

§5. FEM for Stokes Equations. In this section, we discuss finite element error analysis for the Stokes system.

Recall the weak formulation of the Stokes system once again here. Let $V = H_0^1$ and $H = L_0^2$. Find $(\vec{u}, \vec{p}) \in H_0^1 \times L_0^2$ such that

$$(5.1) \quad a(\vec{u}, \vec{v}) - (\vec{p}, \nabla \cdot \vec{v}) = (\vec{F}, \vec{v}) \quad \forall v \in H_0^1$$

$$(5.2) \quad (q, \nabla \cdot \vec{u}) = 0 \quad \forall q \in L_0^2$$

We have verified ~~the~~ ^{hypotheses of} Babuska-Brezzi Theorem and hence, \exists ! pair of weak solutions $(\vec{u}, \vec{p}) \in H_0^1 \times L_0^2$ of (5.1)-(5.2) which satisfies the stability property

$$(5.3) \quad \| \vec{u} \|_{H_0^1} + \| \vec{p} \|_{L_0^2} \leq C \| \vec{F} \| .$$

5.1. Error Analysis.

i) Assume that the finite dimensional subspaces $V_h \subset H_0^1$ and $H_h \subset L_0^2$ have the following approximation properties:

(i) \exists exist ^{linear operators} $\tau_h: H^1 \rightarrow V_h$ and $j_h: L_0^2 \rightarrow H_h$ such that

$$(5.4) \quad (q_h, \nabla \cdot (\vec{v} - \tau_h \vec{v})) = 0 \quad \forall q_h \in H_h \text{ and } \vec{v} \in H^2 \cap H_0^1$$

$$(5.5) \quad \| \vec{v} - \tau_h \vec{v} \|_h \leq C h^{m+1-j} \| \vec{v} \| \quad \forall \vec{v} \in H^{m+1} \cap H^1$$

$$(5.6) \quad \| q_h - j_h q \|_h \leq C h^m \| q \|_m \quad \forall q \in H^m \cap L_0^2, \quad 1 \leq m \leq \ell$$

Theorem 5.1. Under the hypothesis (H₁).

Now the finite element formulation is to find

$(\vec{u}_h, p_h) \in V_h \times H_h$ such that

$$(5.7) \quad a(\vec{u}_h, \vec{v}_h) - (p_h, \nabla \cdot \vec{v}_h) = (\vec{F}, \vec{v}_h) \quad \forall \vec{v}_h \in V_h$$

$$(5.8) \quad (q_h, \nabla \cdot \vec{u}_h) = 0 \quad \forall q_h \in H_h .$$

Theorem 5.1. Under the hypothesis (H₁), the problem (5.7)-(5.8)

has exactly one solution $u_h \in V_h$. Moreover, if

$(\vec{u}, p) \in (H^m \cap H_0^1) \times (H^m \cap L_0^2)$, then there is

fix constant C such that

$$(5.9) \|\nabla(\bar{u} - u_h)\| \leq C h^m (\|\bar{u}\|_{m+1} + \|p\|_m) \quad (1 \leq m \leq 2).$$

Proof: Set

$$(5.10) \quad Z_h = \{w_h \in V_h : (q_h, \nabla \cdot w_h) = 0 \quad \forall q_h \in H_h\}$$

By ~~assumption~~ (ii), note that $Z_h \neq \{\vec{0}\}$ (not a trivial space)
Hence, ~~an application~~ of Lemma 4.1, for wellposedness
of the discrete system, we need to verify ellipticity
of $a(\cdot)$.

$$\text{Now } a(\vec{v}_h, \vec{v}_h) = \vec{v}_h \cdot \nabla \vec{v}_h \geq \alpha_0 \|\nabla \vec{v}_h\|^2 \quad 0 < \alpha_0 \leq \gamma$$

Note that $Z_h(0) = Z_h$. Hence $\exists ! \vec{u}_h \in V_h$

Using first part of theorem 4.2

$$(5.11) \|\vec{u} - \vec{u}_h\|_{H_0^1} = \|\nabla(\vec{u} - \vec{u}_h)\| \leq C \left\{ \inf_{0 \neq \vec{v}_h \in Z_h} \|\nabla(\vec{u} - \vec{v}_h)\| + \inf_{0 \neq q_h \in H_h} \|p - q_h\| \right\}$$

Using (5.6),

$$(5.12) \inf_{0 \neq q_h \in H_h} \|p - q_h\| \leq \|p - j_h p\| \leq C h^m \|p\|_m \quad (1 \leq m \leq 2)$$

For the first term on the right hand side of (5.11),

let $U = Z \cap H^2$. Note that U is dense in Z (this holds!).

