# $C^0$ Interior Penalty Methods

## Geometric Multigrid

Current Research in Finite Element Methods

CIMPA Summer School

Mumbai, July 2015

# Outline

- Ideas
- Set-Up
- Multigrid Algorithms
- Error Propagation Operators
- Smoothing and Approximation
- Convergence of $W$-Cycle
- Convergence of $V$-Cycle
- Other Algorithms

# General References


- Hackbusch, Multi-grid Methods and Applications, Springer-Verlag, 1985.
- Bramble, Multigrid Methods, Longman Scientific & Technical, 1993.
- Bramble and Zhang, *The Analysis of Multigrid Methods*, in Handbook of Numerical Analysis VII, North-Holland, 2000.
- Trottenberg, Oosterlee and Schüller, Multigrid, Academic Press, 2001.
- Briggs, Henson and McCormick, A Multigrid Tutorial (Second Edition), SIAM, 2000.

# Ideas

# Numerical Linear Algebra

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Let $\mathbf{A}$ be a SPD matrix. Suppose we solve

$$\text{(L)} \qquad \mathbf{A}\mathbf{x} = \mathbf{b}$$

by an iterative method (Richardson, Gauss-Seidel, etc.).

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After $m$ iterations (starting with some initial guess), we obtain an approximate solution $\mathbf{x}_m$. Then the error $\mathbf{e}_m = \mathbf{x}_* - \mathbf{x}_m$ satisfies the residual equation

$$(\mathrm{RE}) \qquad \mathbf{A}\mathbf{e}_m = \mathbf{r}_m$$

where $\mathbf{r}_m = \mathbf{b} - \mathbf{A}\mathbf{x}_m$ is the (computable) residual.

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where $\mathbf{r}_m = \mathbf{b} - \mathbf{A}\mathbf{x}_m$ is the (computable) residual.

If we can solve (RE) exactly, then we can recover the exact solution $\mathbf{x}_*$ of (L) by the relation

$$(\mathrm{C}) \qquad \mathbf{x}_* = \mathbf{x}_m + (\mathbf{x}_* - \mathbf{x}_m) = \mathbf{x}_m + \mathbf{e}_m$$

In reality, we will only solve (RE) approximately to obtain an approximation $\mathbf{e}'_m$ of $\mathbf{e}_m$. Then, hopefully, the correction

$$(\mathrm{C}') \qquad \mathbf{x}' = \mathbf{x}_m + \mathbf{e}'_m$$

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$$(\mathrm{FE_h}) \qquad \mathbf{A}_h\mathbf{x}_h = \mathbf{f}_h$$

there is a natural way to carry out this idea.

Smoothing Step    Apply $m$ iterations of a classical iterative method to obtain an approximation $\mathbf{x}_{m,h}$ of $\mathbf{x}_h$ and the corresponding residual equation

$$(\mathrm{RE_h}) \qquad \mathbf{A}_h\mathbf{e}_{m,h} = \mathbf{r}_{m,h}$$

$\mathbf{e}_{m,h} = \mathbf{x}_h - \mathbf{x}_{m,h}, \quad \mathbf{r}_{m,h} = \mathbf{f}_h - \mathbf{A}_h\mathbf{x}_{m,h}$

Correction Step Instead of solving $(\mathrm{RE_h})$, we solve a related equation on a coarser grid $\mathcal{T}_{2h}$ (assuming that $\mathcal{T}_h$ is obtained from $\mathcal{T}_{2h}$ by uniform refinement).

$$(\mathrm{RE_{2h}}) \qquad \mathbf{A}_{2h}\mathbf{e}_{2h} = \mathbf{r}_{2h}$$

$\mathbf{r}_{2h}$ = projection of $\mathbf{r}_{h,m}$ onto the coarse grid space

$\mathbf{A}_{2h}$ = stiffness matrix for the coarse grid

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$\mathbf{A}_{2h} =$ stiffness matrix for the coarse grid

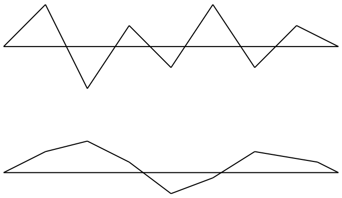
We then use a transfer operator $\mathbf{I}_{2h}^{h}$ to move $\mathbf{e}_{2h}$ to the fine grid $\mathcal{T}_h$ and obtain the final output

$$\mathbf{x}_{m+1,h} = \mathbf{x}_{m,h} + \mathbf{I}_{2h}^{h}\mathbf{e}_{2h}$$

This is known as the two-grid algorithm.

# Numerical Linear Algebra

- Smoothing steps will damp out the highly oscillatory part of the error so that we can capture $\mathbf{e}_{m,h}$ accurately on the coarser grid by the correction step. Together they produce a good approximate solution of $(\mathrm{FE_h})$.



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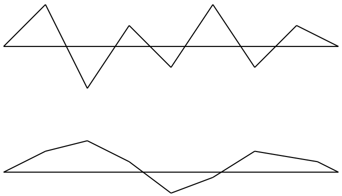

- Moreover, it is cheaper to solve the coarse grid residual equation $(\mathrm{RE_{2h}})$.

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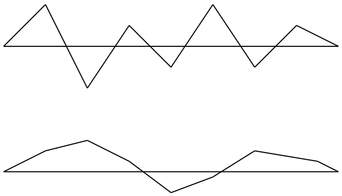


- Moreover, it is cheaper to solve the coarse grid residual equation $(\mathrm{RE_{2h}})$.

Of course we do not have to solve $(\mathrm{RE_{2h}})$ exactly. Instead we can apply the same idea recursively to $(\mathrm{RE_{2h}})$. The resulting algorithm is a multigrid algorithm.

# Set-Up

# Model Problem

Find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = \int_\Omega f\,v\,dx \qquad \forall\, v \in H_0^2(\Omega)$$

$\Omega =$ bounded polygonal domain in $\mathbb{R}^2$

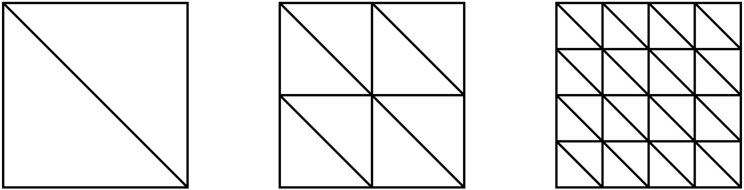
$f \in L_2(\Omega)$

$$a(w,v) = \int_\Omega D^2 w : D^2 v\,dx$$

(Biharmonic equation with the boundary conditions of clamped plates.)

# Discrete Problems

Let $\mathcal{T}_0$ be an initial triangulation of $\Omega$ and $\mathcal{T}_k$ $(k \geq 1)$ be obtained from $\mathcal{T}_{k-1}$ by uniform refinement.



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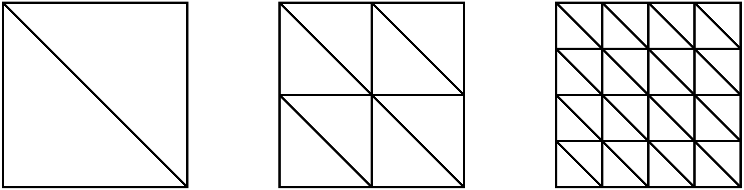

$V_k$ $(k \geq 0)$ is a $\mathbb{P}_j/\mathbb{Q}_j$ $(j \geq 2)$ finite element space associated with $\mathcal{T}_k$ for clamped plates.

$V_0 \subset V_1 \subset \cdots V_k \subset V_{k+1} \subset \cdots$

# Discrete Problems

## $k$-th Level Discrete Problem

Find $u_k \in V_k$ such that

$$a_k(u_k, v) = \int_\Omega f\,v\,dx \qquad \forall\, v \in V_k \qquad (f \in L_2(\Omega))$$

$$\begin{aligned} a_k(w,v) = \sum_{T\in\mathcal{T}_k} \int_T D^2 w : D^2 v\,dx &+ \sum_{e\in\mathcal{E}_k} \int_e \left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds \\ &+ \sum_{e\in\mathcal{E}_k} \int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] ds \\ &+ \sum_{e\in\mathcal{E}_k} \frac{\sigma}{|e|} \int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds \end{aligned}$$

($\mathcal{E}_k$ is the set of the edges of $\mathcal{T}_k$.)

# Discrete Problems

## Discrete Operator $A_k : V_k \longrightarrow V_k'$

$$\langle A_k v, w \rangle = a_k(v, w) \qquad \forall\, v, w \in V_k$$

$\langle \cdot, \cdot \rangle =$ canonical bilinear form on $V_k' \times V_k$

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$A_k$ is SPD in the sense that

$$\begin{aligned} \langle A_k v, w \rangle &= \langle A_k w, v \rangle && \forall\, v, w \in V_k \\ \langle A_k v, v \rangle &> 0 && \forall\, v \in V_k \setminus \{0\}. \end{aligned}$$

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The matrix representing $A_k$ with respect to the natural nodal basis of the finite element space $V_k$ and the dual basis of $V_k'$ is the stiffness matrix, whose condition number is $O(h_k^{-4})$.

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$$A_k u_k = \phi_k$$

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Multigrid algorithms are optimal order iterative methods for

$$(*) \qquad A_k z = \psi$$

where $z \in V_k$ and $\psi \in V_k'$.

# Multigrid Algorithms

# Two Ingredients

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- We need intergrid transfer operators to move data between grids.

- We need a good smoothing scheme for the equation

$$A_k z = \psi$$

  to damp out the highly oscillatory part of the error so that the remaining error can be captured accurately on a coarser grid.

# Intergrid Transfer Operators

$$V_0 \subset V_1 \subset \cdots V_{k-1} \subset V_k \subset \cdots$$

Coarse-to-Fine Operator $\quad I_{k-1}^{k} : V_{k-1} \longrightarrow V_k$

$$I_{k-1}^{k} = \text{natural injection}$$

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Fine-to-Coarse Operator $\quad I_{k}^{k-1} : V_k' \longrightarrow V_{k-1}'$

$$\langle I_k^{k-1}\psi, v\rangle = \langle \psi, I_{k-1}^{k} v\rangle$$

for all $\psi \in V_k'$ and $v \in V_{k-1}$

# Smoother for $A_k z = \psi \qquad (z \in V_k,\ \psi \in V_k')$

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## Preconditioned Iteration

$$(\mathrm{S}) \qquad z_{\mathrm{new}} = z_{\mathrm{old}} + \omega_k B_k^{-1}(\psi - A_k z_{\mathrm{old}})$$

where $B_k : V_k \longrightarrow V'_k$ is SPD and $\omega_k > 0$ is a damping factor chosen so that the spectral radius of the operator $\omega_k B_k^{-1} A_k$ satisfies

$$\rho(\omega_k B_k^{-1} A_k) \leq 1$$

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## Example 1 (standard smoother)

$$\langle B_k v, w \rangle = h_k^2 \sum_{p \in \mathcal{N}_k} v(p) w(p)$$

where $\mathcal{N}_k$ is the set of the (interior) nodes of the finite element space $V_k$.

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$$\langle B_k v, v \rangle \approx \|v\|_{L_2(\Omega)}^2 \qquad \forall\, v \in V_k$$

# Smoother for $A_k z = \psi \qquad (z \in V_k,\ \psi \in V_k')$

Example 2 (nonstandard smoother)

$B_k^{-1} : V_k' \longrightarrow V_k$ is an SPD operator which is an approximate inverse of the discrete Laplace operator $L_k : V_k \longrightarrow V_k'$ defined by

$$\langle L_k v_1, v_2 \rangle = \int_\Omega \nabla v_1 \cdot \nabla v_2 \, dx \qquad \forall\, v \in V_k$$

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We can take $B_k^{-1}$ to be a multigrid Poisson solve, which can be easily implemented because the finite element spaces in the $C^0$ interior penalty methods are standard spaces for second order problems.

