OPTIMAL DESIGN OF STRUCTURES (MAP 562)

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LECTURE III

GEOMETRIC OPTIMIZATION

CIMPA Summer School on Current Research in Finite Element Methods, IIT Mumbai

Geometric optimization of a membrane

A membrane is occupying a variable domain Ω in $\mathbb{R}^N$ with boundary

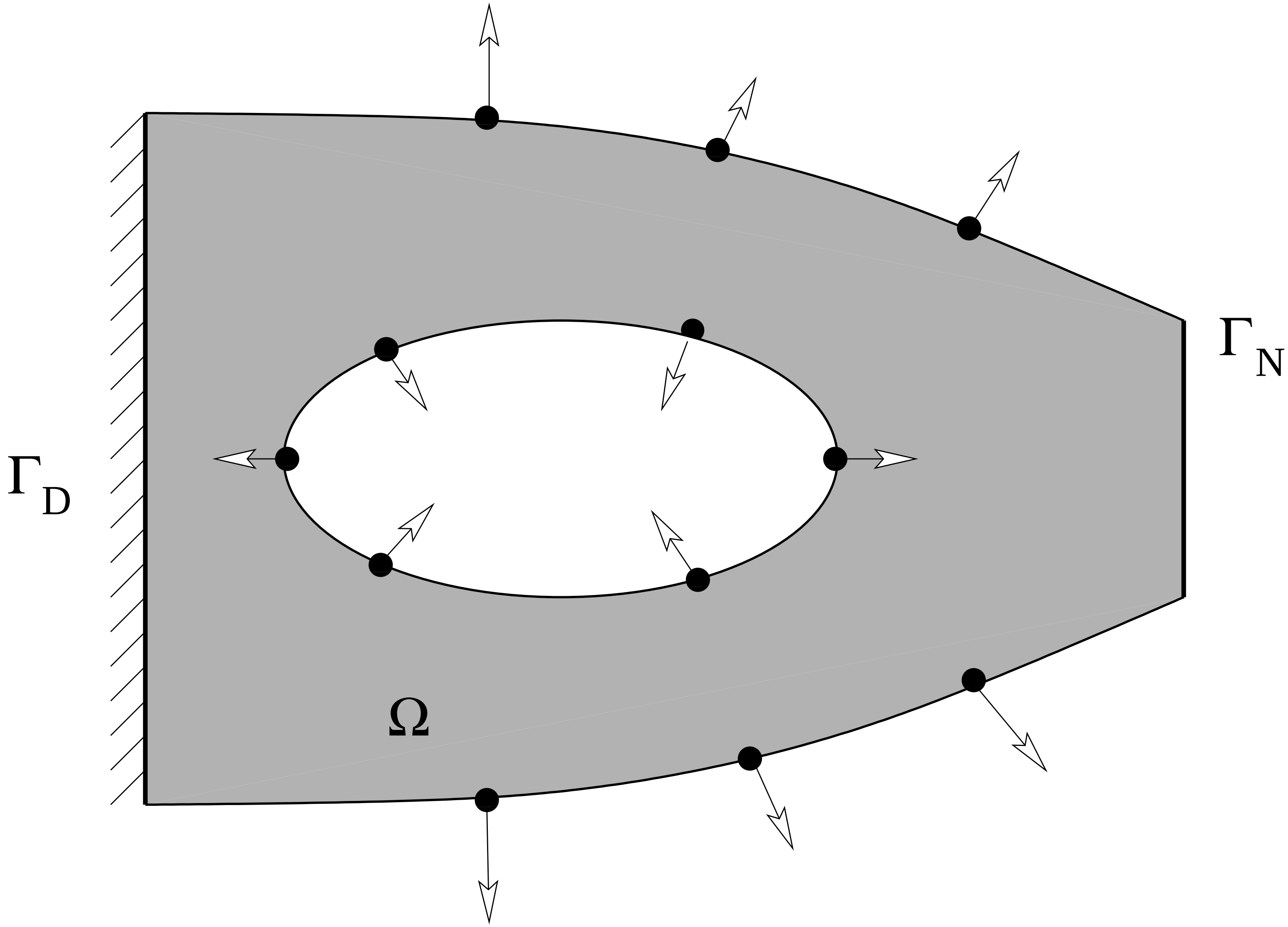
$$\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D,$$

where $\Gamma \neq \emptyset$ is the variable part of the boundary, $\Gamma_D \neq \emptyset$ is a fixed part of the boundary where the membrane is clamped, and $\Gamma_N \neq \emptyset$ is another fixed part of the boundary where the loads $g \in L^2(\Gamma_N)$ are applied.

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma \end{cases}$$

(No bulk forces to simplify)

Boundary variation in geometric optimization



Shape optimization of a membrane

Geometric shape optimization problem

$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega)$$

We must defined the set of admissible shapes $\mathcal{U}_{ad}$. That is the main difficulty.

Examples:

☞ Compliance or work done by the load (rigidity measure)

$$J(\Omega) = \int_{\Gamma_N} g u \, ds$$

☞ Least square criterion for a target displacement $u_0 \in L^2(\Omega)$

$$J(\Omega) = \int_{\Omega} |u - u_0|^2 dx$$

where u depends on Ω through the state equation.

Shape differentiation

Goal: to compute a derivative of $J(\Omega)$ by using the parametrization based on diffeomorphisms T.

