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The Stokes' System

Introduction

In this set of lectures, we discuss mainly weak formulation, existence uniqueness results and finite element approximations with their a priori error analysis.

In general, the motion of an incompressible viscous fluid in domain $\Omega \subset \mathbb{R}^d$ is described by the following system of eqns:

$$(1.1) \quad \begin{cases} \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} - \nabla \cdot \vec{\sigma} = \vec{f} & \text{in } \Omega \\ \text{with incompressibility condition:} \\ \sum_{j=1}^d \frac{\partial u_j}{\partial x_j} = \nabla \cdot \vec{u} = 0 & \text{in } \Omega \end{cases}$$

\vec{u} : velocity, $\sigma = [\sigma_{ij}] \rightarrow$ stress tensor

$$\text{f: volumetric force: (1.1) has } d+1 \text{ equations, but } d+1 \text{ unknowns}$$

constitutive relations: $\sigma_{ij} = -P\delta_{ij} + 2\mu \epsilon_{ij}(\vec{u})$: pressure

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

- Looking for steady state neglecting $\frac{\partial p}{\partial t}$
- neglecting convection effect: $\vec{u} \cdot \nabla \vec{u} \approx 0$ (where \vec{u} is small)
- P constant

then we $f = P$, $\gamma = \frac{N}{P}$, we arrive at Stokes system:

$$(2) \quad \begin{cases} -2\gamma \sum_{j=1}^d \frac{\partial}{\partial x_j} (\epsilon_{ij}(\vec{u})) + \frac{\partial P}{\partial x_i} = f_i \\ \sum_{i=1}^d \epsilon_{ij}(\vec{u}) = 0 \end{cases}$$

when $\operatorname{div} \vec{u} = 0$, then

$$\sum_{j=1}^d \frac{\partial}{\partial x_j} \epsilon_{ij}(\vec{u}) = \frac{1}{2} \sum_{j=1}^d \left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) = \frac{1}{2} \Delta \vec{u}$$

Then (2) reduces to

$$(1.3) \quad \begin{cases} -\Delta \vec{u} + \nabla p = \vec{f} \\ \nabla \cdot \vec{u} = 0 \end{cases} \quad \text{in } \Omega$$

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To close the system, we need boundary condition. So for simplicity assume Dirichlet BC:

$$(1.4) \quad \vec{u} = 0 \quad \text{on } \partial\Omega.$$

Weak formulation:

In weak formulation, we need some vector valued function spaces. Let $H_0^1 = (H^1(\Omega))^d$ with inner product $\langle (\vec{v}, \vec{w}) \rangle = \sum_{j=1}^d (v_j, w_j) = \sum_{j=1}^d \sum_{i=1}^d \int_{\Omega} \frac{\partial v_j}{\partial x_i} \frac{\partial w_j}{\partial x_i} dx$

and induced norm

$$\|\nabla \vec{v}\|_*^2 = \langle (\vec{v}, \vec{v}) \rangle^{1/2} = \left(\sum_{j=1}^d \|\nabla v_j\|^2 \right)^{1/2}$$

For pressure:

$$L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_{\Omega} q dx = 0\}.$$

Then the weak formulation reads: Find $\vec{u} \in H_0^1(\Omega)$, $p \in L_0^2(\Omega)$ such that

~~$a(u, v) = b(v, p)$~~

$$1) \quad \gamma(\nabla \vec{u}, \nabla \vec{v}) + (p, \nabla \vec{v}) = (\vec{f}, \vec{v}) \quad \forall \vec{v} \in H_0^1(\Omega)$$

$$2) \quad (\nabla \cdot \vec{u}, q) = 0 \quad \forall q \in L_0^2(\Omega)$$

Put in abstract form: Set $V = H_0^1$, $H = L_0^2$.

