)$ tridiagonal matrix $K(n)$ with diagonal

$(0,0,\ldots,0)$, subdiagonal $(1,2,\ldots,n)$ and superdiagonal $(n,n-1,\ldots,1)$:

$$K(n) = \begin{bmatrix} 0 & n & & & & \\ 1 & 0 & n-1 & & & \\ & 2 & 0 & n-2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & n-1 & 0 & 1 \\ & & & & n & 0 \end{bmatrix}.$$

The eigenvalues of $K(n)$ are $n-2k$, $k=0,1,\ldots,n$, and its eigenvectors have been written down (see **[1, 2, 3, 8, 15]**).

We define the q-analog of the Kac matrix to be the $(n+1)\times(n+1)$ tridiagonal matrix $K_q(n)$ with diagonal $(0,0,\ldots,0)$, subdiagonal $((1)_q,(2)_q,\ldots,(n)_q)$ and superdiagonal $((n)_q, q(n-1)_q,\ldots,q^{n-1}(1)_q)$:

$$K_q(n) = \begin{bmatrix} 0 & (n)_q & & & & \\ (1)_q & 0 & q(n-1)_q & & & \\ & (2)_q & 0 & q^2(n-2)_q & & \\ & & \ddots & \ddots & \ddots & \\ & & & (n-1)_q & 0 & q^{n-1}(1)_q \\ & & & & (n)_q & 0 \end{bmatrix}.$$

More formally, let us index the rows and columns of $K_q(n)$ by the set $\{0,1,2,\ldots,n\}$. If $c_0,c_1,\ldots,c_n$ denote column vectors in $\mathbb{F}_q^{n+1}$ with c_i having a 1 in the component indexed by i and 0's elsewhere (and we set $c_{-1}=c_{n+1}=0$) then, for $0\le k\le n$, column k of $K_q(n)$ is

$$(k+1)_q\, c_{k+1} + q^{k-1}(n-k+1)_q\, c_{k-1}. \tag{5}$$

We now discuss the significance of $K_q(n)$ for studying $M_q(n)$. The proper framework for studying this are the up (and down) operators on the poset of subspaces.

For a finite set S, we denote the complex vector space with S as basis by $\mathbb{C}[S]$. We denote by r the rank function (given by dimension) of the graded poset $B_q(n)$. Then we have (vector space direct sum)

$$\mathbb{C}[B_q(n)] = \mathbb{C}[B_q(n,0)] \oplus \mathbb{C}[B_q(n,1)] \oplus \cdots \oplus \mathbb{C}[B_q(n,n)].$$

An element $v\in\mathbb{C}[B_q(n)]$ is *homogeneous* if $v\in\mathbb{C}[B_q(n,i)]$ for some i, and if $v\neq 0$, we extend the notion of rank to nonzero homogeneous elements by writing $r(v)=i$. For $0\le k\le n$, the k^{th} *up operator* $U_{n,k}:\mathbb{C}[B_q(n)]\to\mathbb{C}[B_q(n)]$ is defined, for $X\in B_q(n)$, by $U_{n,k}(X)=0$ if $\dim(X)\neq k$ and $U_{n,k}(X)=\sum_Y Y$, where the sum is over all $Y\in B_q(n)$ covering X, if $\dim(X)=k$. Similarly we define the k^{th} *down operator* $D_{n,k}:\mathbb{C}[B_q(n)]\to\mathbb{C}[B_q(n)]$ (we have $U_{n,n}=D_{n,0}=0$). Set $U_n=U_{n,0}+U_{n,1}+\cdots+U_{n,n}$ and $D_n=D_{n,0}+D_{n,1}+\cdots+D_{n,n}$, called, respectively, the *up and down* operators on $\mathbb{C}[B_q(n)]$.

If we think of the elements of $\mathbb{C}[B_q(n)]$ as column vectors with components indexed by the *standard basis* elements $B_q(n)$ then $M_q(n)$ is the matrix of the operator

$$\mathcal{M}_q(n) = U_n + \sum_{k=0}^{n} q^{k-1} D_{n,k}$$

with respect to the basis $B_q(n)$.

For $0 \le k \le n$, define $s_k \in \mathbb{C}[B_q(n,k)]$ by

$$s_k = \sum_{X \in B_q(n,k)} X,$$

and define $R_q(n)$ to be the subspace of $\mathbb{C}[B_q(n)]$ spanned by $s_0, s_1, \ldots, s_n$. Elements of $R_q(n)$ are called *radial vectors*. Clearly, $R_q(n)$ is closed under $\mathcal{M}_q(n)$ and $\dim(R_q(n)) = n+1$. We have

$$\mathcal{M}_q(n)(s_k) = (k+1)_q \; s_{k+1} + q^{k-1}(n-k+1)_q \; s_{k-1}, \quad 0 \le k \le n.$$

It follows from (5) that the matrix of $\mathcal{M}_q(n) : R_q(n) \to R_q(n)$ with respect to the basis $\{s_0, \ldots, s_n\}$ is $K_q(n)$.

Thus, knowing the eigenvalues of $K_q(n)$ would at least tell us some of the eigenvalues of $M_q(n)$. The eigenvalues and eigenvectors of $K_q(n)$ have been determined by Terwilliger **[18]** (we shall not use the eigenvector information given in the result below in this paper).

Theorem 2.1. *(i) The eigenvalues of* $K_q(n)$ *are*

$$(n-k)_q - (k)_q, \; k = 0, 1, \ldots, n.$$

(ii) For $0 \le k \le n$*, there is a right eigenvector of* $K_q(n)$ *corresponding to the eigenvalue* $(n-k)_q - (k)_q$ *whose component* $i, \; 0 \le i \le n$ *is given by*

$${}_3\phi_2 \left(\begin{array}{c} q^{-i}, q^{-k}, -q^{k-n} \\ 0, q^{-n} \end{array} \middle| \, q, q \right)$$

where the basic hypergeometric series notation is from **[4]**.

Proof. This is Lemma 4.20 in **[18]** (replace b by q, d by n, and j by $n-k$). Part (i) also follows from Theorem 2 in Johnson **[7]** (by taking $a = z = 1$ and $h = k = 0$). $\Box$

To show that there are no other eigenvalues of $M_q(n)$ and that the eigenvalue multiplicities are as claimed in Theorem 1.1 we shall use another result of Terwilliger **[16]**.

A *symmetric chain* in $\mathbb{C}[B_q(n)]$ is a sequence

$$s = (v_k, \ldots, v_{n-k}), \quad k \le n/2, \tag{6}$$

of nonzero homogeneous elements of $\mathbb{C}[B_q(n)]$ such that

- $r(v_i) = i$ for $i = k, \ldots, n-k$.
- $U_n(v_i)$ is a nonzero scalar multiple of v_{i+1}, for $i = k, \ldots, n-k-1$ and $U_n(v_{n-k}) = 0$.
- $D_n(v_{i+1})$ is a nonzero scalar multiple of v_i for $i = k, \ldots, n-k-1$ and $D_n(v_k) = 0$.

Note that the elements of the sequence s are linearly independent, being nonzero and of different ranks. We say that s *starts* at rank k and *ends* at rank $n-k$. Note that the subspace spanned by the elements of s is closed under U_n, D_n and also $\mathcal{M}_q(n)$.

The following result was proved (in an equivalent form) by Terwilliger **[16]** (see Item 5 of Theorem 3.3 on top of page 208). For a proof using Proctor's $\mathfrak{sl}(2,\mathbb{C})$ technique **[11]** see Theorem 2.1 in **[12]** (where also the result is stated differently but in an equivalent form to that given below).

