C^0 Interior Penalty Methods

Variational Inequalities

Current Research in Finite Element Methods

CIMPA Summer School

Mumbai, July 2015

References

S.C.Brenner, S., Y. Zhang

Finite element methods for the displacement obstacle problem of clamped plates

Math. Comp. (2012)

A C^0 interior penalty method for an elliptic optimal control problem with state constraints

IMA Vol. Math. Appl. (2014)

Post-processing procedures for an elliptic distributed optimal control problem with pointwise state constraints

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A quadratic C^0 interior penalty method for the displacement obstacle problem of clamped Kirchhoff plates

SIAM J. Numer. Anal. (2012)

An Obstacle Problem for Clamped Kirchhoff Plates

An Obstacle Problem for Clamped Kirchhoff Plates

$\Omega =$ bounded convex polygon (for simplicity) $\qquad f \in L_2(\Omega)$

$\psi \in C^2(\bar{\Omega})$, $\qquad \psi < 0$ on $\partial\Omega$ $\qquad$ (obstacle function)

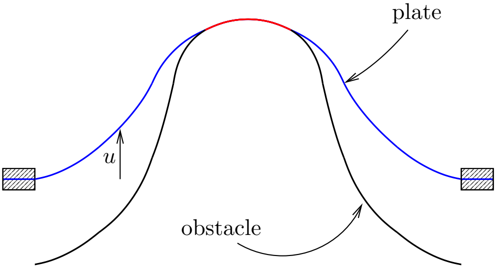
$K = \{v \in H_0^2(\Omega) : v \geq \psi \text{ on } \Omega\}$

Find $u = \underset{v \in K}{\operatorname{argmin}} \left[\frac{1}{2} a(v,v) - (f,v)\right]$

$$a(w,v) = \int_\Omega D^2 w : D^2 v \, dx \qquad D^2 w : D^2 v = \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}$$

$$(f,v) = \int_\Omega f v \, dx$$

An Obstacle Problem for Clamped Kirchhoff Plates



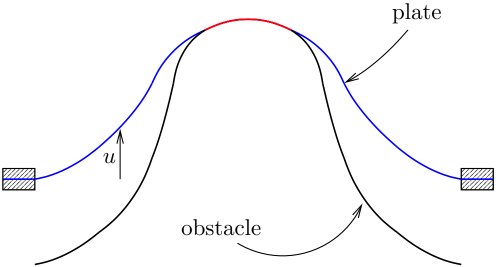
u is the vertical displacement of the midsurface of the thin plate.

f is the vertical load density. (flexural rigidity =1)

$$\frac{1}{2}a(v,v) - (f,v)$$

is the energy of the plate determined by the displacement v.

An Obstacle Problem for Clamped Kirchhoff Plates



u is the vertical displacement of the midsurface of the thin plate.

f is the vertical load density. (flexural rigidity =1)

$$u = \underset{v \in K}{\operatorname{argmin}} \left[\frac{1}{2} a(v,v) - (f,v) \right]$$

The obstacle problem is to find the plate that has minimum energy among all admissible plates.

An Optimal Control Problem with Pointwise State Constraint

minimize $\quad \dfrac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \dfrac{\beta}{2}\|u\|^2_{L_2(\Omega)}$

over $\quad (y, u) \in H^1_0(\Omega) \times L_2(\Omega)$

subject to
$$\begin{cases} -\Delta y = u & \text{in } \Omega \\ \quad\;\; y \leq \psi & \text{a.e. in } \Omega \end{cases}$$

An Optimal Control Problem with Pointwise State Constraint

$$\text{minimize} \quad \frac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|u\|^2_{L_2(\Omega)}$$

$$\text{over} \quad (y, u) \in H^1_0(\Omega) \times L_2(\Omega)$$

$$\text{subject to} \quad \begin{cases} -\Delta y = u & \text{in } \Omega \\ y \leq \psi & \text{a.e. in } \Omega \end{cases}$$

$y \in H^1_0(\Omega)$ is the state.

y_d is the desired state.

$y \leq \psi$ is a pointwise constraint on the state.

$u \in L_2(\Omega)$ is the control.

$\beta > 0$ is related to the cost for implementing the control u.

An Optimal Control Problem with Pointwise State Constraint

$$\text{minimize} \quad \frac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|u\|^2_{L_2(\Omega)}$$

$$\text{over} \quad (y,u) \in H^1_0(\Omega) \times L_2(\Omega)$$

$$\text{subject to} \quad \begin{cases} -\Delta y = u & \text{in } \Omega \\ \quad\; y \leq \psi & \text{a.e. in } \Omega \end{cases}$$

If Ω is convex or smooth, then y belongs to $H^2(\Omega)$ by elliptic regularity.

An Optimal Control Problem with Pointwise State Constraint

$$\text{minimize} \quad \frac{1}{2}\|y - y_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|u\|^2_{L_2(\Omega)}$$

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If Ω is convex or smooth, then y belongs to $H^2(\Omega)$ by elliptic regularity.

