C^0 Interior Penalty Methods

Introduction

Current Research in Finite Element Methods

CIMPA Summer School

Mumbai, July 2015

Overview

Overview

- Introduction
  - Motivations
  - Derivations

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- Convergence Analysis
  - Standard *A Priori* Error Analysis
  - *Medius* Error Analysis
  - *A Posteriori* Error Analysis

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- Fast Solver
  - Multigrid

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  - *A Posteriori* Error Analysis
- Fast Solver
  - Multigrid
- Variational Inequalities

General References

- B. and Scott, The Mathematical Theory of Finite Element Methods (Third Edition), Springer-Verlag, 2008.

  (Springer International Edition 2012)

- B., C^0 Interior Penalty Methods, Frontiers in Numerical Analysis-Durham 2010, Springer-Verlag, 2012.

Outline of Lecture

- Examples of Fourth Order Problems
- Classical Finite Element Methods
- C^0 Interior Penalty Methods
- Enriching Operators
- Extensions

Examples of Fourth Order Problems

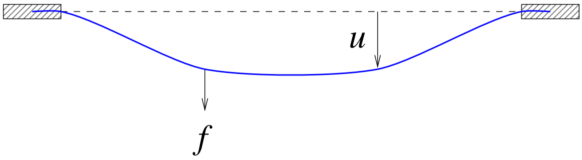
Bernoulli Beam (clamped)

$$\frac{d^2}{dx^2}\left(EI\frac{d^2u}{dx^2}\right) = f \qquad a < x < b$$

$$u(a) = u'(a) = 0 = u(b) = u'(b)$$

$u =$ vertical displacement, $f =$ vertical load density

$E =$ Young's modulus, $I =$ moment of inertia



Kirchhoff Plate (clamped)

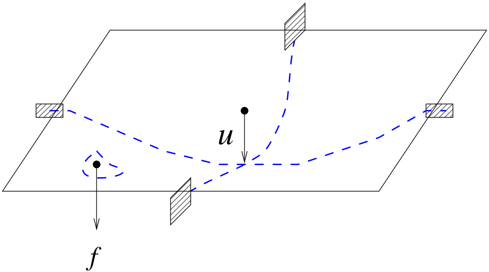
$$\Delta\Big(\frac{2Et^3}{3(1-\sigma^2)}\Delta u\Big) = f \qquad \text{in} \quad \Omega$$

$$u = \partial u/\partial n = 0 \qquad \text{on} \quad \partial\Omega$$

u = vertical displacement, f = vertical load density

E = Young's modulus, σ = Poisson ratio

t = thickness of plate



Kirchhoff Plate (simply supported)

$$\Delta\left(\frac{2Et^3}{3(1-\sigma^2)}\Delta u\right) = f \qquad \text{in} \quad \Omega$$

$$u = \Delta u = 0 \qquad \text{on} \quad \partial\Omega$$

Clamped Plate

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on} \quad \partial\Omega$$

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Clamped Plate

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on} \quad \partial\Omega$$

- Both boundary conditions for the clamped plate are essential boundary conditions, i.e., they are imposed on the space for the variational (or weak) formulation of the problem.

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Clamped Plate

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on} \quad \partial\Omega$$

- Both boundary conditions for the clamped plate are essential boundary conditions, i.e., they are imposed on the space for the variational (or weak) formulation of the problem.
- The condition $\Delta u = 0$ for the simply supported plate is a natural boundary condition, i.e., it is implied by the variational (or weak) formulation of the problem (for sufficiently smooth solutions).

Stokes Problem (2D)

$$
\begin{aligned}
-\nu \Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \qquad \text{in} \quad \Omega \\
\nabla \cdot \boldsymbol{u} &= 0 \qquad \text{in} \quad \Omega \\
\boldsymbol{u} &= \boldsymbol{0} \qquad \text{on} \quad \partial\Omega
\end{aligned}
$$

$\Omega =$ a simply connected domain

$\boldsymbol{u} =$ fluid velocity, $\quad p =$ pressure

$\boldsymbol{f} =$ force density, $\quad \nu =$ viscosity

Stokes Problem (2D)

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Stream Function ψ

$$\nabla \times \psi = \boldsymbol{u}$$

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\end{aligned}
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Stream Function ψ

$$\nabla \times \psi = \boldsymbol{u}$$

Boundary value problem for ψ (after simplification)

$$
\begin{aligned}
\nu \Delta^2 \psi &= \nabla \times \boldsymbol{f} && \text{in} \quad \Omega \\
\psi = \partial\psi/\partial n &= 0 && \text{on} \quad \partial\Omega
\end{aligned}
$$

Strain Gradient Elasticity

$$
\begin{aligned}
-\text{div}\,\boldsymbol{\sigma} &= \mathbf{f} \qquad \text{in} \quad \Omega \subset \mathbb{R}^d \; (d = 2, 3) \\
\boldsymbol{\sigma} &= 2\mu\, \boldsymbol{\epsilon}_*(\boldsymbol{u}) + \lambda\, [\text{tr}\, \boldsymbol{\epsilon}_*(\boldsymbol{u})]\mathbf{I} \\
\boldsymbol{\epsilon}_*(\boldsymbol{u}) &= (1 - \gamma^2 \Delta)\boldsymbol{\epsilon}(\boldsymbol{u}) \\
\boldsymbol{\epsilon}(\boldsymbol{u}) &= \frac{1}{2}(\nabla + \nabla^{\mathrm{T}})\boldsymbol{u}
\end{aligned}
$$

$\boldsymbol{u}$ = displacement of an elastic material

$\boldsymbol{\sigma}$ = stress, $\boldsymbol{f}$ = force density

μ and λ are the Lamé constants.

$\boldsymbol{\epsilon}(\boldsymbol{u})$ = standard strain, $\boldsymbol{\epsilon}_*(\boldsymbol{u})$ = modified strain

γ is a parameter.

Strain Gradient Elasticity

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$$

- $\gamma = 0$: standard elasticity system $(\boldsymbol{\epsilon}_*(\boldsymbol{u}) = \boldsymbol{\epsilon}(\boldsymbol{u}))$
- $\gamma \neq 0$: fourth order system

Strain Gradient Elasticity

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- $\gamma = 0$: standard elasticity system $(\boldsymbol{\epsilon}_*(\boldsymbol{u}) = \boldsymbol{\epsilon}(\boldsymbol{u}))$
- $\gamma \neq 0$: fourth order system

- Strain gradient elasticity is a phenomenological theory for capturing the scale effect and localization due to non-homogeneity at the microscopic level.

Cahn-Hilliard Equation

$$\frac{\partial c}{\partial t} = \nabla \cdot \beta(c)\nabla(\mu(c) - \Delta c) \qquad \text{in} \quad \Omega \times (0,T)$$

plus initial and boundary conditions

$\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$)

$c(x,t) =$ concentration of one of the two substances being tracked ($0 \leq c \leq 1$)

$\beta(c) =$ mobility

$\mu(c) =$ derivative of the free energy

Cahn-Hilliard Equation

$$\frac{\partial c}{\partial t} = \nabla \cdot \beta(c) \nabla (\mu(c) - \Delta c) \qquad \text{in} \quad \Omega \times (0, T)$$

- phase segregation of binary alloys
- two-phase fluid flow
- image processing
- self-assembly of nanovoids
- planet formation

Cahn-Hilliard Equation

$$\frac{\partial c}{\partial t} = \nabla \cdot \beta(c)\nabla(\mu(c) - \Delta c) \qquad \text{in} \quad \Omega \times (0, T)$$

This can be solved numerically by an implicit time discretization combined with the Newton-Raphson scheme for a fourth order nonlinear problem at each time step.

An Obstacle Problem

$\Omega =$ bounded polygon $\quad f \in L_2(\Omega) \quad$ (external force)

$\psi \in C^2(\bar{\Omega})$, $\quad \psi < 0$ on $\partial\Omega \quad$ (obstacle function)

$K = \{v \in H_0^2(\Omega) : v \geq \psi \text{ on } \Omega\}$

($v \in H_0^2(\Omega)$ if and only if $\partial^\alpha u/\partial x^\alpha \in L_2(\Omega)$ for $|\alpha| \leq 2$ and $v = \partial v/\partial n = 0$ on $\partial\Omega$.)

An Obstacle Problem

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$K = \{v \in H_0^2(\Omega) : v \geq \psi \text{ on } \Omega\}$

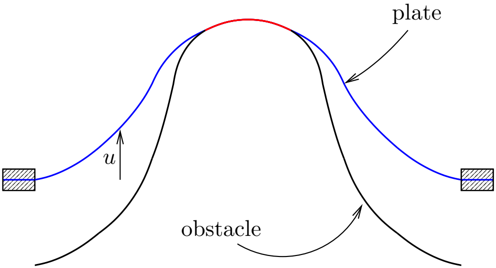
Find $u = \underset{v \in K}{\operatorname{argmin}} \left[\frac{1}{2}a(v,v) - (f,v)\right]$

$$a(w,v) = \int_\Omega D^2 w : D^2 v \, dx \qquad D^2 w : D^2 v = \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}$$

$$(f,v) = \int_\Omega f v \, dx$$

(bending of a clamped Kirchhoff plate over an obstacle)

An Obstacle Problem

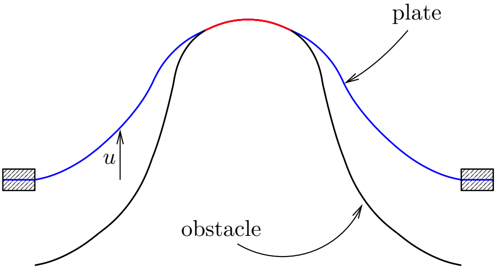


u is the vertical displacement of the midsurface of the thin plate.

$$\frac{1}{2}a(v,v) - (f,v)$$

is the energy of the plate determined by the displacement v.

An Obstacle Problem

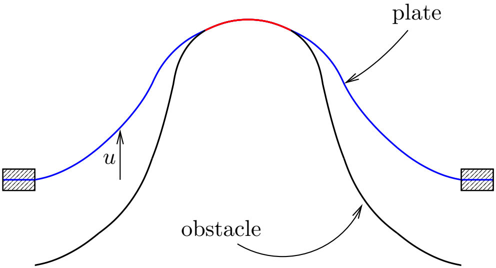


u is the vertical displacement of the midsurface of the thin plate.

$$u = \underset{v \in K}{\operatorname{argmin}} \left[\frac{1}{2} a(v, v) - (f, v) \right]$$

The obstacle problem is to find the plate that has minimum energy among all admissible plates.

An Obstacle Problem



u is the vertical displacement of the midsurface of the thin plate.

Variational Inequality

$$a(u, v - u) \geq (f, v - u) \qquad \forall\, v \in K$$

An Optimal Control Problem with State Constraint

minimize $\quad \dfrac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \dfrac{\beta}{2}\|u\|^2_{L_2(\Omega)}$

over $\quad (y,u) \in H^1_0(\Omega) \times L_2(\Omega)$

subject to
$$\begin{cases} -\Delta y = u & \text{in } \Omega \\ \quad\;\; y \le \psi & \text{a.e. in } \Omega \end{cases}$$

$y \in H^1_0(\Omega)$ is the state (temperature distribution).

y_d is the desired state.

$y \le \psi$ is a pointwise constraint on the state.