We restrict ourselves to diffeomorphisms of the type

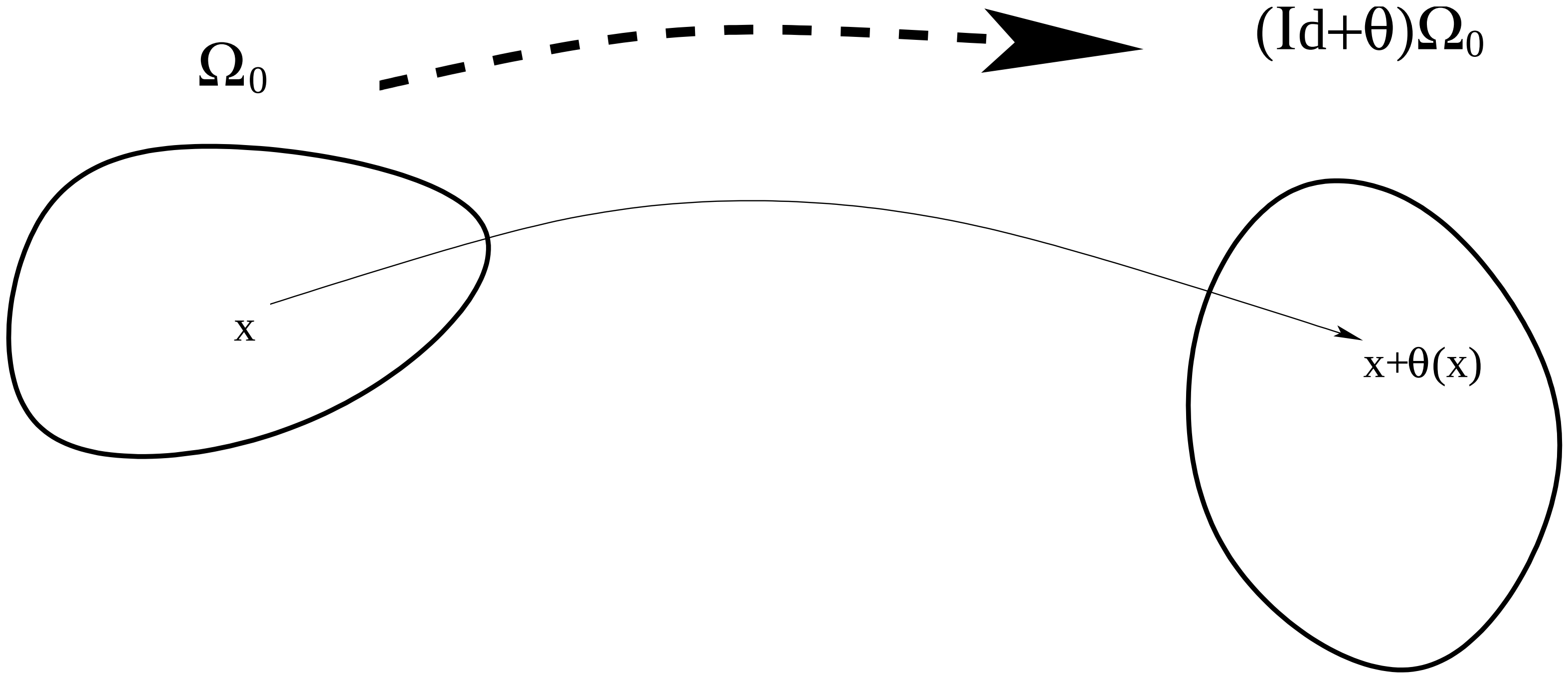
$$T = \text{ Id} + \theta \quad \text{ with } \quad \theta \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$$

Idea: we differentiate $\theta \to J\big((\text{ Id} + \theta)\Omega_0\big)$ at 0.

Remark. This approach generalizes the Hadamard method of boundary shape variations along the normal: $\Omega_0 \to \Omega_t$ for $t \geq 0$

$$\partial\Omega_t = \Big\{ x_t \in \mathbb{R}^N \mid \exists x_0 \in \partial\Omega_0 \mid x_t = x_0 + t\, g(x_0)\, n(x_0) \Big\}$$

with a given incremental function g.



The shape $\Omega = (\,\mathrm{Id} + \theta)(\Omega_0)$ is defined by

$$\Omega = \{x + \theta(x) \mid x \in \Omega_0\}\,.$$

Thus $\theta(x)$ is a vector field which plays the role of the **displacement** of the reference domain Ω_0.

Lemma. For any $\theta \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ satisfying $\|\theta\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)} < 1$, the map $T = \text{Id} + \theta$ is one-to-one into $\mathbb{R}^N$ and belongs to the set $\mathcal{T}$.

Proof. Based on the formula

$$\theta(x) - \theta(y) = \int_0^1 (x-y) \cdot \nabla\theta\big(y + t(x-y)\big) dt\,,$$

we deduce that $| \theta(x) - \theta(y) | \leq \|\theta\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)} | x - y |$ and θ is a strict contraction. Thus, $T = \text{Id} + \theta$ is one-to-one into $\mathbb{R}^N$.

Indeed, $\forall b \in \mathbb{R}^N$ the map $K(x) = b - \theta(x)$ is a contraction and thus admits a unique fixed point y, i.e., $b = T(y)$ and T is therefore one-to-one into $\mathbb{R}^N$.

Since $\nabla T = I + \nabla\theta$ (with $I = \nabla\,\text{Id}$) and the norm of the matrix $\nabla\theta$ is strictly smaller than 1 ($\|I\| = 1$), the map ∇T is invertible. We then check that $(T^{-1} - \text{Id}) \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$.

Definition of the shape derivative

Definition. Let $J(\Omega)$ be a map from the set of admissible shapes $\mathcal{C}(\Omega_0)$ into $\mathbb{R}$. We say that J is shape differentiable at Ω_0 if the function

$$\theta \to J\big((\,\mathrm{Id} + \theta)(\Omega_0)\big)$$

is Fréchet differentiable at 0 in the Banach space $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$, i.e., there exists a linear continuous form $L = J'(\Omega_0)$ on $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ such that