Define the bilinear forms:

$$a(\vec{w}, \vec{v}) = \gamma(\nabla \vec{w}, \nabla \vec{v}) \quad \text{and} \quad b(\vec{v}, q) = -(\nabla \cdot \vec{v}, q).$$

$$\text{Let } F(v) = (\vec{f}, \vec{v})$$

Then (2-1) - (2-2) leads to the abstract formulation as:

Given two Hilbert spaces V and H , and two bilinear forms $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ and

$b(\vec{v}, \cdot) : V \times H \rightarrow \mathbb{R}$, and two linear forms

$$F : V \rightarrow \mathbb{R} \quad \text{and} \quad g : H \rightarrow \mathbb{R}, \quad \text{see } (\vec{u}, p) \in V \times H$$

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such that (\vec{u}, \vec{p}) satisfies.

$$(2.3) \quad a(\vec{u}, \vec{v}) + b(\vec{v}, p) = f(\vec{v}) \quad \forall v \in V$$

$$(2.4) \quad b(\vec{u}, q) = g(q) \quad \forall q \in H$$

The advantage of this abstract formulation is that all mixed formulation will mostly have a form like $(2.3) - (2.4)$. A more natural question is related to the wellposedness of $\begin{cases} (2.3) \\ (2.4) \end{cases}$ which forms the part of the next section.

§ 3. Existence, Uniqueness & Stability Results

In this section, we ~~impose~~ discuss the wellposedness of the abstract formulation $(2.3) - (2.4)$.

Theorem 3.1 (Babuška-Brezzi): Let V and H be two real Hilbert spaces with respective dual spaces V' and H' . Assume that bilinear forms $a(\cdot, \cdot): V \times V \rightarrow \mathbb{R}$ and $b(\cdot, \cdot): V \times H \rightarrow \mathbb{R}$ satisfy the following conditions:

(i) \exists a +ve constant M such that for $v, w \in V$ and $q \in H$

$$|a(\vec{v}, \vec{w})| \leq M \|\vec{v}\|_V \|\vec{w}\|_V$$

and

$$(3.1) \quad |b(\vec{v}, q)| \leq m \|\vec{v}\|_V \|q\|_H$$

(ii) (2-ellipticity) \exists a +ve constant $\alpha_0 > 0$ such that

$$(3.2) \quad a(\vec{v}, \vec{v}) \geq \alpha_0 \|\vec{v}\|_V^2 \quad \forall v \in \mathcal{Z}$$

where

$$\mathcal{Z} = \left\{ w \in V : b(w, q) = 0 \quad \forall q \in H \right\}$$

(iii) (Inf-Sup condition) (LBB condition): $\exists \beta_0 > 0$ such that

$$(3.3) \quad \inf_{q \in H} \sup_{\vec{v} \in V} \frac{b(\vec{v}, q)}{\|\vec{v}\|_V \|q\|_H} \geq \beta_0 \quad \left(\text{or } \sup_{\vec{v} \in V} \frac{b(\vec{v}, q)}{\|\vec{v}\|_V} \geq \beta_0 \|q\|_H \right)$$

Then for given $F \in V'$ and $G \in H'$, there exists a unique pair $(\vec{u}, p) \in V \times H$ of solutions such that

$$(3.4) \quad \begin{cases} a(\vec{u}, \vec{v}) + b(\vec{v}, p) = F(\vec{v}) & \forall \vec{v} \in V \\ (3.5) \quad b(\vec{u}, q) = G(q) & \forall q \in H. \end{cases}$$

Moreover, the following stability result holds:

$$(3.6) \quad \|u\|_V + \|q\|_H \leq C(d_0, m) \|F\|_V + C(d_0, \beta_0, m) \|G\|_H;$$

Proof: Since bilinear forms are continuous, ^{using} of Ritz-Representation theorem ensures the existence of linear operators $A : V \rightarrow V'$ and $B : V \rightarrow H'$ satisfying

$$(3.7) \quad \langle A\vec{w}, \vec{v} \rangle = a(\vec{w}, \vec{v}) \quad \forall \vec{w}, \vec{v} \in V$$

and $V' \quad V$

$$(3.8) \quad \langle B\vec{w}, q \rangle := b(\vec{w}, q) \quad \forall \vec{w} \in V, q \in H$$

