OPTIMAL DESIGN OF STRUCTURES (MAP 562)

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LECTURE II

OPTIMAL CONTROL AND PARAMETRIC (OR SIZING) OPTIMIZATION

CIMPA Summer School on Current Research in Finite Element Methods, IIT Mumbai

Control of an elastic membrane

For $f \in L^2(\Omega)$, the vertical displacement u of the membrane is solution of

$$\begin{cases} -\Delta u = f + v & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where v is a **control force** which is our optimization variable (for example, a piezzo-electric actuator). We define the set of admissible controls

$$K = \left\{ v \in L^2(\omega) \mid v_{min}(x) \le v(x) \le v_{max}(x) \text{ in } \omega \text{ and } v = 0 \text{ in } \Omega \setminus \omega \right\}.$$

We want to control the membrane in order to reach a prescribed displacement $u_0 \in L^2(\Omega)$ by minimizing ($c > 0$)

$$\inf_{v \in K} \left\{ J(v) = \frac{1}{2} \int_\Omega \left(|u - u_0|^2 + c|v|^2 \right) dx \right\}.$$

Existence of an optimal control

Proposition.

There exists a unique optimal control $\overline{v} \in K$.

Proof. $v \to u$ is an affine function from K into $H^1_0(\Omega)$.

The integrand of J is a positive "polynomial" of degree two in v.

$v \to J(v)$ is strongly convex on K which is convex.

Remark. The existence is often more delicate to prove, but the important thing here is to compute a gradient $J'(v)$ for numerical purposes.

Important notice: the solution u of the p.d.e. depends on the control v.

Gradient and optimality condition

The safest and simplest way of computing a gradient is to evaluate the directional derivative

$$j(t) = J(v + tw) \quad \Rightarrow \quad j'(0) = \langle J'(v), w \rangle = \int_\Omega J'(v) w \, dx \, .$$

By linearity, we have $u(v + tw) = u(v) + t\tilde{u}(w)$ with

$$\begin{cases} -\Delta \tilde{u}(w) = w & \text{in } \Omega \\ \tilde{u}(w) = 0 & \text{on } \partial\Omega. \end{cases}$$

In other words, $\tilde{u}(w) = \langle u'(v), w \rangle$.

Since $J(v)$ is quadratic the computation is very simple and we obtain

$$\int_\Omega J'(v) w \, dx = \int_\Omega \Big((u(v) - u_0) \tilde{u}(w) + cvw \Big) dx,$$

Unfortunately $J'(v)$ is not explicit because we cannot factorize out w in $\tilde{u}(w)$!

Adjoint state

To simplify the gradient formula we use the so-called adjoint state p, defined as the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\Delta p = u - u_0 & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega. \end{cases}$$

We multiply the equation for $\tilde{u}(w)$ by p and conversely

$$\text{equation for } p \times \tilde{u}(w) \quad \Rightarrow \quad \int_\Omega \nabla p \cdot \nabla \tilde{u}(w)\, dx = \int_\Omega (u - u_0)\tilde{u}(w)\, dx$$

$$\text{equation for } \tilde{u}(w) \times p \quad \Rightarrow \quad \int_\Omega \nabla \tilde{u}(w) \cdot \nabla p\, dx = \int_\Omega wp\, dx$$

Comparing these two equalities we deduce that

$$\int_\Omega (u - u_0)\tilde{u}(w)\, dx = \int_\Omega wp\, dx \quad \Rightarrow \quad \int_\Omega J'(v)w\, dx = \int_\Omega (p + cv)w\, dx.$$

Conclusion on the adjoint state

We found an explicit formula of the gradient

$$J'(v) = p + cv.$$

- ☞ Adjoint method: computation of the gradient by solving 2 boundary value problems (u and p).
- ☞ If one does not use the adjoint: for **each** direction w one must solve 2 boundary value problems (u and $\tilde{u}(w)$) to evaluate $\langle J'(v), w\rangle$.

  For example, if $J'(v)$ is a vector of dimension n, its n components are obtained by solving $(n+1)$ problems !
- ☞ Very efficient in practice: it is the best possible method.
- ☞ Inconvenient: if one uses a black-box software to compute u, it can be very difficult to modify it in order to get the adjoint state p.

Further remarks on the notion of adjoint state

- ☞ If the state equation is not self-adjoint (the bilinear form is not symmetric), the operator of the adjoint equation is the transposed or adjoint of the direct operator.
- ☞ If the state equation is time dependent with an initial condition, then the adjoint equation is time dependent too, but backward with a final condition.
- ☞ If the state equation is non-linear, the adjoint equation is linear.

The adjoint is not just a trick ! It can be deduced from the Lagrangian of the problem.

General method to find the adjoint equation

We consider the state equation as a constraint and, for any $(\hat{v}, \hat{u}, \hat{p}) \in L^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$, we introduce the Lagrangian of the minimization problem

$$\mathcal{L}(\hat{v}, \hat{u}, \hat{p}) = \frac{1}{2} \int_\Omega \left(|\hat{u} - u_0|^2 + c|\hat{v}|^2 \right) dx + \int_\Omega \hat{p}(\Delta \hat{u} + f + \hat{v})\, dx,$$

where $\hat{p}$ is the Lagrange multiplier for the constraint which links the two independent variables $\hat{v}$ and $\hat{u}$.

Integrating by parts yields

$$\mathcal{L}(\hat{v}, \hat{u}, \hat{p}) = \frac{1}{2} \int_\Omega \left(|\hat{u} - u_0|^2 + c|\hat{v}|^2 \right) dx + \int_\Omega (-\nabla \hat{p} \cdot \nabla \hat{u} + f\hat{p} + \hat{v}\hat{p})\, dx.$$

Proposition. The optimality conditions are equivalent to the stationnarity of the Lagrangian, i.e.,

$$\frac{\partial \mathcal{L}}{\partial v} = \frac{\partial \mathcal{L}}{\partial u} = \frac{\partial \mathcal{L}}{\partial p} = 0.$$

Proof

- $\frac{\partial \mathcal{L}}{\partial p} = 0 \Rightarrow$ by definition, we recover the equation satisfied by the state u.
- $\frac{\partial \mathcal{L}}{\partial u} = 0 \Rightarrow$ equation satisfied by the adjoint state p. Indeed,

$$\ell_u(t) = \mathcal{L}(\hat{v}, \hat{u} + t\phi, \hat{p}) \quad \Rightarrow \quad \ell_u'(0) = \langle \frac{\partial \mathcal{L}}{\partial u}, \phi \rangle = \int_\Omega \left((\hat{u} - u_0)\phi - \nabla \hat{p} \cdot \nabla \phi\right) dx$$

which is the variational formulation of the adjoint equation.

- $\frac{\partial \mathcal{L}}{\partial v} = 0 \Rightarrow$ formula for $J'(v)$. Indeed,

$$\ell_v(t) = \mathcal{L}(\hat{v} + tw, \hat{u}, \hat{p}) \quad \Rightarrow \quad \ell_v'(0) = \langle \frac{\partial \mathcal{L}}{\partial v}, w \rangle = \int_\Omega (c\hat{v} + \hat{p}) w \, dx$$

Simple formula for the derivative

In the preceding proof we obtained

$$J'(v) = \frac{\partial \mathcal{L}}{\partial v}(v, u, p)$$

with the state u and the adjoint p (both depending on v).

It is not a surprise ! Indeed,

$$J(v) = \mathcal{L}(v, u, \hat{p}) \quad \forall \hat{p}$$

because u is the state. Thus, if $u(v)$ is differentiable, we get

$$\langle J'(v), w \rangle = \langle \frac{\partial \mathcal{L}}{\partial v}(v, u, \hat{p}), w \rangle + \langle \frac{\partial \mathcal{L}}{\partial u}(v, u, \hat{p}), \frac{\partial u}{\partial v}(w) \rangle$$

We then take $\hat{p} = p$, the adjoint, to obtain

$$\langle J'(v), w \rangle = \langle \frac{\partial \mathcal{L}}{\partial v}(v, u, p), w \rangle$$

Another interpretation of the adjoint state

The adjoint state p is the Lagrange multiplier for the constraint of the state equation. But it is also a sensitivity function.

