C^0 Interior Penalty Methods

Error Analysis

Current Research in Finite Element Methods

CIMPA Summer School

Mumbai, July 2015

Outline

- Elliptic Regularity
- Standard A Priori Analysis
- *Medius* Error Analysis
- *A Posteriori* Error Analysis

Elliptic Regularity

Biharmonic Equation

$$\Delta^2 u = f \qquad \text{in } \Omega$$

with different boundary conditions

$\Omega =$ bounded polygonal domain in $\mathbb{R}^2$ $\qquad f \in L_2(\Omega)$

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Boundary Conditions of Clamped Plates

$V = H_0^2(\Omega)$

$(u = \partial u / \partial n = 0$ on $\partial\Omega)$

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Boundary Conditions of Simply Supported Plates

$V = H^2(\Omega) \cap H_0^1(\Omega)$

($u = \Delta u = 0$ on $\partial\Omega$)

Biharmonic Equation

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with different boundary conditions

$\Omega =$ bounded polygonal domain in $\mathbb{R}^2$ $\qquad f \in L_2(\Omega)$

Variational/Weak Formulation Find $u \in V$ such that

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Boundary Conditions of the Cahn-Hilliard Type

$V = \{v \in H^2(\Omega) : \partial v/\partial n = 0 \text{ on } \partial\Omega\}$

($\partial u/\partial n = \partial(\Delta u)/\partial n = 0$ on $\partial\Omega$)

Shift Theorem

If Ω is a smooth domain, then $f \in L_2(\Omega)$ implies the solution u of the variational/weak problem belongs to $H^4(\Omega)$.

Agmon-Douglis-Nirenberg (1959)

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There exists $\alpha \leq 2$, depending on the interior angles of Ω and the boundary conditions, such that the solution u of the model problem belongs to $H^{2+\alpha}(\Omega)$ when $f \in L_2(\Omega)$ and we have

$$\|u\|_{H^{2+\alpha}(\Omega)} \leq C_{\Omega,\alpha}\|f\|_{L_2(\Omega)}$$

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$$\|u\|_{H^{2+\alpha}(\Omega)} \leq C_{\Omega,\alpha}\|f\|_{L_2(\Omega)}$$

$\alpha =$ index of elliptic regularity

Shift Theorem

Fractional Order Sobolev Space $H^{2+s}(\Omega)$ $\qquad (0 < s < 1)$

A function v belongs to $H^{2+s}(\Omega)$ if and only if

- v belongs to $H^2(\Omega)$
- if w is a second order (weak) derivative of v, then

$$\int_\Omega \int_\Omega \frac{|w(x) - w(y)|^2}{|x - y|^{2+2s}}\, dx\, dy < \infty$$

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The norm of $H^{2+s}(\Omega)$ is defined by

$$\|v\|_{H^{2+s}(\Omega)}^2 = \|v\|_{H^2(\Omega)}^2 + \sum_{|\beta|=2} \int_\Omega \int_\Omega \frac{|(\partial^\beta v)(x) - (\partial^\beta v)(y)|^2}{|x - y|^{2+2s}} \, dx \, dy$$

Shift Theorem

Boundary Conditions of Clamped Plates

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega$$

The index of elliptic regularity α satisfies

$$\frac{1}{2} < \alpha \leq 2$$

$\alpha > 1$ if Ω is convex.

$\alpha < 1$ if Ω is non-convex.

Shift Theorem

Boundary Conditions of Simply Supported Plates

$$u = \Delta u = 0 \qquad \text{on } \partial\Omega$$

The index of elliptic regularity α satisfies

$$0 < \alpha \leq 2$$

For a rectangle, $\alpha = 2$.

For an L-shaped domain, α can be any number $< \frac{1}{3}$.

α can be close to 0 if there is an interior angle of Ω close to π.
(This can happen for a convex Ω.)

Shift Theorem

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$$\partial u/\partial n = \partial(\Delta u)/\partial n = 0 \qquad \text{on } \partial\Omega$$

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A Second Order Model Problem

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall \, v \in H_0^1(\Omega)$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain and $f \in L_2(\Omega)$.

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If Ω is smooth, then u belongs to $H^2(\Omega)$.

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where $\Omega \subset \mathbb{R}^2$ is a polygonal domain and $f \in L_2(\Omega)$.

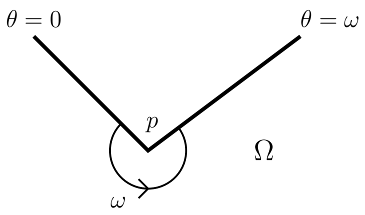
Shift Theorem

If Ω is smooth, then u belongs to $H^2(\Omega)$.

If Ω is a non-convex polygonal domain, then u does not belong to $H^2(\Omega)$ in general.

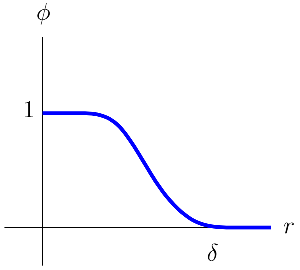
A Second Order Model Problem

Let $\omega > \pi$ be the interior angle at an vertex p of Ω, and (r, θ) be the polar coordinates at p.



Consider the function

$$\psi(r, \theta) = r^{(\pi/\omega)} \sin\big((\pi/\omega)\theta\big)\phi(r)$$



where $\phi(r)$ is a smooth cut-off function that equals 1 near 0 and vanishes for $r \geq \delta > 0$.

It is clear that ψ vanishes on the two edges at p and, if δ is sufficiently small, ψ also vanishes on the rest of $\partial\Omega$.

A Second Order Model Problem

Away from the vertex p the function

$$\psi(r,\theta) = r^{(\pi/\omega)} \sin\big((\pi/\omega)\theta\big)\phi(r)$$

is smooth up to the boundary of Ω. Near the vertex p, ψ is square integrable and

$$(\nabla\psi)(r,\theta) = (\pi/\omega)r^{(\pi/\omega)-1} \begin{bmatrix} \sin\big(\big[(\pi/\omega)-1\big]\theta\big) \\ \cos\big(\big[(\pi/\omega)-1\big]\theta\big) \end{bmatrix}$$

is also square integrable since $(\pi/\omega)-1 > \frac{1}{2}-1 = -\frac{1}{2}$.

Therefore ψ belongs to $H^1(\Omega)$ and hence $\psi \in H^1_0(\Omega)$ since ψ vanishes on $\partial\Omega$.

A Second Order Model Problem

$$\psi(r,\theta) = r^{(\pi/\omega)} \sin\big((\pi/\omega)\theta\big)\phi(r)$$

$$= r^{(\pi/\omega)} \sin\big((\pi/\omega)\theta\big) = \mathrm{Im}(z^{(\pi/\omega)}) \qquad \text{near } p$$

$$(z = re^{i\theta})$$

A Second Order Model Problem

$$\begin{aligned}\psi(r,\theta) &= r^{(\pi/\omega)} \sin\big((\pi/\omega)\theta\big)\phi(r)\\ &= r^{(\pi/\omega)} \sin\big((\pi/\omega)\theta\big) = \mathrm{Im}(z^{(\pi/\omega)}) \qquad \text{near } p\end{aligned}$$

Therefore

$$-\Delta\psi = 0 \quad \text{near } p$$

and hence

$$F = -\Delta\psi \in C^\infty(\bar{\Omega})$$

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Therefore

$$-\Delta\psi = 0 \quad \text{near } p$$

and hence

$$F = -\Delta\psi \in C^\infty(\bar{\Omega})$$

$\psi \in H_0^1(\Omega)$ satisfies

$$\int_\Omega \nabla\psi\cdot\nabla v\,dx = \int_\Omega Fv\,dx \qquad \forall\, v \in H_0^1(\Omega)$$

But ψ does not belong to $H^2(\Omega)$.

A Second Order Model Problem

Near the corner p

$$\frac{\partial^2\psi}{\partial x_1^2} = \frac{\pi}{\omega}\Big(\frac{\pi}{\omega} - 1\Big) r^{(\pi/\omega)-2} \sin\left(\big[(\pi/\omega) - 2\big]\theta\right)$$

is not square integrable because

$$(\pi/\omega) - 2 < 1 - 2 = -1$$

We will refer to ψ as a singular function associated with the reentrant corner.

$$\psi \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$$

for any $\epsilon > 0$, but

$$\psi \notin H^{1+(\pi/\omega)}(\Omega)$$

A Second Order Model Problem

Fractional Order Sobolev Space $H^{1+s}(\Omega)$ $(0 < s < 1)$

A function v belongs to $H^{1+s}(\Omega)$ if and only if

- v belongs to $H^1(\Omega)$
- if w is a first order (weak) derivative of v, then

$$\int_\Omega \int_\Omega \frac{|w(x) - w(y)|^2}{|x - y|^{2+2s}} \, dx \, dy < \infty$$

The norm of $H^{1+s}(\Omega)$ is defined by

$$\|v\|^2_{H^{1+s}(\Omega)} = \|v\|^2_{H^2(\Omega)} + \sum_{|\beta|=1} \int_\Omega \int_\Omega \frac{|(\partial^\beta v)(x) - (\partial^\beta v)(y)|^2}{|x - y|^{2+2s}} \, dx \, dy$$

A Second Order Model Problem

Singular Function Representation   In general we have

$$u = u_R + \sum_{\omega_j > \pi} c_j \psi_j$$

where

$$u_R \in H^2(\Omega) \cap H_0^1(\Omega)$$

$\omega_1, \omega_2, \ldots$ are the interior angles at the vertices of Ω, c_j's are constants, and

$$\psi_j = r_j^{(\pi/\omega_j)} \sin\big((\pi/\omega_j)\theta_j\big)\phi_j(r_j)$$

is the singular function associated with the re-entrant corner with interior angle ω_j.

A Second Order Model Problem

Singular Function Representation   In general we have

$$u = u_R + \sum_{\omega_j > \pi} c_j \psi_j$$

where $u_R \in H^2(\Omega) \cap H^1_0(\Omega)$. In particular

$$u = u_R \in H^2(\Omega)$$

if Ω is convex, but in general

$$u \in H^s(\Omega) \quad \text{for all} \quad s < 1 + \frac{\pi}{\omega}$$

if Ω is nonconvex, where ω is the largest re-entrant corner.

A Second Order Model Problem

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if Ω is nonconvex, where ω is the largest re-entrant corner.

Moreover, we have an elliptic regularity estimate

$$|u_R|_{H^2(\Omega)} + \sum_{\omega_j > \pi} |c_j| \leq C_\Omega \|f\|_{L_2(\Omega)}$$

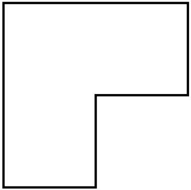
A Second Order Model Problem

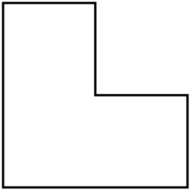
Example

$$\omega = \frac{3\pi}{2}$$

for an L-shaped domain and hence the solution of the Poisson problem in general only belongs to

$$H^s(\Omega) \quad \text{for} \quad s < \frac{5}{3} = 1 + \frac{\pi}{\omega}$$





A Second Order Model Problem

Shift Theorem Revisited

Given $F \in H^{-1}(\Omega) = [H_0^1(\Omega)]'$, there is a unique $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = F(v) \qquad \forall\, v \in H_0^1(\Omega)$$

Moreover we have

$$\|u\|_{H^1(\Omega)} \leq C_\Omega \|F\|_{H^{-1}(\Omega)}$$

where

$$\|F\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{|F(v)|}{\|v\|_{H^1(\Omega)}}$$

Therefore the shift theorem holds for $H^{-1}(\Omega)$. On the other hand, if Ω is nonconvex, then the shift theorem fails for $H^0(\Omega) = L_2(\Omega)$.

A Second Order Model Problem

Shift Theorem Revisited

Let Ω be a nonconvex polygonal domain with maximum reentrant angle ω, and $u \in H_0^1(\Omega)$ satisfy

$$\int_\Omega \nabla u \cdot \nabla v \, dx = F(v) \qquad \forall\, v \in H_0^1(\Omega)$$

where $F \in H^{-1+s}(\Omega)$ for some $s \in (0, \pi/\omega)$. Then we have $u \in H^{1+s}(\Omega)$ and

$$\|u\|_{H^{1+s}(\Omega)} \le C_{\Omega,s} \|F\|_{H^{-1+s}(\Omega)}$$

For $0 < t < 1$, $H^{-t}(\Omega)$ is the subspace of $H^{-1}(\Omega) = [H_0^1(\Omega)]'$ consisting of linear functionals G such that

$$\|G\|_{H^{-t}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{|G(v)|}{\|v\|_{H^t(\Omega)}} < \infty$$

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- Dauge, Elliptic Boundary Value Problems on Corner Domains, Springer-Verlag, 1988
- Nazarov and Plamenevsky, Elliptic Problems in Domains with Piecewise Smooth Boundaries, de Gruyter, 1994
- Blum and Rannacher, *On the boundary value problem of the biharmonic operator on domains with angular corners*, *Math. Methods Appl. Sci.*, 1980.