# $V$-Cycle Algorithm for $A_k z = \psi$

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Output $= MG_V(k, \psi, z_0, m)$

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Pre-smoothing Step $\quad\quad$ For $1 \leq k \leq m$, compute

$$z_k = z_{k-1} + \omega_k B_k^{-1}(\psi - A_k z_{k-1})$$

# $V$-Cycle Algorithm for $A_k z = \psi$

Correction Step  Transfer the residual $\psi - A_k z_m \in V_k'$ to the coarse grid using $I_k^{k-1}$ and solve the coarse grid residual equation

$$A_{k-1} e_{k-1} = I_k^{k-1}(\psi - A_k z_m)$$

by applying the $(k-1)^{\mathrm{st}}$ level algorithm using $0$ as the initial guess, i.e., we compute

$$q = MG_V(k-1, I_k^{k-1}(\psi - A_k z_m), 0, m)$$

as an approximation to $e_{k-1}$. Then we make the correction

$$z_{m+1} = z_m + I_{k-1}^k q$$

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Post-smoothing Step For $m+2 \le k \le 2m+1$, compute

$$z_k = z_{k-1} + \omega_k B_k^{-1}(\psi - A_k z_{k-1})$$

# $V$-Cycle Algorithm for $A_k z = \psi$

## Final Output

$$MG_V(k, \psi, z_0, m) = z_{2m+1}$$

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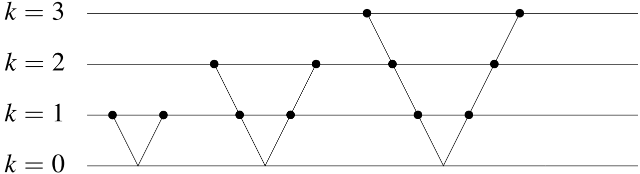


scheduling diagram for the $V$-cycle algorithm

# $W$-Cycle Algorithm for $A_k z = \psi$

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Output $= MG_W(k, \psi, z_0, m)$

$z_0 =$ initial guess $\qquad m =$ number of smoothing steps

# $W$-Cycle Algorithm for $A_k z = \psi$

Output $= MG_W(k, \psi, z_0, m)$

$z_0 =$ initial guess $\qquad m =$ number of smoothing steps

Correction Step (apply the coarse grid algorithm twice)

$$q' = MG_W(k-1, I_k^{k-1}(\psi - A_k z_m), 0, m)$$
$$q = MG_W(k-1, I_k^{k-1}(\psi - A_k z_m), q', m)$$

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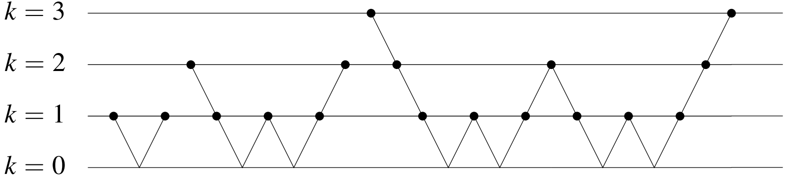


scheduling diagram for the $W$-cycle algorithm

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Output $= MG_F(k, \psi, z_0, m)$

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# $F$-Cycle Algorithm for $A_k z = \psi$

Output $= MG_F(k, \psi, z_0, m)$

$z_0 =$ initial guess $\qquad m =$ number of smoothing steps

Correction Step $\qquad$ (coarse grid algorithm followed by $V$-cycle)

$$q' = MG_F(k-1, I_k^{k-1}(\psi - A_k z_m), 0, m)$$
$$q = MG_V(k-1, I_k^{k-1}(\psi - A_k z_m), q', m)$$

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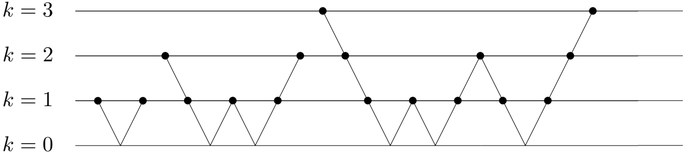


scheduling diagram for the $F$-cycle algorithm

# Operation Counts

$n_k = \dim V_k \qquad (n_k \approx 4^k)$

$W_k =$ number of flops for the $k^{\text{th}}$ level multigrid algorithm

$m =$ number of smoothing steps

$p = 1$ or $2$

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$$W_k \leq C_* m n_k + p W_{k-1}$$

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$$\begin{aligned} W_k &\le C_* m n_k + p W_{k-1} \\ &\le C_* m n_k + p(C_* m n_{k-1}) + p^2(C_* m n_{k-2}) + \cdots p^{k-1}(C_* m n_1) + p^k W \end{aligned}$$

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$m =$ number of smoothing steps

$p = 1$ or $2$

$$\begin{aligned} W_k &\leq C_* m n_k + p W_{k-1} \\ &\leq C_* m n_k + p(C_* m n_{k-1}) + p^2 (C_* m n_{k-2}) + \cdots p^{k-1}(C_* m n_1) + p^k W \\ &\leq C_\dagger m 4^k + p C_\dagger m 4^{k-1} + p^2 C_\dagger m 4^{k-2} + \cdots p^{k-1}(C_\dagger m 4) + p^k W_0 \end{aligned}$$

# Operation Counts

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&= C_\dagger m 4^k \Big(1 + \frac{p}{4} + \frac{p^2}{4^2} + \ldots \frac{p^{k-1}}{4^{k-1}}\Big) + p^k W_0
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&\le \frac{C_\dagger m 4^k}{1 - p/4} + p^k W_0 \le C_\sharp m 4^k + W_0 p^k \le C_\natural m \, n_k
\end{aligned}$$

# Error Propagation Operators

# $V$-Cycle Algorithm

# $V$-Cycle Algorithm

Let $E_k^V : V_k \longrightarrow V_k$ be the error propagation operator that maps the initial error $z - z_0$ to the final error $z - MG_V(k, \psi, z_0, m)$. We want to develop a recursive relation between $E_k^V$ and $E_{k-1}^V$.

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It follows from

$$(\mathrm{S}) \qquad z_{\mathrm{new}} = z_{\mathrm{old}} + \omega_k B_k^{-1}(\psi - A_k z_{\mathrm{old}})$$

and $A_k z = \psi$ that

$$z - z_{\mathrm{new}} = z - z_{\mathrm{old}} - \omega_k B_k^{-1}(\psi - A_k z_{\mathrm{old}}) = (Id_k - \omega_k B_k^{-1} A_k)(z - z_{\mathrm{old}})$$

where $Id_k$ is the identity operator on $V_k$.

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where $Id_k$ is the identity operator on $V_k$.

Therefore the effect of one smoothing step is measured by the operator

$$R_k = Id_k - \omega_k B_k^{-1} A_k$$

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Let $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ be the transpose of the coarse-to-fine operator $I_{k-1}^k$ with respect to the variational forms, i.e.

$$a_{k-1}(P_k^{k-1} v, w) = a_k(v, I_{k-1}^k w) \qquad \forall\, v \in V_k,\ w \in V_{k-1}$$

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&= a_k(z - z_m, I_{k-1}^k v)
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$$\begin{aligned} z - MG_V(k, \gamma, z_0, m) &= z - z_{2m+1} \\ &= R_k^m (z - z_{m+1}) \\ &= R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1}^V P_k^{k-1}) R_k^m (z - z_0) \end{aligned}$$

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## Recursive Relation for $V$-Cycle

$$E_k^V = R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1}^V P_k^{k-1}) R_k^m$$

$$E_0^V = 0$$

# $W$-Cycle Algorithm

Let $E_k^W : V_k \longrightarrow V_k$ be the error propagation operator that maps the initial error $z - z_0$ to the final error $z - MG_W(k, \psi, z_0, m)$.

$$E_k^W = R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m$$

$$E_0^W = 0$$

# $\boldsymbol{F}$-Cycle Algorithm

Let $E_k^F : V_k \longrightarrow V_k$ be the error propagation operator that maps the initial error $z - z_0$ to the final error $z - MG_F(k, \psi, z_0, m)$.

$$E_k^F = R_k^m(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^V E_{k-1}^F) P_k^{k-1}) R_k^m$$

$$E_0^F = 0$$

# Two-Grid Algorithm

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In the two grid algorithm the coarse grid residual equation is solved exactly. The error propagation operator $E_k^{TG}$ is therefore given by

$$E_k^{TG} = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

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$W$-Cycle

$$E_k^W = R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m$$

$F$-Cycle

$$E_k^F = R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^V E_{k-1}^F) P_k^{k-1}) R_k^m$$

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It is clear that the analysis of the two-grid algorithm depends on the property of the operator $R_k^m$ (smoothing property) and the property of the operator $Id_k - I_{k-1}^k P_k^{k-1}$ (approximation property).

# Smoothing and Approximation

# Mesh-Dependent Norms

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Mesh-dependent inner product

$$(v, w)_k = \langle B_k v, w \rangle \qquad \forall\, v, w \in V_k$$

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The operator $B_k^{-1}A_k$ is SPD with respect to $(\cdot,\cdot)_k$.

$$\begin{aligned}((B_k^{-1}A_k)v,w)_k &= \langle A_k v, w\rangle\\ &= a_k(v,w)\\ &= a_k(w,v)\\ &= ((B_k^{-1}A_k)w,v)_k \qquad \forall\, v,w\in V_k\end{aligned}$$

$$((B_k^{-1}A_k)v,v)_k = a_k(v,v) > 0 \qquad \text{if } v \neq 0$$

# Mesh-Dependent Norms

Mesh-dependent inner product

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The operator $B_k^{-1} A_k$ is SPD with respect to $(\cdot, \cdot)_k$.

For $t \in \mathbb{R}$,

$$|\!|\!| v |\!|\!|_{t,k} = \sqrt{((B_k^{-1} A_k)^t v, v)_k} \qquad \forall\, v \in V_k$$

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For $t \in \mathbb{R}$,

$$|||v|||_{t,k} = \sqrt{((B_k^{-1} A_k)^t v, v)_k} \qquad \forall\, v \in V_k$$

In particular

$$|||v|||_{0,k}^2 = (v, v)_k = \langle B_k v, v \rangle \qquad \forall\, v \in V_k$$

$$|||v|||_{1,k}^2 = ((B_k^{-1} A_k) v, v)_k = a_k(v, v) \qquad \forall\, v \in V_k$$

# Mesh-Dependent Norms

**Lemma** (Generalized Cauchy-Schwarz Inequality)

$$a_k(v, w) \leq |||v|||_{1+t,k} |||w|||_{1-t,k} \qquad \forall\, v, w \in V_k$$

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**Lemma** (Generalized Cauchy-Schwarz Inequality)

$$a_k(v,w) \le |\!|\!|v|\!|\!|_{1+t,k} |\!|\!|w|\!|\!|_{1-t,k} \qquad \forall\, v,w \in V_k$$

*Proof.*

$$\begin{aligned}
a_k(v,w) &= \langle A_k v, w\rangle \\
&= \langle B_k(B_k^{-1}A_k)v, w\rangle \\
&= \big((B_k^{-1}A_k)v, w\big)_k \\
&= \big((B_k^{-1}A_k)^{(1+t)/2}v, (B_k^{-1}A_k)^{(1-t)/2}w\big)_k \\
&\le \big((B_k^{-1}A_k)^{(1+t)/2}v, (B_k^{-1}A_k)^{(1+t)/2}v\big)_k^{\frac{1}{2}} \\
&\qquad \times \big((B_k^{-1}A_k)^{(1-t)/2}w, (B_k^{-1}A_k)^{(1-t)/2}w\big)_k^{\frac{1}{2}} \\
&\le \big((B_k^{-1}A_k)^{(1+t)}v, v\big)_k^{\frac{1}{2}} \big((B_k^{-1}A_k)^{(1-t)}w, w\big)_k^{\frac{1}{2}} \\
&= |\!|\!|v|\!|\!|_{1+t,k} |\!|\!|w|\!|\!|_{1-t,k} \qquad \square
\end{aligned}$$

# Spectral Radius of $B_k^{-1}A_k$

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$$\rho(B_k^{-1}A_k) = \max_{v \in V_k} \frac{((B_k^{-1}A_k)v, v)_k}{(v, v)_k} \qquad \text{(Rayleigh Quotient)}$$

$$= \max_{v \in V_k} \frac{\langle A_k v, v\rangle}{\langle B_k v, v\rangle} \qquad ((\cdot,\cdot)_k = \langle B_k v, v\rangle)$$

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## Standard Smoother

$$\langle B_k v, v\rangle \approx \|v\|^2_{L_2(\Omega)} \qquad \forall\, v \in V_k$$

$$\langle A_k v, v\rangle = a_k(v,v) \approx |v|_{H^2(\Omega;\mathcal{T}_k)} \qquad \forall\, v \in V_k$$

$$\rho(B_k^{-1}A_k) \approx h_k^{-4}$$

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## Nonstandard Smoother

$$\langle B_k v, v\rangle \approx |v|^2_{H^1(\Omega)} \qquad \forall\, v \in V_k$$

$$\langle A_k v, v\rangle = a_k(v,v) \approx |v|^2_{H^2(\Omega;\mathcal{T}_k)} \qquad \forall\, v \in V_k$$

$$\rho(B_k^{-1}A_k) \approx h_k^{-2}$$

# Smoothing Property

There exists a constant $C > 0$ (independent of $k$) such that

$$\|R_k^m v\|_{s,k} \leq C \rho_k^{(s-t)/2} m^{(t-s)/2} \|v\|_{t,k} \qquad \forall\, v \in V_k,\ 0 \leq t \leq s \leq 2$$

where $\rho_k = \rho(B_k^{-1} A_k)$.

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where $\rho_k = \rho(B_k^{-1}A_k)$.

*Proof*. Recall

$$R_k^m = (Id_k - \omega_k B_k^{-1} A_k)^m$$

$$\rho(\omega_k B_k^{-1} A_k) = \omega_k \rho_k \le 1$$

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*Proof*. Recall

$$R_k^m = (Id_k - \omega_k B_k^{-1} A_k)^m$$

$$\rho(\omega_k B_k^{-1} A_k) = \omega_k \rho_k \leq 1$$

Therefore we can take

$$\omega_k \approx \rho_k^{-1}$$

# Smoothing Property

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where $\rho_k = \rho(B_k^{-1}A_k)$.

*Proof*. Since $B_k^{-1}A_k$ is SPD with respect to the inner product $(\cdot,\cdot)_k = \langle B_k \cdot, \cdot\rangle$, there exist, by the Spectral Theorem, $v_1, \ldots, v_{n_k}$ ($n_k =$ dim $V_k$) such that

$$B_k^{-1}A_k v_j = \lambda_j v_j \quad \text{and} \quad (v_j, v_\ell)_k = \delta_{j\ell}$$

where $\lambda_1, \ldots, \lambda_{n_k} > 0$ are the eigenvalues of $B_k^{-1}A_k$.