Define the Lagrangian

$$\mathcal{L}(\hat{v},\hat{u},\hat{p},f) = \frac{1}{2}\int_\Omega \left(|\hat{u}-u_0|^2 + c|\hat{v}|^2\right) dx + \int_\Omega (-\nabla\hat{p}\cdot\nabla\hat{u} + f\hat{p} + \hat{v}\hat{p})\, dx.$$

We study the sensitivity of the minimum with respect to variations of f.

We denote by $v(f)$, $u(f)$ and $p(f)$ the optimal values, depending on f. We assume that they are differentiable with respect to f. Then

$$\nabla_f \Big(J(v(f)) \Big) = p(f).$$

p gives the derivative (without further computation) of the minimun with respect to f !

Indeed $J(v(f)) = \mathcal{L}(v(f),u(f),p(f),f)$ and $\frac{\partial \mathcal{L}}{\partial v} = \frac{\partial \mathcal{L}}{\partial u} = \frac{\partial \mathcal{L}}{\partial p} = 0$.

PARAMETRIC (OR SIZING) OPTIMIZATION

Optimization of a membrane thickness

Membrane occupying a bounded domain Ω in $\mathbb{R}^N$. Forces $f \in L^2(\Omega)$, displacement $u \in H_0^1(\Omega)$ which is solution of

$$\begin{cases} -\operatorname{div}(h\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is called parametric (or sizing) optimization because the computational domain Ω is fixed. The thickness $h(x)$ is just a parameter.

Admissible set of thicknesses h, defined by

$$\mathcal{U}_{ad} = \left\{ h \in L^\infty(\Omega)\,, \quad h_{max} \geq h(x) \geq h_{min} > 0 \text{ in } \Omega, \int_\Omega h(x)\,dx = h_0|\Omega| \right\}.$$

Parametric shape optimization problem:

$$\inf_{h \in \mathcal{U}_{ad}} J(h) = \int_\Omega j(u)\, dx$$

where u depends on h through the state equation, and j is a C^1 function from $\mathbb{R}$ to $\mathbb{R}$ such that $|j(u)| \leq C(u^2 + 1)$ and $|j'(u)| \leq C(|u| + 1)$.

Examples:

☞ Compliance or work done by the load (a measure of rigidity)

$$j(u) = fu$$

☞ Least square criteria to reach a target displacement $u_0 \in L^2(\Omega)$

$$j(u) = |u - u_0|^2$$

Computation of a continuous gradient

$$\begin{cases} -\operatorname{div}(h\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

$$\mathcal{U} = \left\{ h \in L^\infty(\Omega) \ , \quad \exists h_0 > 0 \text{ such that } h(x) \geq h_0 \text{ in } \Omega \right\}.$$

Lemma. The application $h \to u(h)$, which gives the solution $u(h) \in H_0^1(\Omega)$ for $h \in \mathcal{U}$, is differentiable and its directional derivative at h in the direction $k \in L^\infty(\Omega)$ is given by

$$\langle u'(h), k \rangle = v,$$

where v is the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}(h\nabla v) = \operatorname{div}(k\nabla u) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Formaly, one simply differentiates the equation with respect to h. However, to be mathematically rigorous one should rather work at the level of the variational formulation.

To compute the directional derivative, we define $h(t) = h + tk$ for $t > 0$. Let $u(t)$ be the solution for the thickness $h(t)$. Deriving with respect to t leads to

$$\begin{cases} -\operatorname{div}\left(h(t)\nabla u'(t)\right) = \operatorname{div}\left(h'(t)\nabla u(t)\right) & \text{in } \Omega \\ u'(t) = 0 & \text{on } \partial\Omega, \end{cases}$$

and, since $h'(0) = k$, we deduce $u'(0) = v$.

Lemma. For $h \in \mathcal{U}$, let $u(h)$ be the state in $H^1_0(\Omega)$ and

$$J(h) = \int_\Omega j\big(u(h)\big)\,dx\ ,$$

where j is a C^1 function from $\mathbb{R}$ into $\mathbb{R}$ such that $|j(u)| \leq C(u^2+1)$ and $|j'(u)| \leq C(|u|+1)$ for any $u \in \mathbb{R}$. The application $J(h)$, from $\mathcal{U}$ into $\mathbb{R}$, is differentiable and its directional derivative at h in the direction $k \in L^\infty(\Omega)$ is given by

$$\langle J'(h), k\rangle = \int_\Omega j'\big(u(h)\big) v\,dx\ ,$$

where $v = \langle u'(h), k\rangle$ is the unique solution in $H^1_0(\Omega)$ of

$$\begin{cases} -\operatorname{div}(h\nabla v) = \operatorname{div}(k\nabla u) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. By simple composition of differentiable applications.

Adjoint state

We introduce an adjoint state p defined as the unique solution in $H^1_0(\Omega)$ of

$$\begin{cases} -\operatorname{div}(h\nabla p) = -j'(u) & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem. The cost function $J(h)$ is differentiable on $\mathcal{U}$ and

$$J'(h) = \nabla u \cdot \nabla p \; .$$

If $h \in \mathcal{U}_{ad}$ is a local minimizer of J in $\mathcal{U}_{ad}$, it satisfies the necessary optimality condition

$$\int_\Omega \nabla u \cdot \nabla p \, (k - h) \, dx \geq 0$$

for any $k \in \mathcal{U}_{ad}$.

Proof. To make explicit $J'(h)$, we must eliminate $v = \langle u'(h), k\rangle$. We use the adjoint state for that: multiplying the equation for v by p and that for p by v, we integrate by parts

$$\int_\Omega h\nabla p \cdot \nabla v \, dx = -\int_\Omega j'(u) v \, dx$$

$$\int_\Omega h\nabla v \cdot \nabla p \, dx = -\int_\Omega k\nabla u \cdot \nabla p \, dx$$

Comparing these two equalities we deduce

$$\langle J'(h), k\rangle = \int_\Omega j'(u) v \, dx = \int_\Omega k\nabla u \cdot \nabla p \, dx,$$

for any $k \in L^\infty(\Omega)$. Since $\nabla u \cdot \nabla p$ belongs to $L^1(\Omega)$, we check that $J'(h)$ is continuous on $L^\infty(\Omega)$.

How to find the adjoint state

For independent variables $(\hat{h}, \hat{u}, \hat{p}) \in L^\infty(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$, we introduce the Lagrangian

$$\mathcal{L}(\hat{h}, \hat{u}, \hat{p}) = \int_\Omega j(\hat{u})\, dx + \int_\Omega \hat{p} \left(- \operatorname{div} \left(\hat{h} \nabla \hat{u} \right) - f \right) dx,$$

where $\hat{p}$ is a Lagrange multiplier (a function) for the constraint which connects u to h. By integration by parts we get

$$\mathcal{L}(\hat{h}, \hat{u}, \hat{p}) = \int_\Omega j(\hat{u})\, dx + \int_\Omega \left(\hat{h} \nabla \hat{p} \cdot \nabla \hat{u} - f \hat{p} \right) dx,$$

The partial derivative of $\mathcal{L}$ with respect to u in the direction $\phi \in H_0^1(\Omega)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial u}(\hat{h}, \hat{u}, \hat{p}), \phi \rangle = \int_\Omega j'(\hat{u}) \phi \, dx + \int_\Omega \left(\hat{h} \nabla \hat{p} \cdot \nabla \phi \right) dx,$$

which, when it vanishes, is nothing else than the variational formulation of the adjoint equation.