It is an easy exercise that $A \in \mathcal{L}(V, V')$ and $B \in \mathcal{L}(V, H')$. ~~The bounded linear operator~~ ^{linear} B induces an adjoint operator $B^* \in \mathcal{L}(H, V')$ defined by

$$(3.9) \quad \langle B^*q, \vec{v} \rangle_{V'} = \cancel{\langle Bq, B\vec{v} \rangle} \quad \forall \vec{v} \in V, q \in H.$$

Then the system (3.4) - (3.5) leads to an equivalent abstract system: Find $(\vec{u}, p) \in V \times H$ such that

$$(3.10) \quad Au + B^*p = F \quad \text{in } V'$$

$$(3.11) \quad Bu = G \quad \text{in } H'$$

From (3.9), we obtain using (3.3) that

$$\sup_{\vec{v} \neq \vec{0} \in V} \frac{\langle B^*q, \vec{v} \rangle}{\|\vec{v}\|_V} = \sup_{\vec{v} \neq \vec{0} \in V} \frac{b(\vec{v}, q)}{\|\vec{v}\|_V} \geq \beta_0 \|q\|_H \quad \forall q \in H$$

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This implies

$$(3.12) \quad \|B^*g\|_{V'} \geq p_0 \|g\|_H \quad \forall g \in H$$

and hence, B^* is ~~one-to-one~~ one-to-one operator from H onto $R(B^*) \subset V'$. Now the continuity of B^* ensures that $R(B^*)$ is closed (by closed graph theorem). As a consequence, the inverse of B^* is continuous linear operator. Since B^* is an isomorphism $\xrightarrow{\text{from } H \text{ onto } R(B^*)}$, an application of closed range theorem of Banach (see, pp 205-208, K.Yosida Functional Analysis, Springer) yields $R(B^*) = (\ker(B))^\circ$, where the polar set $(\ker(B))^\circ$ is defined by

$$(\ker(B))^\circ = \{g \in V' : \forall n \in \ker(B) \quad \langle g, n \rangle = 0\}$$

Note that

$$Z = (\ker(B))^\circ \text{ and hence } R(B^*) = Z^\circ$$

$$\text{Claim: } R(B^*) = Z^\circ.$$

~~since $R(B^*)$ is a closed subspace of V' and B^*~~
Observe that Z° can be identified isometrically with $(Z^\perp)'$, that is, $Z^\circ \approx (Z^\perp)'$.

To see this for $\vec{n} \in V$, let $P\vec{n}$ denote the orthogonal projection of \vec{n} onto Z^\perp . Then for $g \in (Z^\perp)'$ we can associate an element $\tilde{g} \in V'$ defined by

$$\langle \tilde{g}, n \rangle = \langle g, P\vec{n} \rangle \quad \forall \vec{n} \in V$$

clearly, $\tilde{g} \in Z^\circ$ and the map: $g \mapsto \tilde{g}$ maps isometrically $(Z^\perp)'$ onto Z° which shows $(Z^\perp)'' \approx Z^\circ$

Therefore, $B: Z^\perp \xrightarrow{\text{onto}} H'$ is an isomorphism and $\|B\vec{v}\|_{V'} \geq p_0 \|\vec{v}\|_V$

For a given $g \in H'$, \exists ~~unique~~ a unique element $\vec{v}_g \in Z^\perp$

such that

(3.13)

$$B\vec{v}_g = g$$

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Therefore and

$$(3.14) \quad \|u_0\|_V \leq \frac{1}{\beta} \|g\|_H$$

Therefore the problem (3.11) - (3.12) is equivalent to the following problem : Find $\vec{w} := \vec{u} - \vec{u}_0 \in Z$ such that

$$(3.15) \quad a(\vec{w}, \vec{v}) = f(\vec{v}) - a(\vec{u}_0, \vec{v}) \quad \forall \vec{v} \in Z$$