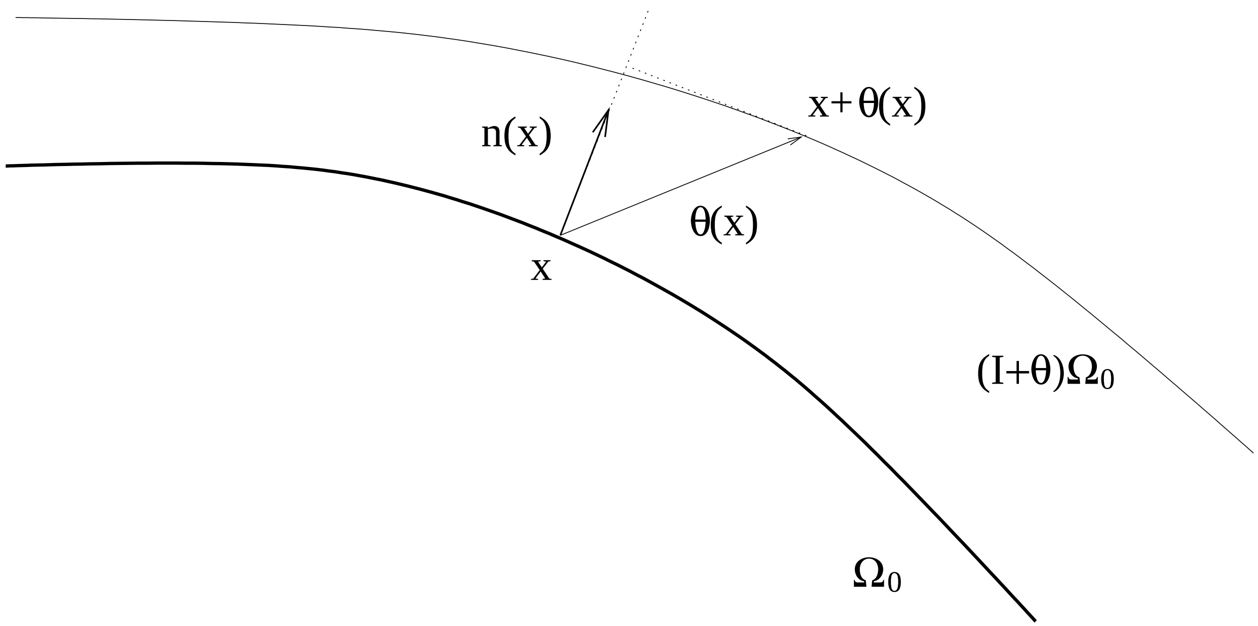
$$J\big((\,\mathrm{Id} + \theta)(\Omega_0)\big) = J(\Omega_0) + L(\theta) + o(\theta) \quad , \quad \text{with} \quad \lim_{\theta \to 0} \frac{|o(\theta)|}{\|\theta\|} = 0\,.$$

$J'(\Omega_0)$ is called the shape derivative and $J'(\Omega_0)(\theta)$ is a directional derivative.

The directional derivative $J'(\Omega_0)(\theta)$ depends only on the **normal component of θ on the boundary** of Ω_0.

This surprising property is linked to the fact that the internal variations of the field θ does not change the shape Ω, i.e.,

$$\theta \in C_c^1(\Omega)^N \text{ and } \|\theta\| << 1 \ \Rightarrow (\,\mathrm{Id} + \theta)\Omega = \Omega.$$



Proposition. Let Ω_0 be a smooth bounded open set of $\mathbb{R}^N$. Let J be a differentiable map at Ω_0 from $\mathcal{C}(\Omega_0)$ into $\mathbb{R}$. Its directional derivative $J'(\Omega_0)(\theta)$ depends only on the normal trace on the boundary of θ, i.e.

$$J'(\Omega_0)(\theta_1) = J'(\Omega_0)(\theta_2)$$

if $\theta_1, \theta_2 \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ satisfy

$$\theta_1 \cdot n = \theta_2 \cdot n \quad \text{ on } \partial\Omega_0.$$

(Proof admitted.)

Examples of shape derivatives

Proposition. Let Ω_0 be a smooth bounded open set of $\mathbb{R}^N$, $f(x) \in W^{1,1}(\mathbb{R}^N)$ and J the map from $\mathcal{C}(\Omega_0)$ into $\mathbb{R}$ defined by

$$J(\Omega) = \int_\Omega f(x)\, dx.$$

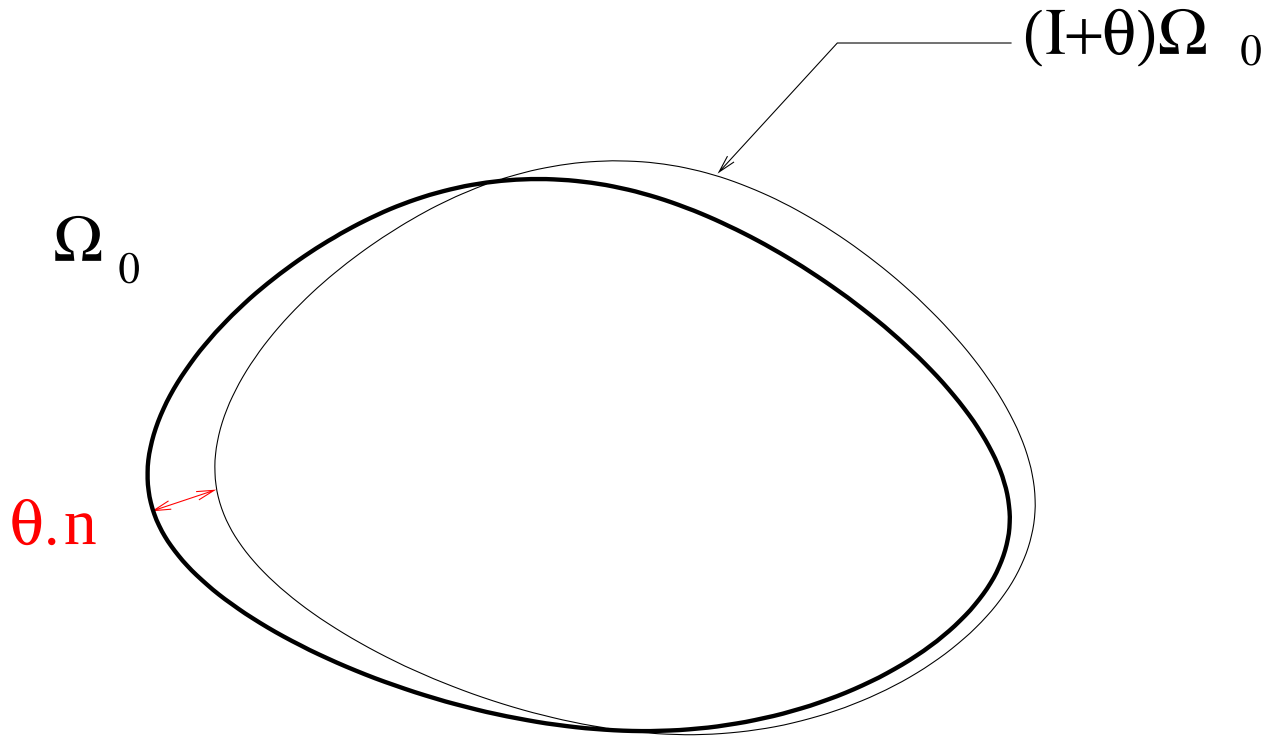
Then J is shape differentiable at Ω_0 and

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} \operatorname{div}\big(\theta(x)\, f(x)\big)\, dx = \int_{\partial\Omega_0} \theta(x) \cdot n(x)\, f(x)\, ds$$

for any $\theta \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$.