A simple formula for the derivative

The Lagrangian yields the following formula

$$J'(h) = \frac{\partial \mathcal{L}}{\partial h}(h, u, p)$$

with the state u and the adjoint p.

This is not a surprise ! Indeed,

$$J(h) = \mathcal{L}(h, u, \hat{p}) \quad \forall \hat{p}$$

because u is the state. Thus, if $u(h)$ is differentiable, we get

$$\langle J'(h), k \rangle = \langle \frac{\partial \mathcal{L}}{\partial h}(h, u, \hat{p}), k \rangle + \langle \frac{\partial \mathcal{L}}{\partial u}(h, u, \hat{p}), \frac{\partial u}{\partial h}(k) \rangle$$

Then, taking $\hat{p} = p$, the adjoint, we obtain

$$\langle J'(h), k \rangle = \langle \frac{\partial \mathcal{L}}{\partial h}(h, u, p), k \rangle$$

The self-adjoint case: the compliance

When $j(u) = fu$, we find $p = -u$ since $j'(u) = f$. This particular case is said to be **self-adjoint**.

We use the dual or complementary energy

$$\int_\Omega f u \, dx = \min_{\substack{\tau \in L^2(\Omega)^N \\ -\operatorname{div}\tau = f \text{ in } \Omega}} \int_\Omega h^{-1} |\tau|^2 dx \; .$$

We can rewrite the optimization problem as a double minimization

$$\inf_{h \in \mathcal{U}_{ad}} \min_{\substack{\tau \in L^2(\Omega)^N \\ -\operatorname{div}\tau = f \text{ in } \Omega}} \int_\Omega h^{-1} |\tau|^2 dx \; ,$$

and the order of minimization is irrelevent.

An existence result

We rewrite the problem under the form

$$\inf_{(h,\tau)\in\mathcal{U}_{ad}\times H} \int_\Omega h^{-1}|\tau|^2 dx \ .$$

with $H = \{\tau \in L^2(\Omega)^N \, , \, -\text{div}\tau = f \text{ in } \Omega\}$.

Lemma. The function $\phi(a,\sigma) = a^{-1}|\sigma|^2$, defined from $\mathbb{R}^+ \times \mathbb{R}^N$ into $\mathbb{R}$, is convex and satisfies

$$\phi(a,\sigma) = \phi(a_0,\sigma_0) + \phi'(a_0,\sigma_0)\cdot(a-a_0,\sigma-\sigma_0) + \phi(a,\sigma - \frac{a}{a_0}\sigma_0),$$

where the derivative is given by

$$\phi'(a_0,\sigma_0)\cdot(b,\tau) = -\frac{b}{a_0^2}|\sigma_0|^2 + \frac{2}{a_0}\sigma_0\cdot\tau.$$

Theorem. There exists a minimizer to the compliance minimization problem.

Optimality conditions for compliance minimization

Lemma. Take $\tau \in L^2(\Omega)^N$. The problem

$$\min_{h\in\mathcal{U}_{ad}} \int_\Omega h^{-1}|\tau|^2 dx$$

admits a minimizer $h(\tau)$ in $\mathcal{U}_{ad}$ given by

$$h(\tau)(x) = \begin{cases} h^*(x) & \text{if } h_{min} < h^*(x) < h_{max} \\ h_{min} & \text{if } h^*(x) \leq h_{min} \\ h_{max} & \text{if } h^*(x) \geq h_{max} \end{cases} \qquad \text{with } h^*(x) = \frac{|\tau(x)|}{\sqrt{\ell}},$$

where $\ell \in \mathbb{R}^+$ is the Lagrange multiplier such that $\displaystyle\int_\Omega h(x)\,dx = h_0|\Omega|$.

Proof. The function $h \to \int_\Omega h^{-1}|\tau|^2 dx$ is convex from $\mathcal{U}_{ad}$ into $\mathbb{R}$ and we easily compute its derivative.

Numerical algorithm

Projected gradient

1. Initialization of the thickness $h_0 \in \mathcal{U}_{ad}$ (by example, a constant function which satisfies the constraints).

2. Iterations until convergence, for $n \geq 0$:

$$h_{n+1} = P_{\mathcal{U}_{ad}}\Big(h_n - \mu J'(h_n)\Big),$$

where $\mu > 0$ is a descent step, $P_{\mathcal{U}_{ad}}$ is the projection operator on the closed convex set $\mathcal{U}_{ad}$ and the derivative is given by

$$J'(h_n) = \nabla u_n \cdot \nabla p_n$$

with the state u_n and the adjoint p_n (associated to the thickness h_n).

To make the algorithm fully explicit, we have to specify what is the projection operator $P_{\mathcal{U}_{ad}}$.

We characterize the projection operator $P_{\mathcal{U}_{ad}}$

$$\Big(P_{\mathcal{U}_{ad}}(h)\Big)(x) = \max\left(h_{min}, \min\left(h_{max}, h(x) + \ell\right)\right)$$

where ℓ is the unique Lagrange multiplier such that

$$\int_\Omega P_{\mathcal{U}_{ad}}(h)\, dx = h_0|\Omega|.$$

The determination of the constant ℓ is not explicit: we must use an iterative algorithm based on the property of the function

$$\ell \to F(\ell) = \int_\Omega \max\left(h_{min}, \min\left(h_{max}, h(x) + \ell\right)\right)\, dx$$

which is strictly increasing on the interval $[\ell^-, \ell^+]$, reciprocal image of $[h_{min}|\Omega|, h_{max}|\Omega|]$. Thanks to this monotonicity property, we propose a simple iterative algorithm: we first bracket the root by an interval $[\ell^1, \ell^2]$ such that

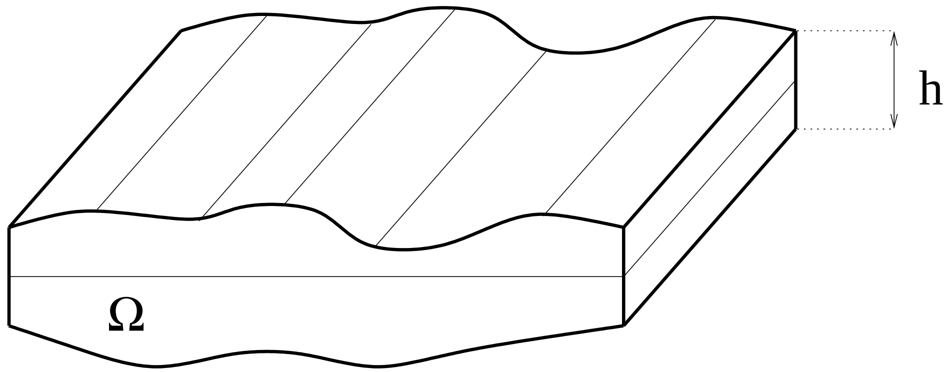
$$F(\ell^1) \leq h_0|\Omega| \leq F(\ell^2),$$

then we proceed by dichotomy to find the root ℓ.

- In practice, we rather use a projected gradient algorithm with a variable step (not optimal) which guarantees the decrease of the functional $J(h_{n+1}) < J(h_n)$.
- The algorithm is rather slow. A possible acceleration is based on the quasi-Newton algorithm.
- The overhead generated by the adjoint computation is very modest : one has to build a new right-hand-side (using the state) and solve the corresponding linear system (with the same rigidity matrix).
- Convergence is detected when the optimality condition is satisfied with a threshold $\epsilon > 0$

$$|h_n - \max\left(h_{min}, \min\left(h_{max}, h_n - \mu_n J'(h_n) + \ell_n\right)\right)| \leq \epsilon \mu_n h_{max}.$$

Thickness optimization of an elastic plate



$$\begin{cases} -\operatorname{div}\sigma = f & \text{in } \Omega \\ \sigma = 2\mu h\, e(u) + \lambda h\, \mathrm{tr}(e(u))\, \mathrm{Id} & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \sigma n = g & \text{on } \Gamma_N \end{cases}$$

with the strain tensor $e(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^t\big)$.