$u \in L_2(\Omega)$ is the control (heat source).

$\beta > 0$ is related to the cost for implementing the control u.

An Optimal Control Problem with State Constraint

$$\text{minimize} \qquad \frac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|u\|^2_{L_2(\Omega)}$$

$$\text{over} \qquad (y, u) \in H^1_0(\Omega) \times L_2(\Omega)$$

$$\text{subject to} \qquad \begin{cases} -\Delta y & = & u & \text{in } \Omega \\ \quad y & \leq & \psi & \text{a.e. in } \Omega \end{cases}$$

If Ω is convex or smooth, then y belongs to $H^2(\Omega)$ by elliptic regularity and we can rewrite the problem as

$$\text{Find} \quad y = \underset{v \in K}{\operatorname{argmin}} \frac{1}{2}\left[\|y - y_d\|^2_{L_2(\Omega)} + \beta\|\Delta y\|^2_{L_2(\Omega)}\right]$$

$$K = \{v \in H^2(\Omega) \cap H^1_0(\Omega) : \ v \leq \psi \quad \text{in } \Omega\}$$

(a fourth order obstacle problem with the boundary conditions of a simply supported plate)

Numerical Solutions for 4th Order Problems

Two main difficulties

- We need to discretize the fourth order problems so that the solutions of the discrete problems are good approximations of the solutions of the continuous problem.

  Such schemes are usually complicated.

- We want to solve the discrete problem accurately and efficiently.

  This is difficult because the discrete system is very ill-conditioned.

Classical Finite Element Methods

Conforming Methods

Conforming Methods

Since the continuous problems are posed on the Sobolev space $H^2(\Omega)$, the finite element spaces of conforming methods must be subspaces of $H^2(\Omega)$, i.e., they are C^1 finite element spaces.

The advantage of conforming methods is that they are always convergent by Galerkin orthogonality. The disadvantage is that they are complicated in 2D (and more so in 3D).

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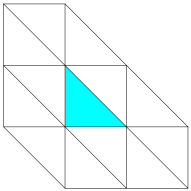
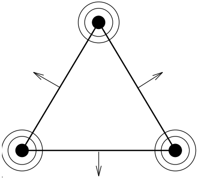


- C^1 continuity imposes many conditions on the vertices and the edges of an element.
- Need many dofs in order to satisfy all these conditions.

Conforming Methods

Argyris (TUBA) triangular elements

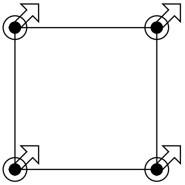


$$\mathbb{P}_5 = \langle x_1^m x_2^n : m + n \le 5 \rangle \qquad \text{(21 dof)}$$

- ● value of the function ↑ value of the normal derivative
- ○ values of the first order derivatives
- ◯ values of the second order derivatives

Conforming Methods

Bogner-Fox-Schmit rectangular elements

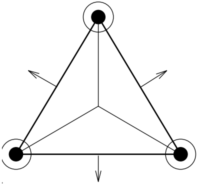


$$\mathbb{Q}_3 = \langle x_1^m x_j^n : m, n \leq 3 \rangle \qquad \text{(16 dof)}$$

- ● value of the function
- ○ values of the first order derivatives
- ⇗ value of the mixed second order derivative

Conforming Methods

Macro elements



C^1 piecewise cubic polynomials (12 dof)

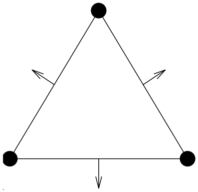
Nonconforming Methods

Nonconforming Methods

Nonconforming finite element methods were invented because C^1 finite element methods are too complicated. Nonconforming methods are simpler since we only require the finite element functions and their derivatives to satisfy some weak continuity conditions.

Nonconforming Methods

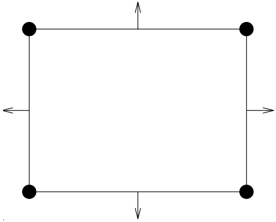
Morley element



$\mathbb{P}_2$ (6 dof)

Nonconforming Methods

Incomplete biquadratic element



$\mathbb{P}_2 + \langle x_1^2 x_2, x_1 x_2^2 \rangle$ (8 dof)

Nonconforming Methods

- It takes a lot of ingenuity to construct nonconforming finite elements that work (especially for more complicated fourth order problems).
- They are only low order elements (no natural hierarchy), which are not efficient for smooth solutions.
- Very little is known about 3D nonconforming elements for fourth order problems.

Mixed Finite Element Methods

Mixed Finite Element Methods

Find $(\omega, u) \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega \omega\mu\,dx - \int_\Omega \nabla\mu\cdot\nabla u\,dx &= 0 && \forall\,\mu \in H^1(\Omega) \\ \int_\Omega \nabla\omega\cdot\nabla v\,dx &= \int_\Omega f\,v\,dx && \forall\,v \in H_0^1(\Omega) \end{aligned}$$

In this mixed formulation, the biharmonic problem with the boundary conditions of clamped plates is split into two second order problems, we only need to use finite element spaces that are subspaces of $H^1(\Omega)$, i.e., C^0 elements.

Mixed Finite Element Methods

- It is not easy to come up with correct mixed formulations for more complicated fourth order problems.

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- In the mixed formulation we use a finite element space for the unknown ω and a finite element space for the unknown u. The mixed method only works if the finite element pair satisfies the Ladyzhenskaya-Babuška-Brezzi condition.

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  It is not easy to find such finite element pairs!

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- For the biharmonic problem with boundary conditions of simply supported plates, some mixed methods can miss the leading singularity and produce a wrong solution (Sapondzhyan paradox).

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- In the mixed formulation we use a finite element space for the unknown ω and a finite element space for the unknown u. The mixed method only works if the finite element pair satisfies the Ladyzhenskaya-Babuška-Brezzi condition.
- For the biharmonic problem with boundary conditions of simply supported plates, some mixed methods can miss the leading singularity and produce a wrong solution (Sapondzhyan paradox).
- At the end one still needs to solve a saddle point problem, which is more complicated than solving an SPD problem.

C^0 Interior Penalty Methods

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- We can compute smooth solutions using higher order elements in this family which are as efficient as higher order C^1 elements and much simpler (in 2D and 3D).

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- We can compute smooth solutions using higher order elements in this family which are as efficient as higher order C^1 elements and much simpler (in 2D and 3D).
- Unlike mixed methods, this approach can be extended in a straight-forward way to more complicated fourth order problems (such as the fourth order elliptic systems that appear in strain-gradient elasticity problems).

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- We can compute smooth solutions using higher order elements in this family which are as efficient as higher order C^1 elements and much simpler (in 2D and 3D).

- Unlike mixed methods, this approach can be extended in a straight-forward way to more complicated fourth order problems (such as the fourth order elliptic systems that appear in strain-gradient elasticity problems).

- The C^0 interior penalty methods belong to the class of discontinuous Galerkin methods (where the discontinuity involves the first order derivatives).

C^0 Interior Penalty Methods

Biharmonic Equation

$$\Delta^2 u = f \qquad \text{in } \Omega$$

with different boundary conditions

$\Omega =$ bounded polygonal domain in $\mathbb{R}^2$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

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Boundary Conditions of Clamped Plates

$$u = 0 \qquad \text{on } \partial\Omega \qquad \text{essential}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega \qquad \text{essential}$$

Biharmonic Equation

$$\Delta^2 u = f \qquad \text{in } \Omega$$

with different boundary conditions

$\Omega =$ bounded polygonal domain in $\mathbb{R}^2$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

Boundary Conditions of Simply Supported Plates

$$u = 0 \qquad \text{on } \partial\Omega \qquad \text{essential}$$

$$\Delta u = 0 \qquad \text{on } \partial\Omega \qquad \text{natural}$$

Biharmonic Equation

$$\Delta^2 u = f \qquad \text{in } \Omega$$

with different boundary conditions

$\Omega =$ bounded polygonal domain in $\mathbb{R}^2$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

Boundary Conditions of the Cahn-Hilliard Type

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega \qquad \text{essential}$$

$$\frac{\partial(\Delta u)}{\partial n} = 0 \qquad \text{on } \partial\Omega \qquad \text{natural}$$

Boundary Conditions of Clamped Plates

$\Omega =$ bounded polygonal domain $\qquad f \in L_2(\Omega)$

$$\Delta^2 u = f \qquad \text{in } \Omega$$

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega$$

Boundary Conditions of Clamped Plates

$\Omega =$ bounded polygonal domain $\qquad f \in L_2(\Omega)$

$$\Delta^2 u = f \qquad \text{in } \Omega$$

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega$$

Variational/Weak Problem Find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = \int_\Omega f\,v\,dx \qquad \forall\, v \in H_0^2(\Omega)$$

$$a(w,v) = \int_\Omega D^2 w : D^2 v\,dx \qquad D^2 w : D^2 v = \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}$$

($u = 0$ and $\partial u/\partial n = 0$ are both essential boundary conditions that are imposed on the solution space $H_0^2(\Omega)$.)

Boundary Conditions of Clamped Plates

$\Omega =$ bounded polygonal domain $\qquad f \in L_2(\Omega)$

$$\Delta^2 u = f \qquad \text{in } \Omega$$

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega$$

Variational/Weak Problem $\qquad$ Find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = \int_\Omega f v \, dx \qquad \forall\, v \in H_0^2(\Omega)$$

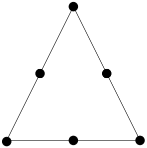
$$a(w,v) = \int_\Omega D^2 w : D^2 v \, dx \qquad D^2 w : D^2 v = \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}$$

Since $a(\cdot,\cdot)$ is bounded and coercive on $H_0^2(\Omega)$, the vari-aitonal/weak problem has a unique solution.

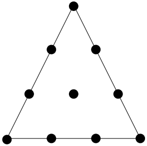
Finite Element Spaces

$\mathcal{T}_h =$ a simplicial triangulation of Ω

V_h $(\subset H^1_0(\Omega)) = \mathbb{P}_k$ $(k \geq 2)$ Lagrange finite element space



k=2

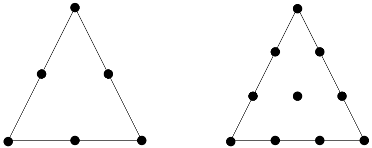


k=3

Finite Element Spaces

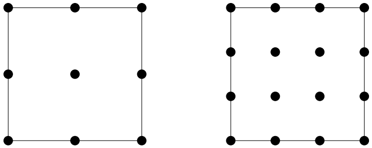
$\mathcal{T}_h$ = a simplicial triangulation of Ω

$V_h\ (\subset H_0^1(\Omega)) = \mathbb{P}_k$ ($k \geq 2$) Lagrange finite element space



k=2 k=3

We can also use $\mathbb{Q}_k$ Lagrange finite element spaces.

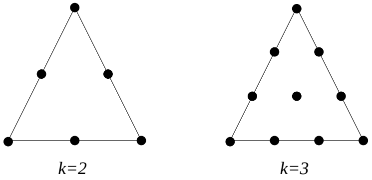


k=2 k=3

Finite Element Spaces

$\mathcal{T}_h =$ a simplicial triangulation of Ω

$V_h \ (\subset H_0^1(\Omega)) = \mathbb{P}_k$ ($k \geq 2$) Lagrange finite element space



These are standard C^0 finite element spaces for second order problems.