Standard *A Priori* Error Analysis

Boundary Conditions of Clamped Plates

$\Omega =$ bounded polygonal domain $\qquad f \in L_2(\Omega)$

$$\Delta^2 u = f \qquad \text{in } \Omega$$

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega$$

Variational/Weak Formulation

Find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = \int_\Omega fv\,dx \qquad \forall\, v \in H_0^2(\Omega)$$

$$a(w,v) = \int_\Omega D^2 w : D^2 v\,dx \qquad D^2 w : D^2 v = \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}$$

Discrete Problem

Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f\, v\, dx \qquad \forall\, v \in V_h$$

$$\begin{aligned} a_h(w,v) = \sum_{T\in\mathcal{T}_h} \int_T D^2 w : D^2 v\, dx &+ \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 w}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] ds \\ &+ \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n} \right]\!\!\right] ds \\ &+ \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \int_e \left[\!\!\left[\frac{\partial w}{\partial n} \right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] ds \end{aligned}$$

$\mathcal{E}_h =$ set of edges $\qquad \{\!\{\cdot\}\!\} =$ average $\qquad [\![\cdot]\!] =$ jump

$|e| =$ length of e $\qquad \sigma =$ penalty parameter

Galerkin Orthogonality

The C^0 interior penalty method is consistent in the sense that

$$(*) \qquad a_h(u,v) = \int_\Omega fv\,dx \qquad \forall\, v \in V_h$$

This was derived earlier under the assumption that u is sufficiently smooth. Since u is not very smooth at the corners of Ω, one must justify carefully the integration by parts involving u around the corners. This can be done by using the singular function representation of u.

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This was derived earlier under the assumption that u is sufficiently smooth. Since u is not very smooth at the corners of Ω, one must justify carefully the integration by parts involving u around the corners. This can be done by using the singular function representation of u.

Since

$$a_h(u_h,v) = \int_\Omega fv\,dx \qquad \forall\, v \in V_h$$

by the definition of the C^0 interior penalty method,

$$a_h(u-u_h,v) = 0 \qquad \forall\, v \in V_h$$

Two Mesh-Dependent Norms

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \Big\| \Big[\!\Big[\frac{\partial v}{\partial n} \Big]\!\Big] \Big\|_{L_2(e)}^2$$

$$|\!|\!| v |\!|\!|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \Big\| \Big[\!\Big[\frac{\partial v}{\partial n} \Big]\!\Big] \Big\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} \frac{|e|}{\sigma} \Big\| \Big\{\!\Big\{ \frac{\partial^2 v}{\partial n^2} \Big\}\!\Big\} \Big\|_{L_2(e)}^2$$

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The first mesh-dependent norm $\|\cdot\|_h$ is well-defined on $H^2(\Omega)$.

The second mesh-dependent norm $|\!|\!|\cdot|\!|\!|_h$ is only well-defined if v is a piecewise H^s function for some $s > \frac{5}{2}$.

Two Mesh-Dependent Norms

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The first mesh-dependent norm $\|\cdot\|_h$ is well-defined on $H^2(\Omega)$.

The second mesh-dependent norm $|\!|\!|\cdot|\!|\!|_h$ is only well-defined if v is a piecewise H^s function for some $s > \frac{5}{2}$.

Since the solution u belongs to $H^{2+\alpha}(\Omega)$ for $\alpha > \frac{1}{2}$, the second mesh dependent norm is well-defined on the space

$$\langle u\rangle + V_h$$

Two Mesh-Dependent Norms

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \Big\| \Big[\!\Big[\frac{\partial v}{\partial n} \Big]\!\Big] \Big\|_{L_2(e)}^2$$

$$|\!|\!| v |\!|\!|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \Big\| \Big[\!\Big[\frac{\partial v}{\partial n} \Big]\!\Big] \Big\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} \frac{|e|}{\sigma} \Big\| \Big\{\!\Big\{ \frac{\partial^2 v}{\partial n^2} \Big\}\!\Big\} \Big\|_{L_2(e)}^2$$

Relations between the Two Norms

Obviously

$$\|v\|_h \le |\!|\!| v |\!|\!|_h \qquad \forall\, v \in V_h$$

Two Mesh-Dependent Norms

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Obviously

$$\|v\|_h \leq |\!|\!| v |\!|\!|_h \qquad \forall\, v \in V_h$$

On the other hand

$$|\!|\!| v |\!|\!|_h \leq C\|v\|_h \qquad \forall\, v \in V_h$$

because

$$\sum_{e\in\mathcal{E}_h} \frac{|e|}{\sigma} \|\{\!\{\partial^2 v/\partial n^2\}\!\}\|_{L_2(e)}^2 \leq C \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2$$

Two Mesh-Dependent Norms

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \Big\| \Big[\!\!\Big[\frac{\partial v}{\partial n} \Big]\!\!\Big] \Big\|_{L_2(e)}^2$$

$$|\!|\!| v |\!|\!|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \Big\| \Big[\!\!\Big[\frac{\partial v}{\partial n} \Big]\!\!\Big] \Big\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} \frac{|e|}{\sigma} \Big\| \Big\{\!\!\Big\{ \frac{\partial^2 v}{\partial n^2} \Big\}\!\!\Big\} \Big\|_{L_2(e)}^2$$

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$$\|v\|_h \leq |\!|\!| v |\!|\!|_h \qquad \forall\, v \in V_h$$

On the other hand

$$|\!|\!| v |\!|\!|_h \leq C\|v\|_h \qquad \forall\, v \in V_h$$

Therefore the two norms are equivalent on V_h.

Properties of $a_h(\cdot, \cdot)$

Properties of $a_h(\cdot,\cdot)$

Boundedness

$$
\begin{aligned}
a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T D^2 w : D^2 v\,dx &+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds \\
&+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] ds \\
&+ \sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds
\end{aligned}
$$

Properties of $a_h(\cdot,\cdot)$

Boundedness

$$a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T D^2w : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$
$$+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] ds$$
$$+ \sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$

$$\sum_{T\in\mathcal{T}_h}\int_T D^2w : D^2v\,dx \le \Big(\sum_{T\in\mathcal{T}_h}|w|^2_{H^2(T)}\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)}\Big)^{\frac{1}{2}}$$

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$$+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial w}{\partial n}\right]\!\right] ds$$

$$+ \sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\int_e \left[\!\left[\frac{\partial w}{\partial n}\right]\!\right]\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds$$

$$\sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\int_e \left[\!\left[\frac{\partial w}{\partial n}\right]\!\right]\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds$$

$$\leq \Big(\sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\Big\|\left[\!\left[\frac{\partial w}{\partial n}\right]\!\right]\Big\|^2_{L_2(e)}\Big)^{\frac{1}{2}} \Big(\sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\Big\|\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]\Big\|^2_{L_2(e)}\Big)^{\frac{1}{2}}$$

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\end{aligned}
$$

$$
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&\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds \\
&\qquad \le \Big(\sum_{e\in\mathcal{E}_h}\frac{|e|}{\sigma}\Big\|\left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\}\Big\|_{L_2(e)}^2\Big)^{\frac12}\Big(\sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\Big\|\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\Big\|_{L_2(e)}^2\Big)^{\frac12}
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$$a_h(w,v) \le 2|\!|\!|w|\!|\!|_h |\!|\!|v|\!|\!|_h \qquad \forall\, v,w \in \langle u\rangle + V_h$$

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In particular

$$(*) \qquad a_h(w,v) \le C\|w\|_h\|v\|_h \qquad \forall\, v,w \in V_h$$

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$$+ \sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$

$$a_h(w,v) \le 2|\!|\!|w|\!|\!|_h |\!|\!|v|\!|\!|_h \qquad \forall\, v,w\in\langle u\rangle + V_h$$

In particular

$$(*) \qquad a_h(w,v) \le C\|w\|_h\|v\|_h \qquad \forall\, v,w\in V_h$$

But the estimate $(*)$ is not valid for $v,w\in\langle u\rangle + V_h$.

Properties of $a_h(\cdot,\cdot)$

Coercivity

We know that

$$a_h(v,v) \geq \frac{1}{2}\|v\|_h^2 \qquad \forall\, v \in V_h$$

provided σ is sufficiently large.

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Therefore $a_h(\cdot,\cdot)$ is also coercive with respect to the norm $\|\!|\cdot|\!\|_h$, i.e.,

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Therefore $a_h(\cdot,\cdot)$ is also coercive with respect to the norm $|\!|\!|\cdot|\!|\!|_h$, i.e.,

$$a_h(v,v) \geq C|\!|\!|v|\!|\!|_h^2 \qquad \forall\, v \in V_h$$

Consequently

$$\|v\|_h \approx |\!|\!|v|\!|\!|_h \approx \sqrt{a_h(v,v)} \qquad \forall\, v \in V_h$$

Error Estimate in the Energy Norm $|||\cdot|||_h$

Error Estimate in the Energy Norm $\|\cdot\|_h$

For any $v \in V_h$

$$\|u - u_h\|_h \leq \|u - v\|_h + \|v - u_h\|_h$$

Error Estimate in the Energy Norm $\|\cdot\|_h$

For any $v \in V_h$

$$\begin{aligned}\|u - u_h\|_h &\le \|u - v\|_h + \|v - u_h\|_h \\ &\le \|u - v\|_h + C \max_{w \in V_h} \frac{a_h(v - u_h, w)}{\|w\|_h}\end{aligned}$$

Coercivity

$$\|v\|_h^2 \le C a_h(v, v)$$

$$\Longrightarrow \|v\|_h \le C \frac{a_h(v, v)}{\|v\|_h}$$

$$\Longrightarrow \|v\|_h \le C \max_{w \in V_h} \frac{a_h(v, w)}{\|w\|_h}$$

Error Estimate in the Energy Norm $\|\cdot\|_h$

For any $v \in V_h$

$$\begin{aligned}
\|u - u_h\|_h &\leq \|u - v\|_h + \|v - u_h\|_h \\
&\leq \|u - v\|_h + C \max_{w \in V_h} \frac{a_h(v - u_h, w)}{\|w\|_h} \\
&= \|u - v\|_h + C \max_{w \in V_h} \frac{a_h(v - u, w)}{\|w\|_h}
\end{aligned}$$

Galerkin Orthogonality

$$a_h(u - u_h, w) = 0 \qquad \forall\, w \in V_h$$

Error Estimate in the Energy Norm $|||\cdot|||_h$

For any $v \in V_h$

$$\begin{aligned}
|||u-u_h|||_h &\le |||u-v|||_h + |||v-u_h|||_h \\
&\le |||u-v|||_h + C \max_{w\in V_h} \frac{a_h(v-u_h,w)}{|||w|||_h} \\
&= |||u-v|||_h + C \max_{w\in V_h} \frac{a_h(v-u,w)}{|||w|||_h} \\
&\le |||u-v|||_h + C|||v-u|||_h
\end{aligned}$$

Boundedness

$$a_h(v-u,w) \le 2|||v-u|||_h|||w|||_h$$

(We cannot use the norm $\|\cdot\|_h$ in this step.)

Error Estimate in the Energy Norm $\|\cdot\|_h$

For any $v \in V_h$

$$
\begin{aligned}
\|u - u_h\|_h &\le \|u - v\|_h + \|v - u_h\|_h \\
&\le \|u - v\|_h + C \max_{w \in V_h} \frac{a_h(v - u_h, w)}{\|w\|_h} \\
&= \|u - v\|_h + C \max_{w \in V_h} \frac{a_h(v - u, w)}{\|w\|_h} \\
&\le \|u - v\|_h + C\|v - u\|_h \\
&\le C\|u - v\|_h
\end{aligned}
$$

Error Estimate in the Energy Norm $\|\cdot\|_h$

For any $v \in V_h$

$$\begin{aligned}
\|u - u_h\|_h &\le \|u - v\|_h + \|v - u_h\|_h \\
&\le \|u - v\|_h + C \max_{w \in V_h} \frac{a_h(v - u_h, w)}{\|w\|_h} \\
&= \|u - v\|_h + C \max_{w \in V_h} \frac{a_h(v - u, w)}{\|w\|_h} \\
&\le \|u - v\|_h + C\|v - u\|_h \\
&\le C\|u - v\|_h
\end{aligned}$$

Therefore

$$\|u - u_h\|_h \le C \inf_{v \in V_h} \|u - v\|_h$$

Error Estimate in the Energy Norm $\|\!|\cdot|\!\|_h$

Let Π_h be the nodal interpolation operator from $C(\bar{\Omega})$ into V_h.

Standard Interpolation Error Estimates

$$\begin{aligned} h_T^{-2(2+\alpha)}\|\zeta-\Pi_h\zeta\|_{L_2(T)}^2 &+ h_T^{-2(1+\alpha)}|\zeta-\Pi_h\zeta|_{H^1(T)}^2 \\ &+ h_T^{-2\alpha}|\zeta-\Pi_h\zeta|_{H^2(T)}^2 \leq C\|\zeta\|_{H^{2+\alpha}(T)}^2 \qquad \forall\, T\in\mathcal{T}_h,\ \zeta\in H^{2+\alpha}(T) \end{aligned}$$

Error Estimate in the Energy Norm $\|\!|\cdot|\!\|_h$

Let Π_h be the nodal interpolation operator from $C(\bar{\Omega})$ into V_h.