Set of admissible thicknesses:

$$\mathcal{U}_{ad} = \left\{ h \in L^\infty(\Omega) \ , \quad h_{max} \geq h(x) \geq h_{min} > 0 \text{ in } \Omega, \int_\Omega h(x)\, dx = h_0 |\Omega| \right\}.$$

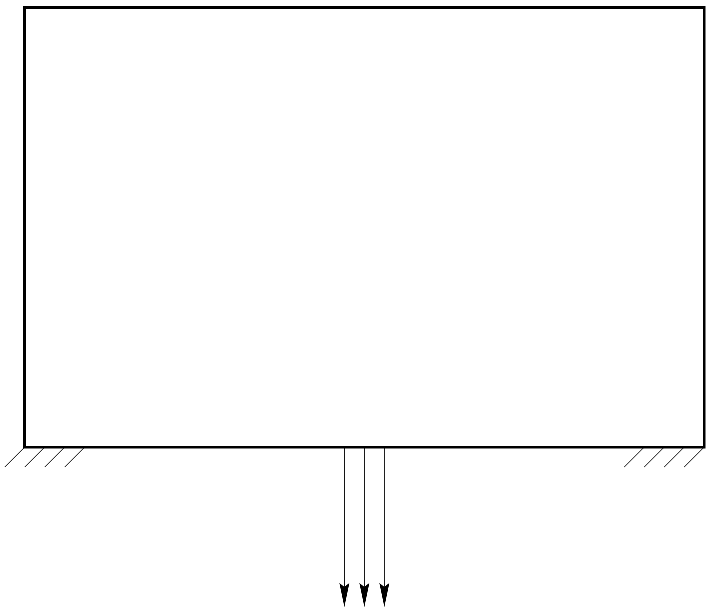
The compliance optimization can be written

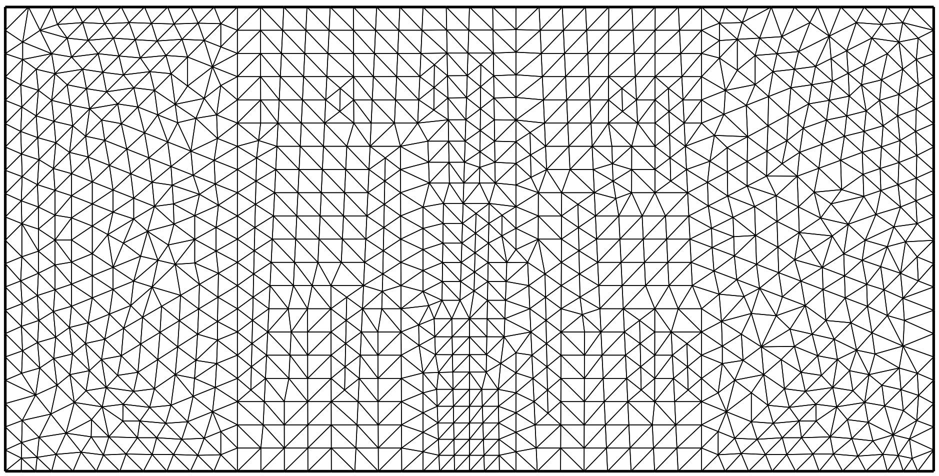
$$\inf_{h \in \mathcal{U}_{ad}} J(h) = \int_\Omega f \cdot u \, dx + \int_{\Gamma_N} g \cdot u \, ds.$$

The theoretical results are the same.

We apply the optimality criteria method.