Discrete Problem

Discrete Problem

Key Observation The solution u of the continuous problem

$$a(u,v) = \int_\Omega f\,v\,dx \qquad \forall\, v \in H_0^2(\Omega)$$

$$a(w,v) = \int_\Omega D^2 w : D^2 v\,dx$$

$$f \in L_2(\Omega)$$

also satisfies a mesh-dependent problem

$$a_h(u,v) = \int_\Omega f\,v\,dx \qquad \forall\, v \in V_h$$

obtained by

- integration by parts
- symmetrization
- stabilization

An Integration by Parts Formula

$$\int_T (\Delta w) z \, dx = \int_{\partial T} \frac{\partial w}{\partial n} z \, ds - \int_T \nabla w \cdot \nabla z \, dx$$

An Integration by Parts Formula

$$\int_T (\Delta w) z \, dx = \int_{\partial T} \frac{\partial w}{\partial n} z \, ds - \int_T \nabla w \cdot \nabla z \, dx$$

$$\int_T (\Delta^2 u) v \, dx = \int_{\partial T} \frac{\partial (\Delta u)}{\partial n} \, v \, ds - \int_T \nabla(\Delta u) \cdot \nabla v \, dx$$

$w = \Delta u$ and $z = v$

An Integration by Parts Formula

$$\int_T (\Delta w) z \, dx = \int_{\partial T} \frac{\partial w}{\partial n} z \, ds - \int_T \nabla w \cdot \nabla z \, dx$$

$$\begin{aligned}\int_T (\Delta^2 u) v \, dx &= \int_{\partial T} \frac{\partial (\Delta u)}{\partial n} \, v \, ds - \int_T \nabla (\Delta u) \cdot \nabla v \, dx \\ &= \int_{\partial T} \frac{\partial (\Delta u)}{\partial n} \, v \, ds - \int_T \Delta (\nabla u) \cdot \nabla v \, dx\end{aligned}$$

An Integration by Parts Formula

$$\int_T (\Delta w) z \, dx = \int_{\partial T} \frac{\partial w}{\partial n} z \, ds - \int_T \nabla w \cdot \nabla z \, dx$$

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\int_T (\Delta^2 u) v \, dx &= \int_{\partial T} \frac{\partial (\Delta u)}{\partial n} \, v \, ds - \int_T \nabla(\Delta u) \cdot \nabla v \, dx \\
&= \int_{\partial T} \frac{\partial (\Delta u)}{\partial n} \, v \, ds - \int_T \Delta(\nabla u) \cdot \nabla v \, dx \\
&= \int_{\partial T} \frac{\partial (\Delta u)}{\partial n} \, v \, ds - \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u \right) \cdot \nabla v \, ds \\
&\quad + \int_T \nabla(\nabla u) : \nabla(\nabla v) \, dx
\end{aligned}$$

$w = \nabla u$ and $z = \nabla v$

An Integration by Parts Formula

$$\int_T (\Delta w) z \, dx = \int_{\partial T} \frac{\partial w}{\partial n} z \, ds - \int_T \nabla w \cdot \nabla z \, dx$$

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&= \int_{\partial T} \frac{\partial (\Delta u)}{\partial n} v \, ds - \int_{\partial T} \Big(\frac{\partial}{\partial n} \nabla u \Big) \cdot \nabla v \, ds \\
&\qquad + \int_T \nabla (\nabla u) : \nabla (\nabla v) \, dx
\end{aligned}$$

$$\Delta^2 u = f \qquad \nabla(\nabla u) = D^2 u \qquad \nabla(\nabla v) = D^2 v$$

An Integration by Parts Formula

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$$\int_T f v \, dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v \, ds - \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u\right) \cdot \nabla v \, ds + \int_T D^2 u : D^2 v \, dx$$

Notation

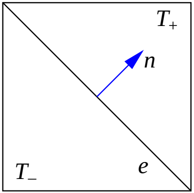
Let v be a piecewise H^2 function.

$e =$ interior edge shared by $T_\pm$

$$\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] = \left(\frac{\partial v_+}{\partial n}\right) - \left(\frac{\partial v_-}{\partial n}\right)$$

The unit normal n points from T_- to T_+.

$v_\pm = v\big|_{T_\pm}$



The definition of $[\![\partial v/\partial n]\!]$ is independent of the choice of $T_\pm$.

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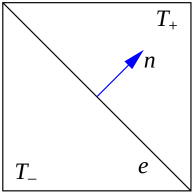
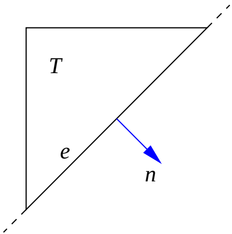


$e =$ boundary edge, $e \subset \partial T$

$$\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] = -\frac{\partial v_T}{\partial n}$$

The unit normal n points outside Ω.

$v_T = v\big|_T$

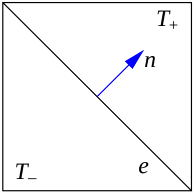


Notation

Let v be a piecewise H^s function for some $s > \frac{5}{2}$.

$e =$ interior edge shared by $T_\pm$

$$\left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} = \frac{1}{2}\left[\left(\frac{\partial^2 v_+}{\partial n^2}\right) + \left(\frac{\partial^2 v_-}{\partial n^2}\right)\right]$$



The unit normal n points from T_- to T_+.

$v_\pm = v\big|_{T_\pm}$

The definition of $\{\!\{\partial^2 v/\partial n^2\}\!\}$ is independent of the choice of $T_\pm$.

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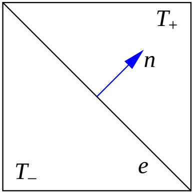
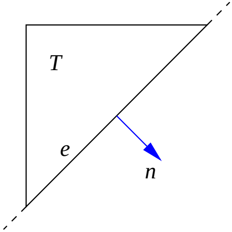


The unit normal n points from T_- to T_+.

$v_\pm = v\big|_{T_\pm}$

$e =$ boundary edge, $e \subset \partial T$

$$\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} = \frac{\partial^2 v_T}{\partial n^2}$$



The unit normal n points outside Ω.

$v_T = v\big|_T$

A Mesh-Dependent Problem for u

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_T f v\,dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n}\,v\,ds - \int_{\partial T} \left(\frac{\partial}{\partial n}\nabla u\right)\cdot\nabla v\,ds + \int_T D^2 u : D^2 v\,dx$$

A Mesh-Dependent Problem for u

$$\int_T f v \, dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v \, ds - \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u \right) \cdot \nabla v \, ds + \int_T D^2 u : D^2 v \, dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v \, ds = 0$$

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Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial(\Delta u)}{\partial n}\,v\,ds = 0$$

Each interior edge is shared by two triangles. Since v and the derivatives of u are continuous and the normals from the two triangles are pointing at opposite directions, the contributions from the two triangles cancel.

There is also no contribution from the boundary edges because $v = 0$ on $\partial\Omega$.

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_T f v\,dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n}\, v\, ds - \int_{\partial T} \left(\frac{\partial}{\partial n}\nabla u\right)\cdot \nabla v\, ds + \int_T D^2 u : D^2 v\, dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial(\Delta u)}{\partial n}\, v\, ds = 0$$

$$-\sum_{T\in\mathcal{T}_h} \int_{\partial T} \left(\frac{\partial}{\partial n}\nabla u\right)\cdot \nabla v\, ds = \sum_{e\in\mathcal{E}_h} \int_e \left(\frac{\partial^2 u}{\partial n^2}\right) \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds$$

$\mathcal{E}_h =$ the set of all the edges in $\mathcal{T}_h$

A Mesh-Dependent Problem for u

$$\int_T f v\,dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v\,ds - \int_{\partial T} \Big(\frac{\partial}{\partial n}\nabla u\Big)\cdot\nabla v\,ds + \int_T D^2 u : D^2 v\,dx$$

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$$\int_\Omega f v\,dx = \sum_{T\in\mathcal{T}_h} \int_T D^2 u : D^2 v\,dx + \sum_{e\in\mathcal{E}_h} \int_e \Big(\frac{\partial^2 u}{\partial n^2}\Big) \Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big] ds$$

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_\Omega f v\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \left(\frac{\partial^2 u}{\partial n^2}\right)\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds$$

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$
\begin{aligned}
\int_\Omega f v \, dx &= \sum_{T\in\mathcal{T}_h} \int_T D^2 u : D^2 v \, dx + \sum_{e\in\mathcal{E}_h} \int_e \left(\frac{\partial^2 u}{\partial n^2}\right) \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds \\
&= \sum_{T\in\mathcal{T}_h} \int_T D^2 u : D^2 v \, dx + \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{\frac{\partial^2 u}{\partial n^2}\right\}\!\!\right\} \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds
\end{aligned}
$$

Assuming that u is sufficiently smooth the trace of $\partial^2 u/\partial n^2$ from the two sides of e are identical.

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_\Omega f v\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \Big(\frac{\partial^2 u}{\partial n^2}\Big)\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big]ds$$

$$= \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \Big\{\!\!\Big\{\frac{\partial^2 u}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big]ds$$

$$+ \sum_{e\in\mathcal{E}_h}\int_e \Big\{\!\!\Big\{\frac{\partial^2 v}{\partial n^2}\Big\}\!\!\Big\}\underbrace{\Big[\!\!\Big[\frac{\partial u}{\partial n}\Big]\!\!\Big]}_{=0}ds$$

symmetrization

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_\Omega f v\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \left(\frac{\partial^2 u}{\partial n^2}\right)\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]ds$$

$$= \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 u}{\partial n^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]ds$$

$$+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial u}{\partial n}\right]\!\right]ds$$

$$+ \sigma\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\int_e \underbrace{\left[\!\left[\frac{\partial u}{\partial n}\right]\!\right]}_{=0}\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]ds$$

stabilization

$|e|$ is the length of e.

$\sigma > 0$ is a penalty parameter.

A Mesh-Dependent Problem for $\boldsymbol{u}$

The solution u of the continuous problem satisfies

$$a_h(u,v) = \int_\Omega f\,v\,dx \qquad \forall\, v \in V_h$$

$$\begin{aligned}
a_h(w,v) = \sum_{T\in\mathcal{T}_h} \int_T D^2 w : D^2 v\,dx &+ \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 w}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] ds \\
&+ \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n} \right]\!\!\right] ds \\
&+ \sigma \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \int_e \left[\!\!\left[\frac{\partial w}{\partial n} \right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] ds
\end{aligned}$$

$\mathcal{E}_h =$ set of edges $\qquad \{\!\{\cdot\}\!\} =$ average $\qquad [\![\cdot]\!] =$ jump

$|e| =$ length of e $\qquad \sigma =$ penalty parameter

Discrete Problem

Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h$$

$$\begin{aligned} a_h(w,v) = \sum_{T \in \mathcal{T}_h} \int_T D^2 w : D^2 v \, dx &+ \sum_{e \in \mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 w}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] ds \\ &+ \sum_{e \in \mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n} \right]\!\!\right] ds \\ &+ \sigma \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \int_e \left[\!\!\left[\frac{\partial w}{\partial n} \right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] ds \end{aligned}$$

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Discrete Problem

Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f\, v\, dx \qquad \forall\, v \in V_h$$

Galerkin Orthogonality

$$a_h(u, v) = \int_\Omega fv\, dx \qquad \forall\, v \in V_h$$

$$a_h(u_h, v) = \int_\Omega fv\, dx \qquad \forall\, v \in V_h$$

$$a_h(u - u_h, v) = 0 \qquad \forall\, v \in V_h$$

Discrete Problem

This is an interior penalty method obtained through integration by parts, symmetrization and stabilization.