With π_h as a linear mapping in hypothesis (H₁),

now (5.5) implies for $\vec{v} \in U$

$$(\vec{u}_h, \operatorname{div}(\vec{v}))$$

$$0 = (q_h, \operatorname{div}(\vec{v} - \pi_h \vec{v})) = (q_h, \operatorname{div}(\pi_h \vec{v}))$$

Thus, $\pi_h \vec{v} \in V_h$ for $\vec{v} \in U$. & we can apply (5.4)
to infer

$$\inf_{0 \neq \vec{v}_h \in Z_h} \|\nabla(\vec{u} - \vec{v}_h)\| \leq \|\nabla(\vec{u} - \pi_h \vec{u})\| \leq C h^m \|\vec{u}\|_{m+1}, \quad (1 \leq m \leq 2).$$

Hence the result.

In order to define optimal L^2 -estimates for $\vec{u} - \vec{u}_h$, we apply the Aubin-Nitsche argument.

Assumption. (H₂): For a given $\vec{g} \in L^2(\Omega)$, assume that the Stokes problem

$$(S-13) \quad -\nabla \cdot \vec{\Phi} + \nabla q = \vec{g}$$

with (S-14) $\vec{\Phi} = \begin{cases} \vec{0} & \text{on } \partial\Omega \\ 0 & \text{on } \partial\Omega \end{cases}$ has a unique pair of solutions $(\vec{\Phi}, q) \in (H^2 \cap H_0) \times (H^1 \times L^2_0)$ satisfying the following regularity condition:

$$(S-15) \quad \|\vec{\Phi}\|_2 + \|q\|_1 \leq C \|\vec{g}\|.$$

Theorem 5.2. Under the hypotheses (H₁) - (H₂), there exists a tve constant $C > 0$ such that

$$(S-16) \quad \|\vec{u} - \vec{u}_h\| \leq C h^{m+1} (\|\vec{u}\|_{m+1} + \|p\|_m) \quad 1 \leq m \leq r$$

Proof. Form L^2 -inner product between (S-13) and $u - u_h$ to obtain

$$(u - u_h, g) = a(\vec{u} - \vec{u}_h, \vec{\Phi}) + b(\vec{u} - \vec{u}_h, q)$$

Since $u_h \in Z_h$ is a solution, note for $\vec{\Phi}_h = \pi_h \vec{\Phi}$.

$$\begin{aligned} a(\vec{u} - \vec{u}_h, \vec{\Phi}_h) &= a(u, \vec{\Phi}_h) - a(u_h, \vec{\Phi}_h) \\ &= a(u, \vec{\Phi}_h) - F(\vec{\Phi}_h) = -b(\vec{\Phi}_h, p) \\ &= -b(\vec{\Phi}_h, p - j_h p) = b(\vec{\Phi} - \vec{\Phi}_h, p - j_h p) \\ &\quad \text{as } (p - j_h p, \vec{\Phi} \cdot \vec{\Phi}) = 0 \end{aligned}$$

Thus

$$(\vec{u} - \vec{u}_h, \vec{g}) = a(u - \vec{u}_h, \vec{\Phi} - \vec{\Phi}_h) - b(\vec{\Phi}_h, p - j_h p) + b(u - \vec{u}_h, q - q_h)$$

$$\text{as } b(\vec{u} - \vec{u}_h, q_h) = \cancel{b(\vec{u} - \vec{u}_h, \nabla \cdot \vec{u})} + \cancel{b(q_h, \nabla \cdot \vec{u}_h)} = 0$$