Remark. To make sure the result is right, the safest way (but not the easiest) is to make a change of variables to get back to the reference domain Ω_0.

Intuitive proof



Surface swept by the transformation: difference between $(\,\mathrm{Id}+\theta)\Omega_0$ and Ω_0 $\approx \partial\Omega_0 \times \Big(\theta \cdot n\Big)$. Thus

$$\int_{(\,\mathrm{Id}+\theta)\Omega_0} f(x)\,dx = \int_{\Omega_0} f(x)\,dx + \int_{\partial\Omega_0} f(x)\theta\cdot n\,ds + o(\theta).$$

Proof. We rewrite $J(\Omega)$ as an integral on the reference domain Ω_0

$$J\big((\,\mathrm{Id}+\theta)\Omega_0\big) = \int_{\Omega_0} f\circ(\,\mathrm{Id}+\theta)\,|\,\det(\,\mathrm{Id}+\nabla\theta)\,|\,dx.$$

The functional $\theta \to \det(\,\mathrm{Id}+\nabla\theta)$ is differentiable from $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$ because

$$\det(\,\mathrm{Id}+\nabla\theta) = \det\,\mathrm{Id} + \,\mathrm{div}\theta + o(\theta) \quad \text{with} \quad \lim_{\theta\to 0}\frac{\|o(\theta)\|_{L^\infty(\mathbb{R}^N;\mathbb{R}^N)}}{\|\theta\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)}} = 0.$$

On the other hand, if $f(x)\in W^{1,1}(\mathbb{R}^N)$, the functional $\theta \to f\circ(\,\mathrm{Id}+\theta)$ is differentiable from $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$ because

$$f\circ(\,\mathrm{Id}+\theta)(x) = f(x) + \nabla f(x)\cdot\theta(x) + o(\theta) \quad \text{with} \quad \lim_{\theta\to 0}\frac{\|o(\theta)\|_{L^1(\mathbb{R}^N)}}{\|\theta\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)}} = 0.$$

By composition of these two derivatives we obtain the result.

Proposition. Let Ω_0 be a smooth bounded open set of $\mathbb{R}^N$, $f(x) \in W^{2,1}(\mathbb{R}^N)$ and J the map from $\mathcal{C}(\Omega_0)$ into $\mathbb{R}$ defined by

$$J(\Omega) = \int_{\partial\Omega} f(x)\, ds.$$

Then J is shape differentiable at Ω_0 and

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \left(\nabla f \cdot \theta + f\big(\,\mathrm{div}\theta - \nabla\theta n \cdot n\big)\right) ds$$

for any $\theta \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$. By a (boundary) integration by parts this formula is equivalent to

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta \cdot n \left(\frac{\partial f}{\partial n} + Hf\right) ds,$$

where H is the mean curvature of $\partial\Omega_0$ defined by $H = \mathrm{div} n$.

Interpretation

Two simple examples:

- ☞ If $\partial\Omega_0$ is an hyperplane, then $H = 0$ and the variation of the boundary integral is proportional to the normal derivative of f.
- ☞ If $f \equiv 1$, then $J(\Omega)$ is the perimeter (in 2-D) or the surface (in 3-D) of the domain Ω and its variation is proportional to the mean curvature.

Proof. A change of variable yields

$$J\big((\mathrm{Id}+\theta)\Omega_0\big) = \int_{\partial\Omega_0} f\circ(\mathrm{Id}+\theta)|\det(\mathrm{Id}+\nabla\theta)|\ \left|\big((\mathrm{Id}+\nabla\theta)^{-1}\big)^t n\right|_{\mathbb{R}^N} ds.$$

We already proved that $\theta \to \det(\mathrm{Id}+\nabla\theta)$ and $\theta \to f\circ(\mathrm{Id}+\theta)$ are differentiables.

On the other hand, $\theta \to \big((\mathrm{Id}+\nabla\theta)^{-1}\big)^t n$ is differentiable from $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ into $L^\infty(\partial\Omega_0;\mathbb{R}^N)$ because

$$\big((\mathrm{Id}+\nabla\theta)^{-1}\big)^t n = n - (\nabla\theta)^t n + o(\theta) \quad \text{with} \quad \lim_{\theta\to 0}\frac{\|o(\theta)\|_{L^\infty(\partial\Omega_0;\mathbb{R}^N)}}{\|\theta\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)}} = 0.$$

By composition with the derivative of $g \to |g|_{\mathbb{R}^N}$, we deduce

$$\left|\big((\mathrm{Id}+\nabla\theta)^{-1}\big)^t n\right|_{\mathbb{R}^N} = 1-(\nabla\theta)^t n\cdot n+o(\theta) \quad \text{with} \quad \lim_{\theta\to 0}\frac{\|o(\theta)\|_{L^\infty(\partial\Omega_0)}}{\|\theta\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)}} = 0.$$

Composing these three derivatives leads to the result. The formula, including the mean curvature, is obtained by an integration by parts on the surface $\partial\Omega_0$.

Shape derivation: the (fast) Lagrangian method

- Rigorous derivation is quite tedious... but there is a simpler and faster (albeit formal) method, called the Lagrangian method (proposed in this context by J. Céa).
- The Lagrangian allows us to find the correct definition of the adjoint state too.
- It relies on the assumption that the state is shape differentiable but it does not require its formula.
- It is easy for Neumann boundary conditions, a little more involved for Dirichlet ones.

Fast derivation for Neumann boundary conditions

If the objective function is

$$J(\Omega) = \int_\Omega j(u(\Omega))\, dx,$$

the Lagrangian is defined as the sum of J and of the variational formulation of the state equation

$$\mathcal{L}(\Omega, v, q) = \int_\Omega j(v)\, dx + \int_\Omega \Big(\nabla v \cdot \nabla q + vq - fq \Big) dx - \int_{\partial\Omega} gq\, ds,$$

with v and $q \in H^1(\mathbb{R}^N)$. It is important to notice that the space $H^1(\mathbb{R}^N)$ does not depend on Ω and thus the three variables in $\mathcal{L}$ are clearly independent.

The partial derivative of $\mathcal{L}$ with respect to q in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial q}(\Omega, v, q), \phi \rangle = \int_\Omega \Big(\nabla v \cdot \nabla \phi + v\phi - f\phi \Big) dx - \int_{\partial\Omega} g\phi \, ds,$$

which, upon equating to 0, gives the variational formulation of the state.