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where $\lambda_1, \ldots, \lambda_{n_k} > 0$ are the eigenvalues of $B_k^{-1} A_k$.

$$(B_k^{-1} A_k)^s v_j = \lambda_j^s v_j \qquad \text{for any } s$$

$$R_k v_j = (Id_k - \omega_k B_k^{-1} A_k) v_j = (1 - \omega_k \lambda_j) v_j$$

# Smoothing Property

$$\|R_k^m v\|_{s,k}^2 = \big((B_k^{-1} A_k)^s R_k^m v, R_k^m v\big)_k$$

# Smoothing Property

$$\begin{aligned}
\|R_k^m v\|_{s,k}^2 &= \left((B_k^{-1}A_k)^s R_k^m v, R_k^m v\right)_k \\
&= \left((B_k^{-1}A_k)^t (B_k^{-1}A_k)^{(s-t)} R_k^m v, R_k^m v\right)_k
\end{aligned}$$

# Smoothing Property

$$\begin{aligned}
|||R_k^m v|||_{s,k}^2 &= \big((B_k^{-1}A_k)^s R_k^m v, R_k^m v\big)_k \\
&= \big((B_k^{-1}A_k)^t (B_k^{-1}A_k)^{(s-t)} R_k^m v, R_k^m v\big)_k \\
&= \omega_k^{(t-s)} \big((B_k^{-1}A_k)^t (\omega_k B_k^{-1}A_k)^{(s-t)} R_k^m v, R_k^m v\big)_k
\end{aligned}$$

# Smoothing Property

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&= \omega_k^{(t-s)} \big((B_k^{-1}A_k)^t (\omega_k B_k^{-1}A_k)^{(s-t)} R_k^m v, R_k^m v\big)_k \\
&= \omega_k^{(t-s)} \sum_{j=1}^{n_k} \lambda_j^t (\omega_k \lambda_j)^{(s-t)} (1-\omega_k\lambda_j)^{2m} c_j^2
\end{aligned}$$

$$v = \sum_{j=1}^{n_k} c_j v_j$$

$$(B_k^{-1}A_k)v_j = \lambda_j v_j \qquad (v_j, v_\ell)_k = \delta_{j\ell}$$

$$R_k^m v_j = (1-\omega_k\lambda_j)^m v_j$$

# Smoothing Property

$$\begin{aligned}
\|R_k^m v\|_{s,k}^2 &= \big((B_k^{-1}A_k)^s R_k^m v, R_k^m v\big)_k \\
&= \big((B_k^{-1}A_k)^t (B_k^{-1}A_k)^{(s-t)} R_k^m v, R_k^m v\big)_k \\
&= \omega_k^{(t-s)} \big((B_k^{-1}A_k)^t (\omega_k B_k^{-1}A_k)^{(s-t)} R_k^m v, R_k^m v\big)_k \\
&= \omega_k^{(t-s)} \sum_{j=1}^{n_k} \lambda_j^t (\omega_k \lambda_j)^{(s-t)} (1-\omega_k\lambda_j)^{2m} c_j^2 \\
&\le \omega_k^{(t-s)} \max_{1\le j\le n_k} \big[(\omega_k \lambda_j)^{(s-t)} (1-\omega_k\lambda_j)^{2m}\big] \sum_{j=1}^{n_k} \lambda_j^t c_j^2
\end{aligned}$$

# Smoothing Property

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&= \big((B_k^{-1}A_k)^t (B_k^{-1}A_k)^{(s-t)} R_k^m v, R_k^m v\big)_k \\
&= \omega_k^{(t-s)} \big((B_k^{-1}A_k)^t (\omega_k B_k^{-1}A_k)^{(s-t)} R_k^m v, R_k^m v\big)_k \\
&= \omega_k^{(t-s)} \sum_{j=1}^{n_k} \lambda_j^t (\omega_k \lambda_j)^{(s-t)} (1-\omega_k\lambda_j)^{2m} c_j^2 \\
&\le \omega_k^{(t-s)} \max_{1\le j\le n_k} \left[(\omega_k\lambda_j)^{(s-t)}(1-\omega_k\lambda_j)^{2m}\right] \sum_{j=1}^{n_k} \lambda_j^t c_j^2 \\
&\le \omega_k^{(t-s)} \max_{0\le x\le 1} \left[x^{(s-t)}(1-x)^{2m}\right] \big((B_k^{-1}A_k)^t v, v\big)_k
\end{aligned}
$$

$\omega_k \lambda_j \le 1$ since $\rho(\omega_k B_k^{-1} A_k) \le 1$

# Smoothing Property

$$
\begin{aligned}
\|R_k^m v\|_{s,k}^2 &= \left((B_k^{-1}A_k)^s R_k^m v, R_k^m v\right)_k \\
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&\le \omega_k^{(t-s)} \max_{1\le j\le n_k} \left[(\omega_k\lambda_j)^{(s-t)}(1-\omega_k\lambda_j)^{2m}\right] \sum_{j=1}^{n_k} \lambda_j^t c_j^2 \\
&\le \omega_k^{(t-s)} \max_{0\le x\le 1} \left[x^{(s-t)}(1-x)^{2m}\right] \left((B_k^{-1}A_k)^t v, v\right)_k \\
&\le \omega_k^{(t-s)} C m^{(t-s)} \|v\|_{t,k}^2
\end{aligned}
$$

$$\max_{0\le x\le 1} \left[x^{(s-t)}(1-x)^{2m}\right] \le C m^{(t-s)} \qquad \text{(calculus)}$$

# Smoothing Property

$$\begin{aligned}
\|R_k^m v\|_{s,k}^2 &= \left((B_k^{-1}A_k)^s R_k^m v, R_k^m v\right)_k \\
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&\leq \omega_k^{(t-s)} C m^{(t-s)} \|v\|_{t,k}^2 \\
&\leq C\rho_k^{(s-t)} m^{(t-s)} \|v\|_{t,k}^2 \qquad \square
\end{aligned}$$

$$\omega_k \approx \rho_k^{-1}$$

# Smoothing Property

In the special case where $t = s$, we have

$$|\!|\!| R_k v |\!|\!|_{s,k} \leq |\!|\!| v |\!|\!|_{s,k} \qquad \forall\, v \in V_k$$

# Approximation Property

Assuming that $\Omega$ is convex, we have

$$\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{0,k} \leq C h_k^2 \|v\|_{2,k} \qquad \forall\, v \in V_k$$

where the mesh-dependent norms are defined in terms of the nonstandard smoother.

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where the mesh-dependent norms are defined in terms of the nonstandard smoother.

*Proof*. Recall that

$$\|v\|_{0,k}^2 = \langle B_k v, v\rangle \approx |v|_{H^1(\Omega)}^2 \qquad \forall\, v \in V_k$$

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where the mesh-dependent norms are defined in terms of the nonstandard smoother.

*Proof*. Recall that

$$\|v\|_{0,k}^2 = \langle B_k v, v\rangle \approx |v|_{H^1(\Omega)}^2 \qquad \forall\, v \in V_k$$

By duality

$$\begin{aligned}\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{0,k} &\approx |(Id_k - I_{k-1}^k P_k^{k-1})v|_{H^1(\Omega)} \\ &= \sup_{\phi \in H^{-1}(\Omega)} \frac{\phi((Id_k - I_{k-1}^k P_k^{k-1})v)}{\|\phi\|_{H^{-1}(\Omega)}}\end{aligned}$$

# Approximation Property

Let $\phi \in H^{-1}(\Omega)$ be arbitrary and define $\zeta \in H_0^2(\Omega)$, $\zeta_k \in V_k$ and $\zeta_{k-1} \in V_{k-1}$ by

$$\begin{aligned} a(\zeta, v) &= \phi(v) \qquad \forall\, v \in H_0^2(\Omega) \\ a_k(\zeta_k, v) &= \phi(v) \qquad \forall\, v \in V_k \\ a_{k-1}(\zeta_{k-1}, v) &= \phi(v) \qquad \forall\, v \in V_{k-1} \end{aligned}$$

$\zeta$ is the solution of the clamped plate problem where the right-hand side $\phi$ belongs to $H^{-1}(\Omega)$.

$\zeta_k$ is the approximation of $\zeta$ obtained by the $k$-th level $C^0$ interior penalty method.

$\zeta_{k-1}$ is the approximation of $\zeta$ obtained by the $(k-1)$-st level $C^0$ interior penalty method.

# Approximation Property

Let $\phi \in H^{-1}(\Omega)$ be arbitrary and define $\zeta \in H_0^2(\Omega)$, $\zeta_k \in V_k$ and $\zeta_{k-1} \in V_{k-1}$ by

$$a(\zeta, v) = \phi(v) \qquad \forall\, v \in H_0^2(\Omega)$$

$$a_k(\zeta_k, v) = \phi(v) \qquad \forall\, v \in V_k$$

$$a_{k-1}(\zeta_{k-1}, v) = \phi(v) \qquad \forall\, v \in V_{k-1}$$

## Error Estimates in $H^1(\Omega)$

$$|\zeta - \zeta_k|_{H^1(\Omega)} \le C h_k^2 |\phi|_{H^{-1}(\Omega)}$$

$$|\zeta - \zeta_{k-1}|_{H^1(\Omega)} \le C h_{k-1}^2 |\phi|_{H^{-1}(\Omega)} \le C h_k^2 |\phi|_{H^{-1}(\Omega)}$$

# Approximation Property

Let $\phi \in H^{-1}(\Omega)$ be arbitrary and define $\zeta \in H_0^2(\Omega)$, $\zeta_k \in V_k$ and $\zeta_{k-1} \in V_{k-1}$ by

$$
\begin{aligned}
a(\zeta, v) &= \phi(v) \qquad \forall\, v \in H_0^2(\Omega) \\
a_k(\zeta_k, v) &= \phi(v) \qquad \forall\, v \in V_k \\
a_{k-1}(\zeta_{k-1}, v) &= \phi(v) \qquad \forall\, v \in V_{k-1}
\end{aligned}
$$

## Error Estimates in $H^1(\Omega)$

$$
\begin{aligned}
|\zeta - \zeta_k|_{H^1(\Omega)} &\le C h_k^2 |\phi|_{H^{-1}(\Omega)} \\
|\zeta - \zeta_{k-1}|_{H^1(\Omega)} &\le C h_{k-1}^2 |\phi|_{H^{-1}(\Omega)} \le C h_k^2 |\phi|_{H^{-1}(\Omega)}
\end{aligned}
$$

## Relation between $\zeta_k$ and $\zeta_{k-1}$

$$
\zeta_{k-1} = P_k^{k-1} \zeta_k
$$

# Approximation Property

$$\phi((Id_k - I_{k-1}^k P_k^{k-1})v) = a_k(\zeta_k, (Id_k - I_{k-1}^k P_k^{k-1})v)$$

$$a_k(\zeta_k, v) = \phi(v) \qquad \forall\, v \in V_k$$

# Approximation Property

$$\begin{aligned}\phi((Id_k - I_{k-1}^k P_k^{k-1})v) &= a_k(\zeta_k, (Id_k - I_{k-1}^k P_k^{k-1})v) \\ &= a_k(\zeta_k, v) - a_k(\zeta_k, I_{k-1}^k P_k^{k-1} v)\end{aligned}$$

# Approximation Property

$$\begin{aligned}\phi((Id_k - I_{k-1}^k P_k^{k-1})v) &= a_k(\zeta_k, (Id_k - I_{k-1}^k P_k^{k-1})v) \\ &= a_k(\zeta_k, v) - a_k(\zeta_k, I_{k-1}^k P_k^{k-1} v) \\ &= a_k(\zeta_k, v) - a_{k-1}(P_k^{k-1}\zeta_k, P_k^{k-1} v)\end{aligned}$$

$$a_{k-1}(P_k^{k-1} v, w) = a_k(v, I_{k-1}^k w) \qquad \forall\, v \in V_k,\ w \in V_{k-1}$$

# Approximation Property

$$\begin{aligned}
\phi((Id_k - I_{k-1}^k P_k^{k-1})v) &= a_k(\zeta_k, (Id_k - I_{k-1}^k P_k^{k-1})v) \\
&= a_k(\zeta_k, v) - a_k(\zeta_k, I_{k-1}^k P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(P_k^{k-1}\zeta_k, P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(\zeta_{k-1}, P_k^{k-1} v)
\end{aligned}$$

$$\zeta_{k-1} = P_k^{k-1}\zeta_k$$

# Approximation Property

$$
\begin{aligned}
\phi((Id_k - I_{k-1}^k P_k^{k-1})v) &= a_k(\zeta_k, (Id_k - I_{k-1}^k P_k^{k-1})v) \\
&= a_k(\zeta_k, v) - a_k(\zeta_k, I_{k-1}^k P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(P_k^{k-1}\zeta_k, P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(\zeta_{k-1}, P_k^{k-1} v) \\
&= a_k(\zeta_k - I_{k-1}^k \zeta_{k-1}, v)
\end{aligned}
$$

$$
a_{k-1}(w, P_k^{k-1} v) = a_k(I_{k-1}^k w, v) \qquad \forall\, v \in V_k,\ w \in V_{k-1}
$$