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$$a_h(w,v) = \underbrace{\sum_{T\in\mathcal{T}_h}\int_T D^2w : D^2v\,dx}_{\text{piecewise version of continuous variational form (ibp)}}$$

$$+ \underbrace{\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds}_{\text{consistency (ibp)}}$$

$$+ \underbrace{\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] ds}_{\text{symmetrization}} + \underbrace{\sigma\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds}_{\text{stabilization}}$$

Discrete Problem

Since the finite element functions are globally continuous, this is a C^0 interior penalty method. It is a discontinuous Galerkin method for fourth order problems, where the discontinuity is in the normal derivative across element boundaries. The discrete problem is a SPD problem when the penalty parameter is sufficiently large. Therefore it preserves the SPD property of the continuous problem.

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Since V_h is not a subspace of $H^2(\Omega)$, this is a nonconforming method. The second essential boundary condition $(\partial u/\partial n) = 0$ is also not imposed on the finite element space V_h.

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These conditions are enforced by the penalty term

$$\sigma \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \int_e \left[\!\left[\frac{\partial w}{\partial n} \right]\!\right] \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] ds$$

as $h \downarrow 0$.

Discrete Problem

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The study of discontinuous Galerkin methods for higher order problems was initiated in a paper by Baker.

Reference

Baker

Finite element methods for elliptic equations using nonconforming elements

Math. Comp. 1977

Discrete Problem

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Reference

Engel, Garikipati, Hughes, Larson, Mazzei, and Taylor

Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity

Comput. Methods Appl. Mech. Engrg. 2002

Coercivity of $\boldsymbol{a}_h(\cdot,\cdot)$

$$
\begin{aligned}
a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T D^2w : D^2v\,dx &+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds \\
&+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] ds \\
&+ \sigma \sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds
\end{aligned}
$$

Coercivity of $a_h(\cdot,\cdot)$

$$
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a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T D^2w : D^2v\,dx &+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds \\
&+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] ds \\
&+ \sigma\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds
\end{aligned}
$$

$$
\begin{aligned}
a_h(v,v) = \sum_{T\in\mathcal{T}_h}\int_T |D^2v|^2\,dx &+ 2\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds \\
&+ \sigma\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\left\|\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\right\|_{L_2(e)}^2
\end{aligned}
$$

Coercivity of $a_h(\cdot,\cdot)$

$$a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T D^2w : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds$$

$$+ \sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial w}{\partial n}\right]\!\right] ds$$

$$+ \sigma\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\int_e \left[\!\left[\frac{\partial w}{\partial n}\right]\!\right]\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds$$

$$a_h(v,v) = \sum_{T\in\mathcal{T}_h}\int_T |D^2 v|^2\,dx + 2\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds$$

$$+ \sigma\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\left\|\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]\right\|^2_{L_2(e)}$$

A Mesh-Dependent Norm

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)} + \sigma\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\left\|\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]\right\|^2_{L_2(e)}$$

Coercivity of $a_h(\cdot,\cdot)$

Lemma For σ sufficiently large, we have

$$a_h(v,v) \geq \frac{1}{2}\|v\|_h^2 \qquad \forall\, v \in V_h$$

Consequently $a_h(\cdot,\cdot)$ is SPD on V_h and hence the discrete problem is well-posed.

Coercivity of $a_h(\cdot,\cdot)$

Lemma For σ sufficiently large, we have

$$a_h(v,v) \geq \frac{1}{2}\|v\|_h^2 \qquad \forall\, v \in V_h$$

Proof.

$$a_h(v,v) = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + 2\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$

$$+ \sigma \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \left\| \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] \right\|_{L_2(e)}^2$$

$$= \|v\|_h^2 + 2\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$

Coercivity of $\boldsymbol{a}_h(\cdot,\cdot)$

Lemma For σ sufficiently large, we have

$$a_h(v,v) \geq \frac{1}{2}\|v\|_h^2 \qquad \forall\, v \in V_h$$

Proof.

$$\begin{aligned} a_h(v,v) &= \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + 2\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds \\ &\quad + \sigma \sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\left\|\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\right\|_{L_2(e)}^2 \\ &= \|v\|_h^2 + 2\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds \end{aligned}$$

It suffices to show that

$$2\left|\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds\right| \leq \frac{1}{2}\|v\|_h^2$$

Coercivity of $\boldsymbol{a}_h(\cdot,\cdot)$

$$2\Big|\sum_{e\in\mathcal{E}_h}\int_e \Big\{\!\!\Big\{\frac{\partial^2 v}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big]\,ds\Big|$$

$$\leq 2\Big(\sum_{e\in\mathcal{E}_h}|e|\|\{\!\{\partial^2 v/\partial n^2\}\!\}\|^2_{L_2(e)}\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}\Big)^{\frac{1}{2}}$$

$$\Big|\int fg\Big| \leq \Big(\int f^2\Big)^{\frac{1}{2}}\Big(\int g^2\Big)^{\frac{1}{2}}$$

$$\Big|\sum a_n b_n\Big| \leq \Big(\sum a_n^2\Big)^{\frac{1}{2}}\Big(\sum b_n^2\Big)^{\frac{1}{2}}$$

Coercivity of $a_h(\cdot,\cdot)$

$$
\begin{aligned}
2\Big|\sum_{e\in\mathcal{E}_h}\int_e \Big\{\!\!\Big\{\frac{\partial^2 v}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big]\,ds\Big| \\
&\leq 2\Big(\sum_{e\in\mathcal{E}_h}|e|\|\{\!\{\partial^2 v/\partial n^2\}\!\}\|_{L_2(e)}^2\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2\Big)^{\frac{1}{2}} \\
&\leq 2\Big(C\sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2\Big)^{\frac{1}{2}}
\end{aligned}
$$

The constant C depends only on the shape regularity of $\mathcal{T}_h$ and the degree of the polynomials in V_h.

Coercivity of $\boldsymbol{a}_h(\cdot,\cdot)$

$$2\Big|\sum_{e\in\mathcal{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds\Big|$$

$$\leq 2\Big(\sum_{e\in\mathcal{E}_h}|e|\|\{\!\{\partial^2 v/\partial n^2\}\!\}\|^2_{L_2(e)}\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}\Big)^{\frac{1}{2}}$$

$$\leq 2\Big(C\sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)}\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}\Big)^{\frac{1}{2}}$$

$$\leq \delta C\sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)}+\frac{1}{\delta}\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}$$

$$2ab\leq \delta a^2+\frac{1}{\delta}b^2$$

Coercivity of $a_h(\cdot,\cdot)$

$$
\begin{aligned}
2\Big|\sum_{e\in\mathcal{E}_h}\int_e \Big\{\!\!\Big\{\frac{\partial^2 v}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big]\,ds\Big|
&\le 2\Big(\sum_{e\in\mathcal{E}_h}|e|\,\|\{\!\{\partial^2 v/\partial n^2\}\!\}\|_{L_2(e)}^2\Big)^{\frac12}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2\Big)^{\frac12}\\
&\le 2\Big(C\sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2\Big)^{\frac12}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2\Big)^{\frac12}\\
&\le \delta C\sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2+\frac{1}{\delta}\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2\\
&= \frac12\sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2+2C\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2
\end{aligned}
$$

$$\delta=\frac{1}{2C}$$

Coercivity of $a_h(\cdot,\cdot)$

$$
\begin{aligned}
2\Big|&\sum_{e\in\mathcal{E}_h}\int_e \Big\{\!\!\Big\{\frac{\partial^2 v}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big]\,ds\Big| \\
&\le 2\Big(\sum_{e\in\mathcal{E}_h}|e|\,\|\{\!\{\partial^2 v/\partial n^2\}\!\}\|_{L_2(e)}^2\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2\Big)^{\frac{1}{2}} \\
&\le 2\Big(C\sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2\Big)^{\frac{1}{2}} \\
&\le \delta C\sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2+\frac{1}{\delta}\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 \\
&= \frac{1}{2}\sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2+2C\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 \\
&\le \frac{1}{2}\sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2+\frac{\sigma}{2}\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2
\end{aligned}
$$

$\sigma \ge 4C$

Coercivity of $a_h(\cdot,\cdot)$

$$
\begin{aligned}
&2\Big|\sum_{e\in\mathcal{E}_h}\int_e \Big\{\!\!\Big\{\frac{\partial^2 v}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big]\,ds\Big| \\
&\quad\le 2\Big(\sum_{e\in\mathcal{E}_h}|e|\,\|\{\!\{\partial^2 v/\partial n^2\}\!\}\|_{L_2(e)}^2\Big)^{\frac12}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2\Big)^{\frac12} \\
&\quad\le 2\Big(C\sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2\Big)^{\frac12}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2\Big)^{\frac12} \\
&\quad\le \delta C\sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2+\frac{1}{\delta}\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 \\
&\quad= \frac12\sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2+2C\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 \\
&\quad\le \frac12\sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2+\frac{\sigma}{2}\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2=\frac12\|v\|_h^2 \qquad \square
\end{aligned}
$$

Boundary Conditions of Simply Supported Plates

$\Omega =$ bounded polygonal domain $\qquad f \in L_2(\Omega)$

$$\begin{aligned} \Delta^2 u = f \qquad & \text{in } \Omega \\ u = \Delta u = 0 \qquad & \text{on } \partial\Omega \end{aligned}$$

Boundary Conditions of Simply Supported Plates

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$$\begin{aligned} \Delta^2 u = f &\qquad \text{in } \Omega \\ u = \Delta u = 0 &\qquad \text{on } \partial\Omega \end{aligned}$$

Variational/Weak Problem Find $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$a(u, v) = \int_\Omega f v \, dx \qquad \forall\, v \in H^2(\Omega) \cap H_0^1(\Omega)$$

$$a(w, v) = \int_\Omega D^2 w : D^2 v \, dx \qquad D^2 w : D^2 v = \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}$$

($u = 0$ is an essential boundary condition that is imposed on the solution space $H^2(\Omega) \cap H_0^1(\Omega)$, whereas $\Delta u = 0$ is a natural boundary condition.)

Boundary Conditions of Simply Supported Plates

$\Omega =$ bounded polygonal domain $\qquad f \in L_2(\Omega)$

$$\Delta^2 u = f \qquad \text{in } \Omega$$
$$u = \Delta u = 0 \qquad \text{on } \partial\Omega$$

Variational/Weak Problem Find $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

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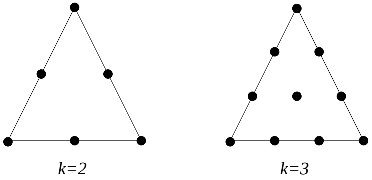
$$a(w,v) = \int_\Omega D^2 w : D^2 v \, dx \qquad D^2 w : D^2 v = \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}$$

Since $a(\cdot,\cdot)$ is bounded and coercive on $H^2(\Omega) \cap H_0^1(\Omega)$, the variational/weak problem has a unique solution.