In operator form : find $\vec{w} \in Z$ such that

$$A\vec{w} = \vec{f} - A\vec{u}_0$$

Since $a(\cdot, \cdot)$ is Z -elliptic, apply Lax-Milgram Theorem to ensure the existence of a unique solution $\vec{w} \in Z$ of (3.15) satisfying

$$(3.16) \quad \|\vec{w}\|_V \leq C \left(\|f\|_V + \|u_0\|_V \right)$$

Thus, the equivalence between (3.15) and (3.11) - (3.12) yields the existence of unique solution $\vec{u} \in \vec{Z}(g)$ where ~~satisfy~~ with $\vec{u} = \vec{w} + \vec{u}_0 = \vec{u}_0 + \vec{v}$ that is

$$\vec{u} \in \vec{Z}(g) = \{ \vec{v} \in V : b(\vec{v}, \vec{q}) = g(\vec{q}) \quad \forall \vec{q} \in H \}$$

and satisfying using (3.16) and (3.14)

$$(3.17) \quad \|\vec{u}\|_V \leq C \underbrace{\left(\|f\|_V + \|g\|_H \right)}_{\text{from (3.16)}}$$

As $F - Au \in Z^0$, $\exists ! p \in H$ such that

$$\vec{B}^* p = F - Au$$

and hence

$$\|p\| \leq \frac{1}{\beta} \|F - Au\|_V \leq C \left(\|f\|_V + \|g\|_H \right)$$

Hence $(u, p) \in V \times H$ is the only soln of (3.4) - (3.5)

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Verification for Stokes system: $V = H_0^1(\Omega)$ and $H = L_0^2(\Omega)$

Now for $\vec{z} \in V$, with $\vec{v} \in \vec{z}$
 $a(\vec{v}, \vec{v}) = \rightarrow \|\nabla \vec{v}\|^2 = \rightarrow \|\vec{v}\|^2$, hence

$a(\cdot, \cdot)$ is \mathbb{R} - coercive where

$$Z = \left\{ \vec{v} \in V : b(\vec{v}, q) = 0 \quad \forall q \in \mathbb{R} \right\}$$

$$= \left\{ \vec{v} \in H_0^1 : -(\nabla \cdot \vec{v}, q) = 0 \quad \forall q \in L_0^2 \right\}$$

To check inf-sup condition, for $q \in L_0^2(\Omega)$, seek $\vec{v} \in H_0^1(\Omega)$ such that $\operatorname{div} \vec{v} = q$ in Ω and $\|\vec{v}\|_{H_0^1} \leq \frac{1}{\beta_0} \|q\|_{L_0^2}$.

Since $q \in L_0^2(\Omega)$ such that $\int_{\Omega} q \, dx = 0$

$$0 = \int_{\Omega} q \, dx = \int_{\Omega} \operatorname{div} \vec{v} \, dx = \int_{\Omega} \vec{v} \cdot \vec{e} \, dx = 0, \text{ then } \vec{v} \in H_0^1(\Omega)$$

Now using regularity result $\|\vec{v}\|_{H_0^1} \leq \frac{1}{\beta_0} \|q\|_{L_0^2}$, we obtain

$$\begin{aligned} \|\vec{v}\|_{H_0^1}^2 &= \int_{\Omega} \nabla \cdot \vec{v} q \, dx \\ \therefore \frac{b(\vec{v}, q)}{\|\vec{v}\|_{H_0^1}} &= \frac{b(\vec{v}, q)}{\|\vec{v}\|_{H_0^1}} \end{aligned}$$

$$\sup_{\vec{w} \in V} \frac{b(\vec{w}, q)}{\|\vec{w}\|_V} \geq \frac{b(\vec{v}, q)}{\|\vec{v}\|_{H_0^1}} = \frac{\int_{\Omega} \nabla \cdot \vec{v} q \, dx}{\|\nabla \vec{v}\|^2} = \frac{\|q\|^2}{\|\nabla \vec{v}\|^2} \geq \frac{\|q\|^2}{\|\vec{v}\|_{H_0^1}^2} \geq \frac{1}{\beta_0^2} \|q\|^2.$$

Hence, the inf-sup condition is satisfied.

By Babuska-Brezzi theorem, \exists unique pair of solutions $(\vec{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ ~~satisfy~~ to the problem (2.1) - (2.2). and

$$\|\vec{u}\| + \|p\| \leq C \|f\|.$$

§ 4. Abstract ~~FEM~~ Finite Dimensional Approximation:

This section deals with finite dimensional approximation to the abstract mixed formulation (3.4) - (3.5).