# Approximation Property

$$\begin{aligned}
\phi((Id_k - I_{k-1}^k P_k^{k-1})v) &= a_k(\zeta_k, (Id_k - I_{k-1}^k P_k^{k-1})v) \\
&= a_k(\zeta_k, v) - a_k(\zeta_k, I_{k-1}^k P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(P_k^{k-1}\zeta_k, P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(\zeta_{k-1}, P_k^{k-1} v) \\
&= a_k(\zeta_k - I_{k-1}^k \zeta_{k-1}, v) \\
&\leq \|\zeta_k - I_{k-1}^k \zeta_{k-1}\|_{0,k} \|v\|_{2,k}
\end{aligned}$$

$$a_k(w, v) \leq \|w\|_{1-t,k} \|v\|_{1+t,k} \qquad \forall\, v, w \in V_k$$

# Approximation Property

$$\begin{aligned}
\phi((Id_k - I_{k-1}^k P_k^{k-1})v) &= a_k(\zeta_k, (Id_k - I_{k-1}^k P_k^{k-1})v) \\
&= a_k(\zeta_k, v) - a_k(\zeta_k, I_{k-1}^k P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(P_k^{k-1}\zeta_k, P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(\zeta_{k-1}, P_k^{k-1} v) \\
&= a_k(\zeta_k - I_{k-1}^k \zeta_{k-1}, v) \\
&\leq \|\zeta_k - I_{k-1}^k \zeta_{k-1}\|_{0,k} \|v\|_{2,k} \\
&\leq C|\zeta_k - \zeta_{k-1}|_{H^1(\Omega)} \|v\|_{2,k}
\end{aligned}$$

$I_{k-1}^k =$ natural injection

$\|\cdot\|_{0,k} \approx |\cdot|_{H^1(\Omega)}$ on $V_k$

# Approximation Property

$$
\begin{aligned}
\phi((Id_k - I_{k-1}^k P_k^{k-1})v) &= a_k(\zeta_k, (Id_k - I_{k-1}^k P_k^{k-1})v) \\
&= a_k(\zeta_k, v) - a_k(\zeta_k, I_{k-1}^k P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(P_k^{k-1}\zeta_k, P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(\zeta_{k-1}, P_k^{k-1} v) \\
&= a_k(\zeta_k - I_{k-1}^k \zeta_{k-1}, v) \\
&\le \|\zeta_k - I_{k-1}^k \zeta_{k-1}\|_{0,k} \|v\|_{2,k} \\
&\le C|\zeta_k - \zeta_{k-1}|_{H^1(\Omega)} \|v\|_{2,k} \\
&\le C\big(|\zeta_k - \zeta|_{H^1(\Omega)} + |\zeta - \zeta_{k-1}|_{H^1(\Omega)}\big) \|v\|_{2,k}
\end{aligned}
$$

# Approximation Property

$$\begin{aligned}
\phi((Id_k - I_{k-1}^k P_k^{k-1})v) &= a_k(\zeta_k, (Id_k - I_{k-1}^k P_k^{k-1})v) \\
&= a_k(\zeta_k, v) - a_k(\zeta_k, I_{k-1}^k P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(P_k^{k-1}\zeta_k, P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(\zeta_{k-1}, P_k^{k-1} v) \\
&= a_k(\zeta_k - I_{k-1}^k \zeta_{k-1}, v) \\
&\leq |||\zeta_k - I_{k-1}^k \zeta_{k-1}|||_{0,k} |||v|||_{2,k} \\
&\leq C|\zeta_k - \zeta_{k-1}|_{H^1(\Omega)} |||v|||_{2,k} \\
&\leq C\big(|\zeta_k - \zeta|_{H^1(\Omega)} + |\zeta - \zeta_{k-1}|_{H^1(\Omega)}\big) |||v|||_{2,k} \\
&\leq C h_k^2 \|\phi\|_{H^{-1}(\Omega)} |||v|||_{2,k}
\end{aligned}$$

$$\begin{aligned}
|\zeta - \zeta_k|_{H^1(\Omega)} &\leq C h_k^2 \|\phi\|_{H^{-1}(\Omega)} \\
|\zeta - \zeta_{k-1}|_{H^1(\Omega)} &\leq C h_k^2 \|\phi\|_{H^{-1}(\Omega)}
\end{aligned}$$

# Approximation Property

$$
\begin{aligned}
\phi((Id_k - I_{k-1}^k P_k^{k-1})v) &= a_k(\zeta_k, (Id_k - I_{k-1}^k P_k^{k-1})v) \\
&= a_k(\zeta_k, v) - a_k(\zeta_k, I_{k-1}^k P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(P_k^{k-1}\zeta_k, P_k^{k-1} v) \\
&= a_k(\zeta_k, v) - a_{k-1}(\zeta_{k-1}, P_k^{k-1} v) \\
&= a_k(\zeta_k - I_{k-1}^k \zeta_{k-1}, v) \\
&\le \|\zeta_k - I_{k-1}^k \zeta_{k-1}\|_{0,k} \|v\|_{2,k} \\
&\le C|\zeta_k - \zeta_{k-1}|_{H^1(\Omega)} \|v\|_{2,k} \\
&\le C\big(|\zeta_k - \zeta|_{H^1(\Omega)} + |\zeta - \zeta_{k-1}|_{H^1(\Omega)}\big) \|v\|_{2,k} \\
&\le C h_k^2 \|\phi\|_{H^{-1}(\Omega)} \|v\|_{2,k}
\end{aligned}
$$

$$
\begin{aligned}
\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{0,k} &\approx \sup_{\phi \in H^{-1}(\Omega)} \frac{\phi((Id_k - I_{k-1}^k P_k^{k-1})v)}{\|\phi\|_{H^{-1}(\Omega)}} \\
&\le C h_k^2 \|v\|_{2,k} \quad \square
\end{aligned}
$$

# Approximation Property

Assuming that $\Omega$ is convex, we have

$$\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{\frac{1}{2},k} \le C h_k^2 \|v\|_{\frac{3}{2},k} \qquad \forall\, v \in V_k$$

where the mesh-dependent norms are defined in terms of the standard smoother.

# Approximation Property

Assuming that $\Omega$ is convex, we have

$$\|\!|(Id_k - I_{k-1}^k P_k^{k-1})v\|\!|_{\frac{1}{2},k} \leq Ch_k^2 \|\!|v\|\!|_{\frac{3}{2},k} \qquad \forall\, v \in V_k$$

where the mesh-dependent norms are defined in terms of the standard smoother.

This approximation property has a similar derivation. The difference is that in this case

$$\|\!|\cdot\|\!|_{\frac{1}{2},k} \approx |\cdot|_{H^1(\Omega)} \quad \text{on } V_k$$

because

$$\|\!|v\|\!|_{0,k} \approx \|v\|_{L_2(\Omega)} \qquad \forall\, v \in V_k$$

$$\|\!|v\|\!|_{1,k} \approx \|v\|_{H^2(\Omega;\mathcal{T}_k)} \qquad \forall\, v \in V_k$$

# Approximation Property

Assuming that $\Omega$ is convex, we have

$$|||(Id_k - I_{k-1}^k P_k^{k-1})v|||_{\frac{1}{2},k} \leq C h_k^2 |||v|||_{\frac{3}{2},k} \qquad \forall\, v \in V_k$$

where the mesh-dependent norms are defined in terms of the standard smoother.

This approximation property has a similar derivation. The difference is that in this case

$$|||\cdot|||_{\frac{1}{2},k} \approx |\cdot|_{H^1(\Omega)} \quad \text{on } V_k$$

Therefore the generalized Cauchy-Schwarz inequality dictates that

$$|||(Id_k - I_{k-1}^k P_k^{k-1})v|||_{\frac{1}{2},k} \leq C h_k^2 |||v|||_{\frac{3}{2},k} \qquad \forall\, v \in V_k$$

# Convergence of $W$- Cycle

# Convergence of the Two-Grid Algorithm

# Convergence of the Two-Grid Algorithm

## Standard Smoother

$(\rho_k = \rho(B_k^{-1} A_k) \approx h_k^{-4})$

$$\|E_k^{TG} v\|_{1,k} = \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k}$$

$$E_k^{TG} = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

# Convergence of the Two-Grid Algorithm

## Standard Smoother $\quad (\rho_k = \rho(B_k^{-1}A_k) \approx h_k^{-4})$

$$\begin{aligned}\|E_k^{TG}v\|_{1,k} &= \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{1,k} \\ &\leq C(\rho_k^{1/4}m^{-1/4})\|(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{\frac{1}{2},k}\end{aligned}$$

## Smoothing Property

$$\|R_k^m v\|_{s,k} \leq C\rho_k^{(s-t)/2} m^{(t-s)/2}\|v\|_{t,k}$$

$$\|R_k^m v\|_{1,k} \leq C\rho_k^{1/4} m^{-1/4}\|v\|_{\frac{1}{2},k}$$

# Convergence of the Two-Grid Algorithm

## Standard Smoother $\quad (\rho_k = \rho(B_k^{-1}A_k) \approx h_k^{-4})$

$$\begin{aligned}
\|\!|E_k^{TG}v\|\!|_{1,k} &= \|\!|R_k^m(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|\!|_{1,k} \\
&\le C(\rho_k^{1/4} m^{-1/4})\|\!|(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|\!|_{\frac{1}{2},k} \\
&\le C(\rho_k^{1/4} m^{-1/4}) h_k^2 \|\!|R_k^m v\|\!|_{\frac{3}{2},k}
\end{aligned}$$

## Approximation Property

$$\|\!|(Id_k - I_{k-1}^k P_k^{k-1})v\|\!|_{\frac{1}{2},k} \le C h_k^2 \|\!|v\|\!|_{\frac{3}{2},k}$$

# Convergence of the Two-Grid Algorithm

## Standard Smoother $\quad (\rho_k = \rho(B_k^{-1} A_k) \approx h_k^{-4})$

$$
\begin{aligned}
\|E_k^{TG} v\|_{1,k} &= \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} \\
&\le C(\rho_k^{1/4} m^{-1/4}) \|(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{\frac{1}{2},k} \\
&\le C(\rho_k^{1/4} m^{-1/4}) h_k^2 \|R_k^m v\|_{\frac{3}{2},k} \\
&\le C(\rho_k^{1/4} m^{-1/4}) h_k^2 (\rho_k^{1/4} m^{-1/4}) \|v\|_{1,k}
\end{aligned}
$$

## Smoothing Property

$$\|R_k^m v\|_{s,k} \le C \rho_k^{(s-t)/2} m^{(t-s)/2} \|v\|_{t,k}$$

$$\|R_k^m v\|_{\frac{3}{2},k} \le C \rho_k^{1/4} m^{-1/4} \|v\|_{1,k}$$

# Convergence of the Two-Grid Algorithm

## Standard Smoother $\quad (\rho_k = \rho(B_k^{-1} A_k) \approx h_k^{-4})$

$$\begin{aligned}
\|E_k^{TG} v\|_{1,k} &= \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} \\
&\leq C(\rho_k^{1/4} m^{-1/4}) \|(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{\frac{1}{2},k} \\
&\leq C(\rho_k^{1/4} m^{-1/4}) h_k^2 \|R_k^m v\|_{\frac{3}{2},k} \\
&\leq C(\rho_k^{1/4} m^{-1/4}) h_k^2 (\rho_k^{1/4} m^{-1/4}) \|v\|_{1,k} \\
&\leq C m^{-1/2} \|v\|_{1,k}
\end{aligned}$$

$$\rho_k \approx h_k^{-4}$$

# Convergence of the Two-Grid Algorithm

Standard Smoother $\quad (\rho_k = \rho(B_k^{-1}A_k) \approx h_k^{-4})$

$$
\begin{aligned}
\|E_k^{TG}v\|_{1,k} &= \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{1,k} \\
&\le C(\rho_k^{1/4}m^{-1/4})\|(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{\frac{1}{2},k} \\
&\le C(\rho_k^{1/4}m^{-1/4})h_k^2\|R_k^m v\|_{\frac{3}{2},k} \\
&\le C(\rho_k^{1/4}m^{-1/4})h_k^2(\rho_k^{1/4}m^{-1/4})\|v\|_{1,k} \\
&\le Cm^{-1/2}\|v\|_{1,k}
\end{aligned}
$$

The two-grid algorithm is a contraction if $m$ is sufficiently large and the contraction number will decrease at the rate of $1/\sqrt{m}$.