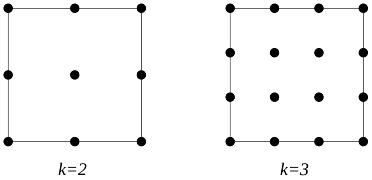
Finite Element Spaces

$\mathcal{T}_h$ = a simplicial triangulation of Ω

V_h $(\subset H_0^1(\Omega)) = \mathbb{P}_k$ $(k \geq 2)$ Lagrange finite element space



We can also use $\mathbb{Q}_k$ Lagrange finite element spaces.



Finite Element Spaces

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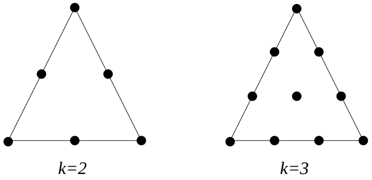

These are the same finite element spaces for the clamped plates.

A Mesh-Dependent Problem for $\boldsymbol{u}$

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_T f\,v\,dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n}\,v\,ds - \int_{\partial T} \left(\frac{\partial}{\partial n}\nabla u\right)\cdot\nabla v\,ds + \int_T D^2 u : D^2 v\,dx$$

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_T f\,v\,dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n}\,v\,ds - \int_{\partial T} \Big(\frac{\partial}{\partial n}\nabla u\Big)\cdot\nabla v\,ds + \int_T D^2 u : D^2 v\,dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial(\Delta u)}{\partial n}\,v\,ds = 0$$

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_T f v \, dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v \, ds - \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u \right) \cdot \nabla v \, ds + \int_T D^2 u : D^2 v \, dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v \, ds = 0$$

$$-\sum_{T \in \mathcal{T}_h} \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u \right) \cdot \nabla v \, ds = \sum_{e \in \mathcal{E}_h} \int_e \left(\frac{\partial^2 u}{\partial n^2} \right) \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] ds$$

($\mathcal{E}_h =$ the set of all the edges in $\mathcal{T}_h$)

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_T f v\,dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n}\,v\,ds - \int_{\partial T}\left(\frac{\partial}{\partial n}\nabla u\right)\cdot\nabla v\,ds + \int_T D^2u : D^2v\,dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial(\Delta u)}{\partial n}\,v\,ds = 0$$

$$-\sum_{T\in\mathcal{T}_h}\int_{\partial T}\left(\frac{\partial}{\partial n}\nabla u\right)\cdot\nabla v\,ds = \sum_{e\in\mathcal{E}_h}\int_e\left(\frac{\partial^2 u}{\partial n^2}\right)\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]ds$$

On any edge $e\subset\Omega$

$$0 = \Delta u$$

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_T f v\,dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v\,ds - \int_{\partial T} \left(\frac{\partial}{\partial n}\nabla u\right)\cdot\nabla v\,ds + \int_T D^2 u : D^2 v\,dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v\,ds = 0$$

$$-\sum_{T\in\mathcal{T}_h} \int_{\partial T} \left(\frac{\partial}{\partial n}\nabla u\right)\cdot\nabla v\,ds = \sum_{e\in\mathcal{E}_h} \int_e \left(\frac{\partial^2 u}{\partial n^2}\right) \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds$$

On any edge $e \subset \Omega$

$$0 = \Delta u = \frac{\partial^2 u}{\partial n^2} + \frac{\partial^2 u}{\partial t^2}$$

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_T f v \, dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v \, ds - \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u \right) \cdot \nabla v \, ds + \int_T D^2 u : D^2 v \, dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v \, ds = 0$$

$$- \sum_{T \in \mathcal{T}_h} \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u \right) \cdot \nabla v \, ds = \sum_{e \in \mathcal{E}_h} \int_e \left(\frac{\partial^2 u}{\partial n^2} \right) \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] ds$$

On any edge $e \subset \Omega$

$$0 = \Delta u = \frac{\partial^2 u}{\partial n^2} + \underbrace{\frac{\partial^2 u}{\partial t^2}}_{=0} = \frac{\partial^2 u}{\partial n^2}$$

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_T f v\,dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n}\, v\, ds - \int_{\partial T} \left(\frac{\partial}{\partial n}\nabla u\right) \cdot \nabla v\, ds + \int_T D^2 u : D^2 v\, dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial(\Delta u)}{\partial n}\, v\, ds = 0$$

$$\begin{aligned} -\sum_{T\in\mathcal{T}_h} \int_{\partial T} \left(\frac{\partial}{\partial n}\nabla u\right) \cdot \nabla v\, ds &= \sum_{e\in\mathcal{E}_h} \int_e \left(\frac{\partial^2 u}{\partial n^2}\right) [\![\frac{\partial v}{\partial n}]\!] ds \\ &= \sum_{e\in\mathcal{E}_h^i} \int_e \left(\frac{\partial^2 u}{\partial n^2}\right) [\![\frac{\partial v}{\partial n}]\!] ds \end{aligned}$$

$\mathcal{E}_h^i$ is the set of the edges of $\mathcal{T}_h$ interior to Ω.

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_T f\,v\,dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n}\,v\,ds - \int_{\partial T} \Big(\frac{\partial}{\partial n}\nabla u\Big)\cdot\nabla v\,ds + \int_T D^2u : D^2v\,dx$$

Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial(\Delta u)}{\partial n}\,v\,ds = 0$$

$$\begin{aligned}-\sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial}{\partial n}\nabla u\Big)\cdot\nabla v\,ds &= \sum_{e\in\mathcal{E}_h}\int_e \Big(\frac{\partial^2 u}{\partial n^2}\Big)\Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big]ds\\ &= \sum_{e\in\mathcal{E}_h^i}\int_e \Big(\frac{\partial^2 u}{\partial n^2}\Big)\Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big]ds\end{aligned}$$

$$\int_\Omega f\,v\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h^i}\int_e \Big(\frac{\partial^2 u}{\partial n^2}\Big)\Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big]ds$$

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_\Omega f v\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h^i}\int_e \left(\frac{\partial^2 u}{\partial n^2}\right)\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds$$

A Mesh-Dependent Problem for u

$$\int_\Omega f v\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h^i}\int_e \left(\frac{\partial^2 u}{\partial n^2}\right)\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds$$

$$= \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h^i}\int_e \left\{\!\!\left\{\frac{\partial^2 u}{\partial n^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] ds$$

$$\frac{\partial^2 u}{\partial n^2} = \left\{\!\!\left\{\frac{\partial^2 u}{\partial n^2}\right\}\!\!\right\}$$

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$
\begin{aligned}
\int_\Omega f\,v\,dx &= \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h^i}\int_e \Big(\frac{\partial^2 u}{\partial n^2}\Big)\Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big]ds \\
&= \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h^i}\int_e \Big\{\!\!\Big\{\frac{\partial^2 u}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big]ds \\
&\quad + \sum_{e\in\mathcal{E}_h^i}\int_e \Big\{\!\!\Big\{\frac{\partial^2 v}{\partial n^2}\Big\}\!\!\Big\}\underbrace{\Big[\!\Big[\frac{\partial u}{\partial n}\Big]\!\Big]}_{=0}ds
\end{aligned}
$$

symmetrization

A Mesh-Dependent Problem for $\boldsymbol{u}$

$$\int_\Omega f\,v\,dx = \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h^i}\int_e \Big(\frac{\partial^2 u}{\partial n^2}\Big)\Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big]ds$$

$$= \sum_{T\in\mathcal{T}_h}\int_T D^2u : D^2v\,dx + \sum_{e\in\mathcal{E}_h^i}\int_e \Big\{\!\!\Big\{\frac{\partial^2 u}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big]ds$$

$$+ \sum_{e\in\mathcal{E}_h^i}\int_e \Big\{\!\!\Big\{\frac{\partial^2 v}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\Big[\frac{\partial u}{\partial n}\Big]\!\Big]ds$$

$$+ \sigma\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\int_e \underbrace{\Big[\!\Big[\frac{\partial u}{\partial n}\Big]\!\Big]}_{=0}\Big[\!\Big[\frac{\partial v}{\partial n}\Big]\!\Big]\,ds$$

stabilization

A Mesh-Dependent Problem for $\boldsymbol{u}$

The solution u of the continuous problem satisfies

$$a_h(u,v) = \int_\Omega f\,v\,dx \qquad \forall\, v \in V_h$$

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$\mathcal{E}_h^i$ = set of interior edges $\{\!\{\cdot\}\!\}$ = average $[\![\cdot]\!]$ = jump

$|e|$ = length of e σ = penalty parameter

Discrete Problem

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$$a_h(u-u_h, v) = 0 \qquad \forall\, v \in V_h$$

Coercivity of $a_h(\cdot,\cdot)$

Lemma For σ sufficiently large, we have

$$a_h(v,v) \ge \frac{1}{2}\|v\|_h^2 \qquad \forall\, v \in V_h$$

where

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sigma \sum_{e\in\mathcal{E}_h^i} \frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2$$

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Consequently $a_h(\cdot,\cdot)$ is SPD on V_h and hence the discrete problem is well-posed.

Boundary Conditions of the Cahn-Hilliard Type

Ω = bounded polygonal domain $\quad f \in L_2(\Omega)$

$$\Delta^2 u = f \qquad \text{in } \Omega$$
$$\partial u/\partial n = \partial(\Delta u)/\partial n = 0 \qquad \text{on } \partial\Omega$$

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Variational/Weak Problem I Find $u \in \mathbb{V}$ such that

$$a(u,v) = \int_\Omega f v\, dx \qquad \forall\, v \in \mathbb{V}$$

$$a(w,v) = \int_\Omega D^2 w : D^2 v\, dx \qquad D^2 w : D^2 v = \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}$$

where

$$\mathbb{V} = \{v \in H^2(\Omega) :\ \partial v/\partial n = 0 \text{ on } \partial\Omega\}$$

($\partial u/\partial n = 0$ is an essential boundary condition imposed on the solution space $\mathbb{V}$.)

Boundary Conditions of the Cahn-Hilliard Type

The bilinear form $a(\cdot,\cdot)$ is not coercive on $\mathbb{V}$ because the constant function 1 belongs to $\mathbb{V}$ and $a(1,1)=0$.

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Under condition $(\dagger)$ the solution of the variational/weak problem I is unique up to an additive constant. In particular it has a unique solution in the subspace

$$\mathbb{V}^* = \{v \in \mathbb{V} : \, v(p_*) = 0\}$$

where p_* is a corner of Ω.

Boundary Conditions of the Cahn-Hilliard Type

Variational/Weak Problem II

Assume that $f \in L_2(\Omega)$ satisfies the constraint

$$(\dagger) \qquad \int_\Omega f\,dx = 0$$

Find $u \in \mathbb{V}^*$ such that

$$a(u,v) = \int_\Omega f v\,dx \qquad \forall\, v \in \mathbb{V}^*$$

where $\mathbb{V}^* = \{v \in H^2(\Omega) :\ \partial v/\partial n = 0$ on $\partial\Omega$ and $v(p_*) = 0\}$.

Boundary Conditions of the Cahn-Hilliard Type

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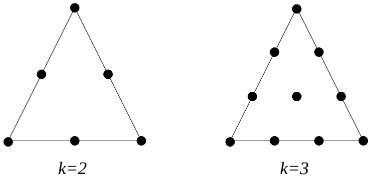
where $\mathbb{V}^* = \{v \in H^2(\Omega) : \partial v/\partial n = 0 \text{ on } \partial\Omega \text{ and } v(p_*) = 0\}$.