Standard Interpolation Error Estimates

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\begin{aligned}
h_T^{-2(2+\alpha)}\|\zeta-\Pi_h\zeta\|_{L_2(T)}^2 &+ h_T^{-2(1+\alpha)}|\zeta-\Pi_h\zeta|_{H^1(T)}^2 \\
&+ h_T^{-2\alpha}|\zeta-\Pi_h\zeta|_{H^2(T)}^2 \le C\|\zeta\|_{H^{2+\alpha}(T)}^2 \qquad \forall\, T\in\mathcal{T}_h,\ \zeta\in H^{2+\alpha}(T)
\end{aligned}
$$

$$
\begin{aligned}
\|\!|u-\Pi_h u|\!\|_h^2 \le \sum_{T\in\mathcal{T}_h}|u-\Pi_h u|_{H^2(T)}^2 &+ \sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\Big\|\Big[\!\!\Big[\frac{\partial(u-\Pi_h u)}{\partial n}\Big]\!\!\Big]\Big\|_{L_2(e)}^2 \\
&+ \sum_{e\in\mathcal{E}_h}\frac{|e|}{\sigma}\Big\|\Big\{\!\!\Big\{\frac{\partial^2(u-\Pi_h u)}{\partial n^2}\Big\}\!\!\Big\}\Big\|_{L_2(e)}^2 \le Ch^{2\alpha}|u|_{H^{2+\alpha}(\Omega)}^2
\end{aligned}
$$

Error Estimate in the Energy Norm $|\!|\!| \cdot |\!|\!|_h$

Let Π_h be the nodal interpolation operator from $C(\bar{\Omega})$ into V_h.

Standard Interpolation Error Estimates

$$h_T^{-2(2+\alpha)}\|\zeta - \Pi_h\zeta\|_{L_2(T)}^2 + h_T^{-2(1+\alpha)}|\zeta - \Pi_h\zeta|_{H^1(T)}^2 + h_T^{-2\alpha}|\zeta - \Pi_h\zeta|_{H^2(T)}^2 \le C\|\zeta\|_{H^{2+\alpha}(T)}^2 \qquad \forall\, T \in \mathcal{T}_h,\ \zeta \in H^{2+\alpha}(T)$$

$$|\!|\!| u - \Pi_h u |\!|\!|_h^2 \le \sum_{T\in\mathcal{T}_h} |u - \Pi_h u|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|}\Big\| \Big[\!\Big[\frac{\partial(u - \Pi_h u)}{\partial n} \Big]\!\Big] \Big\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} \frac{|e|}{\sigma}\Big\| \Big\{\!\Big\{ \frac{\partial^2(u - \Pi_h u)}{\partial n^2} \Big\}\!\Big\} \Big\|_{L_2(e)}^2 \le Ch^{2\alpha}|u|_{H^{2+\alpha}(\Omega)}^2$$

Hence

$$|\!|\!| u - u_h |\!|\!|_h \le C \inf_{v\in V_h} |\!|\!| u - v |\!|\!|_h \le C|\!|\!| u - \Pi_h u |\!|\!|_h \le Ch^\alpha |u|_{H^{2+\alpha}(\Omega)} \le Ch^\alpha \|f\|_{L_2(\Omega)}$$

Other Boundary Conditions

In the standard approach we need to use the mesh-dependent norm $\|\cdot\|_h$ defined by

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \left\| \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] \right\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} \frac{|e|}{\sigma} \left\| \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} \right\|_{L_2(e)}^2$$

in order to exploit the Galerkin orthogonality.

But for the biharmonic equation with boundary conditions of simply supported plates or the Cahn-Hilliard type, the solution u may belong to $H^{2+\alpha}(\Omega)$ for $\alpha \in (0, \frac{1}{2})$, in which case the norm $\|\cdot\|_h$ is not well-defined on $\langle u \rangle + V_h$.

Therefore the standard approach is problematic for these problems.

Error Estimate for the Post-Processed Solution

$$|u - E_h u_h|_{H^2(\Omega)} \leq Ch^{\alpha} \|f\|_{L_2(\Omega)}$$

Error Estimate for the Post-Processed Solution

$$|u - E_h u_h|_{H^2(\Omega)} \le Ch^\alpha \|f\|_{L_2(\Omega)}$$

$$\begin{aligned}
|u - E_h u_h|_{H^2(\Omega)} &\le |u - E_h \Pi_h u|_{H^2(\Omega)} + |E_h(\Pi_h u - u_h)|_{H^2(\Omega)} \\
&\le C\big[h^\alpha |u|_{H^{2+\alpha}(\Omega)} + |\Pi_h u - u_h|_{H^2(\Omega;\mathcal{T}_h)}\big] \\
&\le C\big[h^\alpha |u|_{H^{2+\alpha}(\Omega)} + |\Pi_h u - u|_{H^2(\Omega;\mathcal{T}_h)} \\
&\qquad + |u - u_h|_{H^2(\Omega;\mathcal{T}_h)}\big] \\
&\le C\big[h^\alpha |u|_{H^{2+\alpha}(\Omega)} + \|u - u_h\|_h\big] \\
&\le Ch^\alpha |u|_{H^{2+\alpha}(\Omega)} \\
&\le Ch^\alpha \|f\|_{L_2(\Omega)}
\end{aligned}$$

Error Estimates in Lower Order Norms

Error Estimates in Lower Order Norms

For simplicity we assume that Ω is a convex polygon so that we can take the index of elliptic regularity α to be 1, i.e., $u \in H^3(\Omega)$ and

$$|\!|\!| u - u_h |\!|\!|_h \leq Ch \|f\|_{L_2(\Omega)}$$

Error Estimates in Lower Order Norms

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$$|\!|\!|u - u_h|\!|\!|_h \leq Ch\|f\|_{L_2(\Omega)}$$

Let $F \in H^{-1}(\Omega) = [H_0^1(\Omega)]'$ and $\zeta \in H_0^2(\Omega)$ satisfy the clamped plate problem

$$a(\zeta, v) = F(v) \qquad \forall\, v \in H_0^2(\Omega)$$

$$a(w, v) = \int_\Omega D^2 w : D^2 v\, dx$$

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$$a(\zeta, v) = F(v) \qquad \forall\, v \in H_0^2(\Omega)$$

Then we have the Shift Theorem

$$|\zeta|_{H^3(\Omega)} \leq C_\Omega \|F\|_{H^{-1}(\Omega)}$$

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$$|||u - u_h|||_h \le Ch\|f\|_{L_2(\Omega)}$$

Let $F \in H^{-1}(\Omega) = [H_0^1(\Omega)]'$ and $\zeta \in H_0^2(\Omega)$ satisfy the clamped plate problem

$$a(\zeta, v) = F(v) \qquad \forall\, v \in H_0^2(\Omega)$$

Then we have the Shift Theorem

$$|\zeta|_{H^3(\Omega)} \le C_\Omega \|F\|_{H^{-1}(\Omega)}$$

We can use this Shift Theorem to derive an error estimate for $|u - u_h|_{H^1(\Omega)}$ through a duality argument.

Error Estimates in Lower Order Norms

$$|u - u_h|_{H^1(\Omega)} = \sup_{F \in H^{-1}(\Omega)} \frac{F(u - u_h)}{\|F\|_{H^{-1}(\Omega)}}$$

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$$|u - u_h|_{H^1(\Omega)} = \sup_{F \in H^{-1}(\Omega)} \frac{F(u - u_h)}{\|F\|_{H^{-1}(\Omega)}}$$

Let $\zeta \in H_0^2(\Omega)$ satisfy

$$\int_\Omega D^2\zeta : D^2 v\, dx = F(v) \qquad \forall\, v \in H_0^2(\Omega)$$

and $\zeta_h \in V_h$ satisfy

$$a_h(\zeta_h, v) = F(v) \qquad \forall\, v \in V_h$$

Error Estimates in Lower Order Norms

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and $\zeta_h \in V_h$ satisfy

$$a_h(\zeta_h, v) = F(v) \qquad \forall\, v \in V_h$$

Then $\zeta \in H^3(\Omega)$ by the Shift Theorem,

$$|\zeta|_{H^3(\Omega)} \le C\|F\|_{H^{-1}(\Omega)}$$

and

$$a_h(\zeta-\zeta_h, v) = 0 \qquad \forall\, v \in V_h$$

Error Estimates in Lower Order Norms

$$|u - u_h|_{H^1(\Omega)} = \sup_{F \in H^{-1}(\Omega)} \frac{F(u - u_h)}{\|F\|_{H^{-1}(\Omega)}}$$

$$F(u - u_h) = F(u) - F(u_h)$$

Error Estimates in Lower Order Norms

$$|u - u_h|_{H^1(\Omega)} = \sup_{F \in H^{-1}(\Omega)} \frac{F(u - u_h)}{\|F\|_{H^{-1}(\Omega)}}$$

$$\begin{aligned} F(u - u_h) &= F(u) - F(u_h) \\ &= a(\zeta, u) - a_h(\zeta_h, u_h) \end{aligned}$$

$$\begin{aligned} a(\zeta, v) &= F(v) \qquad \forall\, v \in H_0^2(\Omega) \\ a_h(\zeta_h, v) &= F(v) \qquad \forall\, v \in V_h \end{aligned}$$

Error Estimates in Lower Order Norms

$$|u - u_h|_{H^1(\Omega)} = \sup_{F \in H^{-1}(\Omega)} \frac{F(u - u_h)}{\|F\|_{H^{-1}(\Omega)}}$$

$$\begin{aligned} F(u - u_h) &= F(u) - F(u_h) \\ &= a(\zeta, u) - a_h(\zeta_h, u_h) \\ &= a_h(\zeta, u) - a_h(\zeta, u_h) \end{aligned}$$

$$a_h(v, w) = a(v, w) \qquad \forall\, v, w \in H_0^2(\Omega) \cap H^3(\Omega)$$

$$a_h(\zeta - \zeta_h, v) = 0 \qquad \forall\, v \in V_h$$

Error Estimates in Lower Order Norms

$$|u - u_h|_{H^1(\Omega)} = \sup_{F \in H^{-1}(\Omega)} \frac{F(u - u_h)}{\|F\|_{H^{-1}(\Omega)}}$$

$$\begin{aligned} F(u - u_h) &= F(u) - F(u_h) \\ &= a(\zeta, u) - a_h(\zeta_h, u_h) \\ &= a_h(\zeta, u) - a_h(\zeta, u_h) \\ &= a_h(\zeta, u - u_h) \end{aligned}$$

Error Estimates in Lower Order Norms

$$|u - u_h|_{H^1(\Omega)} = \sup_{F \in H^{-1}(\Omega)} \frac{F(u - u_h)}{\|F\|_{H^{-1}(\Omega)}}$$

$$\begin{aligned} F(u - u_h) &= F(u) - F(u_h) \\ &= a(\zeta, u) - a_h(\zeta_h, u_h) \\ &= a_h(\zeta, u) - a_h(\zeta, u_h) \\ &= a_h(\zeta, u - u_h) \\ &= a_h(\zeta - \Pi_h \zeta, u - u_h) \end{aligned}$$

$$a_h(u - u_h, v) = 0 \qquad \forall\, v \in V_h$$

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&\leq C \|\zeta - \Pi_h \zeta\|_h \|u - u_h\|_h
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&\leq C \|\zeta - \Pi_h \zeta\|_h \|u - u_h\|_h \\
&\leq Ch|\zeta|_{H^3(\Omega)}\, h\|f\|_{L_2(\Omega)} \\
&\leq Ch^2 \|F\|_{H^{-1}(\Omega)} \|f\|_{L_2(\Omega)}
\end{aligned}$$

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&\leq Ch |\zeta|_{H^3(\Omega)}\, h \|f\|_{L_2(\Omega)} \\
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\end{aligned}$$

Therefore

$$|u - u_h|_{H^1(\Omega)} \leq Ch^2 \|f\|_{L_2(\Omega)}$$

Error Estimates in Lower Order Norms

We can also derive an error estimate for the post-processed solution $E_h u_h$ in the H^1 norm.

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Lemma $|E_h v|_{H^1(\Omega)} \approx |v|_{H^1(\Omega)} \quad \forall\, v \in V_h$

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Lemma $\quad |E_h v|_{H^1(\Omega)} \approx |v|_{H^1(\Omega)} \quad \forall\, v \in V_h$

$$\begin{aligned}
|u - E_h u|_{H^1(\Omega)} &\le |u - E_h \Pi_h u|_{H^1(\Omega)} + |E_h(\Pi_h u - u_h)|_{H^1(\Omega)} \\
&\le C\big[h^2 |u|_{H^3(\Omega)} + |\Pi_h u - u_h|_{H^1(\Omega)}\big] \\
&\le C\big[h^2 |u|_{H^3(\Omega)} + |\Pi_h u - u|_{H^1(\Omega)} + |u - u_h|_{H^1(\Omega)}\big] \\
&\le C\big[h^2 |u|_{H^3(\Omega)} + h^2 \|f\|_{L_2(\Omega)}\big] \\
&\le C h^2 \|f\|_{L_2(\Omega)}
\end{aligned}$$

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Proof.

$$\begin{aligned} |E_h v|_{H^1(\Omega)} &\leq |E_h v - v|_{H^1(\Omega)} + |v|_{H^1(\Omega)} \\ &\leq Ch|v|_{H^2(\Omega;\mathcal{T}_h)} + |v|_{H^1(\Omega)} \\ &\leq C|v|_{H^1(\Omega)} \end{aligned}$$

Error Estimates in Lower Order Norms

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Proof.