Boundary conditions and mesh for an elastic plate

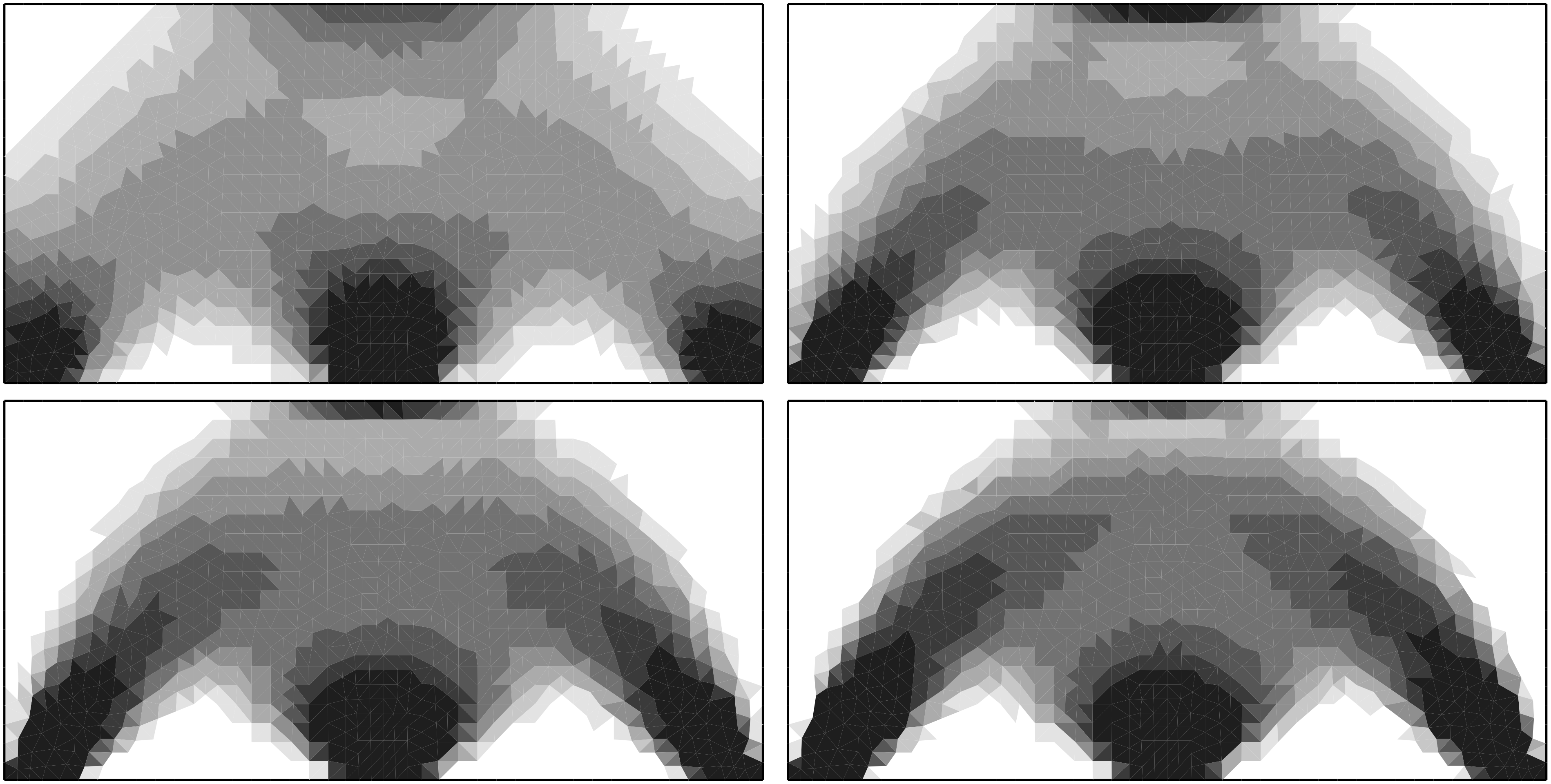




FreeFem++ computations ; scripts available on the web page

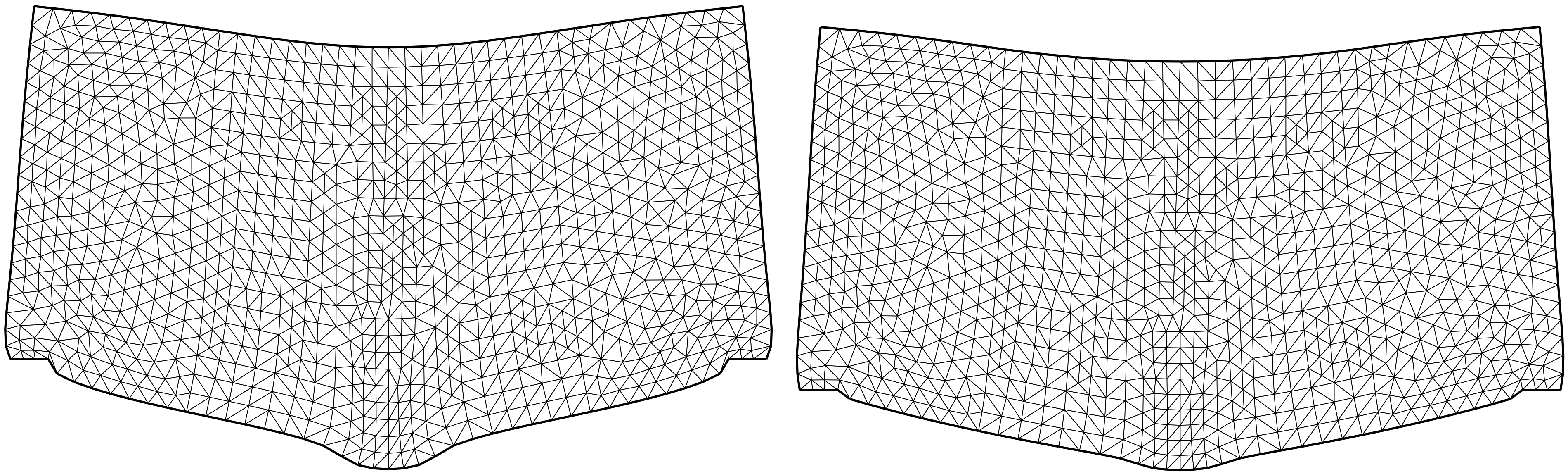
`http://www.cmap.polytechnique.fr/~allaire/cours_X_annee3.html`

Thickness at iterations 1, 5, 10, 30 (uniform initialization).

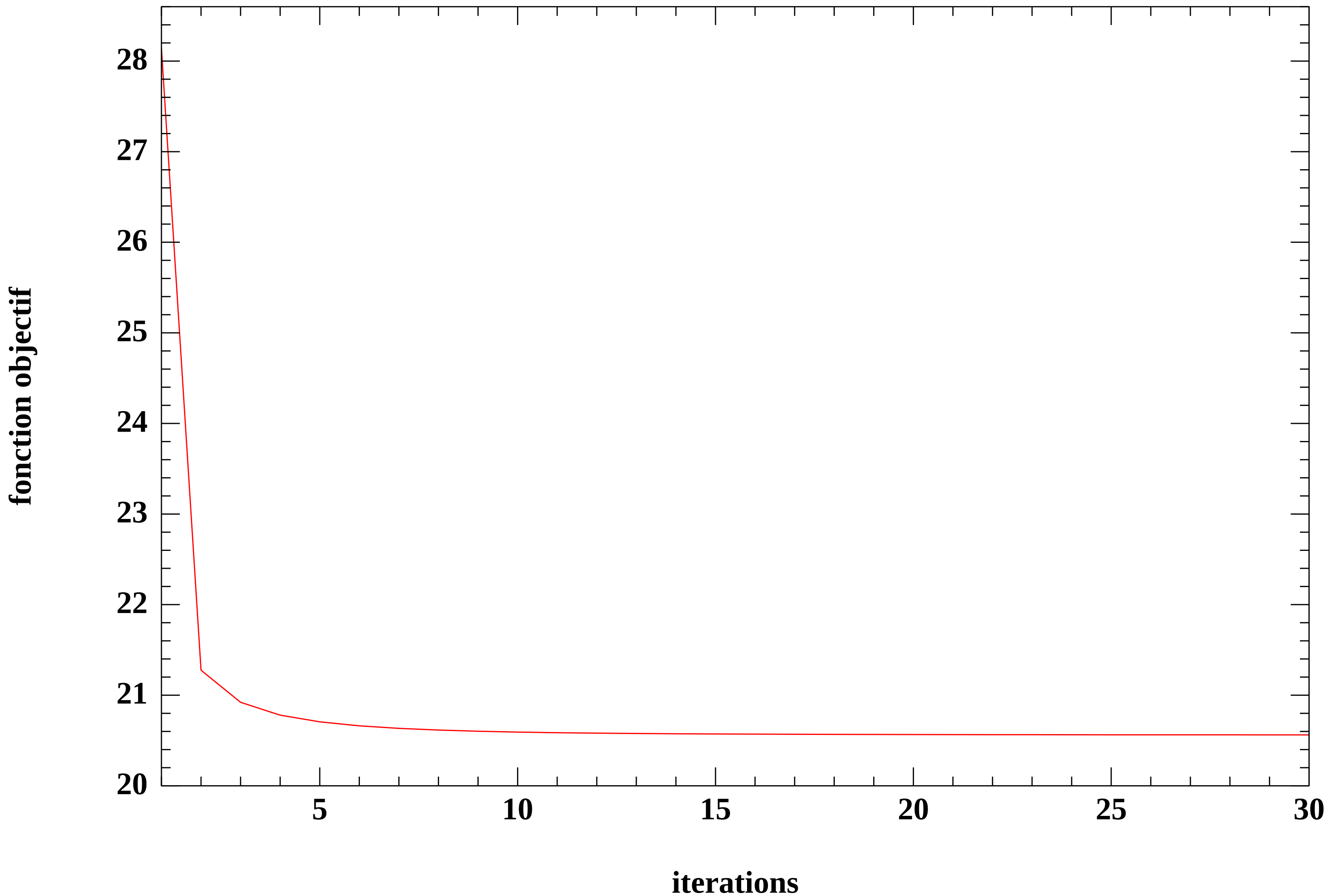


$h_{min} = 0.1$, $h_{max} = 1.0$, $h_0 = 0.5$ (increasing thickness from white to black)