Theorem 2.2. *There exists a basis $T_q(n)$ of $\mathbb{C}[B_q(n)]$ such that*

1. *$T_q(n)$ is a disjoint union of symmetric chains in $\mathbb{C}[B_q(n)]$.*
2. *Let $0 \leq k \leq n/2$ and let $(v_k, \ldots, v_{n-k})$ be any symmetric chain in $T_q(n)$ starting at rank k and ending at rank $n-k$. Then*

$$\begin{aligned} U_n(v_u) &= q^k (u+1-k)_q \, v_{u+1}, \ k \leq u < n-k. \\ D_n(v_{u+1}) &= (n-k-u)_q \, v_u, \ k \leq u < n-k. \end{aligned}$$

We now give the

Proof of Theorem 1.1 Observe the following.

(i) The number of symmetric chains in $T_q(n)$ starting at rank k and ending at rank $n-k$, for $0 \leq k \leq n/2$, is $\binom{n}{k}_q - \binom{n}{k-1}_q$.

(ii) Let $s = (v_k, \ldots, v_{n-k})$ be a symmetric chain in $T_q(n)$ starting at rank k, where $0 \leq k \leq n/2$. Then the subspace spanned by $\{v_k, \ldots, v_{n-k}\}$ is closed under $\mathcal{M}_q(n)$ and, by Theorem 2.2, the matrix of $\mathcal{M}_q(n)$ with respect to the basis s is $q^k K_q(n-2k)$.

(iii) By Theorem 2.1 the eigenvalues of $q^k K_q(n-2k)$ are

$$\begin{aligned} & q^k((n-2k-i)_q - (i)_q), \ i = 0, 1, \ldots, n-2k \\ = \ & ((n-i)_q - (i)_q), \ i = k, \ldots, n-k. \end{aligned}$$

(iv) It now follows from items (i), (ii), (iii) above that the eigenvalues of $\mathcal{M}_q(n)$ are

$$(n-j)_q - (j)_q, \ j = 0, \ldots, n,$$

with respective multiplicities

$$\sum_{i=0}^{\min\{j,n-j\}} \binom{n}{i}_q - \binom{n}{i-1}_q = \binom{n}{j}_q.$$

That completes the proof. $\square$

3 A decomposition of $\mathbb{C}[B_q(n)]$

In this and the next section we give proofs of Theorems 1.1 and 1.2 by inductively writing down an eigenbasis for the operator $\mathcal{M}_q(n)$. This is based on a direct sum decomposition of the vector space $\mathbb{C}[B_q(n)]$ that was given in the paper **[13]**. This decomposition yields a linear algebraic interpretation of the Goldman-Rota recurrence (see equations (13), (14), (16) and Remark 3.4) and is of independent interest. In **[13]** it was used to inductively write down an explicit eigenbasis for the Bose-Mesner algebra of the Grassmann scheme. Here we recall the relevant definitions and results from **[13]**. All the omitted proofs may be found in Section 2 of **[13]**.

We can write down the Goldman-Rota identity in terms of the q-binomial coefficients,

$$\binom{n+1}{k}_q \;=\; \binom{n}{k}_q + \binom{n}{k-1}_q + (q^n-1)\binom{n-1}{k-1}_q,\;\; n,k \geq 1, \tag{7}$$

with $\binom{0}{k}_q = \delta(0,k)$ and $\binom{n}{0}_q = 1$. Note that (3) follows by summing (7) over k.

We shall now give a linear algebraic interpretation to (3) and (7). Denote the standard basis vectors of $\mathbb{F}_q^n$ by the column vectors $e_1, \ldots, e_n$. We identify $\mathbb{F}_q^k$, for $k < n$, with the subspace of $\mathbb{F}_q^n$ consisting of all vectors with the last $n-k$ components zero. So $B_q(k)$ consists of all subspaces of $\mathbb{F}_q^n$ contained in the subspace spanned by $e_1, \ldots, e_k$.

Define $A_q(n+1)$ to be the collection of all subspaces in $B_q(n+1)$ not contained in the hyperplane $\mathbb{F}_q^n$, i.e.,

$$A_q(n+1) = B_q(n+1) - B_q(n) = \{X \in B_q(n+1) : X \not\subseteq \mathbb{F}_q^n\},\; n \geq 0.$$

For $1 \leq k \leq n+1$, let $A_q(n+1,k)$ denote the set of all subspaces in $A_q(n+1)$ with dimension k. We consider $A_q(n+1)$ as an induced subposet of $B_q(n+1)$.

We have a direct sum decomposition

$$\mathbb{C}[B_q(n+1)] = \mathbb{C}[B_q(n)] \oplus \mathbb{C}[A_q(n+1)]. \tag{8}$$

We shall now give a further decomposition of $\mathbb{C}[A_q(n+1)]$.

Let $H(n+1, \mathbb{F}_q)$ denote the subgroup of $GL(n+1, \mathbb{F}_q)$ consisting of all matrices of the form

$$\begin{bmatrix} & & a_1 \\ & I & \cdot \\ & & \cdot \\ & & a_n \\ 0 \cdots 0 & & 1 \end{bmatrix},$$

where I is the $n \times n$ identity matrix.

The additive abelian group $\mathbb{F}_q^n$ is isomorphic to $H(n+1, \mathbb{F}_q)$ via $\phi : \mathbb{F}_q^n \to H(n+1, \mathbb{F}_q)$ given by

$$\phi\left(\begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ a_n \end{bmatrix}\right) \to \begin{bmatrix} & a_1 \\ & \cdot \\ I & \cdot \\ & a_n \\ 0 \cdots 0 & 1 \end{bmatrix}.$$

Let $\mathcal{I}_q(n)$ denote the set of all distinct irreducible characters (all of degree 1) of the finite abelian group $H(n+1, \mathbb{F}_q)$ and let $\mathcal{N}_q(n)$ denote the set of all distinct nontrivial irreducible characters of $H(n+1, \mathbb{F}_q)$.

There is a natural (left) permutation action of $H(n+1, \mathbb{F}_q)$ on $A_q(n+1)$ and $A_q(n+1, k)$. This permutation action induces representations of $H(n+1, \mathbb{F}_q)$ on $\mathbb{C}[A_q(n+1)]$ and $\mathbb{C}[A_q(n+1, k)]$.

For $\chi \in \mathcal{I}_q(n)$, let $W(\chi)$ (respectively, $W(\chi, k)$) denote the isotypical component of $\mathbb{C}[A_q(n+1)]$ (respectively, $\mathbb{C}[A_q(n+1, k)]$) corresponding to the irreducible representation of $H(n+1, \mathbb{F}_q)$ with character χ. When χ is the trivial character we denote $W(\chi)$ (respectively, $W(\chi, k)$) by $W(0)$ (respectively, $W(0, k)$). We have the following decompositions, (note that $W(\chi, n+1)$, for $\chi \in \mathcal{N}_q(n)$, is the zero module).

$$\begin{aligned} W(0) &= W(0,1) \oplus \cdots \oplus W(0, n+1), && (9) \\ W(\chi) &= W(\chi, 1) \oplus \cdots \oplus W(\chi, n), \quad \chi \in \mathcal{N}_q(n), && (10) \\ \mathbb{C}[A_q(n+1)] &= W(0) \oplus \left(\oplus_{\chi \in \mathcal{N}_q(n)} W(\chi)\right). && (11) \end{aligned}$$

Now $GL(n+1, \mathbb{F}_q)$ acts on $B_q(n+1)$ and $\mathbb{C}[B_q(n+1)]$ and the action of U_{n+1} commutes with the action of $GL(n+1, \mathbb{F}_q)$ (and hence with the action of $H(n+1, \mathbb{F}_q)$). Also, $\mathbb{C}[A_q(n+1)]$ is clearly closed under U_{n+1}. Thus

$$W(0), \quad W(\chi), \chi \in \mathcal{N}_q(n) \text{ are } U_{n+1}\text{-closed}. \tag{12}$$

Define an equivalence relation $\sim$ on $A_q(n+1)$ by $X \sim Y$ iff $X \cap \mathbb{F}_q^n = Y \cap \mathbb{F}_q^n$. Denote the equivalence class of $X \in A_q(n+1)$ by $[X]$. For a subspace $X \in B_q(n)$, define $\widehat{X}$ to be the subspace in $A_q(n+1)$ spanned by X and e_{n+1}.