The partial derivative of $\mathcal{L}$ with respect to v in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial v}(\Omega, v, q), \phi \rangle = \int_\Omega j'(v)\phi \, dx + \int_\Omega \Big(\nabla \phi \cdot \nabla q + \phi q \Big) dx,$$

which, upon equating to 0, gives the variational formulation of the adjoint.

The partial derivative of $\mathcal{L}$ with respect to Ω in the direction θ is

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, v, q)(\theta) = \int_{\partial\Omega} \theta \cdot n \left(j(v) + \nabla v \cdot \nabla q + vq - fq - \frac{\partial (gq)}{\partial n} - Hgq \right) ds.$$

When evaluating this derivative with the state $u(\Omega_0)$ and the adjoint $p(\Omega_0)$, we precisely find the derivative of the objective function

$$\boxed{\frac{\partial \mathcal{L}}{\partial \Omega}\Big(\Omega_0, u(\Omega_0), p(\Omega_0)\Big)(\theta) = J'(\Omega_0)(\theta)}$$

Indeed, if we differentiate the equality

$$\mathcal{L}(\Omega, u(\Omega), q) = J(\Omega) \quad \forall\, q \in H^1(\mathbb{R}^N),$$

and if we assume that the state has a shape derivative $u'(\Omega_0)(\theta)$, the chain rule lemma yields

$$J'(\Omega_0)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, u(\Omega_0), q)(\theta) + \langle \frac{\partial \mathcal{L}}{\partial v}(\Omega_0, u(\Omega_0), q), u'(\Omega_0)(\theta) \rangle$$

Taking $q = p(\Omega_0)$, the last term cancels since $p(\Omega_0)$ is the solution of the adjoint equation.

Thanks to this computation, the "correct" result can be guessed for $J'(\Omega_0)$ without knowing the formula for $u'(\Omega_0)(\theta)$.

Nevertheless, in full rigor, this "fast" computation of the shape derivative $J'(\Omega_0)$ is valid only if we know that u is shape differentiable.

The compliance case (self-adjoint)

Theorem. The functional $J(\Omega) = \int_\Omega f u \, dx + \int_{\partial\Omega} g u \, ds$ is shape-differentiable

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta \cdot n \left(-|\nabla u(\Omega_0)|^2 - |u(\Omega_0)|^2 + 2u(\Omega_0) f \right) ds$$

$$+ \int_{\partial\Omega_0} \theta \cdot n \left(2 \frac{\partial (g u(\Omega_0))}{\partial n} + 2 H g u(\Omega_0) \right) ds,$$

Interpretation: assume $f = 0$ and $g = 0$ where $\theta \cdot n \neq 0$. The formula simplifies in

$$J'(\Omega_0)(\theta) = - \int_{\partial\Omega_0} \theta \cdot n \left(|\nabla u|^2 + u^2 \right) ds \leq 0$$

It is always advantageous to increase the domain (i.e., $\theta \cdot n > 0$) for decreasing the compliance.

Fast derivation for Dirichlet boundary conditions

It is more involved ! Let $u \in H_0^1(\Omega)$ be the solution of

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \phi \in H_0^1(\Omega).$$

The "usual" Lagrangian is

$$\mathcal{L}(\Omega, v, q) = \int_\Omega j(v) \, dx + \int_\Omega \Big(\nabla v \cdot \nabla q - f q \Big) dx,$$

for $v, q \in H_0^1(\Omega)$. The variables (Ω, v, q) are not independent !

Indeed, the functions v and q satisfy

$$v \; = \; q \; = \; 0 \quad \text{on } \partial\Omega.$$

Another Lagrangian has to be introduced.

Lagrangian for Dirichlet boundary conditions

The Dirichlet boundary condition is penalized

$$\mathcal{L}(\Omega, v, q, \lambda) = \int_\Omega j(v)\, dx - \int_\Omega (\Delta v + f) q \, dx + \int_{\partial\Omega} \lambda v \, ds$$

where λ is the Lagrange multiplier for the boundary condition. It is now possible to differentiate since the 4 variables $v, q, \lambda \in H^1(\mathbb{R}^N)$ are independent.

Of course, we recover

$$\sup_{q,\lambda} \mathcal{L}(\Omega, v, q, \lambda) = \begin{cases} \int_\Omega j(u)\, dx = J(\Omega) & \text{if } v \equiv u, \\ +\infty & \text{otherwise.} \end{cases}$$

By definition of the Lagrangian:

the partial derivative of $\mathcal{L}$ with respect to q in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial q}(\Omega, v, q, \lambda), \phi \rangle = -\int_\Omega \phi \Big(\Delta v + f \Big) dx,$$

which, upon equating to 0, gives the state equation,

the partial derivative of $\mathcal{L}$ with respect to λ in the direction $\phi \in H^1(\mathbb{R}^N)$ is

$$\langle \frac{\partial \mathcal{L}}{\partial \lambda}(\Omega, v, q, \lambda), \phi \rangle = \int_{\partial\Omega} \phi v \, dx,$$

which, upon equating to 0, gives the Dirichlet boundary condition for the state equation.

To compute the partial derivative of $\mathcal{L}$ with respect to v, we perform a first integration by parts

$$\mathcal{L}(\Omega, v, q, \lambda) = \int_\Omega j(v)\,dx + \int_\Omega (\nabla v \cdot \nabla q - fq)\,dx + \int_{\partial\Omega} \left(\lambda v - \frac{\partial v}{\partial n} q\right) ds,$$

then a second integration by parts

$$\mathcal{L}(\Omega, v, q, \lambda) = \int_\Omega j(v)\,dx - \int_\Omega (v\Delta q - fq)\,dx + \int_{\partial\Omega} \left(\lambda v - \frac{\partial v}{\partial n} q + \frac{\partial q}{\partial n} v\right) ds.$$

We now can differentiate in the direction $\phi \in H^1(\mathbb{R}^N)$

$$\langle \frac{\partial \mathcal{L}}{\partial v}(\Omega, v, q), \phi\rangle = \int_\Omega j'(v)\phi\,dx - \int_\Omega \phi\Delta q\,dx + \int_{\partial\Omega} \left(-q\frac{\partial \phi}{\partial n} + \phi\left(\lambda + \frac{\partial q}{\partial n}\right)\right) ds$$

which, upon equating to 0, gives three relationships, the two first ones being the adjoint problem.