With \vec{h} as discretizing parameter, note let V_h and H_h be two finite dimensional subspaces of V and H , respectively.

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Then the finite dimensional mixed formulation of (3.4) - (3.5) is to find a pair of approximate solution $(\vec{u}_h, p_h) \in V_h \times H_h$ such that

$$(4.1) \quad a(\vec{u}_h, \vec{v}_h) + b(\vec{v}_h, p_h) = F(\vec{v}_h) \quad \forall \vec{v}_h \in V_h$$

$$(4.2) \quad b(\vec{u}_h, q_h) = g(q_h) \quad \forall q_h \in H_h$$

As in continuous case, define

$$(4.3) \quad Z_h = \{\vec{v}_h \in V_h : b(\vec{v}_h, q_h) = 0 \quad \forall q_h \in H_h\}$$

and for $\varsigma \in H'$, set

$$(4.4) \quad Z_h(\varsigma) := \{\vec{v}_h \in V_h : b(\vec{v}_h, q_h) = \varsigma(q_h) \quad \forall q_h \in H_h\}$$

Lemma

Lemma 4.1. Assume that

$$(i) \quad Z_h(\varsigma) \neq \emptyset$$

$$(ii) \quad \exists \alpha_0^* > 0 \text{ such that}$$

$$\boxed{(iii)} \quad a(\vec{v}_h, \vec{v}_h) \geq \alpha_0^* \|\vec{v}_h\|_V^2 \quad \forall \vec{v}_h \in Z_h$$

Then, the ^{discrete} problem

$$(4.5) \quad a(\vec{u}_h, \vec{v}_h) = F(\vec{v}_h) \quad \forall \vec{v}_h \in Z_h$$

has a unique discrete solution $\vec{u}_h \in Z_h(\varsigma)$ satisfying

$$\|\vec{u}_h\|_V \leq C (\|F\|_{V'} + \|g\|_{H'})$$

Hint: As $Z_h(\varsigma) \neq \emptyset$, choose $\vec{v}_h^0 \in Z_h(\varsigma)$ to solve

$\vec{z}_h \in Z_h$ such that

$$a(\vec{z}_h, \vec{v}_h) = F(\vec{v}_h) - a(\vec{v}_h^0, \vec{v}_h) \quad \forall \vec{v}_h \in Z_h$$

Applying Lax-Milgram and infer the required result.

Theorem 4.1 (Solvability of (4.1) - (4.2)) Assume that

$$(i) \quad \exists \alpha_0^* > 0 \text{ such that}$$

$$(4.6) \quad a(\vec{v}_h, \vec{v}_h) \geq \alpha_0^* \|\vec{v}_h\|_V^2 \quad \forall \vec{v}_h \in Z_h$$

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(ii) $\exists \beta_0^* > 0$ such that

$$(4.7) \quad \sup_{\vec{v}_h \in V_h} \frac{b(\vec{v}_h, q_h)}{\|\vec{v}_h\|_V} \geq \beta_0^* \|q_h\|_H \quad \forall q_h \in H_h.$$

Then $Z_h(G) \neq \emptyset$ and \exists 1 pair of discrete solution $(\vec{u}_h, p_h) \in V_h \times H_h$ satisfying abstract of the discrete problem

(4.1) - (4.2). Moreover, the following stability result holds:

$$(4.8) \quad \|\vec{u}_h\|_V + \|p_h\|_H \leq C(\|F\|_V + \|G\|_H)$$

Modify the Babuska-Brezzi theorem to infer $Z_h(G) \neq \emptyset$ and proceed similarly to fit complete the existence and uniqueness result. and then as a consequence prove (4.8). However, we follow a different proof now.