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$$\begin{aligned} |v|_{H^1(\Omega)} &= |\Pi_h E_h v|_{H^1(\Omega)} \\ &= |\Pi_h E_h v - E_h v|_{H^1(\Omega)} + |E_h v|_{H^1(\Omega)} \\ &\le Ch|E_h v|_{H^2(\Omega)} + |E_h v|_{H^1(\Omega)} \\ &\le C|E_h v|_{H^1(\Omega)} \quad \square \end{aligned}$$

Error Estimates in Lower Order Norms

For a general polygonal domain with elliptic regularity index $\alpha \in (\frac{1}{2}, 2]$, we have

$$|u - u_h|_{H^{2-\alpha}(\Omega)} \leq Ch^{2\alpha}\|f\|_{L_2(\Omega)}$$

$$|u - E_h u|_{H^{2-\alpha}(\Omega)} \leq Ch^{2\alpha}\|f\|_{L_2(\Omega)}$$

Error Estimates in Lower Order Norms

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$$|u - E_h u|_{H^{2-\alpha}(\Omega)} \leq Ch^{2\alpha}\|f\|_{L_2(\Omega)}$$

It follows from the Sobolev Embedding Theorem that

$$\|u - u_h\|_{L_\infty(\Omega)} \leq Ch^{2\alpha}\|f\|_{L_2(\Omega)}$$

$$\|u - E_h u\|_{L_\infty(\Omega)} \leq Ch^{2\alpha}\|f\|_{L_2(\Omega)}$$

Reference

- Brenner and S.

  C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains

  J. Sci. Comput., 2005

Medius Error Analysis

Convergence of C^1 Finite Element Methods

If we solve the clamped plate problem using a C^1 finite element space $V_h \subset H_0^2(\Omega)$, then it follows from the Galerkin orthogonality

$$a(u - u_h, v) = 0 \qquad \forall\, v \in V_h$$

that, for any $v \in V_h$,

$$\begin{aligned} |u - u_h|^2_{H^2(\Omega)} &= a(u - u_h, u - u_h) \\ &= a(u - u_h, u - v) + a(u - u_h, v - u_h) \\ &= a(u - u_h, u - v) \\ &\leq |u - u_h|_{H^2(\Omega)} |u - v|_{H^2(\Omega)} \end{aligned}$$

and hence

$$|u - u_h|_{H^1(\Omega)} \leq |u - v|_{H^1(\Omega)} \qquad \forall\, v \in V_h$$

Convergence of C^1 Finite Element Methods

$$|u - u_h|_{H^2(\Omega)} \leq \inf_{v \in V_h} |u - v|_{H^2(\Omega)}$$

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On the other hand, to show that the C^0 interior penalty method converges by the standard approach, we must know some additional elliptic regularity of u. Otherwise we cannot use the mesh-dependent norm $\|\!|\cdot|\!\|_h$ on u.

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On the other hand, to show that the C^0 interior penalty method converges by the standard approach, we must know some additional elliptic regularity of u. Otherwise we cannot use the mesh-dependent norm $\|\cdot\|_h$ on u.

Question Can we prove the convergence of C^0 interior penalty methods without using any additional elliptic regularity of u?

Goal

Let u_h be the solution of a C^0 interior penalty method for the biharmonic equation with the boundary conditions of clamped plates. We want to derive an estimate for $u - u_h$ using only the fact that $u \in H_0^2(\Omega)$ satisfies the variational/weak formulation of the boundary value problem.

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Let u_h be the solution of a C^0 interior penalty method for the biharmonic equation with the boundary conditions of clamped plates. We want to derive an estimate for $u - u_h$ using only the fact that $u \in H_0^2(\Omega)$ satisfies the variational/weak formulation of the boundary value problem.

- We will use the mesh-dependent norm

$$\|v\|_h = \Big(\sum_{T \in \mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e \in \mathcal{E}_h} \frac{\sigma}{|e|} \Big\| \Big[\!\Big[\frac{\partial v}{\partial n} \Big]\!\Big] \Big\|_{L_(e)}^2 \Big)^{\frac{1}{2}}$$

  that is well-defined on $H^2(\Omega)$.

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  that is well-defined on $H^2(\Omega)$.

- We will not perform integration by parts that involve u.

Goal

Main Theorem

$$\|u-u_h\|_h \leq C\Big[\inf_{v\in V_h}\|u-v\|_h + \operatorname{Osc}(f)\Big]$$

where

$$\operatorname{Osc}(f) = \Big(\sum_{T\in\mathcal{T}_h} h_T^4 \inf_{q\in\mathbb{P}_{k-2}(T)} \|f-q\|_{L_2(T)}^2\Big)^{1/2}$$

is of higher order. (k = degree of the polynomials in V_h)

(quasi-optimal up to a higher order term)

Goal

Main Theorem

$$\|u-u_h\|_h \le C\Big[\inf_{v\in V_h}\|u-v\|_h + \mathrm{Osc}(f)\Big]$$

where

$$\mathrm{Osc}(f) = \Big(\sum_{T\in\mathcal{T}_h} h_T^4 \inf_{q\in\mathbb{P}_{k-2}(T)} \|f-q\|^2_{L_2(T)}\Big)^{1/2}$$

is of higher order. ($k=$ degree of the polynomials in V_h)

Corollary

$$\lim_{h\downarrow 0}\|u-u_h\|_h = 0$$

An Integration by Parts Formula

Let v and w be finite element functions that vanish on $\partial\Omega$.

$$\sum_{T\in\mathcal{T}_h}\int_T D^2 w : D^2 v\,dx$$

$$= \sum_{T\in\mathcal{T}_h}\int_T (\Delta^2 w)v\,dx - \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial \Delta w}{\partial n}\Big)v\,ds$$

$$+ \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial}{\partial n}\nabla w\Big)\cdot\nabla v\,ds$$

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&\qquad\qquad + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial}{\partial n}\nabla w\Big)\cdot\nabla v\,ds \\
&\qquad = \sum_{T\in\mathcal{T}_h}\int_T (\Delta^2 w) v\,dx - \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial \Delta w}{\partial n}\Big) v\,ds \\
&\qquad\qquad + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial^2 w}{\partial n^2}\Big)\Big(\frac{\partial v}{\partial n}\Big) ds + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial^2 w}{\partial n\partial t}\Big)\Big(\frac{\partial v}{\partial t}\Big) ds
\end{aligned}
$$

An Integration by Parts Formula

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\int_T D^2w : D^2v\,dx\\ &= \sum_{T\in\mathcal{T}_h}\int_T(\Delta^2 w)v\,dx - \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial\Delta w}{\partial n}\Big)v\,ds\\ &\quad + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial^2 w}{\partial n^2}\Big)\Big(\frac{\partial v}{\partial n}\Big)ds + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial^2 w}{\partial n\partial t}\Big)\Big(\frac{\partial v}{\partial t}\Big)ds\end{aligned}$$

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&\qquad\qquad + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial^2 w}{\partial n^2}\Big)\Big(\frac{\partial v}{\partial n}\Big)ds + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial^2 w}{\partial n\partial t}\Big)\Big(\frac{\partial v}{\partial t}\Big)ds \\
&\qquad = \sum_{T\in\mathcal{T}_h}\int_T(\Delta^2 w)v\,dx + \sum_{e\in\mathcal{E}_h^i}\int_e\Big[\!\!\Big[\frac{\partial\Delta w}{\partial n}\Big]\!\!\Big]v\,ds \\
&\qquad\qquad - \sum_{e\in\mathcal{E}_h}\int_e\Big\{\!\!\Big\{\frac{\partial^2 w}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big]ds - \sum_{e\in\mathcal{E}_h^i}\int_e\Big[\!\!\Big[\frac{\partial^2 w}{\partial n^2}\Big]\!\!\Big]\Big\{\!\!\Big\{\frac{\partial v}{\partial n}\Big\}\!\!\Big\}ds \\
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\end{aligned}
$$

An Alternative Expression for $a_h(\cdot,\cdot)$

$$
\begin{aligned}
a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T D^2 w : D^2 v\,dx &+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds \\
&+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] ds \\
&+ \sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds
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v and w are finite element functions that vanish on $\partial\Omega$.

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&+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] ds \\
&+ \sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds
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= &\sum_{T\in\mathcal{T}_h}\int_T (\Delta^2 w)v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] ds \\
&+ \sum_{e\in\mathcal{E}_h^i}\int_e \left(\left[\!\!\left[\frac{\partial \Delta w}{\partial n}\right]\!\!\right]v - \left[\!\!\left[\frac{\partial^2 w}{\partial n^2}\right]\!\!\right]\left\{\!\!\left\{\frac{\partial v}{\partial n}\right\}\!\!\right\} - \left[\!\!\left[\frac{\partial^2 w}{\partial n\partial t}\right]\!\!\right]\frac{\partial v}{\partial t}\right) ds \\
&+ \sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds
\end{aligned}
$$

Strategy for Deriving the Main Theorem

Strategy for Deriving the Main Theorem

For any $v \in V_h$

$$\begin{aligned} \|u - u_h\|_h &\leq \|u - v\|_h + \|v - u_h\|_h \\ &\leq \|u - v\|_h + C \max_{v \in V_h} \frac{a_h(v - u_h, w)}{\|w\|_h} \end{aligned}$$

Strategy for Deriving the Main Theorem

For any $v \in V_h$

$$\begin{aligned}\|u - u_h\|_h &\le \|u - v\|_h + \|v - u_h\|_h \\ &\le \|u - v\|_h + C \max_{v \in V_h} \frac{a_h(v - u_h, w)}{\|w\|_h}\end{aligned}$$

If we can show that

$$\max_{v \in V_h} \frac{a_h(v - u_h, w)}{\|w\|_h} \le C\big[\|u - v\|_h + \mathrm{Osc}(f)\big]$$

then

$$\|u - u_h\|_h \le C\big[\|u - v\|_h + \mathrm{Osc}(f)\big] \qquad \forall\, v \in V_h$$

and hence

$$\|u - u_h\|_h \le C\big[\inf_{v \in V_h} \|u - v\|_h + \mathrm{Osc}(f)\big]$$

Preliminary Estimates

$$a_h(v - u_h, w) = a_h(v, E_h w) + a_h(v, w - E_h w) - \int_\Omega f\, w\, dx$$

$$a_h(u_h, w) = \int_\Omega f\, w\, dx$$

E_h is an enriching operator that maps V_h into $H_0^2(\Omega)$.

Preliminary Estimates

$$a_h(v-u_h,w)=a_h(v,E_hw)+a_h(v,w-E_hw)-\int_\Omega f\,w\,dx$$

First Term on the Right-Hand Side

$$\begin{aligned}a_h(v,E_hw)&=\sum_{T\in\mathcal{T}_h}\int_T D^2v:D^2(E_hw)\,dx+\sum_{e\in\mathcal{E}_h}\int_e\left\{\!\!\left\{\frac{\partial^2(E_hw)}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]ds\\&=\sum_{T\in\mathcal{T}_h}\int_T D^2(v-u):D^2(E_hw)\,dx+\int_\Omega f(E_hw)\,dx\\&\qquad+\sum_{e\in\mathcal{E}_h}\int_e\left\{\!\!\left\{\frac{\partial^2(E_hw)}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]ds.\end{aligned}$$

$$\int_\Omega D^2u:D^2(E_hw)\,dx=\int_\Omega f(E_hw)\,dx\quad\text{since }\ E_hw\in H_0^2(\Omega)$$

Preliminary Estimates

Second Term on the Right-Hand side

$$
\begin{aligned}
& a_h(v, w - E_h w) \\
&= \sum_{T \in \mathcal{T}_h} \int_T (\Delta^2 v)(w - E_h w)\,dx + \sum_{e \in \mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 (w - E_h w)}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] ds \\
&\quad + \sum_{e \in \mathcal{E}_h^i} \int_e \left(\left[\!\!\left[\frac{\partial \Delta v}{\partial n} \right]\!\!\right] (w - E_h w) - \left[\!\!\left[\frac{\partial^2 v}{\partial n^2} \right]\!\!\right] \left\{\!\!\left\{ \frac{\partial (w - E_h w)}{\partial n} \right\}\!\!\right\} \right) ds \\
&\qquad - \sum_{e \in \mathcal{E}_h^i} \int_e \left[\!\!\left[\frac{\partial^2 v}{\partial n \partial t} \right]\!\!\right] \frac{\partial (w - E_h w)}{\partial t} ds \\
&\qquad\quad + \sigma \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \int_e \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] \left[\!\!\left[\frac{\partial (w - E_h w)}{\partial n} \right]\!\!\right] ds
\end{aligned}
$$

alternative expression for $a_h(\cdot, \cdot)$

Preliminary Estimates

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\begin{aligned}
&a_h(v - u_h, w) \\
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&\quad + \sum_{e\in\mathcal{E}_h^i} \int_e \left(\left[\!\!\left[\frac{\partial \Delta v}{\partial n} \right]\!\!\right] (w - E_h w) - \left[\!\!\left[\frac{\partial^2 v}{\partial n^2} \right]\!\!\right] \left\{\!\!\left\{ \frac{\partial (w - E_h w)}{\partial n} \right\}\!\!\right\} \right) ds \\
&\quad - \sum_{T\in\mathcal{T}_h} \int_T (f - \Delta^2 v)(w - E_h w) dx.
\end{aligned}
$$