Lemma 3.1. *Let $X, Y \in A_q(n+1)$ and $Z, T \in B_q(n)$. Then*

(i) $\dim(X \cap \mathbb{F}_q^n) = \dim X - 1$ *and* $\widehat{X \cap \mathbb{F}_q^n} \in [X]$.

(ii) $Z \leq T$ *iff* $\widehat{Z} \leq \widehat{T}$.

(iii) Y *covers* X *iff*

(a) $Y \cap \mathbb{F}_q^n$ *covers* $X \cap \mathbb{F}_q^n$ *and*

(b) $Y =$ *span* $((Y \cap \mathbb{F}_q^n) \cup \{v\})$ *for any* $v \in X - \mathbb{F}_q^n$.

(iv) The number of subspaces $Z' \in A_q(n+1)$ with $Z' \cap \mathbb{F}_q^n = Z$ is q^l, where $l = n - \dim Z$. Thus, $|[X]| = q^{n+1-k}$, where $k = \dim X$. $\Box$

For $X \in A_q(n+1)$, let $G_X \subseteq H(n+1, \mathbb{F}_q)$ denote the stabilizer of X.

Lemma 3.2. *Let $X, Y \in A_q(n+1)$. Then*

(i) The orbit of X under the action of $H(n+1, \mathbb{F}_q)$ is $[X]$.

(ii) Suppose Y covers X. Then the bipartite graph of the covering relations between $[Y]$ and $[X]$ is regular with degrees q (on the $[Y]$ side) and 1 (on the $[X]$ side).

(iii) Suppose $X \subseteq Y$. Then $G_X \subseteq G_Y$. $\Box$

Consider $\mathbb{C}[B_q(n+1)]$. For $X \in B_q(n)$ define

$$\theta_n(X) = \sum_Y Y,$$

where the sum is over all $Y \in A_q(n+1)$ covering X. Equivalently, the sum is over all $Y \in A_q(n+1)$ with $Y \cap \mathbb{F}_q^n = X$, i.e., $Y \in [\widehat{X}]$. It follows from Lemma 3.2(i) that

$$\theta_n : \mathbb{C}[B_q(n)] \to W(0) \tag{13}$$

is a linear isomorphism.

Combining (8) and (11) we have the decomposition

$$\mathbb{C}[B_q(n+1)] \quad = \quad (\mathbb{C}[B_q(n)] \oplus W(0)) \oplus \left(\oplus_{\chi \in \mathcal{N}_q(n)} W(\chi)\right), \tag{14}$$

where, by (12) and (13),

$$\mathbb{C}[B_q(n)] \oplus W(0) \text{ is } U_{n+1}\text{-closed.} \tag{15}$$

Let ψ_k (respectively, ψ) denote the character of the permutation representation of $H(n+1, \mathbb{F}_q)$ on $\mathbb{C}[A_q(n+1,k)]$ (respectively, $\mathbb{C}[A_q(n+1)]$) corresponding to the left action. Clearly $\psi = \sum_{k=1}^{n+1} \psi_k$. Below $[,]$ denotes character inner product and the q-binomial coefficient $\binom{n}{k}_q$ is taken to be zero when n or k is < 0.

Theorem 3.3. *(i) Let $\chi \in \mathcal{I}_q(n)$ be the trivial character. Then $[\chi, \psi_k] = \binom{n}{k-1}_q,\ 1 \le k \le n+1$.*

(ii) Let $\chi \in \mathcal{N}_q(n)$. Then $[\chi, \psi_k] = \binom{n-1}{k-1}_q,\ 1 \le k \le n+1$. $\Box$

Remark 3.4. Using Theorem 3.3(ii) we see that

$$\dim(W(\chi)) \quad = \quad \sum_{k=1}^{n+1} \binom{n-1}{k-1}_q \; = \; G_q(n-1), \quad \chi \in \mathcal{N}_q(n). \tag{16}$$

Now, by taking dimensions on both sides of (14) and using (13), (16) we get the Goldman-Rota identity (3). More generally, by restricting to dimension k on both sides of (14), we get the identity (7).

For $\chi \in \mathcal{I}_q(n)$, define the following element of the group algebra of $H(n+1, \mathbb{F}_q)$:

$$p(\chi) = \sum_g \overline{\chi(g)}\, g,$$

where the sum is over all $g \in H(n+1, \mathbb{F}_q)$. For $1 \leq k \leq n+1$, the map

$$p(\chi) : \mathbb{C}[A_q(n+1,k)] \to \mathbb{C}[A_q(n+1,k)], \tag{17}$$

given by $v \mapsto \sum_{g \in H(n+1,\mathbb{F}_q)} \overline{\chi(g)}\, gv$, is a nonzero multiple of the $H(n+1, \mathbb{F}_q)$-linear projection onto $W(\chi, k)$. Similarly for $p(\chi) : \mathbb{C}[A_q(n+1)] \to \mathbb{C}[A_q(n+1)]$.

For future reference we record the following observation:

$$p(\chi)(\widehat{Y}) \text{ and } p(\chi)(\widehat{Z}) \text{ have disjoint supports, for } Y \neq Z \in B_q(n). \tag{18}$$

Lemma 3.5. *Let $X \in A_q(n+1)$ and $\chi \in \mathcal{I}_q(n)$. Then $p(\chi)(X) = 0$ iff $\chi : G_X \to \mathbb{C}^*$ is a nontrivial character of G_X.* $\Box$

Theorem 3.6. *(i) Let $\chi \in \mathcal{I}_q(n),\ X, Y \in A_q(n+1)$ with $X = hY$ for some $h \in H(n+1, \mathbb{F}_q)$. Then*

$$p(\chi)(X) = \overline{\chi(h^{-1})}\, p(\chi)(Y).$$

(ii) Let $\chi \in \mathcal{I}_q(n)$. Then $\{p(\chi)(\widehat{X}) : X \in B_q(n, k-1)$ with $p(\chi)(\widehat{X}) \neq 0\}$ is a basis of $W(\chi, k)$, $1 \leq k \leq n+1$.