1. If ϕ has compact support in Ω_0, we get

$$-\Delta p = -j'(u) \quad \text{dans} \quad \Omega_0.$$

2. If $\phi = 0$ on $\partial\Omega_0$ with any value of $\frac{\partial \phi}{\partial n}$ in $L^2(\partial\Omega_0)$, we deduce

$$p = 0 \quad \text{sur} \quad \partial\Omega_0.$$

3. If ϕ is now varying in the full $H^1(\Omega_0)$, we find

$$\frac{\partial p}{\partial n} + \lambda = 0 \quad \text{sur} \quad \partial\Omega_0.$$

The adjoint problem has actually been recovered but furthermore the optimal Lagrange multiplier λ has been characterized.

Eventually, the shape partial derivative is

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, u, p, \lambda)(\theta) = \int_{\partial \Omega_0} \theta \cdot n \Big(j(u) - (\Delta u + f)p + \frac{\partial (u\lambda)}{\partial n} + H u \lambda \Big) ds$$

Knowing that $u = p = 0$ on $\partial \Omega_0$ and $\lambda = -\frac{\partial p}{\partial n}$ we deduce

$$\frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, u, p, \lambda)(\theta) = \int_{\partial \Omega_0} \theta \cdot n \Big(j(0) - \frac{\partial u}{\partial n} \frac{\partial p}{\partial n} \Big) ds = J'(\Omega_0)(\theta)$$

$$\boxed{J'(\Omega_0)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}\Big(\Omega_0, u(\Omega_0), p(\Omega_0)\Big)(\theta)}$$

This formula is not a surprise because differentiating

$$\mathcal{L}(\Omega, u(\Omega), q, \lambda) = J(\Omega) \quad \forall q, \lambda$$

yields

$$J'(\Omega_0)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega_0, u(\Omega_0), q, \lambda)(\theta) + \langle \frac{\partial \mathcal{L}}{\partial v}(\Omega_0, u(\Omega_0), q, \lambda), u'(\Omega_0)(\theta)\rangle.$$

Then, taking $q = p(\Omega_0)$ (the adjoint state) and $\lambda = -\frac{\partial p}{\partial n}(\Omega_0)$, the last term cancels and we obtain the desired formula.

Application to compliance minimization

We minimize $J(\Omega) = \int_\Omega f u \, dx$ with $u \in H_0^1(\Omega)$ solution of

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \, \phi \in H_0^1(\Omega).$$

The adjoint state is just $p = -u$. The shape derivative is

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta \cdot n \Big(f u - \frac{\partial u}{\partial n} \frac{\partial p}{\partial n} \Big) ds = \int_{\partial\Omega_0} \theta \cdot n \left(\frac{\partial u}{\partial n} \right)^2 ds \leq 0$$

It is always advantageous to shrink the domain (i.e., $\theta \cdot n < 0$) to decrease the compliance.

This is the opposite conclusion compared to Neumann b.c., but it is logical !

Numerical algorithms in the elasticity setting

Free boundary Γ. Fixed boundary Γ_N and Γ_D.

$$\begin{cases} -\,\mathrm{div}\sigma = 0 & \text{in } \Omega \\ \sigma = 2\mu e(u) + \lambda\,\mathrm{tr}(e(u))\,\mathrm{Id} & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \sigma n = g & \text{on } \Gamma_N \\ \sigma n = 0 & \text{on } \Gamma, \end{cases}$$

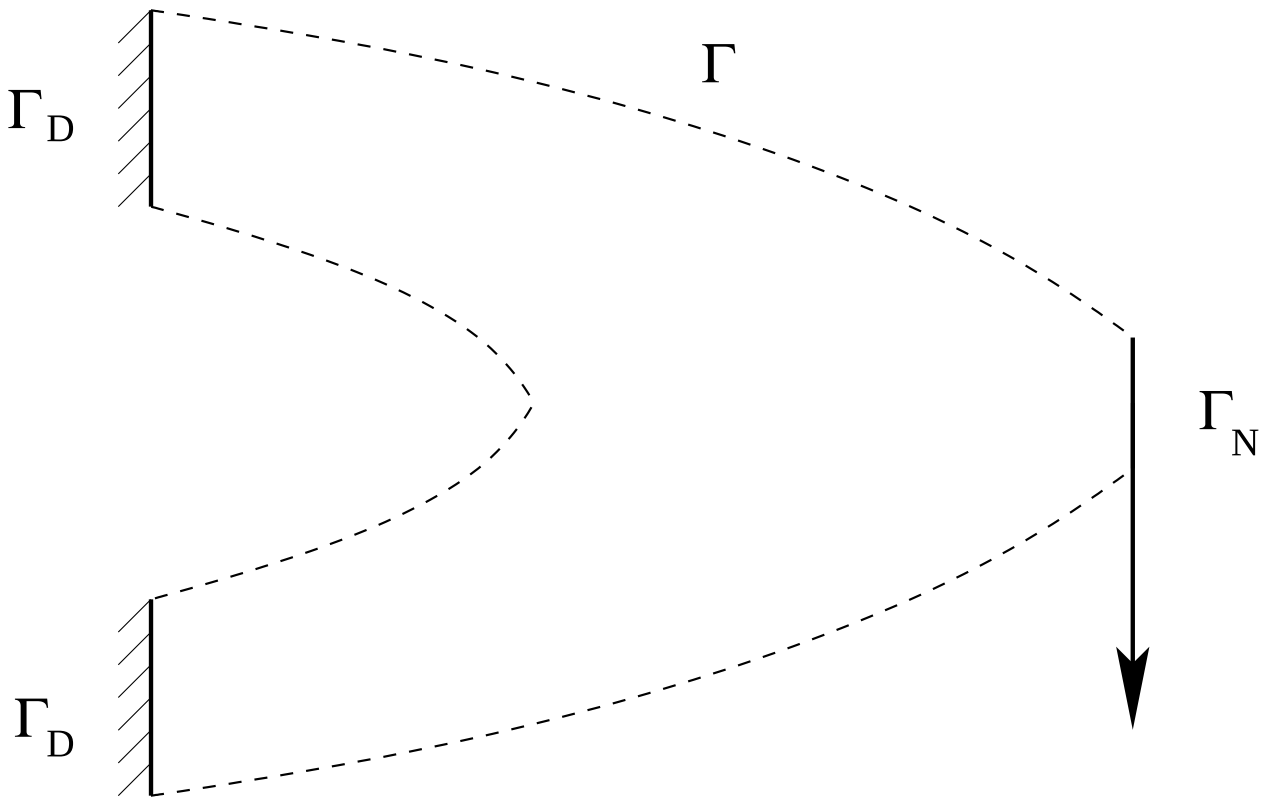
with $e(u) = (\nabla u + (\nabla u)^t)/2$. Compliance is minimized

$$J(\Omega) = \int_{\Gamma_N} g \cdot u\, dx.$$

In such a (self-adjoint) case we get

$$J'(\Omega_0)(\theta) = -\int_{\Gamma} \theta \cdot n\,\left(2\mu|e(u)|^2 + \lambda(\,\mathrm{tr}\, e(u))^2\right) ds.$$

Boundary conditions for an elastic cantilever: Γ_D is the left vertical side, Γ_N is the right vertical side, and Γ (dashed line) is the remaining boundary.