Since the bilinear form $a(\cdot,\cdot)$ is bounded and coercive on $\mathbb{V}^*$, the variational/weak problem II has a unique solution, which is also a solution of the variational/weak problem I due to the condition $(\dagger)$.

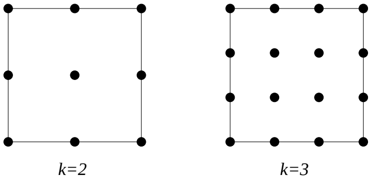
Finite Element Spaces

$\mathcal{T}_h =$ a simplicial triangulation of Ω

$V_h = \mathbb{P}_k$ $(k \geq 2)$ Lagrange finite element space



We can also use $\mathbb{Q}_k$ Lagrange finite element spaces.



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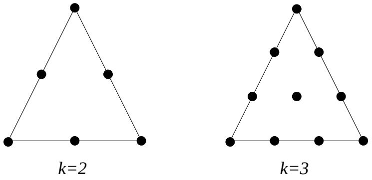

Note that no boundary condition is imposed on the finite element space because, as in the case of the clamped plates, the essential boundary condition $\partial u/\partial n = 0$ can be enforced by the penalty term as $h \downarrow 0$.

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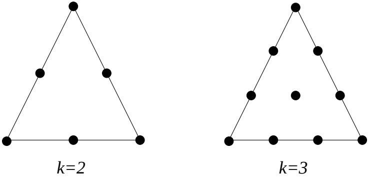

Subspace V_h^*

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where p_* is a corner of Ω.

A Mesh-Dependent Problem for $\boldsymbol{u}$

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$$\int_T f v \, dx = \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v \, ds - \int_{\partial T} \left(\frac{\partial}{\partial n} \nabla u \right) \cdot \nabla v \, ds + \int_T D^2 u : D^2 v \, dx$$

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Summing up over all the triangles in $\mathcal{T}_h$,

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial(\Delta u)}{\partial n} v\,ds = 0$$

Each interior edge is shared by two triangles. Since v and the derivatives of u are continuous and the normals from the two triangles are pointing at opposite directions, the contributions from the two triangles cancel.

There is also no contribution from the boundary edges because $\partial(\Delta u)/\partial n = 0$ on $\partial\Omega$.

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$[\![\partial v / \partial t]\!] = 0$ over interior edges.

$\dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial n}\right) = 0$ on boundary edges.

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Discrete Problem

Find $u_h \in V_h^*$ such that $\qquad (V_h^* = \{v \in V_h : v(p_*) = 0\})$

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Consequently $a_h(\cdot,\cdot)$ is SPD on V_h^* and hence the discrete problem is well-posed.

Summary

- The C^0 interior penalty method for the biharmonic equation with the boundary conditions of simply supported plates share the same finite element spaces with the C^0 interior penalty method for the biharmonic equation with the boundary conditions of clamped plates, but uses a slightly different bilinear form ($\mathcal{E}_h^i$ instead of $\mathcal{E}_h$).
- The C^0 interior penalty method for the biharmonic equation with the boundary conditions of the Cahn-Hilliard type shares the same bilinear form with the C^0 interior penalty method for the biharmonic equation with the boundary conditions of clamped plates, but uses a slightly different finite element space (no boundary condition $+$ vanishing at a corner).

Enriching Operators

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Clamped Plates

Solution space for the continuous problem is $H_0^2(\Omega)$.

Solution space for the discrete problem is a $\mathbb{P}_k$ (or $\mathbb{Q}_k$) Lagrange finite element space $V_h \subset H_0^1(\Omega)$.

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Solution space for the discrete problem is a $\mathbb{P}_k$ (or $\mathbb{Q}_k$) Lagrange finite element space $V_h \subset H_0^1(\Omega)$.

The goal is to construct an embedding (i.e., one-to-one map) E_h from V_h into $H_0^2(\Omega)$ so that

$$|v - E_h v|_{H^2(\Omega;\mathcal{T}_h)} = \Big(\sum_{T\in\mathcal{T}_h} |v - E_h v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \Big\| \Big[\!\Big[\frac{\partial(v - E_h v)}{\partial n} \Big]\!\Big] \Big\|^2_{L_2(e)} \Big)^{\frac{1}{2}}$$

provides a measure for the distance between $v \in V_h$ and $H_0^2(\Omega)$ with respect to $|\cdot|_{H^2(\Omega;\mathcal{T}_h)}$.

(The piecewise H^2 (semi-) norm $|\cdot|_{H^2(\Omega;\mathcal{T}_h)}$ is identical to the mesh-dependent norm $\|\cdot\|_h$ when $\sigma = 1$.)

C^1 Relatives

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We say that a C^1 element is a (rich) relative of a Lagrange element if

- they have the same element domain,
- the shape functions of the Lagrange finite element are also shape functions for the C^1 element,
- the dofs for the Lagrange element are also dofs for the C^1 element.

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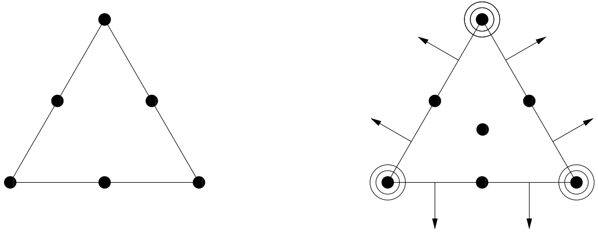

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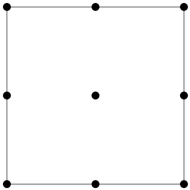

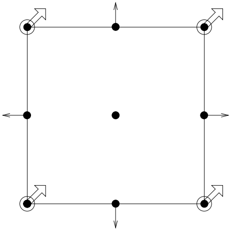

Averaging

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$V_h = Q_2$ Lagrange Finite Element Space

$$E_h : V_h \longrightarrow W_h \subset H_0^2(\Omega)$$

where W_h is the Q_4 BFS finite element space.

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Let $v \in V_h$. Since the dofs of $E_h v$ along the boundary of Ω must vanish by the condition that $E_h v \in H_0^2(\Omega)$, we only need to specify the dofs at the nodes interior to Ω.

Averaging

Let N be a dof for the Q_4 BFS finite element space associated with the interior node p. We define

$$N(E_h v) = \frac{1}{|\mathcal{T}_p|} \sum_{T \in \mathcal{T}_p} N(v_T)$$

where $\mathcal{T}_p$ is the set of elements that share the node p and v_T is the restriction of v to T.

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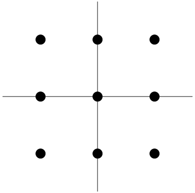

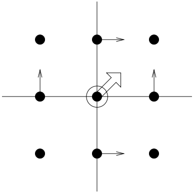

$$\big(E_h v\big)(p) = v(p)$$

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where $\mathcal{T}_p$ is the set of elements that share the node p and v_T is the restriction of v to T.

$$\big(\nabla(E_h v)\big)(p) = \frac{1}{4} \sum_{i=1}^{4} \big(\nabla v_i\big)(p)$$

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$$\left(\frac{\partial(E_h v)}{\partial n}\right)(p) = \frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial v_i}{\partial n}\right)(p)$$

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$$\left(\frac{\partial^2 (E_h v)}{\partial x_1 \partial x_2}\right)(p) = \frac{1}{4} \sum_{i=1}^{4} \left(\frac{\partial^2 v_i}{\partial x_1 \partial x_2}\right)(p)$$

Properties of $\boldsymbol{E}_h$

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E_h is one-to-one.

This is due to the fact that $E_h v$ equals v at all the nodes for the $\mathbb{Q}_2$ Lagrange finite element space.

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Let $\Pi_h : H_0^2(\Omega) \longrightarrow V_h$ be the nodal interpolation operator defined by

$$(\Pi_h w)(p) = w(p)$$

where p is any node for the $\mathbb{Q}_2$ Lagrange finite element space.

Then Π_h is a left inverse for E_h, i.e.

$$\Pi_h \circ E_h = Id_{V_h}$$

$$[\Pi_h(E_h v)](p) = (E_h v)(p) = v(p) \qquad \forall\, v \in V_h$$

Properties of $\boldsymbol{E_h}$

Estimates for $E_h v$

$$
\begin{aligned}
&|v - E_h v|^2_{H^2(\Omega;\mathcal{T}_h)} + \sum_{T\in\mathcal{T}_h} \Big[h_T^{-4} \|v - E_h v\|^2_{L_2(T)} + h_T^{-2} |v - E_h v|^2_{H^1(T)} \Big] \\
&\qquad \le C|v|^2_{H^2(\Omega;\mathcal{T}_h)} \quad \forall\, v \in V_h
\end{aligned}
$$

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The derivation of these estimates uses the fact that all norms on a finite dimensional vector space are equivalent together with scaling arguments.

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The derivation of these estimates uses the fact that all norms on a finite dimensional vector space are equivalent together with scaling arguments.

Corollary

$$\begin{aligned} |E_h v|_{H^2(\Omega)} &= |E_h v|_{H^2(\Omega;\mathcal{T}_h)} \\ &\leq |E_h v - v|_{H^2(\Omega;\mathcal{T}_h)} + |v|_{H^2(\Omega;\mathcal{T}_h)} \\ &\leq C|v|_{H^2(\Omega;\mathcal{T}_h)} \qquad \forall\, v \in V_h \end{aligned}$$

Properties of E_h

Estimates for $E_h \circ \Pi_h : H_0^2(\Omega) \longrightarrow H_0^2(\Omega)$ $\qquad (\Pi_h \circ E_h = Id_{V_h})$

$$\sum_{T\in\mathcal{T}_h} \Big[h_T^{-4} \|\zeta - E_h \Pi_h \zeta\|_{L_2(T)}^2 + h_T^{-2} |\zeta - E_h \Pi_h \zeta|_{H^1(T)}^2 + |\zeta - E_h \Pi_h \zeta|_{H^2(T)}^2 \Big] \leq C |\zeta|_{H^2(\Omega)}^2 \qquad \forall\, \zeta \in H_0^2(\Omega)$$

$$\sum_{T\in\mathcal{T}_h} \Big[h_T^{-6} \|\zeta - E_h \Pi_h \zeta\|_{L_2(T)}^2 + h_T^{-4} |\zeta - E_h \Pi_h \zeta|_{H^1(T)}^2 + h_T^{-2} |\zeta - E_h \Pi_h \zeta|_{H^2(T)}^2 \Big] \leq C |\zeta|_{H^3(\Omega)}^2 \qquad \forall\, \zeta \in H^3(\Omega) \cap H_0^2(\Omega)$$

Properties of $\boldsymbol{E}_h$

Estimates for $E_h \circ \Pi_h : H_0^2(\Omega) \longrightarrow H_0^2(\Omega)$ $\qquad (\Pi_h \circ E_h = Id_{V_h})$

$$\sum_{T\in\mathcal{T}_h} \Big[h_T^{-4} \|\zeta - E_h\Pi_h\zeta\|_{L_2(T)}^2 + h_T^{-2} |\zeta - E_h\Pi_h\zeta|_{H^1(T)}^2 + |\zeta - E_h\Pi_h\zeta|_{H^2(T)}^2 \Big] \leq C|\zeta|_{H^2(\Omega)}^2 \qquad \forall\, \zeta \in H_0^2(\Omega)$$

$$\sum_{T\in\mathcal{T}_h} \Big[h_T^{-6} \|\zeta - E_h\Pi_h\zeta\|_{L_2(T)}^2 + h_T^{-4} |\zeta - E_h\Pi_h\zeta|_{H^1(T)}^2 + h_T^{-2} |\zeta - E_h\Pi_h\zeta|_{H^2(T)}^2 \Big] \leq C|\zeta|_{H^3(\Omega)}^2 \qquad \forall\, \zeta \in H^3(\Omega) \cap H_0^2(\Omega)$$

The derivation of the first set of estimates uses standard interpolation estimates for Π_h and the estimates for E_h.