(iii) Let $\chi \in \mathcal{I}_q(n)$ and let $X, Y \in B_q(n)$ with X covering Y.

$$p(\chi)(\widehat{X}) \neq 0 \textit{ implies } p(\chi)(\widehat{Y}) \neq 0. \quad \Box$$

4 Eigenvectors of $\mathcal{M}_q(n)$

We begin with a brief description of the contents of this section. In the initial part (up to and including Corollary 4.3) we show that the decomposition of $\mathbb{C}[B_q(n+1)]$ given in Section 3 is well adapted to an inductive approach to studying $\mathcal{M}_q(n)$. These results are then used to prove Theorems 1.1 and 1.2 with Theorems 4.4 and 4.5. In applications (see Section 1.9, in particular the upper bound lemma, in **[2]**) the lengths of vectors in an orthogonal basis and the absolute value of the components of these vectors play an important role. With this in view we collect, in Lemma 4.6, such information about our orthogonal basis even though we have no immediate use for this in our paper. Finally, we single out a special set of 2^n eigenvectors of $\mathcal{M}_q(n)$ and write down an explicit expression for them, see Lemma 4.7. These can be seen as the q-analogs of the classical eigenvectors of $M(n)$, written down in **[2, 14]**. Each of the classical eigenvectors has support of size 2^n (i.e., all the coordinates in the standard basis are nonzero). Similarly, these 2^n special eigenvectors of $\mathcal{M}_q(n)$ have support of size $G_q(n)$ (and these are the only eigenvectors with that property). At the end of this section we pose a problem on radial eigenvectors of $\mathcal{M}_q(n)$. In the classical $q = 1$ case this problem has an elementary and well known solution.

Consider the decomposition

$$\mathbb{C}[B_q(n+1)] \;\;=\;\; (\mathbb{C}[B_q(n)] \oplus W(0)) \oplus \left(\oplus_{\chi\in\mathcal{N}_q(n)} W(\chi)\right). \tag{19}$$

We claim that

$$\mathbb{C}[B_q(n)] \oplus W(0) \text{ and } W(\chi), \chi \in \mathcal{N}_q(n) \text{ are } D_{n+1}\text{-closed.} \tag{20}$$

This can be seen as follows. Consider the *standard inner product* on $\mathbb{C}[B_q(n+1)]$ (i.e., declare $B_q(n+1)$ to be an orthonormal basis), which is $GL(n+1, \mathbb{F}_q)$-invariant (and hence $H(n+1, \mathbb{F}_q)$-invariant). It follows that $W(0)$, $W(\chi)$, $\chi \in \mathcal{N}_q(n)$ are orthogonal and hence the decomposition (19) is orthogonal. Since D_{n+1} is the adjoint of U_{n+1}, the claim now follows from (12) and (15).

Thus $\mathbb{C}[B_q(n)] \oplus W(0)$ and $W(\chi), \chi \in \mathcal{N}_q(n)$ are closed under $\mathcal{M}_q(n+1)$.

Let $X \in B_q(n,k)$. By Lemma 3.1(iv) we see that $\theta_n(X)$ is a sum of q^{n-k} subspaces in $A_q(n+1, k+1)$. So we can express $D_{n+1,k+1}(\theta_n(X))$ in the following form

$$D_{n+1,k+1}(\theta_n(X)) = q^{n-k}X + v, \tag{21}$$

for some $v \in \mathbb{C}[A_q(n+1,k)]$. A little reflection shows that $v \in W(0)$. Defining

$$D'_{n+1,k+1}(\theta_n(X)) = v$$

gives a linear map

$$D'_{n+1,k+1} : W(0) \to W(0), \;\; 0 \le k \le n,$$

that takes the vector $\theta_n(X)$ for $X \in B_q(n,k)$ to the vector v above.

Define $\mathcal{M}'_q(n) : W(0) \to W(0)$ by

$$\mathcal{M}'_q(n) = U_{n+1} + \sum_{k=0}^{n} q^k D'_{n+1,k+1}.$$

We have the following relations (the first of which follows from (21)):

$$\mathcal{M}_q(n+1)(\theta_n(v)) \;\;=\;\; q^n v + \mathcal{M}'_q(n)(\theta_n(v)), \;\; v \in \mathbb{C}[B_q(n)], \tag{22}$$

$$\mathcal{M}_q(n+1)(v) \;\;=\;\; \mathcal{M}_q(n)(v) + \theta_n(v), \;\; v \in \mathbb{C}[B_q(n)]. \tag{23}$$

We now write down the matrix of $\mathcal{M}'_q(n)$ with respect to the basis $\{\theta_n(X) \mid X \in B_q(n)\}$ of $W(0)$.

It follows from Lemma 3.1(iii) and Lemma 3.2(ii) that

$$U_{n+1}(\theta_n(X)) \;=\; \sum_{Y} q\,\theta_n(Y), \quad X \in B_q(n),$$

where the sum is over all $Y \in B_q(n)$ covering X. Similarly, it follows that

$$q^k D'_{n+1,k+1}(\theta_n(Y)) \;=\; \sum_{X} q\left\{q^{k-1}\theta_n(X)\right\}, \quad Y \in B_q(n,k),$$

where the sum is over all $X \in B_q(n)$ covered by Y.

Thus we see that

$$\text{Matrix of } \mathcal{M}'_q(n) \text{ with respect to the basis } \{\theta_n(X) \mid X \in B_q(n)\} \text{ is } qM_q(n). \tag{24}$$

For a finite vector space X over $\mathbb{F}_q$ we denote by $B_q(X)$ the set of all subspaces of X and we denote by U_X (respectively, D_X) the up operator (respectively, down operator) on $\mathbb{C}[B_q(X)]$.

Let $\chi \in \mathcal{N}_q(n)$. By Theorem 3.3(ii) we have $\dim(W(\chi,n)) = 1$. It thus follows by Theorem 3.6(ii) and (18) above that there is a unique element $X(\chi) \in B_q(n, n-1)$ such that $p(\chi)(\widehat{X(\chi)}) \neq 0$. Moreover (see Section 2 in **[13]**),

Lemma 4.1. *Let $Y \in B_q(n, n-1)$. Then*

$$|\{\chi \in \mathcal{N}_q(n) \mid X(\chi) = Y\}| = q - 1. \quad \square$$

Let (V_1, f_1) be a pair consisting of a finite dimensional vector space V_1 (over $\mathbb{C}$) and a linear operator f_1 on V. Let (V_2, f_2) be another such pair. By an isomorphism of pairs (V_1, f_1) and (V_2, f_2) we mean a linear isomorphism $\tau : V_1 \to V_2$ such that $\tau(f_1(v)) = f_2(\tau(v)),\ v \in V_1$.

Theorem 4.2. *Let $\chi \in \mathcal{N}_q(n)$ and $X = X(\chi)$. Define*

$$\lambda(\chi) : \mathbb{C}[B_q(X)] \to W(\chi)$$

by $Y \mapsto q^{-\dim(Y)}p(\chi)(\widehat{Y}),\ \ Y \in B_q(X)$.

Then

(i) $\lambda(\chi)$ is an isomorphism of pairs $(\mathbb{C}[B_q(X)], qU_X)$ and $(W(\chi), U_{n+1})$.

(ii) $\lambda(\chi)$ is an isomorphism of pairs $(\mathbb{C}[B_q(X)], D_X)$ and $(W(\chi), D_{n+1})$.

Proof. By Theorem 3.6(iii) it follows that $\lambda(\chi)(Y) \neq 0$ for all $Y \in B_q(X)$. By (16) the dimensions of $\mathbb{C}[B_q(X)]$ and $W(\chi)$ are the same. Thus, it follows from (18) that $\lambda(\chi)$ is a vector space isomorphism.

(i) Let $Y \in B_q(X)$ with $\dim(Y) = k$.

We have (below the sum is over all Z covering Y in $B_q(X)$)

$$
\begin{aligned}
\lambda(\chi)(qU_X(Y)) &= q\lambda(\chi)\left(\sum_Z Z\right) \\
&= q^{-k}\sum_Z p(\chi)(\widehat{Z}).
\end{aligned}
$$

Before calculating $U_{n+1}\lambda(\chi)(Y)$ we make the following observation. By Lemma 3.1(ii) every element covering $\widehat{Y}$ is of the form $\widehat{Z}$, for some Z covering Y in $B_q(n)$. Suppose $Z \in B_q(n) - B_q(X)$. Since $\dim(W(\chi)) = G_q(n-1)$, it follows by parts (ii) and (iii) of Theorem 3.6 and (18) that $p(\chi)(\widehat{Z}) = 0$.