# Convergence of the Two-Grid Algorithm

## Nonstandard Smoother

$(\rho_k = \rho(B_k^{-1} A_k) \approx h_k^{-2})$

$$\|E_k^{TG} v\|_{1,k} = \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k}$$

$$E_k^{TG} = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

# Convergence of the Two-Grid Algorithm

## Nonstandard Smoother $(\rho_k = \rho(B_k^{-1}A_k) \approx h_k^{-2})$

$$
\begin{aligned}
\| E_k^{TG} v \|_{1,k} &= \| R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v \|_{1,k} \\
&\leq C(\rho_k^{1/2} m^{-1/2}) \| (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v \|_{0,k}
\end{aligned}
$$

## Smoothing Property

$$
\| R_k^m v \|_{s,k} \leq C \rho_k^{(s-t)/2} m^{(t-s)/2} \| v \|_{t,k}
$$

$$
\| R_k^m v \|_{1,k} \leq C \rho_k^{1/2} m^{-1/2} \| v \|_{0,k}
$$

# Convergence of the Two-Grid Algorithm

## Nonstandard Smoother $(\rho_k = \rho(B_k^{-1}A_k) \approx h_k^{-2})$

$$\begin{aligned}
\|E_k^{TG}v\|_{1,k} &= \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{1,k} \\
&\le C(\rho_k^{1/2}m^{-1/2})\|(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{0,k} \\
&\le C(\rho_k^{1/2}m^{-1/2})h_k^2\|R_k^m v\|_{2,k}
\end{aligned}$$

## Approximation Property

$$\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{0,k} \le Ch_k^2\|v\|_{2,k}$$

# Convergence of the Two-Grid Algorithm

## Nonstandard Smoother $\quad (\rho_k = \rho(B_k^{-1}A_k) \approx h_k^{-2})$

$$
\begin{aligned}
|\!|\!| E_k^{TG} v |\!|\!|_{1,k} &= |\!|\!| R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v |\!|\!|_{1,k} \\
&\le C(\rho_k^{1/2} m^{-1/2}) |\!|\!| (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v |\!|\!|_{0,k} \\
&\le C(\rho_k^{1/2} m^{-1/2}) h_k^2 |\!|\!| R_k^m v |\!|\!|_{2,k} \\
&\le C(\rho_k^{1/2} m^{-1/2}) h_k^2 (\rho_k^{1/2} m^{-1/2}) |\!|\!| v |\!|\!|_{1,k}
\end{aligned}
$$

## Smoothing Property

$$|\!|\!| R_k^m v |\!|\!|_{s,k} \le C \rho_k^{(s-t)/2} m^{(t-s)/2} |\!|\!| v |\!|\!|_{t,k}$$

$$|\!|\!| R_k^m v |\!|\!|_{2,k} \le C \rho_k^{1/2} m^{-1/2} |\!|\!| v |\!|\!|_{1,k}$$

# Convergence of the Two-Grid Algorithm

## Nonstandard Smoother ($\rho_k = \rho(B_k^{-1}A_k) \approx h_k^{-2}$)

$$
\begin{aligned}
\|E_k^{TG}v\|_{1,k} &= \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{1,k} \\
&\le C(\rho_k^{1/2}m^{-1/2})\|(Id_k - I_{k-1}^k P_k^{k-1})R_k^m v\|_{0,k} \\
&\le C(\rho_k^{1/2}m^{-1/2})h_k^2\|R_k^m v\|_{2,k} \\
&\le C(\rho_k^{1/2}m^{-1/2})h_k^2(\rho_k^{1/2}m^{-1/2})\|v\|_{1,k} \\
&\le Cm^{-1}\|v\|_{1,k}
\end{aligned}
$$

$$\rho_k \approx h_k^{-2}$$

# Convergence of the Two-Grid Algorithm

## Nonstandard Smoother $\quad (\rho_k = \rho(B_k^{-1} A_k) \approx h_k^{-2})$

$$
\begin{aligned}
\|E_k^{TG} v\|_{1,k} &= \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} \\
&\le C(\rho_k^{1/2} m^{-1/2}) \|(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{0,k} \\
&\le C(\rho_k^{1/2} m^{-1/2}) h_k^2 \|R_k^m v\|_{2,k} \\
&\le C(\rho_k^{1/2} m^{-1/2}) h_k^2 (\rho_k^{1/2} m^{-1/2}) \|v\|_{1,k} \\
&\le C m^{-1} \|v\|_{1,k}
\end{aligned}
$$

The two-grid algorithm is a contraction if $m$ is sufficiently large and the contraction number will decrease at the rate of $1/m$.

# Convergence of the Two-Grid Algorithm

Convex Polygonal Domain

## Standard Smoother

$$\|E_k^{TG} v\|_{1,k} \le C m^{-1/2} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

## Nonstandard Smoother

$$\|E_k^{TG} v\|_{1,k} \le C m^{-1} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

# Convergence of the Two-Grid Algorithm

General Polygonal Domain

## Standard Smoother

$$\|E_k^{TG} v\|_{1,k} \le C m^{-\alpha/2} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

## Nonstandard Smoother

$$\|E_k^{TG} v\|_{1,k} \le C m^{-\alpha} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

where $\alpha \in (\frac{1}{2}, 2]$ is the index of elliptic regularity for clamped plates.

# Convergence of the Two-Grid Algorithm

For the mesh-dependent norms defined in terms of the the non-standard smoother, the proof of the approximation property in the case of convex polygonal domains depends on the fact that

$$|||v|||_{0,k} \approx |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k$$

The proof of the approximation property for general polygonal domains requires the relation

$$|||v|||_{1-\alpha,k} \approx |v|_{H^{2-\alpha}(\Omega)} \qquad \forall\, v \in V_k$$

whose proof depends on the existence of a one-to-one enriching operator $E_h : V_h \longrightarrow H_0^2(\Omega)$.

# Convergence of the Two-Grid Algorithm

General Polygonal Domain

## Standard Smoother

$$\| E_k^{TG} v \|_{1,k} \le C m^{-\alpha/2} \| v \|_{1,k} \qquad \forall\, v \in V_k$$

## Nonstandard Smoother

$$\| E_k^{TG} v \|_{1,k} \le C m^{-\alpha} \| v \|_{1,k} \qquad \forall\, v \in V_k$$

Comparing these two estimates we see that the effect of 100 smoothing steps by the standard smoother is (roughly) equivalent to the effect of 10 smoothing steps by the nonstandard smoother.

# Convergence of the Two-Grid Algorithm

General Polygonal Domain

## Standard Smoother

$$\|E_k^{TG} v\|_{1,k} \le C m^{-\alpha/2} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

## Nonstandard Smoother

$$\|E_k^{TG} v\|_{1,k} \le C m^{-\alpha} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

This difference in performance is due to the difference in the spectral radius of the preconditioned operators.

$$\rho(B_k^{-1} A_k) \approx \begin{cases} h_k^{-4} & \text{standard smoother} \\ h_k^{-2} & \text{nonstandard smoother} \end{cases}$$

# Convergence of the $W$-Cycle Algorithm

# Convergence of the $W$-Cycle Algorithm

Estimate for the Operator $I_{k-1}^{k} : V_{k-1} \longrightarrow V_k$

$$\|I_{k-1}^{k} v\|_{1,k} \leq C_{\mathrm{CF}} \|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}$$

where the positive constant $C_{\mathrm{CF}}$ is independent of $k$.

# Convergence of the $W$-Cycle Algorithm

Estimate for the Operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$

$$\|I_{k-1}^k v\|_{1,k} \leq C_{\mathrm{CF}} \|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}$$

where the positive constant $C_{\mathrm{CF}}$ is independent of $k$.

$$\|I_{k-1}^k v\|_{1,k}^2 = a_k(I_{k-1}^k v, I_{k-1}^k v)$$

# Convergence of the $W$-Cycle Algorithm

Estimate for the Operator $I_{k-1}^{k} : V_{k-1} \longrightarrow V_k$

$$\|I_{k-1}^{k} v\|_{1,k} \leq C_{\mathrm{CF}} \|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}$$

where the positive constant $C_{\mathrm{CF}}$ is independent of $k$.

$$\begin{aligned} \|I_{k-1}^{k} v\|_{1,k}^2 &= a_k(I_{k-1}^{k} v, I_{k-1}^{k} v) \\ &\approx \sum_{T \in \mathcal{T}_k} |I_{k-1}^{k} v|_{H^2(T)}^2 + \sum_{e \in \mathcal{E}_k} \frac{\sigma}{|e|} \|I_{k-1}^{k} v\|_{L_2(e)}^2 \end{aligned}$$

# Convergence of the $W$-Cycle Algorithm

Estimate for the Operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$

$$\|\!| I_{k-1}^k v \|\!|_{1,k} \leq C_{\mathrm{CF}} \|\!| v \|\!|_{1,k-1} \qquad \forall\, v \in V_{k-1}$$

where the positive constant $C_{\mathrm{CF}}$ is independent of $k$.

$$\begin{aligned}
\|\!| I_{k-1}^k v \|\!|_{1,k}^2 &= a_k(I_{k-1}^k v, I_{k-1}^k v) \\
&\approx \sum_{T \in \mathcal{T}_k} |I_{k-1}^k v|_{H^2(T)}^2 + \sum_{e \in \mathcal{E}_k} \frac{\sigma}{|e|} \|I_{k-1}^k v\|_{L_2(e)}^2 \\
&= \sum_{T \in \mathcal{T}_k} |v|_{H^2(T)}^2 + \sum_{e \in \mathcal{E}_k} \frac{\sigma}{|e|} \|v\|_{L_2(e)}^2
\end{aligned}$$

# Convergence of the $W$-Cycle Algorithm

Estimate for the Operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$

$$\|I_{k-1}^k v\|_{1,k} \le C_{\mathrm{CF}} \|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}$$

where the positive constant $C_{\mathrm{CF}}$ is independent of $k$.

$$\begin{aligned}
\|I_{k-1}^k v\|_{1,k}^2 &= a_k(I_{k-1}^k v, I_{k-1}^k v) \\
&\approx \sum_{T\in\mathcal{T}_k} |I_{k-1}^k v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_k} \frac{\sigma}{|e|} \|I_{k-1}^k v\|_{L_2(e)}^2 \\
&= \sum_{T\in\mathcal{T}_k} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_k} \frac{\sigma}{|e|} \|v\|_{L_2(e)}^2 \\
&= \sum_{T\in\mathcal{T}_{k-1}} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_k} \frac{\sigma}{|e|} \|v\|_{L_2(e)}^2
\end{aligned}$$

# Convergence of the $W$-Cycle Algorithm

Estimate for the Operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$

$$\|I_{k-1}^k v\|_{1,k} \le C_{\mathrm{CF}} \|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}$$

where the positive constant $C_{\mathrm{CF}}$ is independent of $k$.

$$\begin{aligned}
\|I_{k-1}^k v\|_{1,k}^2 &= a_k(I_{k-1}^k v, I_{k-1}^k v) \\
&\approx \sum_{T\in\mathcal{T}_k} |I_{k-1}^k v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_k} \frac{\sigma}{|e|} \|I_{k-1}^k v\|_{L_2(e)}^2 \\
&= \sum_{T\in\mathcal{T}_k} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_k} \frac{\sigma}{|e|} \|v\|_{L_2(e)}^2 \\
&= \sum_{T\in\mathcal{T}_{k-1}} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_k} \frac{\sigma}{|e|} \|v\|_{L_2(e)}^2 \\
&= \sum_{T\in\mathcal{T}_{k-1}} |v|_{H^2(T)}^2 + 2 \sum_{e\in\mathcal{E}_{k-1}} \frac{\sigma}{|e|} \|v\|_{L_2(e)}^2
\end{aligned}$$

# Convergence of the $W$-Cycle Algorithm

Estimate for the Operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$

$$\|I_{k-1}^k v\|_{1,k} \leq C_{\mathrm{CF}} \|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}$$

where the positive constant $C_{\mathrm{CF}}$ is independent of $k$.

$$\begin{aligned}
\|I_{k-1}^k v\|_{1,k}^2 &= a_k(I_{k-1}^k v, I_{k-1}^k v) \\
&\approx \sum_{T\in\mathcal{T}_k} |I_{k-1}^k v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_k} \frac{\sigma}{|e|} \|I_{k-1}^k v\|_{L_2(e)}^2 \\
&= \sum_{T\in\mathcal{T}_k} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_k} \frac{\sigma}{|e|} \|v\|_{L_2(e)}^2 \\
&= \sum_{T\in\mathcal{T}_{k-1}} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_k} \frac{\sigma}{|e|} \|v\|_{L_2(e)}^2 \\
&= \sum_{T\in\mathcal{T}_{k-1}} |v|_{H^2(T)}^2 + 2 \sum_{e\in\mathcal{E}_{k-1}} \frac{\sigma}{|e|} \|v\|_{L_2(e)}^2 \\
&\approx \|v\|_{1,k-1}^2
\end{aligned}$$

# Convergence of the $W$-Cycle Algorithm

Estimate for the Operator $I_{k-1}^{k} : V_{k-1} \longrightarrow V_k$

$$\|I_{k-1}^{k} v\|_{1,k} \leq C_{\mathrm{CF}} \|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}$$

where the positive constant $C_{\mathrm{CF}}$ is independent of $k$.

Estimate for the Operator $P_{k}^{k-1} : V_k \longrightarrow V_{k-1}$

$$\|P_{k}^{k-1} v\|_{1,k-1} \leq C_{\mathrm{CF}} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

# Convergence of the $W$-Cycle Algorithm

Two-Grid Estimates

$$\|E_k^{TG}v\|_{1,k} \leq C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

where $\gamma = 1$ for the nonstandard smoother and $\gamma = 2$ for the standard smoother.

# Convergence of the $W$-Cycle Algorithm

Two-Grid Estimates

$$\|E_k^{TG} v\|_{1,k} \leq C_{\text{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

where $\gamma = 1$ for the nonstandard smoother and $\gamma = 2$ for the standard smoother.

**Theorem** Given any $C_* > C_{\text{TG}}$, there exists a positive integer $m_*$ independent of $k$ such that

$$\|E_k^{W} v\|_{1,k} \leq C_* m^{-\alpha/\gamma} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

provided $m \geq m_*$.

# Convergence of the $W$-Cycle Algorithm

Two-Grid Estimates

$$\|E_k^{TG} v\|_{1,k} \le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

where $\gamma = 1$ for the nonstandard smoother and $\gamma = 2$ for the standard smoother.

Theorem Given any $C_* > C_{\mathrm{TG}}$, there exists a positive integer $m_*$ independent of $k$ such that

$$\|E_k^{W} v\|_{1,k} \le C_* m^{-\alpha/\gamma} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

provided $m \ge m_*$.