Hint (Alternate). Since V_h & H_h are finite dimensional, let $\{\vec{\phi}_j\}_{j=1}^{N_h}$ & $\{\psi_j\}_{j=1}^{M_h}$ be two basis of V_h & H_h , respectively. Then

$$\vec{u}_h = \sum_{j=1}^{N_h} \alpha_j \vec{\phi}_j \quad \text{and} \quad p_h = \sum_{j=1}^{M_h} \beta_j \psi_j$$

On substitution in (4.1) - (4.2) and using

$$\vec{g}_h = \vec{\phi}_h \text{ with } q_h = \psi_h$$

$$(4.9) \quad A_h \alpha_h + B_h^T \beta_h = F_h \Rightarrow \begin{bmatrix} A_h & B_h^T \\ 0 & B_h \end{bmatrix} \begin{bmatrix} \alpha_h \\ \beta_h \end{bmatrix} = \begin{bmatrix} F_h \\ G_h \end{bmatrix}$$

$$(4.10) \quad B_h \alpha_h = g_h$$

This system has a solution iff it has a unique solution. To prove uniqueness, it is enough to show that (4.9) - (4.10) with $F_h = 0$ & $G_h = 0$ has identically zero solution. Now, we need to show (4.8) to claim that when $F_h = 0, G_h = 0$, then

A careful observation of Babuska-Brezzi theorem helps us to infer that. Say for example from

~~4.1 Abstract Error Analysis~~

(ii), one infers that B_h in ~~(4.2)~~ is $\stackrel{(4.2) \text{ defined by}}{=} b(\vec{v}_h, q_h)$ an isomorphism from $Z_h^\perp \subset V_h \xrightarrow{\text{onto}} H_h$. Thus $Z_h^*(S)$ is nonempty. Then proceed as in Lemma 4.1 to conclude the result ~~passing~~ for unique solution u_h . Then follow and $\|u_h\|_{V'} \leq C(\|F\|_{V'} + \|G\|_{H'})$

Then ~~GE~~

B_h^* defined by $b(B_h^* q_h, \vec{v}_h) = b(\vec{v}_h, q_h)$ and A_h defined by $(A_h \vec{v}_h, \vec{v}_h) = a(\vec{v}_h, \vec{v}_h)$
 F_h by $\langle F_h, v_h \rangle = F(v_h)$ and G_h by $\langle G_h, v_h \rangle = G(v_h)$ satisfies (4.1)

$$B_h^* p_h = F_h - A_h \vec{v}_h$$

and using inf-sup

$$\|p_h\| \leq \|B_h^* p_h\| = \|F_h - A_h \vec{v}_h\|_{V'} \leq C(\|F\|_{V'} + \|G\|_{H'})$$

~~4.1 Abstract Error Analysis~~. We discuss in this section only for Stokes case, that is when $b = 0$, that is for the continuous abstract problem (3.4) - (3.5) ^{with $b=0$} and (4.1) - (4.2) ^{with $b=0$} abstract.

4. This subsection deals with the error analysis.

~~Case I: when $b \in \mathbb{R}$~~ for the Stokes system, when $b = 0$, that is, find $(\vec{v}_h, p_h) \in V_h \times H_h$ such that

$$(4.11) \quad a(\vec{v}_h, \vec{v}_h) + b(\vec{v}_h, p_h) = F(v_h) \quad \forall \vec{v}_h \in V_h$$

$$(4.12) \quad b(u_h, q_h) = 0 \quad \forall q_h \in H_h$$

Thus on Z_h , the problem (4.11) - (4.12) is equivalent to the following problem: Find $u_h \in Z_h$

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such that $u_h \in Z_h$

$$(4.13) \quad a(u_h, v_h) = F(v_h) \quad \forall v_h \in Z_h$$

Theorem 4.2 (Cea's type Lemma). Let $v_h \in V$ and $H_h \subset H$ with Z and Z_h defined earlier. Suppose (4.6) holds for $z \in Z \oplus Z_h$ with coercive constant α_0 . Then

$$(4.14) \quad \|u - u_h\|_V \leq \left(1 + \frac{M}{\alpha_0}\right) \inf_{y \in Z_h} \|u - y\|_V + \inf_{\substack{q \in H_h \\ q \in H}} \|p - q\|_H$$

Further, if inf-sup condition is satisfied then

$$(4.15) \quad \|p - p_h\|_H \leq M (\|u - u_h\|_V + \inf_{q \in H_h} \|p - q\|_H)$$

Here M is the constant appear in the boundedness property of $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$.