We now calculate $U_{n+1}\lambda(\chi)(Y)$. In the second step below we have used the fact that U_{n+1} is $H(n+1, \mathbb{F}_q)$-linear and in the third step, using the observation in the paragraph above, we may restrict the sum to all Z covering Y in $B_q(X)$.

We have

$$
\begin{aligned}
U_{n+1}(\lambda(\chi)(Y)) &= U_{n+1}\left(q^{-k}p(\chi)(\widehat{Y})\right) \\
&= q^{-k}p(\chi)(U_{n+1}(\widehat{Y})) \\
&= q^{-k}\sum_Z p(\chi)(\widehat{Z}).
\end{aligned}
$$

(ii) Let $Y \in B_q(X)$ with $\dim(Y) = k$.

We have (below the sum is over all Z covered by Y in $B_q(X)$)

$$
\begin{aligned}
\lambda(\chi)(D_X(Y)) &= \lambda(\chi)\left(\sum_Z Z\right) \\
&= q^{-k+1}\sum_Z p(\chi)(\widehat{Z}).
\end{aligned}
$$

Before calculating $D_{n+1}\lambda(\chi)(Y)$ we make two observations:

(a) Let Y cover Z, $Z \in B_q(X)$. Then, by Lemma 3.2(ii), there are q subspaces in $[\widehat{Z}]$ which are covered by $\widehat{Y}$. Let $Z_1 \in [\widehat{Z}]$ with $\widehat{Y}$ covering Z_1. Then, there exists $g \in H(n+1, \mathbb{F}_q)$ with $g\widehat{Z} = Z_1$. It follows by Lemma 3.1(iii) that $g\widehat{Y} = \widehat{Y}$. Thus, from Lemma 3.5 we have $\chi(g) = 1$.

(b) Let $\widehat{Y}$ cover Z, where $Z \in B_q(n)$. Then $Z = Y$ and $p(\chi)(Z) = 0$, since χ is nontrivial and every element of $H(n+1, \mathbb{F}_q)$ fixes Z.

Now we compute (using Lemma 3.2(ii), Theorem 3.6(i), and (a), (b) above)

$$\begin{aligned} D_{n+1}(\lambda(\chi)(Y)) &= q^{-k}\left\{D_{n+1}\left(p(\chi)(\widehat{Y})\right)\right\} \\ &= q^{-k}p(\chi)(D_{n+1}(\widehat{Y})) \\ &= q^{-k+1}\sum_{Z} p(\chi)(\widehat{Z}). \end{aligned}$$

where the sum is over all $Z \in B_q(X)$ covered by Y. □

Before proceeding further we introduce some notation. Let $X \in B_q(n, n-1)$. The pairs $(\mathbb{C}[B_q(X)], U_X)$ and $(\mathbb{C}[B_q(n-1)], U_{n-1})$ are clearly isomorphic with many possible isomorphisms. We now define a canonical isomorphism, based on the concept of a matrix in Schubert normal form.

A $n \times k$ matrix M over $\mathbb{F}_q$ is in *Schubert normal form* (or, *column reduced echelon form*) provided

(i) Every column is nonzero.

(ii) The last nonzero entry in every column is a 1. Let the last nonzero entry in column j occur in row r_j.

(iii) We have $r_1 < r_2 < \cdots < r_k$ and the submatrix of M formed by the rows $r_1, r_2, \ldots, r_k$ is the $k \times k$ identity matrix. We call $\{r_1, \ldots, r_k\}$ the *pivotal indices* of M.

It is well known that every k dimensional subspace of $\mathbb{F}_q^n$ is the column space of a unique $n \times k$ matrix in Schubert normal form. Given $X \in B_q(n, k)$, define $P(X) \subseteq \{1, 2, \ldots, n\}$ to be the pivotal indices of the $n \times k$ matrix in Schubert normal form with column space X. It is easy to see that $P(X)$ can also be defined as follows

$$P(X) = \{j \in \{1, 2, \ldots, n\} \ : \ X \cap \mathbb{F}_q^j \in A_q(j)\}.$$

Let $X \in B_q(n, n-1)$ and let $M(X)$ be the $n \times (n-1)$ matrix in Schubert normal form with column space X. The map $\tau(X) : \mathbb{F}_q^{n-1} \to X$ given by $e_j \mapsto$ column j of $M(X)$ is clearly a linear isomorphism and this isomorphism gives rise to an isomorphism

$$\mu(X) : \mathbb{C}[B_q(n-1)] \to \mathbb{C}[B_q(X)]$$

of pairs $(\mathbb{C}[B_q(n-1)], U_{n-1})$ and $(\mathbb{C}[B_q(X)], U_X)$ (and also of pairs $(\mathbb{C}[B_q(n-1)], D_{n-1})$ and $(\mathbb{C}[B_q(X)], D_X)$) given by $\mu(X)(Y) = \tau(X)(Y),\ Y \in B_q(n-1)$. It now follows from Theorem 4.2 that

Corollary 4.3. *Let $\chi \in \mathcal{N}_q(n)$ and $X = X(\chi)$. Then the composition $\lambda(\chi)\mu(X)$ is an isomorphism of pairs $(\mathbb{C}[B_q(n-1)], q\mathcal{M}_q(n-1))$ and $(W(\chi), \mathcal{M}_q(n+1))$.* □

We shall now prove Theorems 1.1 and 1.2 by inductively writing down eigenvectors of $\mathcal{M}_q(n)$. We shall need an indexing set for the eigenvectors. Given the fact that the eigenvalue multiplicities are the q-binomial coefficients it might appear that the set of subspaces $B_q(n)$ may be used as an indexing set. We do not know of any natural way to index the eigenvectors of $\mathcal{M}_q(n)$ by $B_q(n)$ (unlike the $q = 1$ case, where the eigenvectors of $M(n)$ may be naturally indexed by $B(n)$). A more useful indexing set, defined below, for the eigenvectors of $\mathcal{M}_q(n)$ is suggested by the decomposition (19).

For $n \geq 0$, inductively define a set $\mathcal{E}_q(n)$ consisting of sequences as follows (here $()$ denotes the empty sequence):

$$\begin{aligned}
\mathcal{E}_q(0) &= \{()\}, \\
\mathcal{E}_q(1) &= \{(0),(1)\}, \\
\mathcal{E}_q(n) &= \{(\alpha_1,\ldots,\alpha_t) \mid (\alpha_1,\ldots\alpha_{t-1}) \in \mathcal{E}_q(n-1),\ \alpha_t \in \{0,1\}\} \\
&\quad \cup \{(\alpha_1,\ldots,\alpha_t) \mid (\alpha_1,\ldots\alpha_{t-1}) \in \mathcal{E}_q(n-2),\ \alpha_t \in \mathcal{N}_q(n-1)\},\ n \geq 2.
\end{aligned}$$

Given $\alpha \in \mathcal{E}_q(n)$, let $N(\alpha)$ denote the number of nonzero entries in the sequence α (note that a nonzero entry is either 1 or an element of $\mathcal{N}_q(m)$ for some m). Set

$$\begin{aligned}
\mathcal{E}_q(n,k) &= \{\alpha \in \mathcal{E}_q(n) \mid N(\alpha) = k\}, \\
e_q(n,k) &= |\mathcal{E}_q(n,k)|.
\end{aligned}$$

It is easy to see that

$$e_q(n+1,k) = e_q(n,k) + e_q(n,k-1) + (q^n - 1)e_q(n-1,k-1),\ n,k \geq 1, \quad (25)$$

with $e_q(0,k) = \delta(0,k)$ and $e_q(n,0) = 1$, the same recurrence (with the same initial conditions) as (7), Thus $e_q(n,k) = \binom{n}{k}_q$ and $|\mathcal{E}_q(n)| = |B_q(n)|$.