The derivation of the second set of estimates uses the Bramble-Hilbert lemma.

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Estimates for $E_h \circ \Pi_h : H_0^2(\Omega) \longrightarrow H_0^2(\Omega)$ $\qquad (\Pi_h \circ E_h = Id_{V_h})$

$$\sum_{T\in\mathcal{T}_h} \Big[h_T^{-4}\|\zeta - E_h\Pi_h\zeta\|_{L_2(T)}^2 + h_T^{-2}|\zeta - E_h\Pi_h\zeta|_{H^1(T)}^2 + |\zeta - E_h\Pi_h\zeta|_{H^2(T)}^2\Big] \leq C|\zeta|_{H^2(\Omega)}^2 \qquad \forall\, \zeta \in H_0^2(\Omega)$$

$$\sum_{T\in\mathcal{T}_h} \Big[h_T^{-6}\|\zeta - E_h\Pi_h\zeta\|_{L_2(T)}^2 + h_T^{-4}|\zeta - E_h\Pi_h\zeta|_{H^1(T)}^2 + h_T^{-2}|\zeta - E_h\Pi_h\zeta|_{H^2(T)}^2\Big] \leq C|\zeta|_{H^3(\Omega)}^2 \qquad \forall\, \zeta \in H^3(\Omega)\cap H_0^2(\Omega)$$

These estimates indicate that $E_h \circ \Pi_h$ is a quasi-local interpolation operator from $H_0^2(\Omega)$ into a C^1 finite element space that satisfies the correct discretization error estimates.

Properties of $\boldsymbol{E}_h$

Estimates for $E_h \circ \Pi_h : H_0^2(\Omega) \longrightarrow H_0^2(\Omega)$ $\qquad (\Pi_h \circ E_h = Id_{V_h})$

$$\sum_{T\in\mathcal{T}_h} \Big[h_T^{-4}\|\zeta - E_h\Pi_h\zeta\|_{L_2(T)}^2 + h_T^{-2}|\zeta - E_h\Pi_h\zeta|_{H^1(T)}^2 + |\zeta - E_h\Pi_h\zeta|_{H^2(T)}^2\Big] \leq C|\zeta|_{H^2(\Omega)}^2 \qquad \forall\, \zeta \in H_0^2(\Omega)$$

$$\sum_{T\in\mathcal{T}_h} \Big[h_T^{-6}\|\zeta - E_h\Pi_h\zeta\|_{L_2(T)}^2 + h_T^{-4}|\zeta - E_h\Pi_h\zeta|_{H^1(T)}^2 + h_T^{-2}|\zeta - E_h\Pi_h\zeta|_{H^2(T)}^2\Big] \leq C|\zeta|_{H^3(\Omega)}^2 \qquad \forall\, \zeta \in H^3(\Omega) \cap H_0^2(\Omega)$$

If $v = \Pi_h\zeta$ for some $\zeta \in H^3(\Omega)$, then

$$|v - E_h v|_{H^2(\Omega;\mathcal{T}_h)} \leq |\Pi_h\zeta - \zeta|_{H^2(\Omega;\mathcal{T}_h)} + |\zeta - E_h\Pi_h\zeta|_{H^2(\Omega;\mathcal{T}_h)} \leq Ch|\zeta|_{H^3(\Omega)}$$

which indicates that $|v - E_h v|_{H^2(\Omega;\mathcal{T}_h)}$ provides a measure for the distance between v and $H_0^2(\Omega)$ with respect to $|\cdot|_{H^2(\Omega;\mathcal{T}_h)}$.

Other Boundary Conditions

Other Boundary Conditions

Boundary Conditions of Simply Supported Plates

We can construct an enriching operator from the Lagrange finite element space V_h $(\subset H_0^1(\Omega))$ (same finite element space for clamped plates) into $H^2(\Omega)\cap H_0^1(\Omega)$ with similar properties. But the piecewise H^2 norm $|\cdot|_{H^2(\Omega;\mathcal{T}_h)}$ is defined by

$$|v|^2_{H^2(\Omega;\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h^i} \frac{1}{|e|} \|[\![v]\!]\|^2_{L_2(e)}$$

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Boundary Conditions of the Cahn-Hilliard Type

We can construct an enriching operator from the Lagrange finite element space V_h^* (different from the finite element space for clamped plates) into

$$\mathbb{V}^* = \{v \in H^2(\Omega) : \partial v/\partial n = 0 \text{ on } \partial\Omega \text{ and } v(p_*) = 0\}$$

with similar properties. The piecewise H^2 norm $|\cdot|_{H^2(\Omega;\mathcal{T}_h)}$ is identical to the one for clamped plates.

Post-Processing by $\boldsymbol{E}_h$

Post-Processing by $\boldsymbol{E_h}$

Let $u_h \in V_h$ be the solution of a C^0 interior penalty method for the biharmonic equation with the boundary condition of the clamped plates (respectively simply supported plates or Cahn-Hilliard type) and E_h be an enriching operator from E_h into $H_0^2(\Omega)$ (respectively $H^2(\Omega) \cap H_0^1(\Omega)$ or $\mathbb{V}^*$).

Then $E_h u_h$ is a conforming approximation of the boundary value problem obtained from u_h by post-processing.

Post-Processing by $\boldsymbol{E_h}$

Let $u_h \in V_h$ be the solution of a C^0 interior penalty method for the biharmonic equation with the boundary condition of the clamped plates (respectively simply supported plates or Cahn-Hilliard type) and E_h be an enriching operator from E_h into $H_0^2(\Omega)$ (respectively $H^2(\Omega) \cap H_0^1(\Omega)$ or $\mathbb{V}^*$).

Then $E_h u_h$ is a conforming approximation of the boundary value problem obtained from u_h by post-processing.

It can be shown, by using the error estimates of the C^0 interior penalty methods (to be discussed) and properties of E_h, that $E_h u_h$ is a C^1 approximate solution with correct estimates.

Therefore C^0 interior penalty methods are relevant even if we only want C^1 approximate solutions.

A Poincaré-Friedrichs Inequality

Let $V_h\,(\subset H_0^1(\Omega))$ be a $\mathbb{P}_k$ (or $\mathbb{Q}_k$) Lagrange finite element space. We have

$$\|v\|_{L_2(\Omega)} + |v|_{H^1(\Omega)} \leq C|v|_{H^2(\Omega;\mathcal{T}_h)} \qquad \forall\, v \in V_h$$

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Let $V_h\,(\subset\; H_0^1(\Omega))$ be a $\mathbb{P}_k$ (or $\mathbb{Q}_k$) Lagrange finite element space. We have

$$\|v\|_{L_2(\Omega)} + |v|_{H^1(\Omega)} \le C|v|_{H^2(\Omega;\mathcal{T}_h)} \qquad \forall\, v \in V_h$$

The piecewise H^2 norm is the one for simply supported plates:

$$|v|^2_{H^2(\Omega;\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h^i} \frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}$$

Therefore it also holds for the stronger norm for clamped plates:

$$|v|^2_{H^2(\Omega;\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}$$

A Poincaré-Friedrichs Inequality

Let $V_h\,(\subset H_0^1(\Omega))$ be a $\mathbb{P}_k$ (or $\mathbb{Q}_k$) Lagrange finite element space. We have

$$\|v\|_{L_2(\Omega)} + |v|_{H^1(\Omega)} \leq C|v|_{H^2(\Omega;\mathcal{T}_h)} \qquad \forall\, v \in V_h$$

Proof.

$$\begin{aligned}\|v\|_{L_2(\Omega)} + |v|_{H^1(\Omega)} \leq \|v - E_h v\|_{L_2(\Omega)} + |v - E_h v|_{H^1(\Omega)} \\ + \|E_h v\|_{L_2(\Omega)} + |E_h v|_{H^1(\Omega)}\end{aligned}$$

E_h is an enriching operator for simply supported plates that maps V_h into $H^2(\Omega) \cap H_0^1(\Omega)$.

A Poincaré-Friedrichs Inequality

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$$\|v\|_{L_2(\Omega)} + |v|_{H^1(\Omega)} \le C|v|_{H^2(\Omega;\mathcal{T}_h)} \qquad \forall\, v \in V_h$$

Proof.

$$\begin{aligned}\|v\|_{L_2(\Omega)} + |v|_{H^1(\Omega)} \le \|v - E_h v\|_{L_2(\Omega)} + |v - E_h v|_{H^1(\Omega)} \\ + \|E_h v\|_{L_2(\Omega)} + |E_h v|_{H^1(\Omega)}\end{aligned}$$

$$\begin{aligned}\|v - E_h v\|_{L_2(\Omega)} &\le Ch^2|v|_{H^2(\Omega;\mathcal{T}_h)} \\ |v - E_h v|_{H^1(\Omega)} &\le Ch|v|_{H^2(\Omega;\mathcal{T}_h)}\end{aligned}$$

$$\sum_{T\in\mathcal{T}_h}\Big[h_T^{-4}\|v - E_h v\|_{L_2(T)}^2 + h_T^{-2}|v - E_h v|_{H^1(T)}^2\Big] \le C|v|_{H^2(\Omega;\mathcal{T}_h)}^2$$

A Poincaré-Friedrichs Inequality

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$$\|v\|_{L_2(\Omega)} + |v|_{H^1(\Omega)} \le C|v|_{H^2(\Omega;\mathcal{T}_h)} \qquad \forall\, v \in V_h$$

Proof.

$$\begin{aligned}\|v\|_{L_2(\Omega)} + |v|_{H^1(\Omega)} \le \|v - E_h v\|_{L_2(\Omega)} &+ |v - E_h v|_{H^1(\Omega)} \\ &+ \|E_h v\|_{L_2(\Omega)} + |E_h v|_{H^1(\Omega)}\end{aligned}$$

$$\begin{aligned}\|v - E_h v\|_{L_2(\Omega)} &\le Ch^2|v|_{H^2(\Omega;\mathcal{T}_h)} \\ |v - E_h v|_{H^1(\Omega)} &\le Ch|v|_{H^2(\Omega;\mathcal{T}_h)}\end{aligned}$$

$$\|E_h v\|_{L_2(\Omega)} + |E_h v|_{H^1(\Omega)} \le C_\Omega |E_h v|_{H^2(\Omega)}$$

$E_h v \in H^2(\Omega) \cap H_0^1(\Omega)$

classical Poincaré-Friedrichs inequality

A Poincaré-Friedrichs Inequality

Let $V_h\,(\subset\ H_0^1(\Omega))$ be a $\mathbb{P}_k$ (or $\mathbb{Q}_k$) Lagrange finite element space. We have

$$\|v\|_{L_2(\Omega)} + |v|_{H^1(\Omega)} \le C|v|_{H^2(\Omega;\mathcal{T}_h)} \qquad \forall\, v \in V_h$$