Thus

$$\begin{aligned} (\vec{u} - \vec{u}_h, \vec{g}) &= a(\vec{u} - \vec{u}_h, \vec{\Phi} - \vec{\Phi}_h) + b(\vec{\Phi} - \vec{\Phi}_h, p - j_h p) \\ &\quad + b(\vec{u} - \vec{u}_h, q - q_h) \quad q_h = j_h q \\ &\leq M (\|\nabla(u - u_h)\| + \|p - j_h p\|) \|\vec{\Phi} - \vec{\Phi}_h\| \end{aligned}$$

$$+ M h \|\nabla(u - u_h)\| \|q - q_h\|$$

Using approximation properties (5.4) - (5.6)
we arrive at

$$\|\vec{u} - \vec{u}_h, \vec{g}\| \leq Mch (\|\nabla(u - u_h)\| (\|\Phi\|_2 + \|\Omega\|_2) \\ + Mh^{2+\eta} \|p\|_m \|\vec{\Phi}\|_2)$$

Use regularity result (5.15) to obtain

$$\|\vec{u} - \vec{u}_h\| = \sup_{0 \neq \vec{g} \in L^2} \frac{\|(\vec{u} - \vec{u}_h, \vec{g})\|}{\|\vec{g}\|} \leq Ch \|\nabla(u - u_h)\| + Ch^2 \|p\|_1$$

Using Thm 5.1, we obtain the desired result. \blacksquare

Next question is: What about error estimates in $p - p_h$?

Hypothesis: (H₃) (reflecting discrete condition):

For each $q_h \in H_h$, $\exists \vec{v}_h \in V_h$ such that

$$(5.17) \quad (\vec{v}_h - \vec{v}_h, x_h) = 0 \quad \forall x_h \in H_h.$$

(5.18) $\|\vec{v}_h\| \leq C \|q_h\|$
Thm: Under the hypotheses (H₁) \wedge (H₃), the discrete problem (5.1) - (5.2) has exactly one pair of solutions $(\vec{u}_h, p_h) \in V_h \times H_h$.

With further regularity assumptions: $(\vec{u}, p) \in (H^{m+1} \cap H_0) \times (H^m \cap L^2)$, there holds:

$$(5.19) \quad \|\nabla(\vec{u} - \vec{u}_h)\| + \|p - p_h\| \leq Ch^m (\|\vec{u}\|_{m+1} + \|p\|_m), \quad 1 \leq m \leq l$$

Sketch of the proof: In order to apply Thm 4.1, we need to check inf-sup condn. From hypothesis H₃, i.e., for $q_h \in H_h$, $\exists \vec{v}_h \in V_h$ such that

$$(q_h, \operatorname{div} \vec{v}_h) = \|q_h\|^2 \geq \frac{1}{C} \|q_h\| \|\nabla \vec{v}_h\|$$

$\therefore \inf_{\vec{v}_h \in V_h} \operatorname{div} \vec{v}_h = \inf_{\vec{v}_h \in V_h} (q_h, \operatorname{div} \vec{v}_h) / \|q_h\|$

Therefore

$$\sup_{\vec{V}_h \in V_h} \frac{(q_h, \nabla \cdot \vec{V}_h)}{\|\nabla \vec{V}_h\|} \geq \frac{1}{C} \|q_h\| \quad \forall q_h \in H_h$$

Then apply Thm 1 to infer existence of unique pair of solntns $(\vec{u}_h, p_h) \in V_h \times H_h$ of (5.1)-(5.1).

Again from Thm 2

$$(5.20) \|p - p_h\| \leq C(\|\nabla(u - u_h)\| + \inf_{q_h \in H_h} \|p - q_h\|)$$

Use of approximation property (5.6) and ^{estimate} Theorem 5.1
We complete the rest of the proof. \blacksquare

5.2 Construction of V_h & H_h satisfying (H₁) and (H₃)

Assume that Ω is a ^{bounded} polygonal domain in \mathbb{R}^2 .

Let $\{\mathcal{O}_h\}$ be a family of regular triangulation of $\bar{\Omega}$ which is made of closed triangles K with $h_K = \text{diam}(K)$. Let $h = \max_{K \in \mathcal{O}_h} h_K$

Easiest choice

$$V_h := \{w_h \in (\mathcal{C}^0(\bar{\Omega}))^2 : w_h|_K \in (\mathcal{P}_1(K))^2 \quad \forall K \in \mathcal{O}_h \text{ and } w_h = 0 \text{ on } \partial\Omega\}$$

$$H_h := \{q_h \in L^2(\Omega) : q_h|_K \in \mathcal{P}_0(K) \quad \forall K \in \mathcal{O}_h\} \cap L^2_0(\Omega)$$

Unfortunately

$$\mathcal{Z}_h = \{\vec{0}\} \quad (\text{trivial space})$$

In order to avoid this situation, Choose V_h as higher order space

$$V_h := \{w_h \in (\mathcal{C}^0(\bar{\Omega}))^2 : w_h|_K \in (\mathcal{P}_2(K))^2 \quad \forall K \in \mathcal{O}_h \text{ and } w_h = 0 \text{ on } \partial\Omega\}$$

use H_h as before.