The following result implies Theorem 1.1 and the first part of Theorem 1.2.

Theorem 4.4. *For each $\alpha \in \mathcal{E}_q(n)$ we define a vector $v_\alpha \in \mathbb{C}[B_q(n)]$ such that*

(i) $\mathcal{M}_q(n)(v_\alpha) = ((n-k)_q - (k)_q)v_\alpha$, where $k = N(\alpha)$.

(ii) $\{v_\alpha \mid \alpha \in \mathcal{E}_q(n)\}$ is a basis of $\mathbb{C}[B_q(n)]$.

Proof. The proof is by induction on n, the cases $n = 0, 1$ being clear by taking ($\{0\}$ denotes the zero subspace)

$$v_{()} = \{0\},\ \ v_{(0)} = \{0\} + \mathbb{F}_q,\ \ v_{(1)} = \{0\} - \mathbb{F}_q.$$

Let $n \geq 1$ and consider $\alpha = (\alpha_1,\ldots,\alpha_t) \in \mathcal{E}_q(n+1)$. Set $\beta = (\alpha_1,\ldots,\alpha_{t-1})$ and $k = N(\beta)$. We have three cases:

(a) $\alpha_t = 0$: We have $v_\beta \in \mathbb{C}[B_q(n)]$. Define

$$v_\alpha = q^k v_\beta + \theta_n(v_\beta) \in \mathbb{C}[B_q(n)] \oplus W(0). \quad (26)$$

(b) $\alpha_t = 1$: We have $v_\beta \in \mathbb{C}[B_q(n)]$. Define

$$v_\alpha = q^{n-k} v_\beta - \theta_n(v_\beta) \in \mathbb{C}[B_q(n)] \oplus W(0). \tag{27}$$

(c) $\alpha_t = \chi,\ \chi \in \mathcal{N}_q(n)$: We have $v_\beta \in \mathbb{C}[B_q(n-1)]$. Set $X = X(\chi)$ and define

$$v_\alpha = \lambda(\chi)\mu(X)(v_\beta) \in W(\chi). \tag{28}$$

Let us now check assertions (i) and (ii) in the statement of the theorem, beginning with (i). We have three cases.

(a) $\alpha_t = 0$: By the induction hypothesis and (24) we have

$$\mathcal{M}_q(n)(v_\beta) = ((n-k)_q - (k)_q)(v_\beta), \quad \mathcal{M}_q^{'}(n)(\theta_n(v_\beta)) = q((n-k)_q - (k)_q)(\theta_n(v_\beta)).$$

We have, by (22) and (23),

$$\begin{aligned}
\mathcal{M}_q(n+1)(v_\alpha) &= \mathcal{M}_q(n+1)(q^k v_\beta + \theta_n(v_\beta)) \\
&= q^k \mathcal{M}_q(n+1)(v_\beta) + \mathcal{M}_q(n+1)(\theta_n(v_\beta)) \\
&= q^k(\mathcal{M}_q(n)(v_\beta) + \theta_n(v_\beta)) + q^n v_\beta + \mathcal{M}_q^{'}(n)(\theta_n(v_\beta)) \\
&= (q^{n-k} + (n-k)_q - (k)_q) q^k v_\beta + (q^k + q((n-k)_q - (k)_q))\theta_n(v_\beta) \\
&= ((n+1-k)_q - (k)_q)(q^k v_\beta + \theta_n(v_\beta)).
\end{aligned}$$

(b) $\alpha_t = 1$: By the induction hypothesis and (24) we have

$$\mathcal{M}_q(n)(v_\beta) = ((n-k)_q - (k)_q)(v_\beta), \quad \mathcal{M}_q^{'}(n)(\theta_n(v_\beta)) = q((n-k)_q - (k)_q)(\theta_n(v_\beta)).$$

We have, by (22) and (23),

$$\begin{aligned}
\mathcal{M}_q(n+1)(v_\alpha) &= \mathcal{M}_q(n+1)(q^{n-k} v_\beta - \theta_n(v_\beta)) \\
&= q^{n-k} \mathcal{M}_q(n+1)(v_\beta) - \mathcal{M}_q(n+1)(\theta_n(v_\beta)) \\
&= q^{n-k}(\mathcal{M}_q(n)(v_\beta) + \theta_n(v_\beta)) - q^n v_\beta - \mathcal{M}_q^{'}(n)(\theta_n(v_\beta)) \\
&= (-q^k + (n-k)_q - (k)_q) q^{n-k} v_\beta - (-q^{n-k} + q((n-k)_q - (k)_q))\theta_n(v_\beta) \\
&= ((n+1-(k+1))_q - (k+1)_q)(q^{n-k} v_\beta - \theta_n(v_\beta)).
\end{aligned}$$

(c) $\alpha_t = \chi,\ \chi \in \mathcal{N}_q(n)$: Set $X = X(\chi)$. It follows from Corollary 4.3 that

$$\begin{aligned}
\mathcal{M}_q(n+1)(v_\alpha) &= q((n-1-k)_q - (k)_q) v_\alpha \\
&= ((n+1-(k+1))_q - (k+1)_q) v_\alpha.
\end{aligned}$$

Assertion (ii) follows from the induction hypothesis using the decomposition (19), the isomorphism (13) and observing that the determinant of the 2×2 matrix

$$\begin{bmatrix} q^k & 1 \\ q^{n-k} & -1 \end{bmatrix}$$

is nonzero. □

We denote the basis given in part (ii) of Theorem 4.4 by $\mathcal{B}_q(n)$. Note that (up to scalars) this basis is canonical in the sense that we have not made any choices anywhere.

Remark Note that equations (26), (27), (28) give an inductive procedure to write down $\mathcal{B}_q(n+1)$ given $\mathcal{B}_q(n)$ and $\mathcal{B}_q(n-1)$.

The following result completes the proof of Theorem 1.2.

Theorem 4.5. *The basis $\mathcal{B}_q(n)$ of $\mathbb{C}[B_q(n)]$ is orthogonal with respect to the inner product (2).*

Proof. The proof is by induction on n, the cases $n = 0, 1$ being clear.