Proof. Since we have not assumed that $Z_h \subset Z$, then rewrite $\vec{u} - \vec{u}_h = (u - x) + (\vec{v}_h - x)$, for arbitrary $x \in \mathbb{R}_h$.

* Note that for ~~$u_h \neq x + u_h$~~ with $\vec{v}_h - \vec{x} \in Z_h$

$$\begin{aligned} \alpha_0 \frac{\|u_h - x\|_V}{V} &\leq a(u_h - x, \vec{v}_h - x) = a(u_h - x, \vec{v}_h - \vec{x}) \\ &\leq \sup_{\vec{w} \in Z_h} \frac{a(u_h - x, \vec{w})}{\|\vec{w}\|_V} \end{aligned}$$

$$\begin{aligned} \alpha_0 \frac{\|u_h - x\|_V}{V} &\leq \frac{a(u_h - x, u_h - x)}{\|u_h - x\|_V} \leq \sup_{0 \neq w_h \in Z_h} \frac{|a(u_h - x, w_h)|}{\|w_h\|_V} \\ &= \sup_{0 \neq w_h \in Z_h} \frac{|a(x - u, w_h)| + |a(u - u_h, w_h)|}{\|w_h\|_V} \\ &\leq \sup_{0 \neq w_h \in Z_h} \frac{|a(u - x, w_h)|}{\|w_h\|_V} + \sup_{0 \neq w_h \in Z_h} \frac{|a(u - u_h, w_h)|}{\|w_h\|_V} \\ (4.16) \quad &\leq M \frac{\|\vec{u} - x\|_V}{V} + \sup_{0 \neq w_h \in Z_h} \frac{|a(u - u_h, w_h)|}{\|w_h\|_V} \end{aligned}$$

Thus

$$\|u - u_h\|_V \leq \|u - x\|_V + \|u_h - x\|_V$$

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thus and on substitution of (4.16) yields

$$4.17) \|\vec{u} - \vec{u}_h\|_V \leq \left(1 + \frac{M}{\beta_0}\right) \inf_{x \in Z_h} \|u - x\|_V + \frac{1}{\beta_0} \sup_{w_h \in Z_h} \frac{\alpha(u - u_h, w_h)}{\|w_h\|_V}$$

For the last term in (4.17), we observe that

$$\begin{aligned} \alpha(u - u_h, w_h) &= \alpha(u, w_h) - \alpha(u_h, w_h) \\ &= \alpha(u, w_h) - F(w_h) \end{aligned}$$

Using (3.4)-(3.5) we find that

$$\alpha(u - u_h, w_h) = -b(w_h, p) = -b(w_h, p - q_h) + q_h \in$$

Hence \triangle

$$4.18) |\alpha(u - u_h, w_h)| \leq |b(w_h, p - q_h)| \leq M \|w_h\|_V \|p - q_h\|$$

On substitution (4.18) in (4.17), we obtain the desired estimate (4.14). Now to derive (4.15), note that for $v_h \in V_h$

$$b(v_h, p_h) = \alpha(u - u_h, v_h) + b(v_h, p)$$

Hence for any $q_h \in H_h$

$$b(v_h, p_h - q_h) = \alpha(u - u_h, v_h) + b(v_h, p - q_h)$$

and using inf-sup condition (4.7)

$$\begin{aligned} \beta_0 \|p_h - q_h\|_H &\leq \sup_{v_h \in V_h} \frac{b(v_h, p_h - q_h)}{\|v_h\|_V} \leq \sup_{v_h \in V_h} \frac{|\alpha(u - u_h, v_h) + b(v_h, p - q_h)|}{\|v_h\|_V} \\ &\leq M (\|u - u_h\|_V + \|p - q_h\|_H) \end{aligned}$$

But

$$p - p_h = (p - q_h) + (q_h - p_h)$$

and thus

$$\|p - p_h\|_H \leq \left(1 + \frac{M}{\beta_0}\right) \inf_{q_h \in H_h} \|p - q_h\| + \frac{M}{\beta_0} \|u - u_h\|_V$$

thus completes the rest of the proof.