Comparing the initial and final deformed shapes



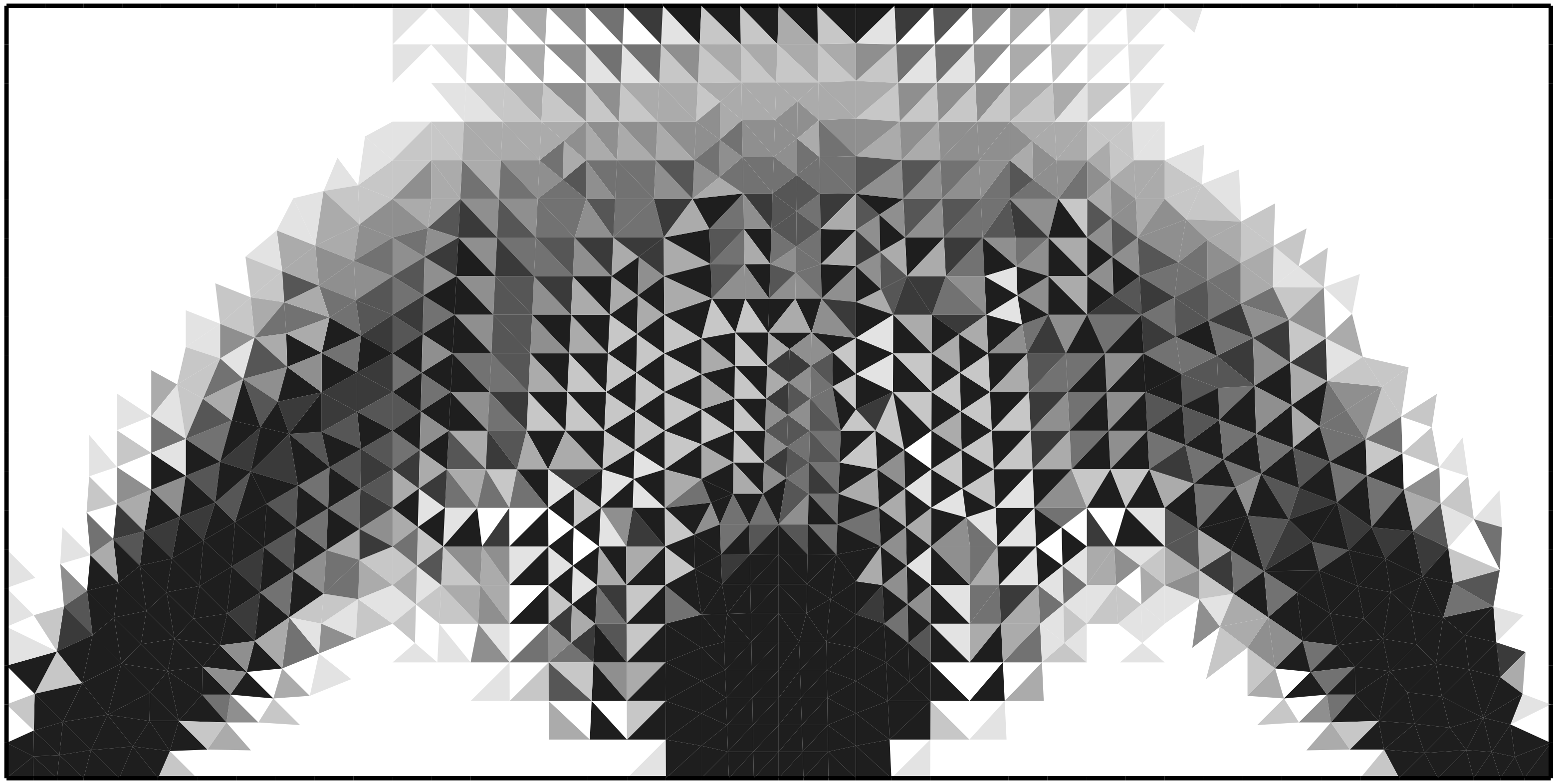
Convergence history



Numerical instabilities (checkerboards)

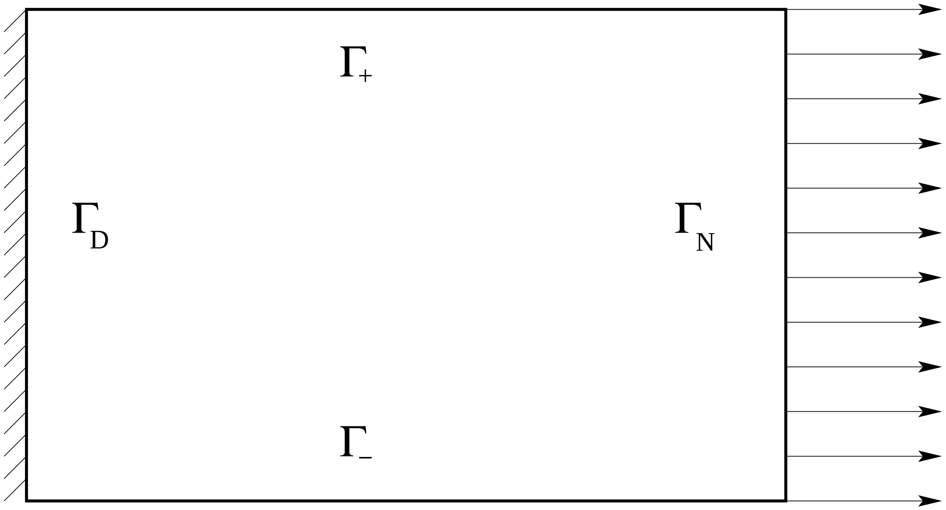
☞ Finite elements $P2$ for u and $P0$ for $h \Rightarrow$ OK

☞ Finite elements $P1$ for u and $P0$ for $h \Rightarrow$ unstable !

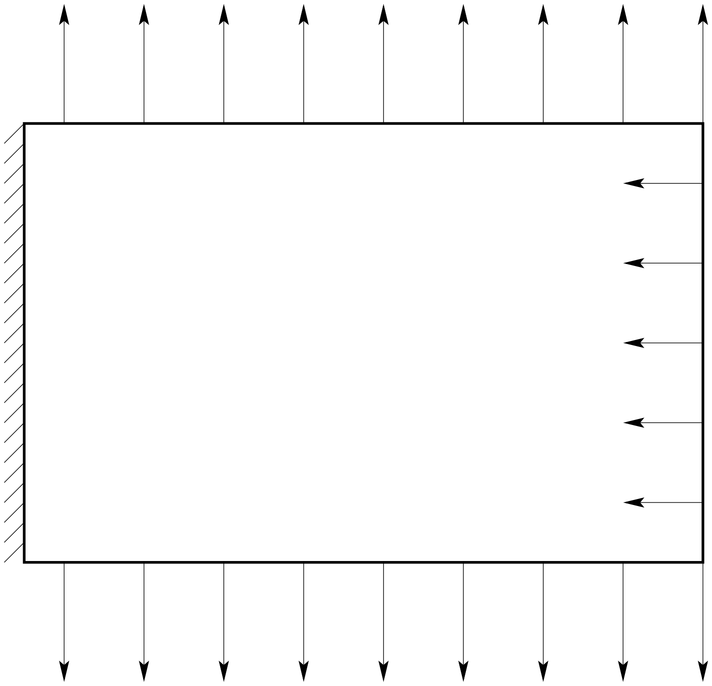


Numerical counter-example of non-existence of an optimal shape (in elasticity)

We look for the design which horizontally is less deformed and vertically more deformed.