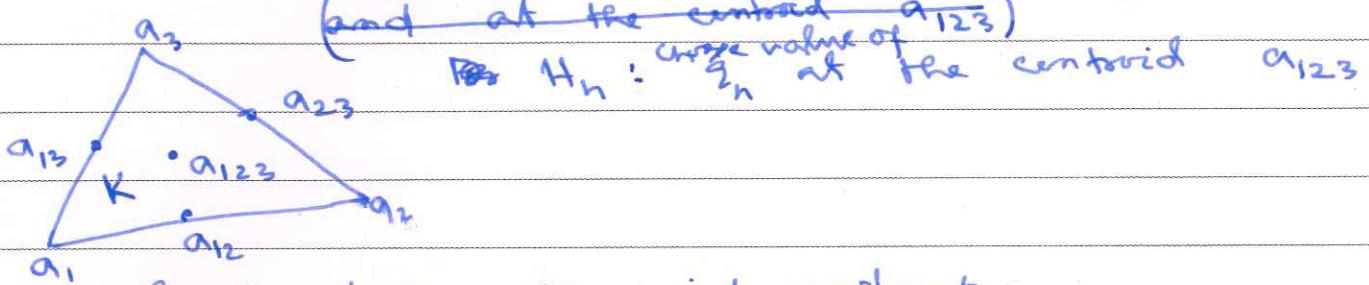
~~Construction of Interpolant: Let $K \in \mathcal{O}_h$. Then $\mathcal{P}_2(\mathcal{C}(K))$ is a function $\Pi_K^N \in \mathcal{P}_2$ defined by:~~

degree of freedom of V_h on each K : for each

components w_{hj} , $j=1, 2$, choose values at the

vertices a_i , $i=1, 2, 3$, midpoint a_{ij} of sides of K

(and at the centroid a_{123})



Construction of interpolant:

For $K \in \mathcal{O}_h$ & each $v \in C^0(\bar{K})$, $\exists !$ interpolant

$\Pi_K v \in \mathbb{P}_2(K)$ defined by

$$(5.21) \quad \begin{cases} \Pi_K v(a_i) = v(a_i) & i=1, 2, 3 \\ \int_{[a_i, a_j]} (\Pi_K v - v) d\sigma = 0 & 1 \leq i < j \leq 3 \end{cases}$$

where $[a_i, a_j]$ denotes the side of K going a_i to a_j .

Hints: write the help of $\sqrt{\lambda_i} \quad i=1, 2, 3$ with
barrycentric coordinates,

$$x_i \in \mathbb{P}, \quad \sum x_i = 1 \quad x_i(a_j) = \delta_{ij} \quad 1 \leq i, j \leq 3$$

define any $p \in \mathbb{P}_2$ on K ~~is~~ is of the form

$$p = \sum_{i=1}^3 p(a_i) x_i + 4 \left(\sum_{1 \leq i < j \leq 3} p(a_{ij}) x_i x_j \right)$$

On K : this system has six unknowns; So claim

if $v=0$ then $\Pi_K v=0$. Because of $\sqrt{(5.21)}$, it is
enough to show $\Pi_K v(a_{ij})=0 \quad 1 \leq i < j \leq 3$

On $[a_i, a_j]$, $\Pi_K v = 4 \Pi_K v(a_{ij}) x_i x_j$. Since

$$\int_{[a_i, a_j]} \Pi_K v d\sigma = 0 \quad \text{and as } \int_{[a_i, a_j]} x_i x_j d\sigma \geq 0, \text{ then}$$