The $W$-cycle algorithm is a contraction if the number of smoothing steps is sufficiently large (but independent of $k$). The contraction numbers are bounded away from $1$ on all levels, i.e., the $W$-cycle algorithm is uniformly convergent.

# Convergence of the $W$-Cycle Algorithm

Two-Grid Estimates

$$\|E_k^{TG} v\|_{1,k} \le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

where $\gamma = 1$ for the nonstandard smoother and $\gamma = 2$ for the standard smoother.

Theorem Given any $C_* > C_{\mathrm{TG}}$, there exists a positive integer $m_*$ independent of $k$ such that

$$\|E_k^{W} v\|_{1,k} \le C_* m^{-\alpha/\gamma} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

provided $m \ge m_*$.

As in the case of the two-grid algorithm, the contraction numbers decrease at the rate of $m^{-\alpha/\gamma}$, where $\gamma = 1$ for the nonstandard smoother and $\gamma = 2$ for the standard smoother. The algorithm based on the nonstandard smoother is therefore more efficient.

# Convergence of the $W$-Cycle Algorithm

Two-Grid Estimates

$$\|E_k^{TG}v\|_{1,k} \leq C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

where $\gamma = 1$ for the nonstandard smoother and $\gamma = 2$ for the standard smoother.

**Theorem** Given any $C_* > C_{\mathrm{TG}}$, there exists a positive integer $m_*$ independent of $k$ such that

$$\|E_k^{W}v\|_{1,k} \leq C_* m^{-\alpha/\gamma} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

provided $m \geq m_*$.

*Proof*. (mathematical induction)

The case $k = 0$ is obvious since $E_k^W = 0$.

# Convergence of the $W$-Cycle Algorithm

$$\|E_k^W v\|_{1,k} = \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m v\|_{1,k}$$

## Recursive Relation for $W$-cycle

$$E_k^W = R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m$$

# Convergence of the $W$-Cycle Algorithm

$$\begin{aligned}
\|E_k^W v\|_{1,k} &= \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m v\|_{1,k} \\
&\leq \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R
\end{aligned}$$

# Convergence of the $W$-Cycle Algorithm

$$
\begin{aligned}
\|E_k^W v\|_{1,k} &= \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m v\|_{1,k} \\
&\leq \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R \\
&= \|E_k^{TG} v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k}
\end{aligned}
$$

## Two-Grid Error Propagation Operator

$$
E_k^{TG} = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m
$$

# Convergence of the $W$-Cycle Algorithm

$$
\begin{aligned}
\|E_k^W v\|_{1,k} &= \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m v\|_{1,k} \\
&\leq \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k \\
&= \|E_k^{TG} v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\leq C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k}
\end{aligned}
$$

## Two-Grid Estimate

$$
\|E_k^{TG} v\|_{1,k} \leq C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} \qquad \forall\, v \in V_k
$$

# Convergence of the $W$-Cycle Algorithm

$$
\begin{aligned}
\|E_k^W v\|_{1,k} &= \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m v\|_{1,k} \\
&\leq \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R \\
&= \|E_k^{TG} v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\leq C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\leq C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}} \|(E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k-1}
\end{aligned}
$$

$$
\|R_k v\|_{1,k} \leq \|v\|_{1,k} \qquad \forall\, v \in V_k
$$

$$
\|I_{k-1}^k v\|_{1,k} \leq C_{\mathrm{CF}} \|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}
$$

# Convergence of the $W$-Cycle Algorithm

$$
\begin{aligned}
\|E_k^W v\|_{1,k} &= \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m v\|_{1,k} \\
&\le \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R \\
&= \|E_k^{TG} v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}} \|(E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k-1} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}} C_*^2 m^{-2\alpha/\gamma} \|P_k^{k-1} R_k^m v\|_{1,k-1}
\end{aligned}
$$

## Induction Hypothesis

$$
\|E_{k-1}^W v\|_{1,k} \le C_* m^{-\alpha/\gamma} \|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}
$$

# Convergence of the $W$-Cycle Algorithm

$$
\begin{aligned}
\|E_k^W v\|_{1,k} &= \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m v\|_{1,k} \\
&\le \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R \\
&= \|E_k^{TG} v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}} \|(E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k-1} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}} C_*^2 m^{-2\alpha/\gamma} \|P_k^{k-1} R_k^m v\|_{1,k-1} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}}^2 C_*^2 m^{-2\alpha/\gamma} \|v\|_{1,k}
\end{aligned}
$$

$$\|P_k^{k-1} v\|_{1,k-1} \le C_{\mathrm{CF}} \|v\|_{1,k} \qquad \forall\, v \in V_k$$

$$\|R_k v\|_{1,k} \le \|v\|_{1,k} \qquad \forall\, v \in V_k$$

# Convergence of the $W$-Cycle Algorithm

$$
\begin{aligned}
\|E_k^W v\|_{1,k} &= \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m v\|_{1,k} \\
&\leq \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R \\
&= \|E_k^{TG} v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\leq C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\leq C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}} \|(E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k-1} \\
&\leq C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}} C_*^2 m^{-2\alpha/\gamma} \|P_k^{k-1} R_k^m v\|_{1,k-1} \\
&\leq C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}}^2 C_*^2 m^{-2\alpha/\gamma} \|v\|_{1,k} \\
&\leq \left(C_{\mathrm{TG}} + C_{\mathrm{CF}}^2 C_*^2 m^{-\alpha/\gamma}\right) m^{-\alpha/\gamma} \|v\|_{1,k}
\end{aligned}
$$

# Convergence of the $W$-Cycle Algorithm

$$
\begin{aligned}
\|E_k^W v\|_{1,k} &= \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m v\|_{1,k} \\
&\le \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R \\
&= \|E_k^{TG} v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}} \|(E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k-1} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}} C_*^2 m^{-2\alpha/\gamma} \|P_k^{k-1} R_k^m v\|_{1,k-1} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}}^2 C_*^2 m^{-2\alpha/\gamma} \|v\|_{1,k} \\
&\le \left(C_{\mathrm{TG}} + C_{\mathrm{CF}}^2 C_*^2 m^{-\alpha/\gamma}\right) m^{-\alpha/\gamma} \|v\|_{1,k} \\
&\le \left(C_{\mathrm{TG}} + (C_* - C_{\mathrm{TG}})\right) m^{-\alpha/\gamma} \|v\|_{1,k}
\end{aligned}
$$

If $m \ge m_*$ and $m_*$ is chosen so that

$$C_{\mathrm{CF}}^2 C_*^2 m_*^{-\alpha/\gamma} \le C_* - C_{\mathrm{TG}}$$

# Convergence of the $W$-Cycle Algorithm

$$
\begin{aligned}
\|E_k^W v\|_{1,k} &= \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1}) R_k^m v\|_{1,k} \\
&\le \|R_k^m(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R \\
&= \|E_k^{TG} v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + \|R_k^m I_{k-1}^k (E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}} \|(E_{k-1}^W)^2 P_k^{k-1} R_k^m v\|_{1,k-1} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}} C_*^2 m^{-2\alpha/\gamma} \|P_k^{k-1} R_k^m v\|_{1,k-1} \\
&\le C_{\mathrm{TG}} m^{-\alpha/\gamma} \|v\|_{1,k} + C_{\mathrm{CF}}^2 C_*^2 m^{-2\alpha/\gamma} \|v\|_{1,k} \\
&\le \left(C_{\mathrm{TG}} + C_{\mathrm{CF}}^2 C_*^2 m^{-\alpha/\gamma}\right) m^{-\alpha/\gamma} \|v\|_{1,k} \\
&\le \left(C_{\mathrm{TG}} + (C_* - C_{\mathrm{TG}})\right) m^{-\alpha/\gamma} \|v\|_{1,k} \\
&\le C_* m^{-\alpha/\gamma} \|v\|_{1,k} \qquad \square
\end{aligned}
$$

# Summary

The convergence analysis of the $W$-cycle algorithm for $C^0$ interior penalty methods is based on the convergence analysis of the two-grid algorithm and a perturbation argument.

# Summary

The convergence analysis of the $W$-cycle algorithm for $C^0$ interior penalty methods is based on the convergence analysis of the two-grid algorithm and a perturbation argument.

## Ingredients for the Convergence Analysis

- calculus
- spectral theorem
- error estimates in lower order norms
  - elliptic regularity
  - enriching operator
  - interpolation of operators

# References


- Bank and Dupont, An optimal order process for solving finite element equations

  *Math. Comp.*, 1981

- B., Convergence of nonconforming multigrid methods without full elliptic regularity

  *Math. Comp.*, 1999

# Convergence of $V$- Cycle

# Conforming Finite Element Methods

# Conforming Finite Element Methods

Second Order Model Problem   Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall \, v \in H_0^1(\Omega)$$

# Conforming Finite Element Methods

Second Order Model Problem  Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in H_0^1(\Omega)$$

Discrete Problem  Find $u_h \in V_h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in V_h$$

where $V_h \subset H_0^1(\Omega)$ is the $P_1$ finite element space associated with a triangulation $\mathcal{T}_h$.

# Conforming Finite Element Methods

Second Order Model Problem  Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall \, v \in H_0^1(\Omega)$$

Discrete Problem  Find $u_h \in V_h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall \, v \in V_h$$

where $V_h \subset H_0^1(\Omega)$ is the $P_1$ finite element space associated with a triangulation $\mathcal{T}_h$.

Multigrid Setting  There is a sequence of triangulations generated by uniform refinements and a corresponding sequence of nested conforming finite element spaces

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset H_0^1(\Omega)$$

# Conforming Finite Element Methods

## With Full Elliptic Regularity ($\Omega$ convex)

$$\|u\|_{H^2(\Omega)} \leq C_\Omega \|f\|_{L_2(\Omega)}$$

1983 Braess-Hackbusch

$$|E_k v|_{H^1(\Omega)} \leq \frac{C}{C+m} |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k$$

for $m, k \geq 1$, where $C$ is independent of $m$ and $k$. In particular, the $V$-cycle is a contraction with only one smoothing step.

# Conforming Finite Element Methods

## Without Full Elliptic Regularity ($\Omega$ nonconvex)

$$\|u\|_{H^{1+\alpha}(\Omega)} \le C_\Omega \|f\|_{L_2(\Omega)}$$

for $\frac{1}{2} < \alpha < 1$

# Conforming Finite Element Methods

## Without Full Elliptic Regularity ($\Omega$ nonconvex)

$$\|u\|_{H^{1+\alpha}(\Omega)} \leq C_\Omega \|f\|_{L_2(\Omega)}$$

for $\frac{1}{2} < \alpha < 1$

1987 Bramble-Pasciak

1988 Decker-Mandel-Parter

$$|E_k v|_{H^1(\Omega)} \leq \Big(1 - \frac{1}{Ck^{(1-\alpha)/\alpha}}\Big)|v|_{H^1(\Omega)} \qquad \forall\, v \in V_k,\ k \geq 1$$

# Conforming Finite Element Methods

## Without Full Elliptic Regularity ($\Omega$ nonconvex)

$$\|u\|_{H^{1+\alpha}(\Omega)} \le C_\Omega \|f\|_{L_2(\Omega)}$$

for $\frac{1}{2} < \alpha < 1$

1987 Bramble-Pasciak

1988 Decker-Mandel-Parter

$$|E_k v|_{H^1(\Omega)} \le \left(1 - \frac{1}{Ck^{(1-\alpha)/\alpha}}\right)|v|_{H^1(\Omega)} \qquad \forall\, v \in V_k,\ k \ge 1$$

1991 Bramble-Pasciak-Wang-Xu (no regularity assumption)

$$|E_k v|_{H^1(\Omega)} \le \left(1 - \frac{1}{Ck}\right)|v|_{H^1(\Omega)} \qquad \forall\, v \in V_k,\ k \ge 1$$

# Conforming Finite Element Methods

1992 Zhang, Xu

1993 Bramble-Pasciak

There exists $\delta \in (0,1)$ such that

$$|E_k v|_{H^1(\Omega)} \le \delta |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k$$

for $m, k \ge 1$. In particular, the $V$-cycle is a contraction with one smoothing step.

# Conforming Finite Element Methods

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There exists $\delta \in (0, 1)$ such that

$$|E_k v|_{H^1(\Omega)} \leq \delta |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k$$

for $m, k \geq 1$. In particular, the $V$-cycle is a contraction with one smoothing step.

Braess-Hackbusch

$$|E_k v|_{H^1(\Omega)} \leq \frac{C}{C + m} |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k$$

for $m, k \geq 1$, where $C$ is independent of $m$ and $k$. In particular, the $V$-cycle is a contraction with only one smoothing step.

# Conforming Finite Element Methods

2002 B.

$$|E_k v|_{H^1(\Omega)} \leq \frac{C}{C + m^\alpha} |v|_{H^1(\Omega)}$$

for $m, k \geq 1$, where $C$ is independent of $m$ and $k$.

# Conforming Finite Element Methods

2002 B.

$$|E_k v|_{H^1(\Omega)} \leq \frac{C}{C + m^\alpha} |v|_{H^1(\Omega)}$$

for $m, k \geq 1$, where $C$ is independent of $m$ and $k$.

This is a complete generalization of the Braess-Hackbusch result:

$$|E_k v|_{H^1(\Omega)} \leq \frac{C}{C + m} |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k$$

for $m, k \geq 1$, where $C$ is independent of $m$ and $k$.