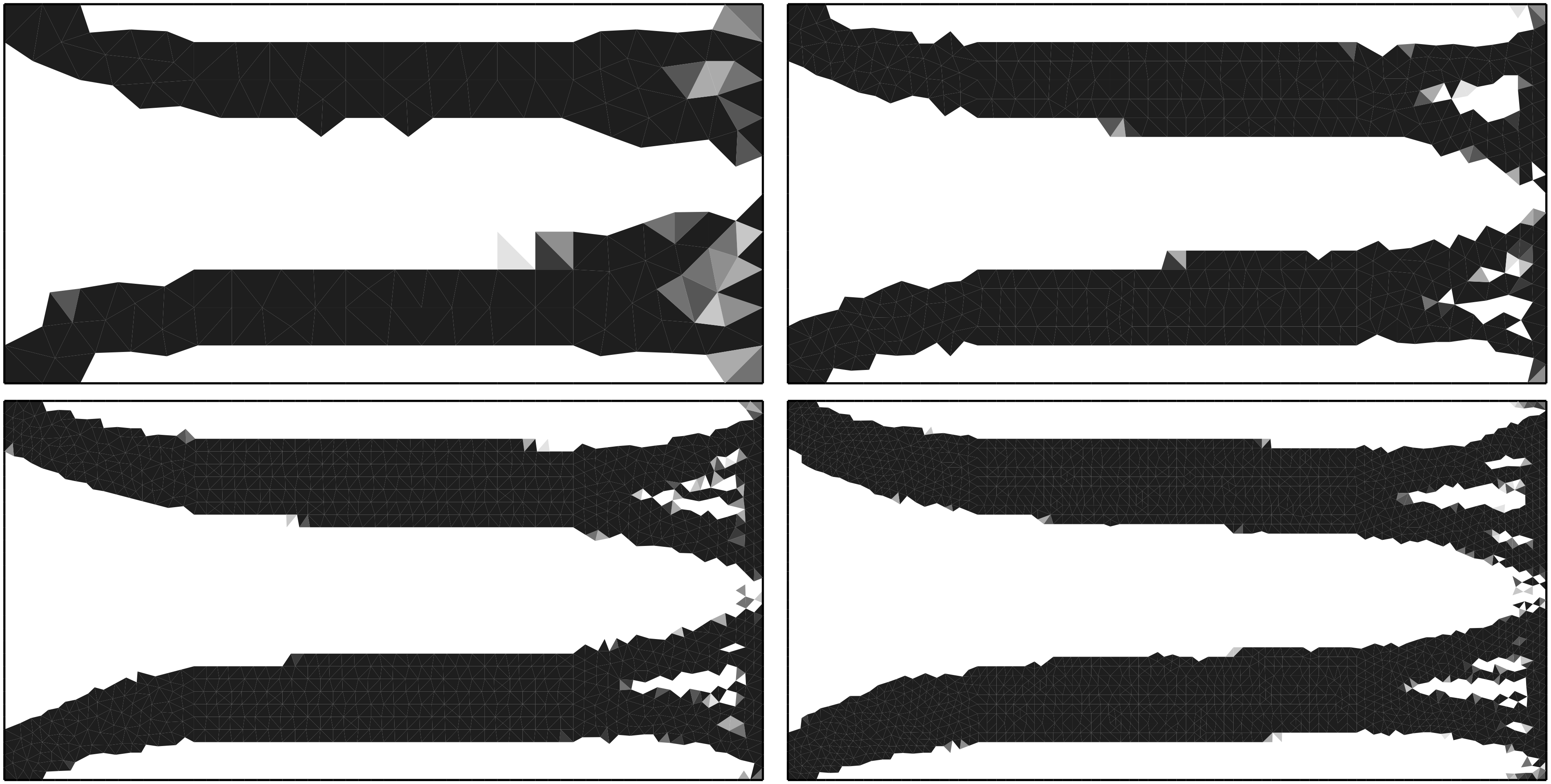


boundary conditions



target displacement

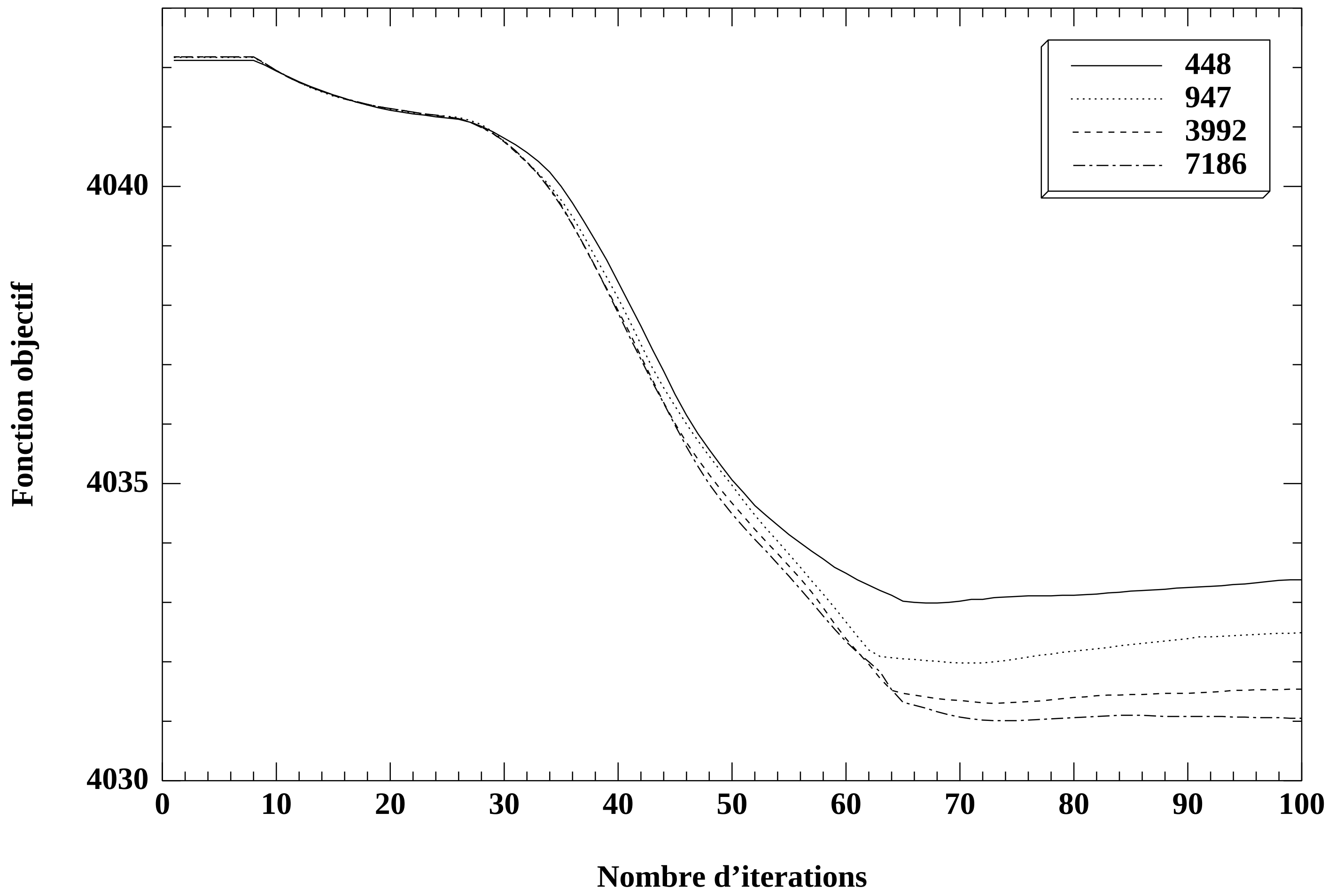
Optimal shapes for meshes with 448, 947, 3992, 7186 triangles



No convergence under mesh refinement !

More and more details appear when the mesh size is decreased.

The value of the objective function decreases with the mesh size.



Regularization

Triple motivation:

- ☞ To avoid instabilities when using $P1$ finite elements for u and $P0$ for h (less expensive than $P2$-$P0$).
- ☞ To obtain an algorithm which converges by mesh refinement.
- ☞ To adhere to a "regularized" framework (with existence of optimal solutions).

Main idea: we change the scalar product

$$\langle J'(h), k\rangle = \int_\Omega k \nabla u \cdot \nabla p \, dx \quad \forall\, k \in \mathcal{U}_{ad}.$$

Previously we identified $\mathcal{U}_{ad}$ to a subspace of $L^2(\Omega)$, thus

$$\langle J'(h), k\rangle = \int_\Omega J'(h)\, k \, dx \qquad \Rightarrow \qquad J'(h) = \nabla u \cdot \nabla p \, .$$