$$\Pi_K v(a_{ij}) = 0 \quad \text{They completed the part.}$$

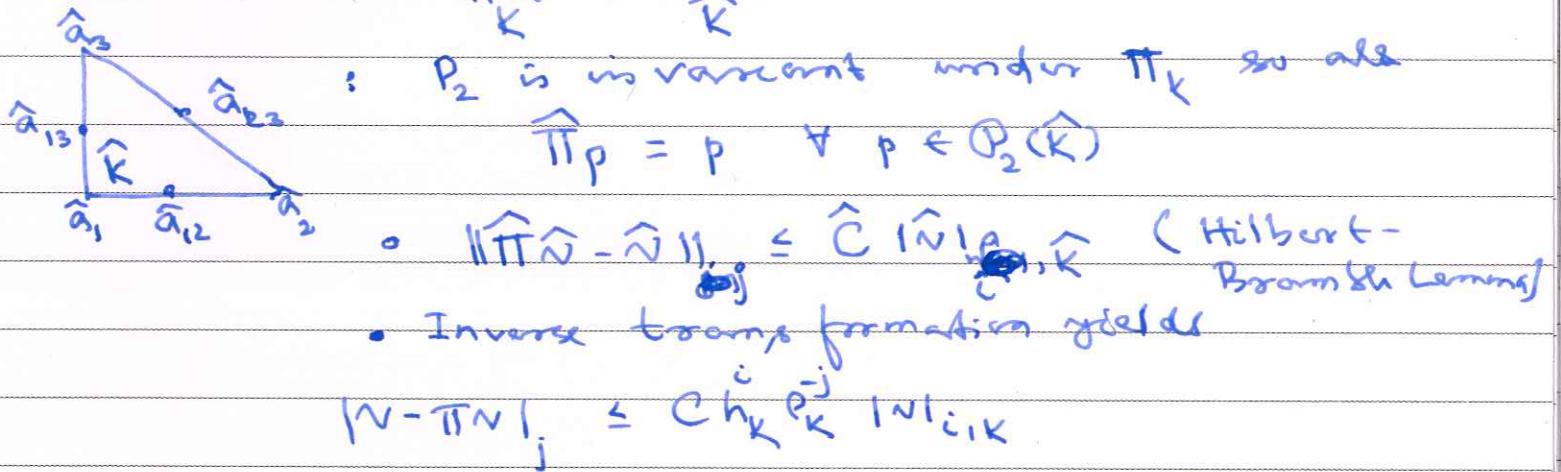
(19)

Lemma 5.1 For $\vec{v} \in H^i(K)$, $i=2 \text{ or } 3$, $\exists C > 0$ such that

$$\|\vec{v} - \Pi_K \vec{v}\|_{j,K} \leq Ch_K^{i-j} \|v\|_{i,K} \quad 0 \leq j \leq i$$

Hints: Use change of variable to transform the inf to a master element \hat{K} .

Check: $\widehat{\Pi_K \vec{v}} = \Pi_{\hat{K}} \widehat{\vec{v}}$



Construction of \vec{v}_h : for $\vec{v} \in (\mathcal{C}^0(\bar{\Omega}))^2$, set

$$(\vec{v}_h \vec{v})|_K = \Pi_K \vec{v} \quad \ell = 1, 2 \quad \forall K \in \mathfrak{G}_h$$

Then we can prove (5.4) $\ell=2$. To see it satisfies (5.5)

For $q_h \in H_h$, work at

$$\begin{aligned} (\nabla \cdot \vec{v}_h \vec{v}, q_h) &= \sum_{K \in \mathfrak{G}_h} q_h|_K \int_K \operatorname{div}(\vec{v}_h \vec{v}) \, dx \\ &= - \sum_K q_h|_K \int_K \Pi_K \vec{v} \cdot \vec{v} \, dx \\ &= - \sum_K q_h|_K \int_K \vec{v} \cdot \vec{v} \, dx = \sum_K q_h|_K \int_K \vec{v} \cdot \vec{v} \, dx \end{aligned}$$

Hence,

$$(\nabla \cdot \vec{v}_h \vec{v}, q_h) = (\nabla \cdot \vec{v}, q_h)$$

and (5.6) is satisfied

$$\boxed{\sigma_K = \frac{h_K}{P_K} \leq \sigma}$$

$$\|\vec{v} - \vec{v}_h \vec{v}\|_j^2 = \sum_K \|\vec{v} - \Pi_K \vec{v}\|_{j,K}^2 \leq C \sum_K h_K^{2(i-j)} \|v\|_{i,K}^2$$

Easy to check $\|\vec{v} - \vec{v}_h \vec{v}\|_j \leq Ch \|\vec{v}\|_i$, $i=1, 2$