Proof.

$$\begin{aligned}\|v\|_{L_2(\Omega)} + |v|_{H^1(\Omega)} \le{}& \|v - E_h v\|_{L_2(\Omega)} + |v - E_h v|_{H^1(\Omega)} \\ &+ \|E_h v\|_{L_2(\Omega)} + |E_h v|_{H^1(\Omega)}\end{aligned}$$

$$\begin{aligned}\|v - E_h v\|_{L_2(\Omega)} &\le Ch^2|v|_{H^2(\Omega;\mathcal{T}_h)} \\ |v - E_h v|_{H^1(\Omega)} &\le Ch|v|_{H^2(\Omega;\mathcal{T}_h)}\end{aligned}$$

$$\|E_h v\|_{L_2(\Omega)} + |E_h v|_{H^1(\Omega)} \le C_\Omega |E_h v|_{H^2(\Omega)}$$

$$|E_h v|_{H^2(\Omega)} \le C|v|_{H^2(\Omega;\mathcal{T}_h)} \qquad \square$$

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Let $V_h\,(\subset H_0^1(\Omega))$ be a $\mathbb{P}_k$ (or $\mathbb{Q}_k$) Lagrange finite element space. We have

$$\|v\|_{L_2(\Omega)} + |v|_{H^1(\Omega)} \leq C|v|_{H^2(\Omega;\mathcal{T}_h)} \qquad \forall\, v \in V_h$$

This inequality also holds for $v \in V_h^*$ (finite element space for boundary conditions of the Cahn-Hilliard type), where the piecewise H^2 norm is given by

$$|v|^2_{H^2(\Omega;\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}$$

An Application of the Poincaré-Friedrichs Inequality

An Application of the Poincaré-Friedrichs Inequality

C^0 Interior Penalty Method for Clamped Plates

$$a_h(u_h, v) = \int_\Omega f\, v\, dx \qquad \forall\, v \in V_h$$

An Application of the Poincaré-Friedrichs Inequality

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$$\begin{aligned} |u_h|^2_{H^2(\Omega;\mathcal{T}_h)} &= \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h} |e|^{-1} \| [\![\partial u_h/\partial n]\!] \|^2_{L_2(e)} \\ &\leq \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sigma \sum_{e\in\mathcal{E}_h} |e|^{-1} \| [\![\partial u_h/\partial n]\!] \|^2_{L_2(e)} \end{aligned}$$

$$\sigma \geq 1$$

An Application of the Poincaré-Friedrichs Inequality

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An Application of the Poincaré-Friedrichs Inequality

C^0 Interior Penalty Method for Clamped Plates

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h$$

$$\begin{aligned}
|u_h|^2_{H^2(\Omega;\mathcal{T}_h)} &= \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h} |e|^{-1} \|[\![\partial u_h/\partial n]\!]\|^2_{L_2(e)} \\
&\leq \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sigma \sum_{e\in\mathcal{E}_h} |e|^{-1} \|[\![\partial u_h/\partial n]\!]\|^2_{L_2(e)} \\
&= \|u_h\|_h^2 \\
&\leq 2a_h(u_h, u_h)
\end{aligned}$$

$$a_h(v, v) \geq \tfrac{1}{2} \|v\|_h^2$$

An Application of the Poincaré-Friedrichs Inequality

C^0 Interior Penalty Method for Clamped Plates

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad \forall \, v \in V_h$$

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|u_h|^2_{H^2(\Omega;\mathcal{T}_h)} &= \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h} |e|^{-1} \| [\![\partial u_h/\partial n]\!] \|^2_{L_2(e)} \\
&\leq \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sigma \sum_{e\in\mathcal{E}_h} |e|^{-1} \| [\![\partial u_h/\partial n]\!] \|^2_{L_2(e)} \\
&= \|u_h\|^2_h \\
&\leq 2 a_h(u_h, u_h) \\
&= 2 \int_\Omega f u_h \, dx
\end{aligned}$$

An Application of the Poincaré-Friedrichs Inequality

C^0 Interior Penalty Method for Clamped Plates

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h$$

$$\begin{aligned}
|u_h|^2_{H^2(\Omega;\mathcal{T}_h)} &= \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h} |e|^{-1} \| [\![\partial u_h/\partial n]\!] \|^2_{L_2(e)} \\
&\leq \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sigma \sum_{e\in\mathcal{E}_h} |e|^{-1} \| [\![\partial u_h/\partial n]\!] \|^2_{L_2(e)} \\
&= \|u_h\|^2_h \\
&\leq 2a_h(u_h, u_h) \\
&= 2\int_\Omega f u_h \, dx \\
&\leq 2\|f\|_{L_2(\Omega)} \|u_h\|_{L_2(\Omega)}
\end{aligned}$$

An Application of the Poincaré-Friedrichs Inequality

C^0 Interior Penalty Method for Clamped Plates

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h$$

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|u_h|^2_{H^2(\Omega;\mathcal{T}_h)} &= \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h} |e|^{-1} \|[\![\partial u_h/\partial n]\!]\|^2_{L_2(e)} \\
&\leq \sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sigma \sum_{e\in\mathcal{E}_h} |e|^{-1} \|[\![\partial u_h/\partial n]\!]\|^2_{L_2(e)} \\
&= \|u_h\|_h^2 \\
&\leq 2a_h(u_h, u_h) \\
&= 2\int_\Omega f u_h \, dx \\
&\leq 2\|f\|_{L_2(\Omega)} \|u_h\|_{L_2(\Omega)} \leq C\|f\|_{L_2(\Omega)} |u_h|_{H^2(\Omega;\mathcal{T}_h)}
\end{aligned}$$

Poincaré-Friedrichs Inequality

An Application of the Poincaré-Friedrichs Inequality

C^0 Interior Penalty Method for Clamped Plates

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad \forall \, v \in V_h$$

After cancellation we have $|u_h|_{H^2(\Omega;\mathcal{T}_h)} \leq C\|f\|_{L_2(\Omega)}$.

Equivalently

$$\sum_{T\in\mathcal{T}_h} |u_h|^2_{H^2(T)} + \sigma \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \|[\![\partial u_h/\partial n]\!]\|^2_{L_2(e)} \leq C\|f\|^2_{L_2(\Omega)}$$

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After cancellation we have $|u_h|_{H^2(\Omega;\mathcal{T}_h)} \le C\|f\|_{L_2(\Omega)}$.

Equivalently

$$\sum_{T\in\mathcal{T}_h} |u_h|^2_{H^2(T)} + \sigma \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \|[\![\partial u_h/\partial n]\!]\|^2_{L_2(e)} \le C\|f\|^2_{L_2(\Omega)}$$

Since $1/|e| \uparrow \infty$ as $h \downarrow 0$,

$$\|[\![\partial u_h/\partial n]\!]\|_{L_2(e)} \to 0 \text{ as } h \downarrow 0$$

In other words, C^1 continuity and the essential boundary condition $\partial u/\partial n = 0$ are enforced by the penalty term as $h \downarrow 0$.

Reference

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  Poincaré-Friedrichs inequalities for piecewise H^2 functions

  Numer. Funct. Anal. Optim., 2004

Extensions

Problems with Lower Order Terms

Find $u \in V$ such that

$$\int_\Omega \Big[(D^2u : D^2v) + \beta(x)\nabla u \cdot \nabla v + \gamma(x)uv\Big]dx = \int_\Omega f v\,dx \qquad \forall\, v \in V$$

where $\beta, \gamma \in L_\infty(\Omega)$ are nonnegative functions and

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where $\beta, \gamma \in L_\infty(\Omega)$ are nonnegative functions and

- $V = H_0^2(\Omega)$ for the boundary conditions of clamped plates,
- $V = H^2(\Omega) \cap H_0^1(\Omega)$ for the boundary conditions of simply supported plates,

Problems with Lower Order Terms

Find $u \in V$ such that

$$\int_\Omega \Big[(D^2u : D^2v) + \beta(x)\nabla u \cdot \nabla v + \gamma(x)uv\Big]dx = \int_\Omega f\,v\,dx \qquad \forall\, v \in V$$

where $\beta, \gamma \in L_\infty(\Omega)$ are nonnegative functions and

- $V = H_0^2(\Omega)$ for the boundary conditions of clamped plates,
- $V = H^2(\Omega) \cap H_0^1(\Omega)$ for the boundary conditions of simply supported plates,
- $V = \mathbb{V}$ if $\gamma \neq 0$ and $V = \mathbb{V}^*$ if $\gamma = 0$ (and $\int_\Omega f\,dx = 0$) for the boundary conditions of the Cahn-Hilliard type.

$$\mathbb{V} = \{v \in H^2(\Omega) : \partial v/\partial n = 0 \text{ on } \partial\Omega\}$$
$$\mathbb{V}^* = \{v \in H^2(\Omega) : \partial v/\partial n = 0 \text{ on } \partial\Omega \text{ and } v(p_*) = 0\}$$

Problems with Lower Order Terms

Find $u \in V$ such that

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where $\beta, \gamma \in L_\infty(\Omega)$ are nonnegative functions and

Modifications

- Include

  $$\int_\Omega \Big[\beta(x)\nabla w \cdot \nabla v + \gamma(x)wv\Big]dx$$

  in the definition of the bilinear form $a_h(w, v)$.

Problems with Lower Order Terms

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where $\beta, \gamma \in L_\infty(\Omega)$ are nonnegative functions and

Modifications

- Include

  $$\int_\Omega \Big[\beta(x)\nabla w \cdot \nabla v + \gamma(x)wv\Big]dx$$

  in the definition of the bilinear form $a_h(w, v)$.

- For boundary conditions of the Cahn-Hilliard type, remove the condition $v(p_*) = 0$ in the definition of the finite element space if $\gamma \neq 0$.

Problems with Less Regular Data

Problems with Less Regular Data

Given any $F \in V'$, we have a well-posed continuous problem:

Find $u \in V$ such that

$$\int_\Omega \Big[(D^2 u : D^2 v) + \beta(x)\nabla u \cdot \nabla v + \gamma(x) u v\Big] dx = F(v) \qquad \forall\, v \in V$$

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Discrete Problem   Find $u_h \in V_h$ such that

$$a_h(u,v) = F(v) \qquad \forall\, v \in V_h$$

does not make sense in general, since $V_h \not\subset V$.

Problems with Less Regular Data

Given any $F \in V'$, we have a well-posed continuous problem:

Find $u \in V$ such that

$$\int_\Omega \Big[(D^2u : D^2v) + \beta(x)\nabla u \cdot \nabla v + \gamma(x)uv\Big]dx = F(v) \qquad \forall\, v \in V$$

Discrete Problem   Find $u_h \in V_h$ such that

$$a_h(u,v) = F(v) \qquad \forall\, v \in V_h$$

does not make sense in general, since $V_h \not\subset V$.

Modification   Find $u_h \in V_h$ such that

$$a_h(u,v) = F(E_h v) \qquad \forall\, v \in V_h$$

where E_h is the enriching operator from V_h into V.

Problems on Smooth Domains

We can also combine the C^0 interior penalty approach with the isoparametric approach to solve fourth order boundary value problems on smooth domains.

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