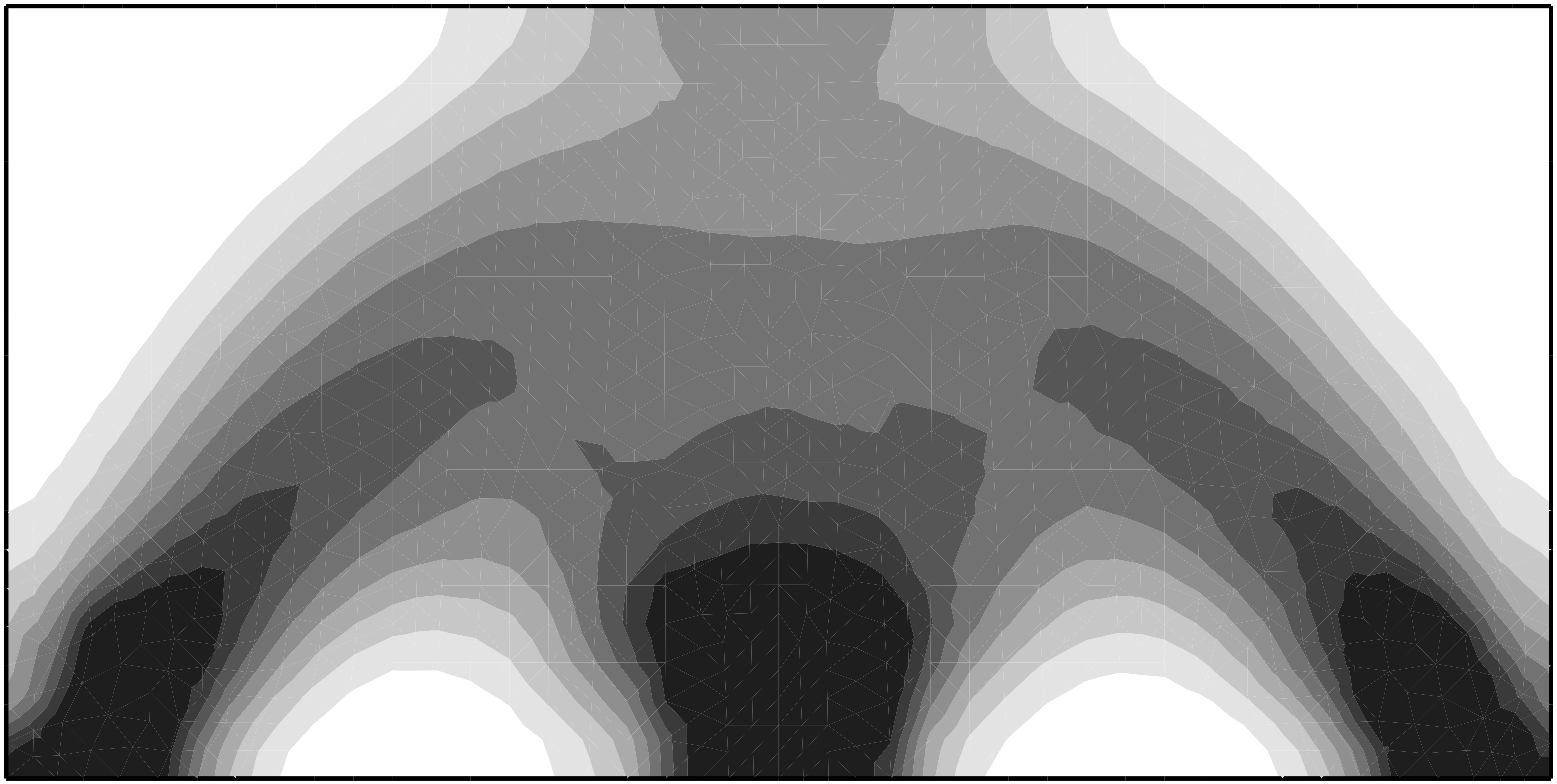
Now, we identify a "regularized" admissible set $\mathcal{U}_{ad}^{reg}$ to a subspace $H^1(\Omega)$, thus

$$\langle J'(h), k\rangle = \int_\Omega \left(\nabla J'(h) \cdot \nabla k + J'(h) k\right) dx \ ,$$

and we deduce a new formula for the gradient

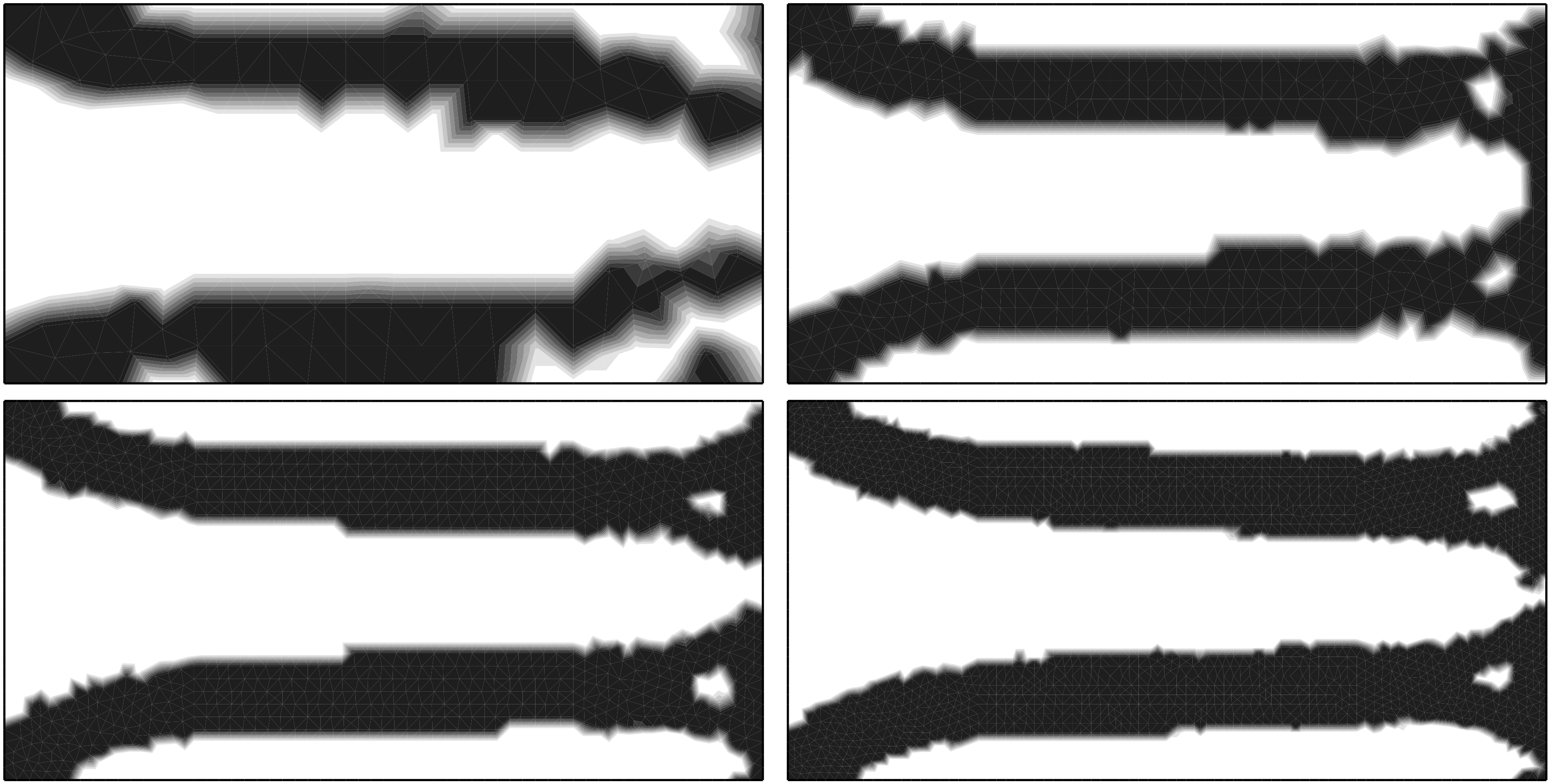
$$\begin{cases} -\Delta J'(h) + J'(h) = \nabla u \cdot \nabla p & \text{in } \Omega, \\ \dfrac{\partial J'(h)}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Regularized optimal shape



Finite elements P_1-P_0. Compliance minimization. Alternate directions algorithm.

Convergence by mesh refinement



Same case as the “numerical counter-examples” (meshes 448, 947, 3992, 7186).

Conclusion

☞ Regularization works !

☞ It costs a bit more (solving an additional Laplacian to compute the gradient).

☞ Difficulty in choosing the regularization parameter $\epsilon > 0$ (which can be interpreted as a lengthscale)

$$-\epsilon^2 \Delta J'(h) + J'(h) = \nabla u \cdot \nabla p \quad \text{in } \Omega$$

☞ It has a tendency to smooth the geometric details.