# Multiplicative Theory

## Recursive Relation for $V$-Cycle

$$E_k = R_k^m(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1} P_k^{k-1}) R_k^m$$

$$E_0 = 0$$

# Multiplicative Theory

## Recursive Relation for $V$-Cycle

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1} P_k^{k-1}) R_k^m$$
$$E_0 = 0$$

## Notation $(j \le \ell)$

- $I_j^\ell : V_j \longrightarrow V_\ell$ is the natural injection
- $P_\ell^j : V_\ell \longrightarrow V_j$ is the transpose of $I_j^\ell$ with respect to the variational form, i.e.,

$$a(P_\ell^j v, w) = a(v, I_j^\ell w) \qquad \forall\, v \in V_\ell,\ w \in V_j$$

- $I_j^j = Id_j = P_j^j$

# Multiplicative Theory

## Properties of $I_j^\ell$ and $P_\ell^j$

For $j \leq i \leq \ell$

- $I_j^\ell = I_i^\ell \circ I_j^i$
- $P_\ell^j = P_i^j \circ P_\ell^i$
- $I_j^i = P_\ell^i \circ I_j^\ell$ (in particular $Id_j = I_j^j = P_\ell^j \circ I_j^\ell$)
- $P_i^j = P_\ell^j \circ I_i^\ell$
- $(I_j^\ell P_\ell^j)^2 = I_j^\ell P_\ell^j$
- $(Id_\ell - I_j^\ell P_\ell^j)^2 = (Id_\ell - I_j^\ell P_\ell^j)$

# Multiplicative Theory

A Recursive Relation for $(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k E_{k-1} P_k^{k-1}$

# Multiplicative Theory

A Recursive Relation for $(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k E_{k-1} P_k^{k-1}$

$$
\begin{aligned}
&\big[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k E_{k-1} P_k^{k-1}\big] \\
&= (Id_k - I_{k-1}^k P_k^{k-1}) \\
&\quad + I_{k-1}^k \big[R_{k-1}^m (Id_{k-1} - I_{k-2}^{k-1} P_{k-1}^{k-2} + I_{k-2}^{k-1} E_{k-2} P_{k-1}^{k-2}) R_{k-1}^m\big] P_k^{k-1}
\end{aligned}
$$

# Multiplicative Theory

A Recursive Relation for $(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k E_{k-1} P_k^{k-1}$

$$
\begin{aligned}
&\left[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k E_{k-1} P_k^{k-1}\right] \\
&= (Id_k - I_{k-1}^k P_k^{k-1}) \\
&\quad + I_{k-1}^k \left[R_{k-1}^m (Id_{k-1} - I_{k-2}^{k-1} P_{k-1}^{k-2} + I_{k-2}^{k-1} E_{k-2} P_{k-1}^{k-2}) R_{k-1}^m\right] P_k^{k-1} \\
&= \left[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1}\right] \\
&\quad \cdot \left[(Id_k - I_{k-2}^k P_k^{k-2}) + I_{k-2}^k E_{k-2} P_k^{k-2})\right] \\
&\quad\quad \cdot \left[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1}\right]
\end{aligned}
$$

# Multiplicative Theory

A Recursive Relation for $(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k E_{k-1} P_k^{k-1}$

$$
\begin{aligned}
&\left[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k E_{k-1} P_k^{k-1}\right] \\
&= (Id_k - I_{k-1}^k P_k^{k-1}) \\
&\quad + I_{k-1}^k \left[R_{k-1}^m (Id_{k-1} - I_{k-2}^{k-1} P_{k-1}^{k-2} + I_{k-2}^{k-1} E_{k-2} P_{k-1}^{k-2}) R_{k-1}^m\right] P_k^{k-1} \\
&= \left[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1}\right] \\
&\quad \cdot \left[(Id_k - I_{k-2}^k P_k^{k-2}) + I_{k-2}^k E_{k-2} P_k^{k-2})\right] \\
&\qquad \cdot \left[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1}\right] \\
&= \left[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1}\right] \\
&\quad \cdot \left[(Id_k - I_{k-2}^k P_k^{k-2}) + I_{k-2}^k R_{k-2}^m P_k^{k-2}\right] \\
&\qquad \cdot \left[(Id_k - I_{k-3}^k P_k^{k-3}) + I_{k-3}^k E_{k-3} P_k^{k-3}\right] \\
&\qquad\quad \cdot \left[(Id_k - I_{k-2}^k P_k^{k-2}) + I_{k-2}^k R_{k-2}^m P_k^{k-2}\right] \\
&\qquad\qquad \cdot \left[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1}\right]
\end{aligned}
$$

# Multiplicative Theory

$$
\begin{aligned}
E_k &= R_k^m \big[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k E_{k-1} P_k^{k-1}\big] R_k^m \\
&= R_k^m \big[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1}\big] \cdots \\
&\qquad \cdot \big[(Id_k - I_1^k P_k^1) + I_1^k R_1^m P_k^1\big] (Id_k - I_0^k P_k^0) \big[(Id_k - I_1^k P_k^1) + I_1^k R_1 \\
&\qquad\quad \cdots \big[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1}\big] R_k^m
\end{aligned}
$$

# Multiplicative Theory

$$
\begin{aligned}
E_k &= R_k^m \big[ (Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k E_{k-1} P_k^{k-1} \big] R_k^m \\
&= R_k^m \big[ (Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1} \big] \cdots \\
&\qquad \cdot \big[ (Id_k - I_1^k P_k^1) + I_1^k R_1^m P_k^1 \big] (Id_k - I_0^k P_k^0) \big[ (Id_k - I_1^k P_k^1) + I_1^k R_1^m \\
&\qquad\quad \cdots \big[ (Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1} \big] R_k^m
\end{aligned}
$$

**Notation** For $1 \le j \le k$

$$T_j \stackrel{\text{def}}{=} I_j^k (Id_j - R_j^m) P_k^j$$

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&\qquad \cdot \big[(Id_k - I_1^k P_k^1) + I_1^k R_1^m P_k^1\big] (Id_k - I_0^k P_k^0) \big[(Id_k - I_1^k P_k^1) + I_1^k R_1 \\
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\end{aligned}
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In particular

$$T_k = I_k^k (Id_k - R_k^m) P_k^k = Id_k - R_k^m \Longrightarrow R_k^m = Id_k - T_k$$

# Multiplicative Theory

$$
\begin{aligned}
E_k &= R_k^m \big[ (Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k E_{k-1} P_k^{k-1} \big] R_k^m \\
&= R_k^m \big[ (Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1} \big] \cdots \\
&\qquad \cdot \big[ (Id_k - I_1^k P_k^1) + I_1^k R_1^m P_k^1 \big] (Id_k - I_0^k P_k^0) \big[ (Id_k - I_1^k P_k^1) + I_1^k R_1^m \\
&\qquad\quad \cdots \big[ (Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1} \big] R_k^m
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$$(Id_k - I_j^k P_k^j) + I_j^k R_j^m P_k^j = Id_k - I_j^k (Id_j - R_j^m) P_k^j = Id_k - T_j$$

# Multiplicative Theory

$$\begin{aligned} E_k &= R_k^m \big[ (Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k E_{k-1} P_k^{k-1} \big] R_k^m \\ &= R_k^m \big[ (Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1} \big] \cdots \\ &\quad \cdot \big[ (Id_k - I_1^k P_k^1) + I_1^k R_1^m P_k^1 \big] (Id_k - I_0^k P_k^0) \big[ (Id_k - I_1^k P_k^1) + I_1^k R_1^m \\ &\quad \cdots \big[ (Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k R_{k-1}^m P_k^{k-1} \big] R_k^m \end{aligned}$$

Notation For $1 \leq j \leq k$

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# Multiplicative Theory

## Multiplicative Expression for $E_k$

$$E_k = (Id_k - T_k)(Id_k - T_{k-1})\dots(Id_k - T_1)$$
$$\cdot (Id_k - T_0)(Id_k - T_1)\dots(Id_k - T_{k-1})(Id_k - T_k)$$

# Multiplicative Theory

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$$E_k = (Id_k - T_k)(Id_k - T_{k-1}) \dots (Id_k - T_1) \cdot (Id_k - T_0)(Id_k - T_1) \dots (Id_k - T_{k-1})(Id_k - T_k)$$

## Strengthened Cauchy-Schwarz Inequality

For $0 \leq j \leq k$, $v_j \in V_j$ and $v_k \in V_k$,

$$\int_\Omega \nabla v_j \cdot \nabla v_k \, dx \leq C 2^{-(k-j)/2} |v_j|_{H^1(\Omega)} h_k^{-1} \|v_k\|_{L_2(\Omega)}$$

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For $0 \le j \le k$, $v_j \in V_j$ and $v_k \in V_k$,

$$\int_\Omega \nabla v_j \cdot \nabla v_k \, dx \le C 2^{-(k-j)/2} |v_j|_{H^1(\Omega)} h_k^{-1} \|v_k\|_{L_2(\Omega)}$$

Standard Cauchy-Schwarz Inequality

$$\int_\Omega \nabla v_j \cdot \nabla v_k \, dx \le |v_j|_{H^1(\Omega)} |v_k|_{H^1(\Omega)}$$

which implies

$$\int_\Omega \nabla v_j \cdot \nabla v_k \, dx \le C |v_j|_{H^1(\Omega)} h_k^{-1} \|v_k\|_{L_2(\Omega)}$$

# Multiplicative Theory

**Theorem** There exists $\delta \in (0,1)$ such that

$$|E_k v|_{H^1(\Omega)} \leq \delta |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k$$

for $m, k \geq 1$.

# Multiplicative Theory

**Theorem** There exists $\delta \in (0,1)$ such that

$$|E_k v|_{H^1(\Omega)} \leq \delta |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k$$

for $m, k \geq 1$.

Details can be found in the book by Bramble and the survey article by Bramble and Zhang. Refinements of the multiplicative theory can be found in the following papers.

## References


- Xu and Zikatanov, The method of alternating projections and the method of subspace corrections in Hilbert space

  *J. Amer. Math. Soc.*, 2002

- B., An additive analysis of multiplicative Schwarz methods

  *Numer. Math.*, 2013

# Multiplicative Theory

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for $m, k \geq 1$.

The multiplicative theory cannot be applied to nonconforming finite element methods since many of the algebraic relations are no longer valid. For example,

$$(Id_k - I_\ell^k P_k^\ell)^2 \neq (Id_k - I_\ell^k P_k^\ell)$$

# Additive Theory

## Recursive Relation

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1} P_k^{k-1}) R_k^m$$
$$E_0 = 0$$

# Additive Theory

## Recursive Relation

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$$E_0 = 0$$

$$
\begin{aligned}
& E_k \\
= & R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m \\
& + R_k^m I_{k-1}^k R_{k-1}^m (Id_{k-1} - I_{k-2}^{k-1} P_{k-1}^{k-2} + I_{k-2}^{k-1} E_{k-2} P_{k-1}^{k-2}) R_{k-1}^m P_k^{k-1} R_k^m \\
= & R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m \\
& + R_k^m I_{k-1}^k R_{k-1}^m (Id_{k-1} - I_{k-2}^{k-1} P_{k-1}^{k-2}) R_{k-1}^m P_k^{k-1} R_k^m \\
& \quad + R_k^m I_{k-1}^k R_{k-1}^m I_{k-2}^{k-1} R_{k-2}^m (Id_{k-2} - I_{k-3}^{k-2} P_{k-2}^{k-3}) R_{k-2}^m P_{k-1}^{k-2} R_{k-1}^m P_k^{k-1} \\
& \qquad + \cdots
\end{aligned}
$$

# Additive Theory

## Recursive Relation

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$$E_0 = 0$$

$$\begin{aligned}
& E_k \\
&= R_k^m(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m \\
&\quad + R_k^m I_{k-1}^k R_{k-1}^m (Id_{k-1} - I_{k-2}^{k-1} P_{k-1}^{k-2} + I_{k-2}^{k-1} E_{k-2} P_{k-1}^{k-2}) R_{k-1}^m P_k^{k-1} R_k^m \\
&= R_k^m(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m \\
&\quad + R_k^m I_{k-1}^k R_{k-1}^m (Id_{k-1} - I_{k-2}^{k-1} P_{k-1}^{k-2}) R_{k-1}^m P_k^{k-1} R_k^m \\
&\qquad + R_k^m I_{k-1}^k R_{k-1}^m I_{k-2}^{k-1} R_{k-2}^m (Id_{k-2} - I_{k-3}^{k-2} P_{k-2}^{k-3}) R_{k-2}^m P_{k-1}^{k-2} R_{k-1}^m P_k^{k-1} \\
&\qquad\quad + \cdots \\
&= \sum_{j=1}^{k} R_k^m I_{k-1}^k \cdots R_{j+1}^m I_j^{j+1} \left[ R_j^m (Id_j - I_{j-1} P_j^{j-1}) R_j^m \right] P_{j+1}^j R_j^m \cdots P_k^{k-1}
\end{aligned}$$

# Additive Theory

## Ideas

- The operator $R_j^m(Id_j - I_{j-1}P_j^{j-1})R_j^m$ has already been analyzed in the two-grid analysis.

# Additive Theory

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- The operator $R_j^m(Id_j - I_{j-1}P_j^{j-1})R_j^m$ has already been analyzed in the two-grid analysis.