Preliminary Estimates

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T D^2(v-u) : D^2(E_h w)\,dx\Big|$$

$$\leq \Big(\sum_{T\in\mathcal{T}_h}|u-v|^2_{H^2(T)}\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}|E_h w|^2_{H^2(T)}\Big)^{\frac{1}{2}}$$

$$= \Big(\sum_{T\in\mathcal{T}_h}|u-v|^2_{H^2(T)}\Big)^{\frac{1}{2}}|E_h w|_{H^2(\Omega)}$$

$$\leq C\Big(\sum_{T\in\mathcal{T}_h}|u-v|^2_{H^2(T)}\Big)^{\frac{1}{2}}\|w\|_h$$

$$|E_h w|_{H^2(\Omega)} \leq C|w|_{H^2(\Omega;\mathcal{T}_h)} \leq C\|w\|_h$$

Preliminary Estimates

$$\Big|\sum_{e\in\mathcal{E}_h}\int_e \Big\{\!\!\Big\{\frac{\partial^2 w}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big]ds\Big| = \Big|\sum_{e\in\mathcal{E}_h}\int_e \Big\{\!\!\Big\{\frac{\partial^2 w}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial (v-u)}{\partial n}\Big]\!\!\Big]ds\Big|$$

$$\leq \Big(\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\Big\|\Big[\!\!\Big[\frac{\partial (u-v)}{\partial n}\Big]\!\!\Big]\Big\|^2_{L_2(e)}\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h}|e|\Big\|\Big\{\!\!\Big\{\frac{\partial^2 w}{\partial n^2}\Big\}\!\!\Big\}\Big\|^2_{L_2(e)}\Big)^{\frac{1}{2}}$$

$$\leq C\Big(\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\Big\|\Big[\!\!\Big[\frac{\partial (u-v)}{\partial n}\Big]\!\!\Big]\Big\|^2_{L_2(e)}\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}|w|^2_{H^2(T)}\Big)^{\frac{1}{2}}$$

$$\leq C\Big(\sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\Big\|\Big[\!\!\Big[\frac{\partial (u-v)}{\partial n}\Big]\!\!\Big]\Big\|^2_{L_2(e)}\Big)^{\frac{1}{2}}\|w\|_h$$

$$\sum_{e\in\mathcal{E}_h}|e|\Big\|\Big\{\!\!\Big\{\frac{\partial^2 w}{\partial n^2}\Big\}\!\!\Big\}\Big\|^2_{L_2(e)} \leq C\sum_{T\in\mathcal{T}_h}|w|^2_{H^2(T)} \leq C\|w\|_h^2$$

Preliminary Estimates

$$
\begin{aligned}
&\Big|\sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\int_e [\![\frac{\partial v}{\partial n}]\!] [\![\frac{\partial(w-E_hw)}{\partial n}]\!] ds\Big| \\
&\qquad \le C\Big(\sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\Big\|[\![\frac{\partial(u-v)}{\partial n}]\!]\Big\|_{L_2(e)}^2\Big)^{\frac{1}{2}} \\
&\qquad\qquad \times\Big(\sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\Big\|[\![\frac{\partial(w-E_hw)}{\partial n}]\!]\Big\|_{L_2(e)}^2\Big)^{\frac{1}{2}} \\
&\qquad \le C\Big(\sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\Big\|[\![\frac{\partial(u-v)}{\partial n}]\!]\Big\|_{L_2(e)}^2\Big)^{\frac{1}{2}}\|w\|_h
\end{aligned}
$$

$$
\|w\|_h^2=\sum_{T\in\mathcal{T}_h}|w|_{H^2(T)}^2+\sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\Big\|[\![\frac{\partial w}{\partial n}]\!]\Big\|_{L_2(e)}^2
$$

Preliminary Estimates

$$\Big|\sum_{e\in\mathcal{E}_h^i}\int_e \Big[\!\!\Big[\frac{\partial^2 v}{\partial n\partial t}\Big]\!\!\Big]\frac{\partial(w-E_hw)}{\partial t}ds\Big|$$

$$\le \Big(\sum_{e\in\mathcal{E}_h}|e|\Big\|\Big[\!\!\Big[\frac{\partial^2 v}{\partial n\partial t}\Big]\!\!\Big]\Big\|_{L_2(e)}^2\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\Big\|\frac{\partial(w-E_hw)}{\partial t}\Big\|_{L_2(e)}^2\Big)^{\frac{1}{2}}$$

$$\le C\Big(\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\Big\|\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big]\Big\|_{L_2(e)}^2\Big)^{\frac{1}{2}}\|w\|_h$$

$$\le C\Big(\sum_{e\in\mathcal{E}_h^i}\frac{\sigma}{|e|}\Big\|\Big[\!\!\Big[\frac{\partial(u-v)}{\partial n}\Big]\!\!\Big]\Big\|_{L_2(e)}^2\Big)^{\frac{1}{2}}\|w\|_h$$

$$\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\Big\|\frac{\partial(w-E_hw)}{\partial t}\Big\|_{L_2(e)}^2 \le C\sum_{T\in\mathcal{T}_h}h_T^{-2}|w-E_hw|_{H^1(T)}^2 \le C\|w\|_h^2$$

Preliminary Estimates

$$\Big|\sum_{e\in\mathcal{E}_h^i}\int_e \Big[\!\!\Big[\frac{\partial\Delta v}{\partial n}\Big]\!\!\Big](w-E_hw)\,ds\Big|$$

$$\leq \Big(\sum_{e\in\mathcal{E}_h^i}|e|^3\Big\|\Big[\!\!\Big[\frac{\partial\Delta v}{\partial n}\Big]\!\!\Big]\Big\|_{L_2(e)}^2\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|^3}\|w-E_hw\|_{L_2(e)}^2\Big)^{\frac{1}{2}}$$

$$\leq C\Big(\sum_{e\in\mathcal{E}_h^i}|e|^3\Big\|\Big[\!\!\Big[\frac{\partial\Delta v}{\partial n}\Big]\!\!\Big]\Big\|_{L_2(e)}^2\Big)^{\frac{1}{2}}\|w\|_h$$

$$\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|^3}\|w-E_hw\|_{L_2(e)}^2 \leq C\sum_{T\in\mathcal{T}_h}h_T^{-4}\|w-E_hw\|_{L_2(T)}^2 \leq C\|w\|_h^2$$

Preliminary Estimates

$$\Big|\sum_{e\in\mathcal{E}_h^i}\int_e \Big[\!\Big[\frac{\partial^2 v}{\partial n^2}\Big]\!\Big]\Big\{\!\!\Big\{\frac{\partial(w-E_h w)}{\partial n}\Big\}\!\!\Big\} ds\Big|$$

$$\leq \Big(\sum_{e\in\mathcal{E}_h^i}|e|\Big\|\Big[\!\Big[\frac{\partial^2 v}{\partial n^2}\Big]\!\Big]\Big\|_{L_2(e)}^2\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\Big\|\Big\{\!\!\Big\{\frac{\partial(w-E_h w)}{\partial n}\Big\}\!\!\Big\}\Big\|_{L_2(e)}^2\Big)^{\frac{1}{2}}$$

$$\leq C\Big(\sum_{e\in\mathcal{E}_h^i}|e|\Big\|\Big[\!\Big[\frac{\partial^2 v}{\partial n^2}\Big]\!\Big]\Big\|_{L_2(e)}^2\Big)^{\frac{1}{2}}\|w\|_h$$

$$\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\Big\|\Big\{\!\!\Big\{\frac{\partial(w-E_h w)}{\partial n}\Big\}\!\!\Big\}\Big\|_{L_2(e)}^2 \leq C\sum_{T\in\mathcal{T}_h} h_T^{-2}|w-E_h w|_{H^1(T)}^2 \leq C\|w\|_h^2$$

Preliminary Estimates

$$
\begin{aligned}
&\Big|\sum_{T\in\mathcal{T}_h}\int_T (f-\Delta^2 v)(w-E_h w)\,dx\Big| \\
&\qquad \le \Big(\sum_{T\in\mathcal{T}_h} h_T^4\|f-\Delta^2 v\|_{L_2(T)}^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-4}\|w-E_h w\|_{L_2(T)}^2\Big)^{\frac{1}{2}} \\
&\qquad \le \Big(\sum_{T\in\mathcal{T}_h} h_T^4\|f-\Delta^2 v\|_{L_2(T)}^2\Big)^{\frac{1}{2}}\|w\|_h
\end{aligned}
$$

Preliminary Estimates

$$
\begin{aligned}
a_h(v-u_h,w) \leq C\Big(&\sum_{T\in\mathcal{T}_h} |u-v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|}\Big\| \Big[\!\Big[\frac{\partial(u-v)}{\partial n} \Big]\!\Big] \Big\|^2_{L_2(e)} \\
&+ \sum_{e\in\mathcal{E}_h^i} |e| \Big\| \Big[\!\Big[\frac{\partial^2 v}{\partial n^2} \Big]\!\Big] \Big\|^2_{L_2(e)} + \sum_{e\in\mathcal{E}_h^i} |e|^3 \Big\| \Big[\!\Big[\frac{\partial \Delta v}{\partial n} \Big]\!\Big] \Big\|^2_{L_2(e)} \\
&+ \sum_{T\in\mathcal{T}_h} h_T^4 \|f-\Delta^2 v\|^2_{L_2(T)} \Big)^{\frac{1}{2}} \|w\|_h
\end{aligned}
$$

Preliminary Estimates

$$a_h(v-u_h,w) \le C\Big(\sum_{T\in\mathcal{T}_h}|u-v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\Big\|\Big[\!\Big[\frac{\partial(u-v)}{\partial n}\Big]\!\Big]\Big\|^2_{L_2(e)}$$
$$+\sum_{e\in\mathcal{E}_h^i}|e|\Big\|\Big[\!\Big[\frac{\partial^2 v}{\partial n^2}\Big]\!\Big]\Big\|^2_{L_2(e)} + \sum_{e\in\mathcal{E}_h^i}|e|^3\Big\|\Big[\!\Big[\frac{\partial\Delta v}{\partial n}\Big]\!\Big]\Big\|^2_{L_2(e)}$$
$$+\sum_{T\in\mathcal{T}_h}h_T^4\|f-\Delta^2 v\|^2_{L_2(T)}\Big)^{\frac{1}{2}}\|w\|_h$$

Hence

$$\max_{v\in V_h}\frac{a_h(v-u_h,w)}{\|w\|_h}$$
$$\le C\Big(\|u-v\|_h^2 + \sum_{e\in\mathcal{E}_h^i}|e|\Big\|\Big[\!\Big[\frac{\partial^2 v}{\partial n^2}\Big]\!\Big]\Big\|^2_{L_2(e)} + \sum_{e\in\mathcal{E}_h^i}|e|^3\Big\|\Big[\!\Big[\frac{\partial\Delta v}{\partial n}\Big]\!\Big]\Big\|^2_{L_2(e)}$$
$$+\sum_{T\in\mathcal{T}_h}h_T^4\|f-\Delta^2 v\|^2_{L_2(T)}\Big)^{\frac{1}{2}}$$

Preliminary Estimates

According to our strategy, it only remains to show that

$$\sum_{e\in\mathcal{E}_h^i} |e| \Big\| \Big[\!\Big[\frac{\partial^2 v}{\partial n^2} \Big]\!\Big] \Big\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h^i} |e|^3 \Big\| \Big[\!\Big[\frac{\partial \Delta v}{\partial n} \Big]\!\Big] \Big\|_{L_2(e)}^2 + \sum_{T\in\mathcal{T}_h} h_T^4 \|f - \Delta^2 v\|_{L_2(T)}^2$$
$$\leq C\big[\|u - v\|_h^2 + \mathrm{Osc}(f)^2\big]$$

Preliminary Estimates

According to our strategy, it only remains to show that

$$\sum_{e\in\mathcal{E}_h^i}|e|\Big\|\Big[\!\!\Big[\frac{\partial^2 v}{\partial n^2}\Big]\!\!\Big]\Big\|_{L_2(e)}^2+\sum_{e\in\mathcal{E}_h^i}|e|^3\Big\|\Big[\!\!\Big[\frac{\partial \Delta v}{\partial n}\Big]\!\!\Big]\Big\|_{L_2(e)}^2+\sum_{T\in\mathcal{T}_h}h_T^4\|f-\Delta^2 v\|_{L_2(T)}^2$$
$$\leq C\big[\|u-v\|_h^2+\mathrm{Osc}(f)^2\big]$$

These are called efficiency estimates in *a posteriori* error analysis when $v=u_h$, the solution of the discrete problem.