Main idea of the algorithm

Given an inital design Ω_0 we compute a sequence of iterative shapes Ω_k, satisfying the following constraints

$$\partial\Omega_k = \Gamma_k \cup \Gamma_N \cup \Gamma_D$$

where Γ_N and Γ_D are fixed, and the volume (or weight) is fixed

$$V(\Omega_k) = \int_{\Omega_k} dx = V(\Omega_0).$$

To take into account the constraint that only Γ is allowed to move, it is enough to take $\theta \cdot n = 0$ on $\Gamma_N \cup \Gamma_D$.

Because of the volume constraint we rely on a projected gradient algorithm with a fixed step .

The derivative of the volume constraint is $V'(\Omega_k)(\theta) = \int_{\Gamma_k} \theta \cdot n.$

Algorithm

Let $t > 0$ be a given descent step. We compute a sequence $\Omega_k \in \mathcal{U}_{ad}$ by

1. Initialization of the shape Ω_0.

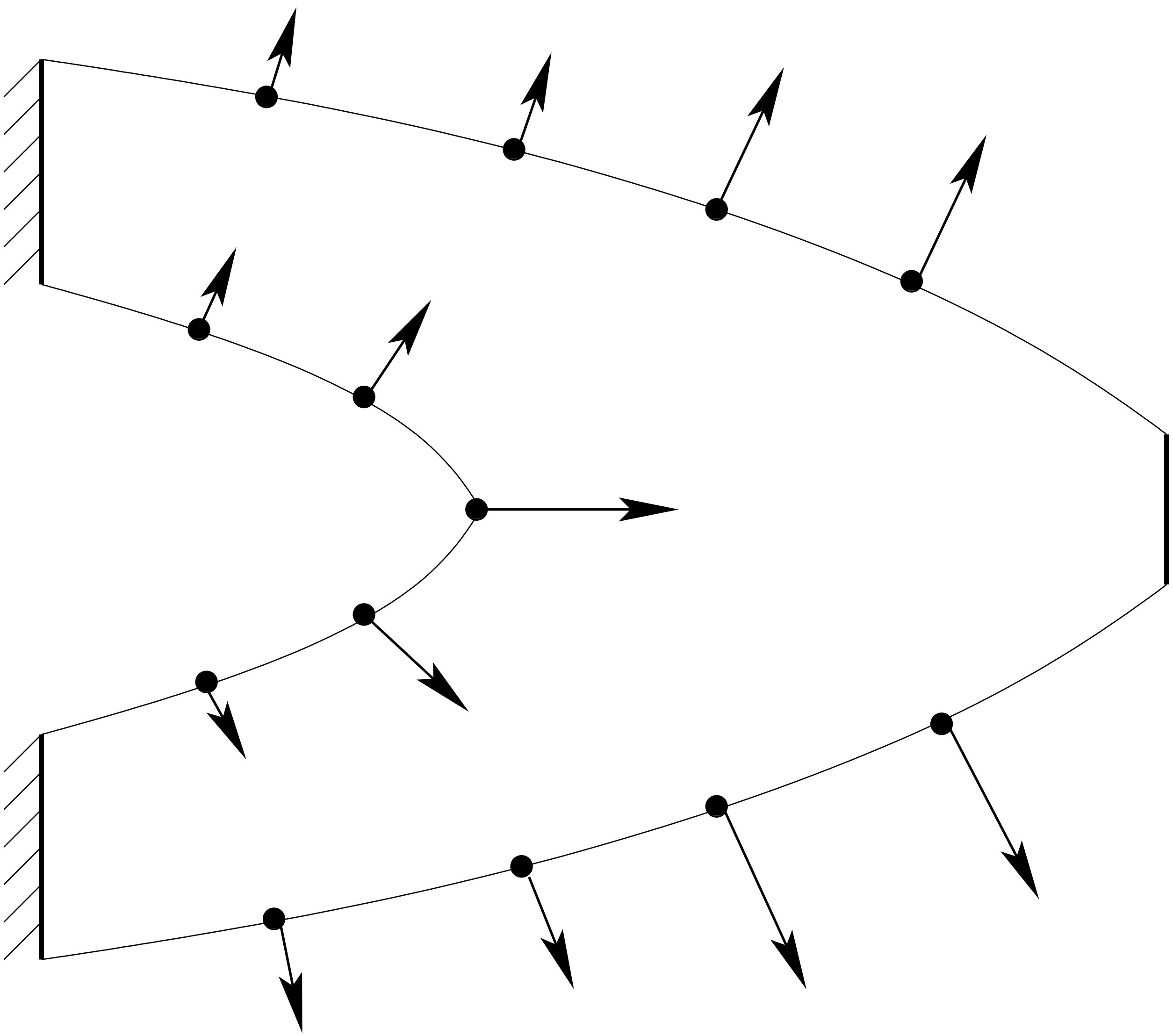
2. Iterations until convergence, for $k \geq 0$:

$$\Omega_{k+1} = (\,\mathrm{Id} + \theta_k)\Omega_k \quad \text{with} \quad \theta_k = t(j_k - \ell_k)n,$$

   where n is the normal to the boundary $\partial\Omega_k$ and $\ell_k \in \mathbb{R}$ is the Lagrange multiplier such that Ω_{k+1} satisfies the volume constraint. The shape derivative is given on the boundary Γ_k by

$$J'(\Omega_k)(\theta) = -\int_\Gamma \theta \cdot n\, j_k\, ds \quad \text{with} \quad j_k = 2\mu|e(u_k)|^2 + \lambda(\,\mathrm{tr}\, e(u_k))^2$$

   where u_k is the solution of the state equation posed in the domain Ω_k.