- The key is to analyze (for $0 \leq j \leq k$) the multi-level operator

$$T_{k,j} \stackrel{\text{def}}{=} R_k^m I_{k-1}^k \cdots R_{j+1}^m I_j^{j+1} : V_j \longrightarrow V_k$$

  and its transpose with respect to the variational forms

$$T_{j,k} \stackrel{\text{def}}{=} P_{j+1}^j R_j^m \cdots P_k^{k-1} R_k^m : V_k \longrightarrow V_j$$

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# Additive Theory

## Ideas

- The operator $R_j^m(Id_j - I_{j-1}P_j^{j-1})R_j^m$ has already been analyzed in the two-grid analysis.
- The key is to analyze (for $0 \leq j \leq k$) the multi-level operator

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$$T_{j,k} \stackrel{\text{def}}{=} P_{j+1}^j R_j^m \cdots P_k^{k-1} R_k^m : V_k \longrightarrow V_j$$

- We will need a strengthened Cauchy-Schwarz inequality with smoothing and estimates that compare the mesh-dependent norms on consecutive levels.
- We need to circumvent the fact that for nonconforming methods in general $(Id_k - I_\ell^k P_k^\ell)^2 \neq (Id_k - I_\ell^k P_k^\ell)$

# Additive Theory

## Strengthened Cauchy-Schwarz Inequality with Smoothing

Let $0 \leq j, \ell \leq k$, $v_j \in V_j$ and $v_\ell \in V_\ell$.

$$a_k(T_{k,j} R_j^m v_j, T_{k,\ell} R_\ell^m v_\ell) \leq \frac{C}{m^\alpha} \delta^{|\ell - j|} |||v_j|||_{1-\alpha,j} |||v_\ell|||_{1-\alpha,\ell}$$

where $C$ is a positive constant, $0 < \delta < 1$ and $\alpha \in (\frac{1}{2}, 1]$ is the index of elliptic regularity, provided the number of smoothing steps $m$ is sufficiently large.

# Additive Theory

## Strengthened Cauchy-Schwarz Inequality with Smoothing

Let $0 \le j, \ell \le k$, $v_j \in V_j$ and $v_\ell \in V_\ell$.

$$a_k(T_{k,j} R_j^m v_j, T_{k,\ell} R_\ell^m v_\ell) \le \frac{C}{m^\alpha} \delta^{|\ell - j|} \| v_j \|_{1-\alpha,j} \| v_\ell \|_{1-\alpha,\ell}$$

where $C$ is a positive constant, $0 < \delta < 1$ and $\alpha \in (\frac{1}{2}, 1]$ is the index of elliptic regularity, provided the number of smoothing steps $m$ is sufficiently large.

## A Nonconforming Estimate

$$\| \big( Id_{k-1} - P_k^{k-1} I_{k-1}^k \big) v \|_{1-\alpha,k-1} \le C h_k^\alpha \| v \|_{1,k-1} \qquad \forall \, v \in V_{k-1}$$

(This will allow us to handle $(Id_k - I_\ell^k P_k^\ell)^2 \ne (Id_k - I_\ell^k P_k^\ell)$.)

# Additive Theory

## Two-Level Estimates $\quad (0 < \theta < 1)$

$$\|I_{k-1}^k v\|_{1,k}^2 \le (1+\theta^2)\|v\|_{1,k-1}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{1+\alpha,k-1}^2 \qquad \forall\, v \in V_{k-1}$$

$$\|I_{k-1}^k v\|_{1-\alpha,k}^2 \le (1+\theta^2)\|v\|_{1-\alpha,k-1}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{1,k-1}^2 \qquad \forall\, v \in V_{k-1}$$

$$\|P_k^{k-1} v\|_{1-\alpha,k-1}^2 \le (1+\theta^2)\|v\|_{1-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{1,k}^2 \qquad \forall\, v \in V_k$$

Important aspect: the constant $C$ is independent of $k$ and $\theta$.

# Additive Theory

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$$\|I_{k-1}^k v\|_{1-\alpha,k}^2 \leq (1+\theta^2)\|v\|_{1-\alpha,k-1}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{1,k-1}^2 \qquad \forall\, v \in V_{k-1}$$

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Important aspect: the constant $C$ is independent of $k$ and $\theta$.

- $\theta$ is a parameter that calibrates the meaning of high/low frequency.
- The freedom to choose different $\theta$ on different levels allows us to build multi-level estimates from these two-level estimates.

# Additive Theory

**Theorem** There exists a positive constant $C$ independent of $k$ and $m$, such that

$$|E_k v|_{H^1(\Omega)} \leq \frac{C}{m^\alpha} |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k$$

provided that the number of smoothing steps $m$ is larger than a number $m_*$ which is independent of $k$. In particular the $V$-cycle algorithm is a contraction with contraction number uniformly bounded away from $1$ if $m$ is sufficiently large.

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This result holds for both conforming and nonconforming finite element methods.

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This result holds for both conforming and nonconforming finite element methods.

In the conforming case we can combine this with the result from the multiplicative theory to show that

$$|E_k v|_{H^1(\Omega)} \le \frac{C}{C + m^\alpha} |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k$$

# Additive Theory

The additive theory has been successfully applied to discontinuous Galerkin methods for second order problems and $C^0$ interior penalty methods for fourth order problems.

# Additive Theory

The additive theory has been successfully applied to discontinuous Galerkin methods for second order problems and $C^0$ interior penalty methods for fourth order problems.

## References

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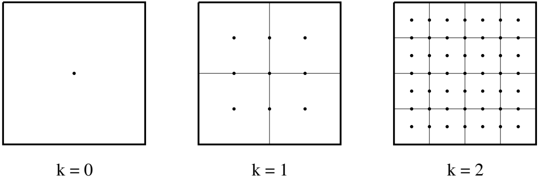
  *Numer. Math.*, 2011

# Numerical Results (Clamped Plates)




Boundary conditions $u = \partial u / \partial n = 0$

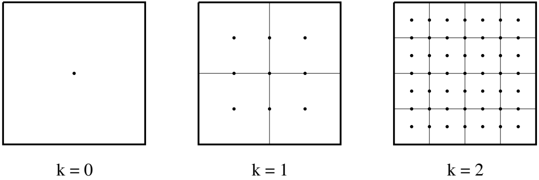
$\sigma = 5$

$V_k (\subset H_0^1(\Omega)) = \mathbb{Q}_2$ rectangular finite element space

$n_k = \dim(V_k) = (2^{k+1} - 1)^2$

$\kappa(A_k) \approx h_k^{-4}$

# Numerical Results (Clamped Plates)




$$n_0 = 1 \qquad \kappa_0 = 1.0 \times 10^0$$
$$n_1 = 9 \qquad \kappa_1 = 4.0 \times 10^1$$
$$n_2 = 49 \qquad \kappa_2 = 1.6 \times 10^3$$
$$n_3 = 225 \qquad \kappa_3 = 2.9 \times 10^4$$
$$n_4 = 961 \qquad \kappa_4 = 4.8 \times 10^5$$
$$n_5 = 3969 \qquad \kappa_5 = 7.8 \times 10^6$$
$$n_6 = 16129 \qquad \kappa_6 = 1.3 \times 10^8$$
$$n_7 = 65025 \qquad \kappa_7 = 2.0 \times 10^9$$

# Contraction Numbers in the Energy Norm

| $k$ \ $m$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| 1 | 0.08 | 0.04 | 0.02 | 0.011 | 0.006 | 0.0032 | 0.0017 |
| 2 | 0.27 | 0.22 | 0.18 | 0.15 | 0.13 | 0.11 | 0.09 |
| 3 | 0.43 | 0.32 | 0.29 | 0.26 | 0.23 | 0.21 | 0.19 |
| 4 | 0.56 | 0.35 | 0.34 | 0.31 | 0.28 | 0.25 | 0.23 |
| 5 | 0.64 | 0.42 | 0.37 | 0.34 | 0.31 | 0.29 | 0.27 |
| 6 | 0.70 | 0.43 | 0.39 | 0.35 | 0.33 | 0.30 | 0.27 |
| 7 | 0.75 | 0.44 | 0.39 | 0.36 | 0.34 | 0.31 | 0.29 |

$V$-Cycle Algorithm With the Nonstandard Preconditioner

# Contraction Numbers in the Energy Norm

| $k$ \ $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.53 | 0.28 | 0.15 | 0.08 | 0.04 | 0.02 | 0.01 | 0.006 | 0.003 | 0.002 |
| 2 | 0.72 | 0.49 | 0.24 | 0.27 | 0.22 | 0.18 | 0.15 | 0.13 | 0.11 | 0.09 |
| 3 | 0.71 | 0.51 | 0.40 | 0.34 | 0.30 | 0.26 | 0.24 | 0.22 | 0.19 | 0.17 |
| 4 | 0.80 | 0.51 | 0.41 | 0.37 | 0.34 | 0.31 | 0.28 | 0.26 | 0.24 | 0.22 |
| 5 | 0.76 | 0.53 | 0.42 | 0.38 | 0.34 | 0.31 | 0.29 | 0.26 | 0.24 | 0.23 |
| 6 | 0.82 | 0.53 | 0.42 | 0.38 | 0.34 | 0.32 | 0.29 | 0.26 | 0.25 | 0.22 |
| 7 | 0.83 | 0.53 | 0.42 | 0.38 | 0.34 | 0.32 | 0.29 | 0.27 | 0.25 | 0.23 |

$W$-Cycle Algorithm With the Nonstandard Preconditioner

# Contraction Numbers in the Energy Norm

| $k$ \ $m$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.28 | 0.15 | 0.08 | 0.04 | 0.02 | 0.01 | 0.0060 | 0.0032 | 0.0017 |
| 2 | 0.50 | 0.35 | 0.27 | 0.22 | 0.18 | 0.15 | 0.13 | 0.11 | 0.09 |
| 3 | 0.52 | 0.40 | 0.34 | 0.30 | 0.27 | 0.24 | 0.22 | 0.19 | 0.18 |
| 4 | 0.53 | 0.42 | 0.37 | 0.34 | 0.31 | 0.28 | 0.26 | 0.24 | 0.22 |
| 5 | 0.53 | 0.43 | 0.37 | 0.34 | 0.31 | 0.29 | 0.27 | 0.25 | 0.23 |
| 6 | 0.53 | 0.44 | 0.38 | 0.34 | 0.32 | 0.29 | 0.27 | 0.25 | 0.23 |
| 7 | 0.54 | 0.46 | 0.38 | 0.35 | 0.32 | 0.29 | 0.27 | 0.25 | 0.23 |

$F$-Cycle Algorithm With the Nonstandard Preconditioner

# Contraction Numbers in the Energy Norm

| $k$ \ $m$ | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.06 | 0.06 | 0.06 | 0.06 | 0.06 | 0.06 | 0.05 | 0.05 | 0.05 |
| 2 | 0.47 | 0.47 | 0.46 | 0.46 | 0.46 | 0.46 | 0.46 | 0.45 | 0.45 |
| 3 | 0.64 | 0.47 | 0.42 | 0.64 | 0.42 | 0.40 | 0.41 | 0.36 | 0.63 |
| 4 | 0.60 | 0.60 | 0.58 | 0.57 | 0.54 | 0.52 | 0.50 | 0.51 | 0.49 |
| 5 | 0.71 | 0.69 | 0.66 | 0.64 | 0.63 | 0.61 | 0.57 | 0.56 | 0.52 |
| 6 | 0.76 | 0.74 | 0.72 | 0.70 | 0.68 | 0.65 | 0.62 | 0.60 | 0.56 |
| 7 | 0.80 | 0.78 | 0.76 | 0.73 | 0.71 | 0.68 | 0.65 | 0.61 | 0.56 |

$V$-Cycle Algorithm With the Standard Preconditioner

# Other Multigrid Algorithms

# Variable $V$-Cycle

This is the $V$-cycle algorithm where the number of smoothing steps can vary from level to level.

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Suppose we want to solve the finite element equation on level $k$. Then $m_j$, the number of smoothing steps for level $j$, is chosen according to the rule

$$\beta_1 m_j \le m_{j-1} \le \beta_2 m_j$$

for $0 \le j \le k$, where $1 < \beta_1 \le \beta_2$.

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The variable $V$-cycle algorithm is mostly used as an optimal preconditioner, and its analysis is based on the same ingredients that appear in the analysis of the $W$-cycle algorithm.

# Full Multigrid

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The $k^{\rm th}$ level multigrid algorithm solves the equation

$$A_k z = \psi$$

with an (arbitrary) initial guess $z_0$.

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Therefore the initial guess for the $k^{\mathrm{th}}$ level multigrid algorithm should come from the solution on the $(k-1)^{\mathrm{st}}$ level.

# Full Multigrid

Finite Element Equation

$$A_k u_k = \phi_k$$
$$\langle \phi_k, v \rangle = \int_\Omega f v \, dx \qquad \forall\, v \in V_k$$

## Full Multigrid Algorithm

For $k = 0$, $\hat{u}_0 = A_0^{-1} \phi_0$

For $k \geq 1$,

$$u_0^k = I_{k-1}^k \hat{u}_{k-1}$$
$$u_\ell^k = MG(k, \phi_k, u_{\ell-1}^k, m) \qquad \text{for} \quad 1 \leq \ell \leq r$$
$$\hat{u}_k = u_r^k$$

# Full Multigrid

Suppose the multigrid algorithm is uniformly convergent. For a sufficiently large $r$, the full multigrid algorithm, which is a nested iteration of the $k^{\mathrm{th}}$ level multigrid algorithms, will produce an approximate solution of the continuous problem that is accurate to the same order as the exact solution of the finite element equation. Moreover the computational cost of the full multigrid algorithm remains proportional to the number of unknowns.