Local Efficiency Estimates

Local Efficiency Estimates

Estimate for $h_T^2\|f-\Delta^2 v\|_{L_2(T)}$

$$\begin{aligned}\mathrm{Osc}(f)^2 &= \sum_{T\in\mathcal{T}_h} h_T^4 \inf_{q\in\mathbb{P}_{k-2}(T)} \|f-q\|_{L_2(T)}^2 \\ &= \sum_{T\in\mathcal{T}_h} h_T^4 \|f-\bar{f}\|_{L_2(T)}^2\end{aligned}$$

$\bar{f}\big|_T = L_2$ projection of f on $\mathbb{P}_{k-2}(T)$

Local Efficiency Estimates

Estimate for $h_T^2\|f-\Delta^2 v\|_{L_2(T)}$

$$\begin{aligned}\mathrm{Osc}(f)^2 &= \sum_{T\in\mathcal{T}_h} h_T^4 \inf_{q\in\mathbb{P}_{k-2}(T)} \|f-q\|_{L_2(T)}^2 \\ &= \sum_{T\in\mathcal{T}_h} h_T^4 \|f-\bar{f}\|_{L_2(T)}^2\end{aligned}$$

$$\bar{f}\big|_T = L_2 \text{ projection of } f \text{ on } \mathbb{P}_{k-2}(T)$$

Let $\zeta\in\mathbb{P}_6(T)$ be the polynomial in $H_0^2(T)$ that equals 1 at the center of T.

$$|\zeta|_{H^2(T)} \leq C h_T^{-2}\|\zeta\|_{L_2(T)} \leq C h_T^{-1}$$

ζ is known as a bubble function. (Verfürth 1994)

Local Efficiency Estimates

Equivalence of norms on finite dimensional spaces

$$C_1 \int_T (\bar{f} - \Delta^2 v)^2 \zeta^2 \, dx \le \|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 \le C_2 \int_T (\bar{f} - \Delta^2 v)^2 \zeta \, dx$$

Local Efficiency Estimates

Equivalence of norms on finite dimensional spaces

$$C_1 \int_T (\bar{f} - \Delta^2 v)^2 \zeta^2 \, dx \le \|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 \le C_2 \int_T (\bar{f} - \Delta^2 v)^2 \zeta \, dx$$

Let the function z be defined by

$$z = \begin{cases} (\bar{f} - \Delta^2 v)\zeta & \text{on } T \\ 0 & \text{on } \Omega \setminus T \end{cases}$$

Then $z \in H_0^2(\Omega)$ and hence

$$\int_T D^2 u : D^2 z \, dx = \int_\Omega D^2 u : D^2 z \, dx = \int_\Omega f z \, dx = \int_T f z \, dx$$

Local Efficiency Estimates

$$\|\bar{f} - \Delta^2 v\|^2_{L_2(T)} \le C \int_T (\bar{f} - \Delta^2 v)^2 \zeta \, dx$$

$$\|\bar{f} - \Delta^2 v\|^2_{L_2(T)} \le C_2 \int_T (\bar{f} - \Delta^2 v)^2 \zeta \, dx$$

Local Efficiency Estimates

$$\|\bar{f} - \Delta^2 v\|^2_{L_2(T)} \leq C \int_T (\bar{f} - \Delta^2 v)^2 \zeta \, dx$$
$$= C \int_T (\bar{f} - \Delta^2 v) z \, dx$$

$$z = (\bar{f} - \Delta^2 v)\zeta$$

Local Efficiency Estimates

$$\begin{aligned}
\|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 \leq C \int_T (\bar{f} - \Delta^2 v)^2 \zeta \, dx \\
= C \int_T (\bar{f} - \Delta^2 v) z \, dx \\
= C \Big[\int_T f z \, dx - \int_T (\Delta^2 v) z \, dx + \int_T (\bar{f} - f) z \, dx \Big]
\end{aligned}$$

Local Efficiency Estimates

$$
\begin{aligned}
\|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 &\le C \int_T (\bar{f} - \Delta^2 v)^2 \zeta \, dx \\
&= C \int_T (\bar{f} - \Delta^2 v) z \, dx \\
&= C \Big[\int_T f z \, dx - \int_T (\Delta^2 v) z \, dx + \int_T (\bar{f} - f) z \, dx \Big] \\
&= C \Big[\int_T D^2 u : D^2 z \, dx - \int_T D^2 v : D^2 z \, dx + \int_T (\bar{f} - f) z \, dx \Big]
\end{aligned}
$$

$$
\begin{aligned}
\int_T D^2 u : D^2 z \, dx &= \int_T f z \, dx \\
\int_T D^2 v : D^2 z \, dx &= \int_T (\Delta^2 v) z \, dx
\end{aligned}
$$

Local Efficiency Estimates

$$
\begin{aligned}
\|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 &\leq C \int_T (\bar{f} - \Delta^2 v)^2 \zeta \, dx \\
&= C \int_T (\bar{f} - \Delta^2 v) z \, dx \\
&= C \Big[\int_T f z \, dx - \int_T (\Delta^2 v) z \, dx + \int_T (\bar{f} - f) z \, dx \Big] \\
&= C \Big[\int_T D^2 u : D^2 z \, dx - \int_T D^2 v : D^2 z \, dx + \int_T (\bar{f} - f) z \, dx \Big] \\
&= C \Big[\int_T (D^2 u - D^2 v) : D^2 z \, dx + \int_T (\bar{f} - f) z \, dx \Big]
\end{aligned}
$$

Local Efficiency Estimates

$$
\begin{aligned}
\|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 &\leq C \int_T (\bar{f} - \Delta^2 v)^2 \zeta \, dx \\
&= C \int_T (\bar{f} - \Delta^2 v) z \, dx \\
&= C \Big[\int_T f z \, dx - \int_T (\Delta^2 v) z \, dx + \int_T (\bar{f} - f) z \, dx \Big] \\
&= C \Big[\int_T D^2 u : D^2 z \, dx - \int_T D^2 v : D^2 z \, dx + \int_T (\bar{f} - f) z \, dx \Big] \\
&= C \Big[\int_T (D^2 u - D^2 v) : D^2 z \, dx + \int_T (\bar{f} - f) z \, dx \Big] \\
&\leq C \big[|u - v|_{H^2(T)} |z|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)} \|z\|_{L_2(T)} \big]
\end{aligned}
$$

Local Efficiency Estimates

$$\begin{aligned}
\|\bar{f}-\Delta^2 v\|_{L_2(T)}^2 &\le C\int_T (\bar{f}-\Delta^2 v)^2\zeta\,dx \\
&= C\int_T (\bar{f}-\Delta^2 v)z\,dx \\
&= C\Big[\int_T f\,z\,dx - \int_T (\Delta^2 v)z\,dx + \int_T (\bar{f}-f)z\,dx\Big] \\
&= C\Big[\int_T D^2u : D^2z\,dx - \int_T D^2v : D^2z\,dx + \int_T (\bar{f}-f)z\,dx\Big] \\
&= C\Big[\int_T (D^2u - D^2v) : D^2z\,dx + \int_T (\bar{f}-f)z\,dx\Big] \\
&\le C\big[|u-v|_{H^2(T)}|z|_{H^2(T)} + \|f-\bar{f}\|_{L_2(T)}\|z\|_{L_2(T)}\big] \\
&\le C\big[h_T^{-2}|u-v|_{H^2(T)} + \|f-\bar{f}\|_{L_2(T)}\big]\,\|z\|_{L_2(T)}
\end{aligned}$$

$$|z|_{H^2(\Omega)} \le Ch_T^{-2}\|z\|_{L_2(T)}$$

Local Efficiency Estimates

$$
\begin{aligned}
\|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 &\leq C \int_T (\bar{f} - \Delta^2 v)^2 \zeta \, dx \\
&= C \int_T (\bar{f} - \Delta^2 v) z \, dx \\
&= C \Big[\int_T f z \, dx - \int_T (\Delta^2 v) z \, dx + \int_T (\bar{f} - f) z \, dx \Big] \\
&= C \Big[\int_T D^2 u : D^2 z \, dx - \int_T D^2 v : D^2 z \, dx + \int_T (\bar{f} - f) z \, dx \Big] \\
&= C \Big[\int_T (D^2 u - D^2 v) : D^2 z \, dx + \int_T (\bar{f} - f) z \, dx \Big] \\
&\leq C \big[|u - v|_{H^2(T)} |z|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)} \|z\|_{L_2(T)} \big] \\
&\leq C \big[h_T^{-2} |u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)} \big] \|z\|_{L_2(T)} \\
&\leq C \big[h_T^{-2} |u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)} \big] \|\bar{f} - \Delta^2 v\|_{L_2(T)}
\end{aligned}
$$

$$
C_1 \|z\|_{L_2(T)}^2 = C_1 \int_T (\bar{f} - \Delta^2 v)^2 \zeta^2 \, dx \leq \|\bar{f} - \Delta^2 v\|_{L_2(T)}^2
$$

Local Efficiency Estimates

$$
\begin{aligned}
\|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 &\leq C \int_T (\bar{f} - \Delta^2 v)^2 \zeta \, dx \\
&= C \int_T (\bar{f} - \Delta^2 v) z \, dx \\
&= C \Big[\int_T f z \, dx - \int_T (\Delta^2 v) z \, dx + \int_T (\bar{f} - f) z \, dx \Big] \\
&= C \Big[\int_T D^2 u : D^2 z \, dx - \int_T D^2 v : D^2 z \, dx + \int_T (\bar{f} - f) z \, dx \Big] \\
&= C \Big[\int_T (D^2 u - D^2 v) : D^2 z \, dx + \int_T (\bar{f} - f) z \, dx \Big] \\
&\leq C \big[|u - v|_{H^2(T)} |z|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)} \|z\|_{L_2(T)} \big] \\
&\leq C \big[h_T^{-2} |u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)} \big] \|z\|_{L_2(T)} \\
&\leq C \big[h_T^{-2} |u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)} \big] \|\bar{f} - \Delta^2 v\|_{L_2(T)}
\end{aligned}
$$

Local Efficiency Estimates

$$\|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 \leq C\big[h_T^{-2}|u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)}\big]\|\bar{f} - \Delta^2 v\|_{L_2(T)}$$

Local Efficiency Estimates

$$\|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 \leq C\big[h_T^{-2}|u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)}\big]\,\|\bar{f} - \Delta^2 v\|_{L_2(T)}$$

which implies

$$\|\bar{f} - \Delta^2 v\|_{L_2(T)} \leq C\big[h_T^{-2}|u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)}\big]$$

Local Efficiency Estimates

$$\|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 \leq C\big[h_T^{-2}|u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)}\big]\,\|\bar{f} - \Delta^2 v\|_{L_2(T)}$$

which implies

$$\|\bar{f} - \Delta^2 v\|_{L_2(T)} \leq C\big[h_T^{-2}|u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)}\big]$$

It follows that

$$\begin{aligned}\|f - \Delta^2 v\|_{L_2(T)} &\leq \|f - \bar{f}\|_{L_2(T)} + \|\bar{f} - \Delta^2 v\|_{L_2(T)} \\ &\leq C\big[h_T^{-2}|u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)}\big]\end{aligned}$$

Local Efficiency Estimates

$$\|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 \leq C\big[h_T^{-2}|u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)}\big]\,\|\bar{f} - \Delta^2 v\|_{L_2(T)}$$

which implies

$$\|\bar{f} - \Delta^2 v\|_{L_2(T)} \leq C\big[h_T^{-2}|u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)}\big]$$

It follows that

$$\begin{aligned}\|f - \Delta^2 v\|_{L_2(T)} &\leq \|f - \bar{f}\|_{L_2(T)} + \|\bar{f} - \Delta^2 v\|_{L_2(T)} \\ &\leq C\big[h_T^{-2}|u - v|_{H^2(T)} + \|f - \bar{f}\|_{L_2(T)}\big]\end{aligned}$$

and hence

$$h_T^2\|f - \Delta^2 v\|_{L_2(T)} \leq C\Big[|u - v|_{H^2(T)} + \underbrace{h_T^2\|f - \bar{f}\|_{L_2(T)}}_{\text{local oscillation}}\Big]$$

Local Efficiency Estimates

Estimate for $|e|^{\frac{1}{2}}\|[\![\partial^2 v/\partial n^2]\!]\|_{L_2(e)}$

$$|e|^{\frac{1}{2}}\|[\![\partial^2 v/\partial n^2]\!]\|_{L_2(e)} \le C\sum_{T\in\mathcal{T}_e}\Big[|u-v|_{H^2(T)} + \underbrace{h_T^2\|f-\bar{f}\|_{L_2(T)}}_{\text{local oscillation}}\Big]$$

Estimate for $|e|^{\frac{3}{2}}\|[\![\partial(\Delta v)/\partial n]\!]\|^2_{L_2(e)}$

$$|e|^{\frac{3}{2}}\|[\![\partial(\Delta v)/\partial n]\!]\|_{L_2(e)}$$
$$\le C\Big(\sum_{T\in\mathcal{T}_e}\Big[|u-v|_{H^2(T)} + \underbrace{h_T^2\|f-\bar{f}\|_{L_2(T)}}_{\text{local oscillation}}\Big] + |e|^{-\frac{1}{2}}\Big\|\Big[\!\!\Big[\frac{\partial(u-v)}{\partial n}\Big]\!\!\Big]\Big\|_{L_2(e)}\Big)$$

$\mathcal{T}_e$ is the set of the triangles that share e as a common edge.