Mesh deformation

To change the shape we need to automatically remesh the new shape, or at least to deform the mesh at each iteration.

- Displacement field θ proportional to n (normal to the boundary), merely defined on the boundary.
- We prefer to deform the mesh (it is less costly).
- In such a case we have to extend θ inside the shape.
- We need to check that the displaced boundaries do not cross...
- Nevertheless, in case of large shape deformations we must remesh (it is computationally costly).
- Often the algorithm stops before convergence because of geometrical constraints.

Implementing geometric optimization on a computer is quite intricate, **especially in 3-d**.

Extension of the displacement field

$$J'(\Omega)(\theta) + \ell V'(\Omega)(\theta) = \int_\Gamma (\ell - j)\, \theta \cdot n\, ds$$

A first possibility to extend $(\ell - j)n$ inside the shape is

$$\begin{cases} -\Delta\theta = 0 & \text{in } \Omega \\ \theta = t(j - \ell)n & \text{on } \Gamma \\ \theta = 0 & \text{on } \Gamma_D \cup \Gamma_N \end{cases}$$

We rather take this opportunity to (furthermore) regularize by solving

$$\begin{cases} -\Delta\theta = 0 & \text{in } \Omega \\ \frac{\partial\theta}{\partial n} = t(j-\ell)n & \text{on } \Gamma \\ \theta = 0 & \text{on } \Gamma_D \cup \Gamma_N \end{cases}$$

Indeed, $j = 2\mu|e(u)|^2 + \lambda\,\mathrm{tr}(e(u))^2$ (for compliance) may be not smooth (not better than in $L^1(\Omega)$) although we always assumed that $\theta \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$).

It can cause boundary oscillations.

Typically, θ admits one order of derivation more than j and one can check that it is actually a descent direction because

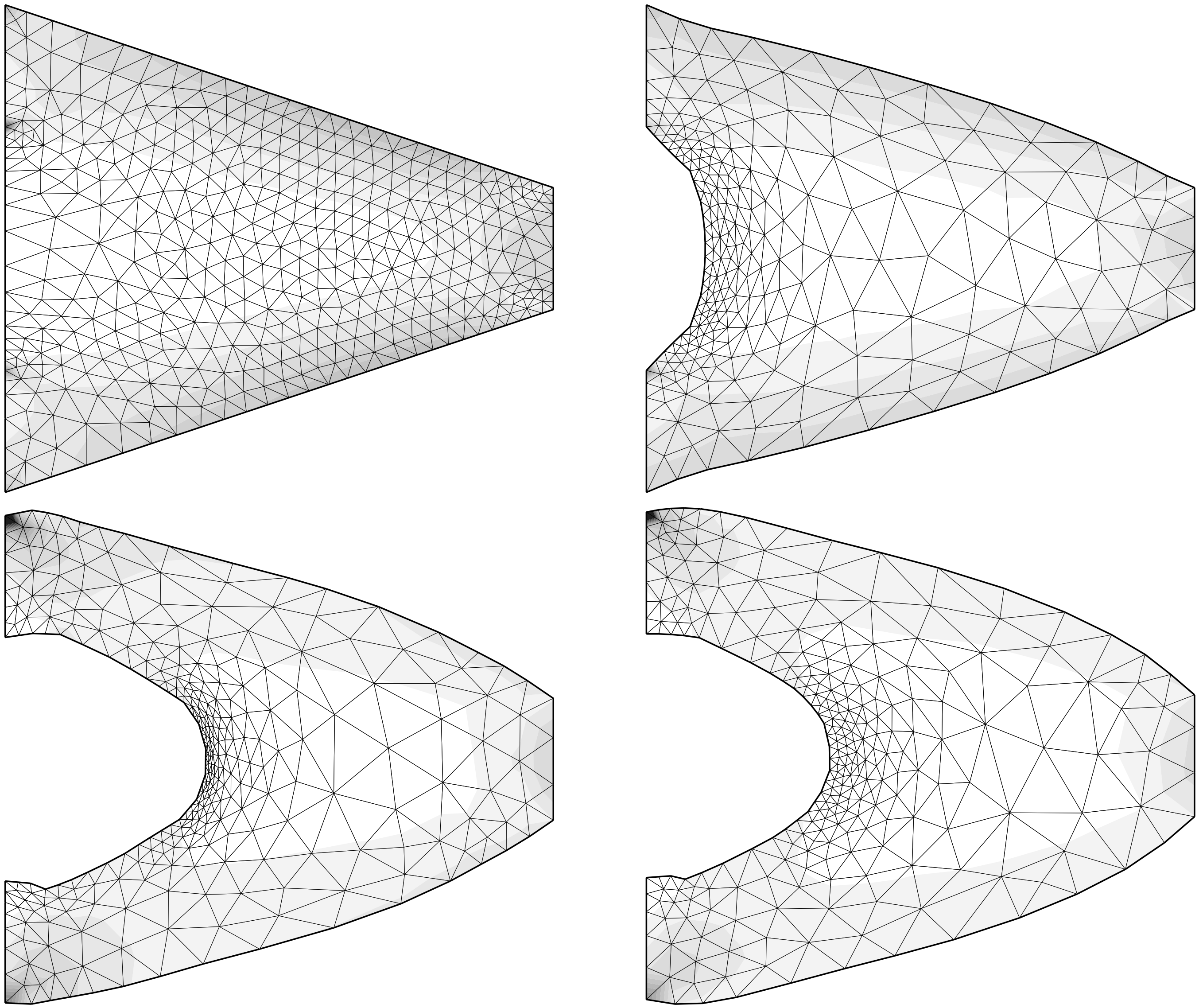
$$-\int_\Omega |\nabla\theta|^2 dx = t\int_\Gamma (\ell - j)\,\theta\cdot n\,ds$$

Technical details

- To check the volume constraint we update "a posteriori" the Lagrange multiplier $\ell_k \in \mathbb{R}$. The volume is thus not exact but it converges to the desired value.
- We step back and diminish the descent step $t > 0$ when $J(\Omega)$ increases.
- To avoid possible oscillations of the boundary, due to numerical instabilities, we use two meshes: a fine one to precisely evaluate u and p, a coarse one which is moved.

FreeFem++ computations ; scripts available on the web page
`http://www.cmap.polytechnique.fr/~allaire/cours_X_annee3.html`

Numerical results: initialization and iterations 5, 10, 20



Influence of the initial topology