Let $n \geq 1$. We consider two cases:

(i) Let $\beta = (\beta_1, \ldots, \beta_{t-1}) \in \mathcal{E}_q(n)$. Set $k = N(\beta)$ and

$$\alpha = (\beta_1, \ldots, \beta_{t-1}, 0), \quad \alpha' = (\beta_1, \ldots, \beta_{t-1}, 1).$$

Given a vectors $u, v \in \mathbb{C}[B_q(n)]$, we shall write $\langle u,\, v\rangle_n$ for the inner product (2) calculated in $\mathbb{C}[B_q(n)]$ and $\langle u,\, v\rangle_{n+1}$ for the inner product calculated in $\mathbb{C}[B_q(n+1)]$. We have, for $X \in B_q(n,k)$,

$$\begin{aligned} \langle X,\, X\rangle_n \;=\; \tfrac{q^{\binom{k}{2}}}{P_q(n)},\;\; \langle X,\, X\rangle_{n+1} \;=\; \tfrac{q^{\binom{k}{2}}}{P_q(n+1)} \;&=\; \tfrac{1}{1+q^n}\langle X,\, X\rangle_n, \\ \langle \theta_n(X),\, \theta_n(X)\rangle_{n+1} \;=\; \tfrac{q^{\binom{k+1}{2}}}{P_q(n+1)}\, q^{n-k} \;&=\; \tfrac{q^n}{1+q^n}\langle X,\, X\rangle_n. \end{aligned}$$

It follows that

$$\langle v,\, v\rangle_{n+1} \;=\; \tfrac{1}{1+q^n}\langle v,\, v\rangle_n, \;\; \langle \theta_n(v),\, \theta_n(v)\rangle_{n+1} \;=\; \tfrac{q^n}{1+q^n}\langle v,\, v\rangle_n, \;\; v \in \mathbb{C}[B_q(n)]. \tag{29}$$

Note that the scalar factors on the right hand side are uniform across all vectors and do not depend on k. Thus, since $\mathbb{C}[B_q(n)]$ and $W(0)$ are orthogonal in $\mathbb{C}[B_q(n+1)]$, it follows by the induction hypothesis that $\{v_\beta, \theta_n(v_\beta) \mid \beta \in \mathcal{E}_q(n)\}$ is an orthogonal basis of $\mathbb{C}[B_q(n)] \oplus W(0)$.

We have

$$v_\alpha \;=\; q^k v_\beta + \theta_n(v_\beta), \;\; v_{\alpha'} \;=\; q^{n-k} v_\beta - \theta_n(v_\beta). \tag{30}$$

Since v_β is orthogonal to $\theta_n(v_\beta)$ we have, using (29),

$$\begin{aligned} \langle v_\alpha,\, v_{\alpha'}\rangle_{n+1} \;&=\; q^n \langle v_\beta,\, v_\beta\rangle_{n+1} - \langle \theta_n(v_\beta),\, \theta_n(v_\beta)\rangle_{n+1} \\ &=\; \frac{q^n}{1+q^n}\langle v_\beta,\, v_\beta\rangle_n - \frac{q^n}{1+q^n}\langle v_\beta,\, v_\beta\rangle_n \\ &=\; 0. \end{aligned}$$

From the isomorphism θ_n we now see that

$$\{v_\alpha, v_{\alpha'} \mid \beta \in \mathcal{E}_n(q)\}$$

is an orthogonal basis of $\mathbb{C}[B_q(n)] \oplus W(0)$.

(ii) Let $\beta = (\beta_1, \ldots, \beta_{t-1}) \in \mathcal{E}_q(n-1)$ and let $\chi \in \mathcal{N}_q(n)$. Set $\alpha = (\beta_1, \ldots, \beta_{t-1}, \chi) \in \mathcal{E}_q(n+1)$ and $X = X(\chi)$, where $X \in B_q(n, n-1)$. We have $v_\alpha = \lambda(\chi)\mu(X)(v_\beta)$.

Let $Y \in B_q(X)$ with $\dim(Y) = k$. We have

$$\langle Y,\, Y\rangle_{n-1} \;=\; \frac{q^{\binom{k}{2}}}{P_q(n-1)}.$$

Now observe the following: $p(\chi)(\widehat{Y})$ is a linear combination of the elements of the orbit $[\widehat{Y}]$, whose cardinality is q^{n-k}. The number of elements $g \in H(n+1, \mathbb{F}_q)$ with $g \cdot \widehat{Y} = \widehat{Y}$ is q^k and by Lemma 3.5 each such g satisfies $\chi(g) = 1$. So, for $Z \in [\widehat{Y}]$, if $g_1 \cdot \widehat{Y} = g_2 \cdot \widehat{Y} = Z$ then $\chi(g_1) = \chi(g_2)$. Thus we have

$$\langle \lambda(\chi)(Y),\, \lambda(\chi)(Y)\rangle_{n+1} \;=\; q^{-2k}\tfrac{q^{\binom{k+1}{2}}}{P_q(n+1)}\, q^{2k} q^{n-k} \;=\; \tfrac{q^n}{(1+q^{n-1})(1+q^n)}\langle Y,\, Y\rangle_{n-1}.$$

It follows that

$$\langle \lambda(\chi)\mu(X)(v),\, \lambda(\chi)\mu(X)(v)\rangle_{n+1} \;=\; \tfrac{q^n}{(1+q^{n-1})(1+q^n)}\langle v,\, v\rangle_{n-1}, \quad v \in \mathbb{C}[B_q(n-1)]. \tag{31}$$

From the isomorphism $\lambda(\chi)\mu(X)$ we now see that

$$\{v_\alpha \mid \beta \in \mathcal{E}_q(n-1)\}$$

is an orthogonal basis of $W(\chi)$.

That completes the proof. □

The following result collects information about the length of the vectors $v_\alpha, \alpha \in \mathcal{E}_q(n)$, under the inner product (2) and the absolute values of their standard coordinates $v_\alpha(Y), Y \in B_q(n)$ (i.e., $v_\alpha = \sum_Y v_\alpha(Y)Y$). Given $X \in A_q(n+1)$ we denote $X \cap \mathbb{F}_q^n$ by X^r. For $\alpha \in \mathcal{E}_q(n)$, we denote by $\overline{\alpha} \in \mathcal{E}_q(n)$ the sequence obtained by interchanging the 0's and 1's in α.

Lemma 4.6. *(a) Let $\beta = (\beta_1, \ldots, \beta_{t-1}) \in \mathcal{E}_q(n)$ with $N(\beta) = k$. We have*

(i) If $\alpha = (\beta_1, \ldots, \beta_{t-1}, 0)$ then, for $Y \in B_q(n+1)$,

$$\begin{aligned} v_\alpha(Y) \;&=\; \begin{cases} q^k v_\beta(Y) & \textit{if } Y \in B_q(n), \\ v_\beta(Y^r) & \textit{if } Y \in A_q(n+1). \end{cases} \\ \langle v_\alpha,\, v_\alpha\rangle_{n+1} \;&=\; \frac{q^n + q^{2k}}{1+q^n}\langle v_\beta,\, v_\beta\rangle_n. \end{aligned}$$

(ii) If $\alpha = (\beta_1, \ldots, \beta_{t-1}, 1)$ then, for $Y \in B_q(n+1)$,

$$\begin{aligned} v_\alpha(Y) \;&=\; \begin{cases} q^{n-k} v_\beta(X) & \textit{if } Y \in B_q(n), \\ -v_\beta(Y^r) & \textit{if } Y \in A_q(n+1). \end{cases} \\ \langle v_\alpha,\, v_\alpha\rangle_{n+1} \;&=\; \frac{q^n + q^{2(n-k)}}{1+q^n}\langle v_\beta,\, v_\beta\rangle_n. \end{aligned}$$

(b) Let $\beta = (\beta_1, \ldots, \beta_{t-1}) \in \mathcal{E}_q(n-1)$ and let $\chi \in \mathcal{N}_q(n)$. We have

If $\alpha = (\beta_1, \ldots, \beta_{t-1}, \chi) \in \mathcal{E}_q(n+1)$ and $X = X(\chi)$, where $X \in B_q(n, n-1)$ then, for $Y \in B_q(n+1)$,

$$
\begin{aligned}
|v_\alpha(Y)| &= \begin{cases} 0 & \text{if } Y \in B_q(n), \\ |v_\beta(Y^r)| & \text{if } Y \in A_q(n+1) \text{ and } Y^r \subseteq X, \\ 0 & \text{if } Y \in A_q(n+1) \text{ and } Y^r \not\subseteq X. \end{cases} \\
\langle v_\alpha,\, v_\alpha \rangle_{n+1} &= \frac{q^n}{(1+q^{n-1})(1+q^n)} \langle v_\beta,\, v_\beta \rangle_{n-1}.
\end{aligned}
$$

(c) For $Y \in B_q(n)$ and $\alpha \in \mathcal{E}_q(n)$ we have

$$|v_\alpha(Y)| = |v_{\overline{\alpha}}(Y)|, \;\; \langle v_\alpha,\, v_\alpha \rangle_{n+1} = \langle v_{\overline{\alpha}},\, v_{\overline{\alpha}} \rangle_{n+1}.$$

Proof. Parts (a)(i) and (a)(ii) follow from (29) and (30).