Local Efficiency Estimates

Estimate for $|e|^{\frac{1}{2}}\|[\![\partial^2 v/\partial n^2]\!]\|_{L_2(e)}$

$$|e|^{\frac{1}{2}}\|[\![\partial^2 v/\partial n^2]\!]\|_{L_2(e)} \le C \sum_{T\in\mathcal{T}_e} \Big[|u-v|_{H^2(T)} + \underbrace{h_T^2\|f-\bar{f}\|_{L_2(T)}}_{\text{local oscillation}}\Big]$$

Estimate for $|e|^{\frac{3}{2}}\|[\![\partial(\Delta v)/\partial n]\!]\|^2_{L_2(e)}$

$$|e|^{\frac{3}{2}}\|[\![\partial(\Delta v)/\partial n]\!]\|_{L_2(e)}$$
$$\le C\Big(\sum_{T\in\mathcal{T}_e}\Big[|u-v|_{H^2(T)} + \underbrace{h_T^2\|f-\bar{f}\|_{L_2(T)}}_{\text{local oscillation}}\Big] + |e|^{-\frac{1}{2}}\Big\|\Big[\!\!\Big[\frac{\partial(u-v)}{\partial n}\Big]\!\!\Big]\Big\|_{L_2(e)}\Big)$$

Estimate for $h_T^2\|f-\Delta^2 v\|_{L_2(T)}$

$$h_T^2\|f-\Delta^2 v\|_{L_2(T)} \le C\Big[|u-v|_{H^2(T)} + \underbrace{h_T^2\|f-\bar{f}\|_{L_2(T)}}_{\text{local oscillation}}\Big]$$

Local Efficiency Estimates

Summing up the local efficiency estimates over all the triangles and edges we have

$$\sum_{e\in\mathcal{E}_h^i}|e|\Big\|\Big[\!\!\Big[\frac{\partial^2 v}{\partial n^2}\Big]\!\!\Big]\Big\|_{L_2(e)}^2+\sum_{e\in\mathcal{E}_h^i}|e|^3\Big\|\Big[\!\!\Big[\frac{\partial \Delta v}{\partial n}\Big]\!\!\Big]\Big\|_{L_2(e)}^2+\sum_{T\in\mathcal{T}_h}h_T^4\|f-\Delta^2 v\|_{L_2(T)}^2$$
$$\leq C\big[\|u-v\|_h^2+\mathrm{Osc}(f)^2\big]$$

which completes the proof of the Main Theorem.

Concrete Error Estimate

$$\begin{aligned}\|u-u_h\|_h &\leq C\Big[\inf_{v\in V_h}\|u-v\|_h+\mathrm{Osc}(f)\Big]\\ &\leq C\Big[\|u-\Pi_h u\|_h+\mathrm{Osc}(f)\Big]\\ &\leq Ch^{\alpha}\|f\|_{L_2(\Omega)}\end{aligned}$$

Concrete Error Estimate

$$\begin{aligned}\|u-u_h\|_h &\leq C\Big[\inf_{v\in V_h}\|u-v\|_h+\mathrm{Osc}(f)\Big]\\ &\leq C\Big[\|u-\Pi_h u\|_h+\mathrm{Osc}(f)\Big]\\ &\leq Ch^{\alpha}\|f\|_{L_2(\Omega)}\end{aligned}$$

Error Estimate in $\|\cdot\|_h$

Concrete Error Estimate

$$
\begin{aligned}
\|u-u_h\|_h &\leq C\Big[\inf_{v\in V_h}\|u-v\|_h + \mathrm{Osc}(f)\Big] \\
&\leq C\Big[\|u-\Pi_h u\|_h + \mathrm{Osc}(f)\Big] \\
&\leq Ch^{\alpha}\|f\|_{L_2(\Omega)}
\end{aligned}
$$

Error Estimate in $\|\!|\cdot|\!\|_h$

$$
\begin{aligned}
\|\!|u-u_h|\!\|_h &\leq \|\!|u-\Pi_h u|\!\|_h + \|\!|\Pi_h u-u_h|\!\|_h \\
&\leq C\Big[h^{\alpha}\|f\|_{L_2(\Omega)} + \|\Pi_h u-u_h\|_h\Big] \\
&\leq C\Big[h^{\alpha}\|f\|_{L_2(\Omega)} + \|\Pi_h u-u\|_h + \|u-u_h\|_h\Big] \\
&\leq Ch^{\alpha}\|f\|_{L_2(\Omega)}
\end{aligned}
$$

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A Posteriori Error Analysis

A Residual Based Error Estimator

A Residual Based Error Estimator

Let $T \in \mathcal{T}_h$ be arbitrary. The residual error

$$\eta_T = h_T^2 \|f - \Delta^2 u_h\|_{L_2(T)}$$

measures the extent to which u_h fails to satisfy the biharmonic equation.

A Residual Based Error Estimator

Let $T \in \mathcal{T}_h$ be arbitrary. The residual error

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measures the extent to which u_h fails to satisfy the biharmonic equation.

Let $e \in \mathcal{E}_h$ be arbitrary. The residual

$$\eta_{e,1} = \frac{\sigma}{|e|^{\frac{1}{2}}} \left\| \left[\!\!\left[\frac{\partial u_h}{\partial n} \right]\!\!\right] \right\|_{L_2(e)}$$

measures the extent to which u_h fails to be in $H_0^2(\Omega)$.

A Residual Based Error Estimator

The residual

$$\eta_{e,2} = |e|^{\frac{1}{2}} \left\| \left[\!\left[\frac{\partial^2 u_h}{\partial n^2} \right]\!\right] \right\|_{L_2(e)}$$

measures the extent that u_h fails to be in $H^3(\Omega)$.

A Residual Based Error Estimator

The residual

$$\eta_{e,2} = |e|^{\frac{1}{2}} \left\| \left[\!\left[\frac{\partial^2 u_h}{\partial n^2} \right]\!\right] \right\|_{L_2(e)}$$

measures the extent that u_h fails to be in $H^3(\Omega)$.

The residual

$$\eta_{e,3} = |e|^{\frac{3}{2}} \left\| \left[\!\left[\frac{\partial(\Delta u_h)}{\partial n_e} \right]\!\right] \right\|_{L_2(e)}$$

measures the extent that u_h fails to be in $H^4(\Omega)$.

A Residual Based Error Estimator

The residual-based error estimator η_h is defined by

$$\eta_h = \Big[\sum_{T \in \mathcal{T}_h} \eta_T^2 + \sum_{e \in \mathcal{E}_h} \eta_{e,1}^2 + \sum_{e \in \mathcal{E}_h^i} \left(\eta_{e,2}^2 + \eta_{e,3}^2 \right) \Big]^{\frac{1}{2}}$$

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Remark We can replace the definition of $\eta_{e,3}$ by

$$|e|^{\frac{3}{2}} [\![\partial^3 u_h / \partial n_e^3]\!] \|_{L_2(e)}$$

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The residual-based error estimator η_h is defined by

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Remark We can replace the definition of $\eta_{e,3}$ by

$$|e|^{\frac{3}{2}} [\![\partial^3 u_h / \partial n_e^3]\!] \|_{L_2(e)}$$

Remark The error estimator $\eta_{e,3}$ is identically 0 for the quadratic C^0 interior penalty method.

Local Efficiency

Local Efficiency

Adaptive Loop

Solve $\rightarrow$ Estimate $\rightarrow$ Mark $\rightarrow$ Refine

In the Estimate step we compute η_T, $\eta_{e,1}$, $\eta_{e,2}$ and $\eta_{e,3}$ over all the triangles and edges. It is important that if one of the residuals is large at a triangle or an edge, then the true error is also large there. Consequently we know that we should refine the mesh at such locations.

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This property is known as the local efficiency of the error estimator.

Local Efficiency

Local Efficiency of η_T

$$\eta_T = h_T^2 \|f - \Delta^2 u_h\|_{L_2(T)} \leq C\big[|u - u_h|_{H^2(T)} + h_T^2 \|f - \bar{f}\|_{L_2(T)}\big]$$

Local Efficiency

Local Efficiency of η_T

$$\eta_T = h_T^2 \|f - \Delta^2 u_h\|_{L_2(T)} \leq C\big[|u - u_h|_{H^2(T)} + h_T^2 \|f - \bar{f}\|_{L_2(T)}\big]$$

Local Efficiency of $\eta_{e,1}$

$$\eta_{e,1} = \frac{\sigma}{|e|^{\frac{1}{2}}} \Big\| \Big[\!\Big[\frac{\partial u_h}{\partial n} \Big]\!\Big] \Big\|_{L_2(e)} = \sqrt{\sigma} \Big(\frac{\sigma^{\frac{1}{2}}}{|e|^{\frac{1}{2}}} \Big\| \Big[\!\Big[\frac{\partial (u - u_h)}{\partial n} \Big]\!\Big] \Big\|_{L_2(e)} \Big)$$

Local Efficiency

Local Efficiency of η_T

$$\eta_T = h_T^2 \|f - \Delta^2 u_h\|_{L_2(T)} \leq C\big[|u - u_h|_{H^2(T)} + h_T^2 \|f - \bar{f}\|_{L_2(T)}\big]$$

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Local Efficiency of $\eta_{e,2}$

$$\eta_{e,2} = |e|^{\frac{1}{2}} \Big\| \Big[\!\Big[\frac{\partial^2 u_h}{\partial n^2} \Big]\!\Big] \Big\|_{L_2(e)} \leq C \sum_{T \in \mathcal{T}_e} \big[|u - u_h|_{H^2(T)} + h_T^2 \|f - \bar{f}\|_{L_2(T)}\big]$$

Local Efficiency

Local Efficiency of $\eta_{e,3}$

$$
\begin{aligned}
\eta_{e,3} &= |e|^{\frac{3}{2}} \Big\| \Big[\!\!\Big[\frac{\partial(\Delta u_h)}{\partial n} \Big]\!\!\Big] \Big\|_{L_2(e)} \\
&\leq C \Bigg(\sum_{T \in \mathcal{T}_e} \big[|u - u_h|_{H^2(T)} + h_T^2 \| f - \bar{f} \|_{L_2(T)} \big] \\
&\qquad + \Big(\frac{\sigma^{\frac{1}{2}}}{|e|^{\frac{1}{2}}} \Big) \Big\| \Big[\!\!\Big[\frac{\partial(u - u_h)}{\partial n} \Big]\!\!\Big] \Big\|_{L_2(e)} \Bigg)
\end{aligned}
$$

Local Efficiency

Local Efficiency of $\eta_{e,3}$

$$
\begin{aligned}
\eta_{e,3} &= |e|^{\frac{3}{2}} \Big\| \Big[\!\!\Big[\frac{\partial(\Delta u_h)}{\partial n} \Big]\!\!\Big] \Big\|_{L_2(e)} \\
&\leq C \Bigg(\sum_{T \in \mathcal{T}_e} \big[|u - u_h|_{H^2(T)} + h_T^2 \|f - \bar{f}\|_{L_2(T)} \big] \\
&\qquad\qquad + \Big(\frac{\sigma^{\frac{1}{2}}}{|e|^{\frac{1}{2}}} \Big) \Big\| \Big[\!\!\Big[\frac{\partial(u - u_h)}{\partial n} \Big]\!\!\Big] \Big\|_{L_2(e)} \Bigg)
\end{aligned}
$$

The local efficiency of η_T, $\eta_{e,2}$ and $\eta_{e,3}$ have already been established in the *medius* analysis.

The local efficiency of $\eta_{e,1}$ is trivial.

Reliability

Reliability

Adaptive Loop

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}$$

By refining the mesh at the locations where the residuals are large, we hope to reduce the error estimator and hence the true error. This requires the true error to be bounded by the error estimator.

Reliability

Adaptive Loop

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}$$

By refining the mesh at the locations where the residuals are large, we hope to reduce the error estimator and hence the true error. This requires the true error to be bounded by the error estimator.

The property that the true error is bounded by the error estimator is known as the reliability of the error estimator.

Reliability

We want to show that

$$\|u - u_h\|_h \leq C\eta_h$$

Reliability

We want to show that

$$\|u - u_h\|_h \leq C\eta_h$$

Recall

$$\|u - u_h\|_h^2 = \sum_{T \in \mathcal{T}_h} |u - u_h|_{H^2(T)}^2 + \sum_{e \in \mathcal{E}_h} \frac{\sigma}{|e|} \Big\| \Big[\!\Big[\frac{\partial (u - u_h)}{\partial n} \Big]\!\Big] \Big\|_{L_2(e)}^2$$

Reliability

We want to show that

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Recall

$$\|u-u_h\|_h^2 = \sum_{T\in\mathcal{T}_h} |u-u_h|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \Big\| \Big[\!\Big[\frac{\partial(u-u_h)}{\partial n} \Big]\!\Big] \Big\|_{L_2(e)}^2$$

Since

$$\sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \Big\| \Big[\!\Big[\frac{\partial(u-u_h)}{\partial n} \Big]\!\Big] \Big\|_{L_2(e)}^2 = \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \Big\| \Big[\!\Big[\frac{\partial u_h}{\partial n} \Big]\!\Big] \Big\|_{L_2(e)}^2 \leq \sum_{e\in\mathcal{E}_h} \eta_{e,1}^2$$

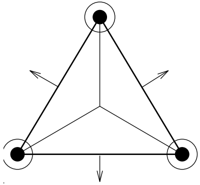
We only need to bound $\sum_{T\in\mathcal{T}_h} |u-u_h|_{H^2(T)}^2$.