Now consider part (b). The formula for $\langle v_\alpha,\, v_\alpha \rangle$ follows from (31) and the formula for $|v_\alpha(Y)|$ follows from the observation in the proof of case (ii) in Theorem 4.5.

Part (c) follows easily by induction from parts (a) and (b) on observing that $N(\overline{\alpha}) = n - N(\alpha)$. $\square$

We now single out a special set of 2^n eigenvectors of $\mathcal{M}_q(n)$, a q-analog of the classical eigenvectors of $M(n)$. Define

$$\mathcal{Z}_q(n) = \{(\alpha_1, \ldots, \alpha_n) \in \mathcal{E}_q(n) \;:\; \alpha_i \in \{0,1\} \text{ for all } i\}.$$

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{Z}_q(n)$ and $i \in \{1, 2, \ldots, n\}$, define

$$d(\alpha, i) = |\{j < i \;:\; \alpha_j \neq \alpha_i\}|.$$

Note that, if $N(\alpha) = k$ then $\sum_{i=1}^n d(\alpha, i) = k(n-k)$. Set $S(\alpha) = \{i \;:\; \alpha_i = 1\}$.

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{Z}_q(n)$ and $X \in B_q(n)$ set $d(\alpha, X) = \sum_i d(\alpha, i)$, where the sum is over all $i \in \{1, 2, \ldots, n\} \setminus P(X)$.

Lemma 4.7. *For $\alpha \in \mathcal{Z}_q(n)$ we have*

$$v_\alpha \;\; = \sum_{X \in B_q(n)} (-1)^{|S(\alpha) \cap P(X)|} \, q^{d(\alpha, X)} \, X.$$

In particular, $v_\alpha(\{0\}) = q^{k(n-k)}$ and $v_\alpha(\mathbb{F}_q^n) = (-1)^k$, where $k = N(\alpha)$.

Remark Note that, when $q = 1$, these are precisely the classical eigenvectors of $M(n)$.

Proof. By induction on n, the cases $n = 0, 1$ being clear. Let $n \geq 1$ and consider $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathcal{Z}_q(n+1)$. Set $\beta = (\alpha_1, \ldots, \alpha_n)$ and consider $X \in B_q(n+1)$. We have the following cases:

(i) $n + 1 \notin P(X)$: We have $X = X \cap \mathbb{F}_q^n$. From the inductive hypothesis and from cases (a), (b) in the proof of Theorem 4.4 we have

$$\begin{aligned} v_\alpha(X) &= q^{d(\alpha,n+1)} \, v_\beta(X) \\ &= (-1)^{|S(\beta)\cap P(X)|} \, q^{d(\alpha,n+1)} \, q^{d(\beta,X)} \\ &= (-1)^{|S(\alpha)\cap P(X)|} \, q^{d(\alpha,X)}. \end{aligned}$$

(ii) $n + 1 \in P(X)$ and $\alpha_{n+1} = 1$: We have

$$\begin{aligned} v_\alpha(X) &= -v_\beta(X^r) \\ &= -(-1)^{|S(\beta)\cap P(X^r)|} \, q^{d(\beta,X^r)} \\ &= (-1)^{|S(\alpha)\cap P(X)|} \, q^{d(\alpha,X)}. \end{aligned}$$

(iii) $n + 1 \in P(X)$ and $\alpha_{n+1} = 0$: Similar to case (ii). $\square$

It is not difficult to see, inductively, that the only eigenvectors in $\mathcal{B}_q(n)$ that have support of size $G_q(n)$ are of the form $v_\alpha,\ \alpha \in \mathcal{Z}_q(n)$. We now relate the space of radial vectors to the span of the 2^n special vectors defined above.

Theorem 4.8. *The space of radial vectors $R_q(n)$ is contained in the subspace spanned by $\{v_\alpha \ : \ \alpha \in \mathcal{Z}_q(n)\}$.*

Proof. By induction on n, the cases $n = 0, 1$ being clear. Let $n \geq 1$. By induction hypothesis and the isomorphism (13) we see that

$$\begin{aligned} R_q(n) &\subseteq \text{Span}\,(\{v_\beta : \beta \in \mathcal{Z}_q(n)\}), \\ R_q(n+1) &\subseteq \text{Span}\,(\{v_\beta : \beta \in \mathcal{Z}_q(n)\}) \oplus \text{Span}\,(\{\theta_n(v_\beta) : \beta \in \mathcal{Z}_q(n)\}), \end{aligned}$$

and the right hand side of the second of these containments is equal to $\text{Span}\,(\{v_\alpha \ : \ \alpha \in \mathcal{Z}_q(n+1)\})$. $\square$

We are thus led to the following

Problem For $0 \leq k \leq n$, there is a unique radial vector (up to scalars) that is an eigenvector of $\mathcal{M}_q(n)$ with eigenvalue $(n-k)_q - (k)_q$. Express this vector as a linear combination of the vectors $\{v_\alpha : \alpha \in \mathcal{Z}_q(n), N(\alpha) = k\}$.

The n-cube case has a well known solution: the radial eigenvector is the sum of the vectors in the (classical) eigenbasis with the same eigenvalue.

5 Weighted count of rooted spanning trees in $C_q(n)$

We now give the proof of the weighted count of rooted spanning trees in $C_q(n)$. We use the definitions of Chapter 10 of **[14]**.

Proof of Theorem 1.3 Form the directed loopless multigraph D with $B_q(n)$ as the vertex set and the following directed edges: for every edge (X, Y) in $C_q(n)$ such that $\dim(Y) = \dim(X) + 1$ create $q^{\dim(X)}$ directed edges from X to Y in D and one directed edge from Y to X in D.

Now observe the following:

(i) The outdegree of a vertex X in D is $q^{\dim(X)}(n - \dim(X))_q + (\dim(X))_q = (n)_q$. Thus the matrix $L(D)$ (the directed analog of the Laplacian) is given by

$$L(D) = (n)_q I - M_q(n).$$

(ii) There is an obvious root preserving onto map from the rooted oriented spanning subtrees of D to the rooted spanning trees in $\mathcal{F}_q(n)$, where the inverse image of $F \in \mathcal{F}_q(n)$ has cardinality $q^{w(F)}$.

(iii) By Theorem 1.1, the eigenvalues of $L(D)$ are

$$(n)_q - ((n-k)_q - (k)_q) = (1 + q^{n-k})(k)_q, \quad k = 0, 1, \ldots, n$$

with multiplicity $\binom{n}{k}_q$.

It follows from Theorem 10.4 in **[14]** (this is Tutte's directed analog of the Matrix-Tree theorem) and item (ii) above that the weighted count of rooted spanning trees in $\mathcal{F}_q(n)$ is the product of the nonzero eigenvalues of $L(D)$ and this agrees with the statement of the Theorem by item (iii) above. $\Box$

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