Another Enriching Operator

We can use averaging to construct an enriching operator

$$E_h : V_h \longrightarrow W_h \,(\subset H_0^2(\Omega))$$

where W_h is the finite element space defined by the Hsieh-Clough-Tocher macro element.



C^1 piecewise cubic polynomials (12 dof)

Another Enriching Operator

Estimates for E_h

$$\sum_{T\in\mathcal{T}_h}\Big(h_T^{-4}\|v-E_hv\|_{L_2(T)}^2+h_T^{-2}|v-E_hv|_{H^1(T)}^2+|v-E_hv|_{H^2(T)}^2\Big)$$

$$\leq C\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\Big\|\Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big]\Big\|_{L_2(e)}^2$$

Another Enriching Operator

Estimates for E_h

$$\sum_{T\in\mathcal{T}_h}\Big(h_T^{-4}\|v-E_hv\|_{L_2(T)}^2+h_T^{-2}|v-E_hv|_{H^1(T)}^2+|v-E_hv|_{H^2(T)}^2\Big)$$

$$\leq C\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\Big\|\Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big]\Big\|_{L_2(e)}^2$$

Remark We can also use this enriching operator in the post-processing procedure.

Another Enriching Operator

Estimates for E_h

$$\sum_{T\in\mathcal{T}_h}\Big(h_T^{-4}\|v-E_hv\|_{L_2(T)}^2+h_T^{-2}|v-E_hv|_{H^1(T)}^2+|v-E_hv|_{H^2(T)}^2\Big)$$
$$\leq C\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\Big\|\Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big]\Big\|_{L_2(e)}^2$$

Remark We can also use this enriching operator in the post-processing procedure.

Remark Since we are not using relatives, the operator E_h is no longer one-to-one and hence not appropriate for the analysis of multigrid algorithms.

Estimate for $\sum_{T\in\mathcal{T}_h} |u - u_h|^2_{H^2(T)}$

Estimate for $\sum_{T\in\mathcal{T}_h} |u-u_h|^2_{H^2(T)}$

$$\sum_{T\in\mathcal{T}_h} |u-u_h|^2_{H^2(T)} \le 2\sum_{T\in\mathcal{T}_h} \left(|u-E_hu_h|^2_{H^2(T)} + |u_h-E_hu_h|^2_{H^2(T)}\right)$$
$$\le 2|u-E_hu_h|^2_{H^2(\Omega)} + C\sum_{e\in\mathcal{E}_h} \eta^2_{e,1},$$

$$\sum_{T\in\mathcal{T}_h} |v-E_hv|^2_{H^2(T)} \le C\sum_{e\in\mathcal{E}_h^i} \frac{1}{|e|}\Big\| \Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big] \Big\|^2_{L_2(e)}$$

Estimate for $\sum_{T\in\mathcal{T}_h} |u - u_h|^2_{H^2(T)}$

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} |u - u_h|^2_{H^2(T)} &\le 2 \sum_{T\in\mathcal{T}_h} \left(|u - E_h u_h|^2_{H^2(T)} + |u_h - E_h u_h|^2_{H^2(T)}\right) \\ &\le 2|u - E_h u_h|^2_{H^2(\Omega)} + C \sum_{e\in\mathcal{E}_h} \eta^2_{e,1},\end{aligned}$$

It only remains to estimate

$$|u - E_h u_h|_{H^2(\Omega)}$$

Estimate for $|u - E_h u_h|_{H^2(\Omega)}$

Estimate for $|u - E_h u_h|_{H^2(\Omega)}$

By duality

$$|u - E_h u_h|_{H^2(\Omega)} = \sup_{\phi \in H_0^2(\Omega)} \frac{a(u - E_h u_h, \phi)}{|\phi|_{H^2(\Omega)}}$$

Estimate for $|u - E_h u_h|_{H^2(\Omega)}$

By duality

$$|u - E_h u_h|_{H^2(\Omega)} = \sup_{\phi \in H_0^2(\Omega)} \frac{a(u - E_h u_h, \phi)}{|\phi|_{H^2(\Omega)}}$$

Strategy Show that

$$\sup_{\phi \in H_0^2(\Omega)} \frac{a(u - E_h u_h, \phi)}{|\phi|_{H^2(\Omega)}} \leq C\eta_h$$

Estimate for $|u - E_h u_h|_{H^2(\Omega)}$

By duality

$$|u - E_h u_h|_{H^2(\Omega)} = \sup_{\phi \in H_0^2(\Omega)} \frac{a(u - E_h u_h, \phi)}{|\phi|_{H^2(\Omega)}}$$

$$\begin{aligned}
a(u - E_h u_h, \phi) &= \int_\Omega D^2(u - E_h u_h) : D^2\phi \, dx \\
&= \sum_{T \in \mathcal{T}_h} \int_T D^2(u_h - E_h u_h) : D^2\phi \, dx - \sum_{T \in \mathcal{T}_h} \int_T D^2 u_h : D^2(\phi - \Pi_h \phi) \, dx \\
&\quad + \int_\Omega D^2 u : D^2\phi \, dx - \sum_{T \in \mathcal{T}_h} \int_T D^2 u_h : D^2(\Pi_h \phi) dx \\
&= \sum_{T \in \mathcal{T}_h} \int_T D^2(u_h - E_h u_h) : D^2\phi \, dx - \sum_{T \in \mathcal{T}_h} \int_T D^2 u_h : D^2(\phi - \Pi_h \phi) \, dx \\
&\quad + a_h(u_h, \Pi_h \phi) - \sum_{T \in \mathcal{T}_h} \int_T D^2 u_h : D^2(\Pi_h \phi) dx + \int_\Omega f(\phi - \Pi_h \phi) dx
\end{aligned}$$

Estimate for $|u - E_h u_h|_{H^2(\Omega)}$

Note that the integration by parts formula

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} \int_T D^2 w : D^2 v\, dx
&= \sum_{T\in\mathcal{T}_h} \int_T (\Delta^2 w) v\, dx + \sum_{e\in\mathcal{E}_h^i} \int_e \left[\!\left[\frac{\partial \Delta w}{\partial n} \right]\!\right] v\, ds \\
&\quad - \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 w}{\partial n^2} \right\}\!\!\right\} \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] ds - \sum_{e\in\mathcal{E}_h^i} \int_e \left[\!\left[\frac{\partial^2 w}{\partial n^2} \right]\!\right] \left\{\!\!\left\{ \frac{\partial v}{\partial n} \right\}\!\!\right\} ds \\
&\quad - \sum_{e\in\mathcal{E}_h^i} \int_e \left[\!\left[\frac{\partial^2 w}{\partial n \partial t} \right]\!\right] \frac{\partial v}{\partial t} ds
\end{aligned}$$

from *medius* analysis is also valid for $w \in V_h$ and $v \in H_0^2(\Omega)$.

Estimate for $|u - E_h u_h|_{H^2(\Omega)}$

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T D^2 u_h : D^2(\phi - \Pi_h\phi)\,dx \\
&\quad = \sum_{T\in\mathcal{T}_h}\int_T (\Delta^2 u_h)(\phi - \Pi_h\phi)dx + \sum_{e\in\mathcal{E}_h^i}\int_e \left[\!\left[\frac{\partial(\Delta u_h)}{\partial n}\right]\!\right](\phi - \Pi_h\phi)ds \\
&\qquad + \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 u_h}{\partial n^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial \Pi_h\phi}{\partial n}\right]\!\right]ds - \sum_{e\in\mathcal{E}_h^i}\int_e \left[\!\left[\frac{\partial^2 u_h}{\partial n^2}\right]\!\right]\left\{\!\!\left\{\frac{\partial(\phi - \Pi_h\phi)}{\partial n}\right\}\!\!\right\}ds \\
&\qquad\quad - \sum_{e\in\mathcal{E}_h^i}\int_e \left[\!\left[\frac{\partial^2 u_h}{\partial n\partial t}\right]\!\right]\frac{\partial(\phi - \Pi_h\phi)}{\partial t}ds
\end{aligned}
$$

Estimate for $|u - E_h u_h|_{H^2(\Omega)}$

$$
\begin{aligned}
a_h(u_h, \Pi_h\phi) &- \sum_{T\in\mathcal{T}_h} \int_T D^2 u_h : D^2(\Pi_h\phi)\,dx \\
&= \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 u_h}{\partial n^2} \right\}\!\!\right\} \left[\!\left[\frac{\partial \Pi_h\phi}{\partial n} \right]\!\right] ds + \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 \Pi_h\phi}{\partial n^2} \right\}\!\!\right\} \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right] ds \\
&\quad + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \int_e \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right] \left[\!\left[\frac{\partial \Pi_h\phi}{\partial n} \right]\!\right] ds.
\end{aligned}
$$

Estimate for $|u - E_h u_h|_{H^2(\Omega)}$

$$
\begin{aligned}
a(u - E_h u_h, \phi) \\
&= \sum_{T\in\mathcal{T}_h} \int_T D^2(u_h - E_h u_h) : D^2\phi \, dx + \sum_{T\in\mathcal{T}_h} \int_T (f - \Delta^2 u_h)(\phi - \Pi_h \phi) dx \\
&\quad - \sum_{e\in\mathcal{E}_h^i} \int_e \left[\!\left[\frac{\partial(\Delta u_h)}{\partial n} \right]\!\right] (\phi - \Pi_h\phi) ds \\
&\quad + \sum_{e\in\mathcal{E}_h^i} \int_e \left[\!\left[\frac{\partial^2 u_h}{\partial n^2} \right]\!\right] \left\{\!\!\left\{ \frac{\partial(\phi - \Pi_h\phi)}{\partial n} \right\}\!\!\right\} ds \\
&\quad + \sum_{e\in\mathcal{E}_h^i} \int_e \left[\!\left[\frac{\partial^2 u_h}{\partial n \partial t} \right]\!\right] \frac{\partial(\phi - \Pi_h\phi)}{\partial t} ds + \sum_{e\in\mathcal{E}_h} \int_e \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right] \left\{\!\!\left\{ \frac{\partial^2 \Pi_h\phi}{\partial n^2} \right\}\!\!\right\} ds \\
&\quad + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \int_e \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right] \left[\!\left[\frac{\partial \Pi_h\phi}{\partial n} \right]\!\right] ds
\end{aligned}
$$

Estimate for $|u - E_h u_h|_{H^2(\Omega)}$

$$a(u - E_h u_h) \leq C\eta_h |\phi|_{H^2(\Omega)}$$

which implies

$$\sup_{\phi \in H_0^2(\Omega)} \frac{a(u - E_h u_h, \phi)}{|\phi|_{H^2(\Omega)}} \leq C\eta_h$$

and completes the proof of reliability.

Numerical Results

We have implemented an adaptive algorithm

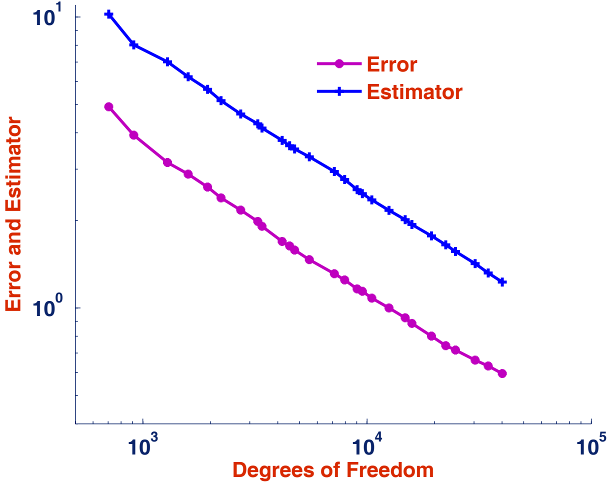
- Solve: compute u_h
- Estimate: compute η_h
- Mark: mark the triangles to be refined
- Refine

using the error estimator η_h and the bulk marking strategy of Dörfler, and tested it on an L-shaped domain using an exact solution with the correct singularity.

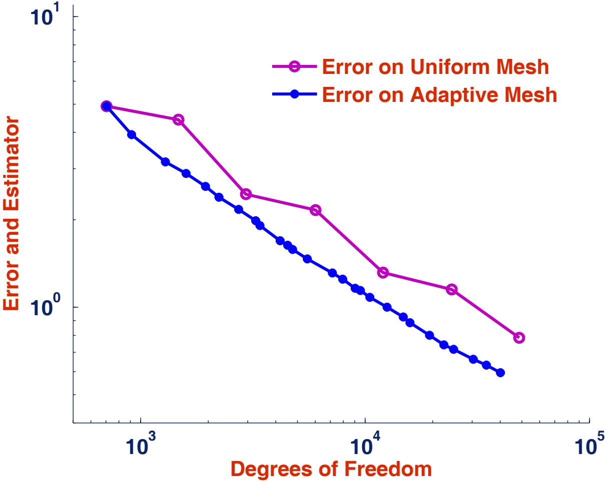
bulk marking strategy

- ♣ choose a number θ between 0 and 1
- ♣ mark enough triangles so that the residuals associated with them and their edges add up to more than $\theta\eta_h$

Numerical Results



Numerical Results



Open Problem

Prove the convergence and optimality of the adaptive algorithm.

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