Obstacle Problem

$$\text{Find} \quad y = \underset{v \in K}{\text{argmin}}\, \frac{1}{2}\Big[\|y - y_d\|^2_{L_2(\Omega)} + \beta\|\Delta y\|^2_{L_2(\Omega)}\Big]$$

$$K = \{v \in H^2(\Omega) \cap H^1_0(\Omega) : v \leq \psi \quad \text{in } \Omega\}$$

An Abstract Constrained Minimization Problem

V is a (real) Hilbert space with inner product $a(\cdot,\cdot)$.

K is a nonempty closed convex subset of V.

$F : V \longrightarrow \mathbb{R}$ is a bounded linear functional on V.

Find

$$u = \underset{v \in K}{\text{argmin}} \left[\frac{1}{2} a(v,v) - F(v) \right]$$

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Find

$$u = \operatorname*{argmin}_{v \in K} \left[\frac{1}{2} a(v,v) - F(v) \right]$$

Riesz Representation Theorem

$$F(v) = a(u_F, v) \qquad \forall\, v \in V$$

for some $u_F \in V$

An Abstract Constrained Minimization Problem

Find

$$u = \operatorname*{argmin}_{v \in K} \left[\frac{1}{2} a(v, v) - F(v) \right]$$

An Abstract Constrained Minimization Problem

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$$= \underset{v \in K}{\text{argmin}} \left[\frac{1}{2} a(v,v) - a(u_F, v) \right]$$

$$F(v) = a(u_F, v) \quad \forall\, v \in V$$

An Abstract Constrained Minimization Problem

Find

$$\begin{aligned} u &= \operatorname*{argmin}_{v \in K} \left[\frac{1}{2} a(v,v) - F(v) \right] \\ &= \operatorname*{argmin}_{v \in K} \left[\frac{1}{2} a(v,v) - a(u_F, v) \right] \\ &= \operatorname*{argmin}_{v \in K} \frac{1}{2} \left[a(v - u_F, v - u_F) - a(u_F, u_F) \right] \end{aligned}$$

completing the square

An Abstract Constrained Minimization Problem

Find

$$\begin{aligned}
u &= \operatorname*{argmin}_{v\in K} \left[\frac{1}{2}a(v,v) - F(v)\right] \\
&= \operatorname*{argmin}_{v\in K} \left[\frac{1}{2}a(v,v) - a(u_F,v)\right] \\
&= \operatorname*{argmin}_{v\in K} \frac{1}{2}\left[a(v-u_F,v-u_F) - a(u_F,u_F)\right] \\
&= \operatorname*{argmin}_{v\in K} a(v-u_F,v-u_F)
\end{aligned}$$

An Abstract Constrained Minimization Problem

Find

$$\begin{aligned}
u &= \operatorname*{argmin}_{v\in K}\left[\frac{1}{2}a(v,v)-F(v)\right]\\
&= \operatorname*{argmin}_{v\in K}\left[\frac{1}{2}a(v,v)-a(u_F,v)\right]\\
&= \operatorname*{argmin}_{v\in K}\frac{1}{2}\Big[a(v-u_F,v-u_F)-a(u_F,u_F)\Big]\\
&= \operatorname*{argmin}_{v\in K} a(v-u_F,v-u_F)
\end{aligned}$$

In other words we are looking for the element of K closest to u_F with respect to the norm $\|\cdot\|_a=\sqrt{a(\cdot,\cdot)}$ of V.

An Abstract Constrained Minimization Problem

Find

$$
\begin{aligned}
u &= \operatorname*{argmin}_{v\in K} \left[\frac{1}{2}a(v,v) - F(v)\right] \\
&= \operatorname*{argmin}_{v\in K} \left[\frac{1}{2}a(v,v) - a(u_F,v)\right] \\
&= \operatorname*{argmin}_{v\in K} \frac{1}{2}\left[a(v-u_F, v-u_F) - a(u_F,u_F)\right] \\
&= \operatorname*{argmin}_{v\in K} a(v-u_F, v-u_F)
\end{aligned}
$$

In other words we are looking for the element of K closest to u_F with respect to the norm $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$ of V.

Since K is a nonempty closed convex subset of V, there is a unique $u \in K$ closest to u_F by the Projection Theorem.

A Variational Inequality

A Variational Inequality

Let $v \in K$ be arbitrary. The line segment

$$(1-t)u + tv \quad \text{for} \quad 0 \leq t \leq 1$$

connecting u to v is in the convex set K and hence the function

$$\Phi(t) = \frac{1}{2}a\big((1-t)u + tv, (1-t)u + tv)\big) - F\big((1-t)u + tv)\big)$$

has a minimum at $t = 0$.

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$$\Phi(t) = \frac{1}{2}a\big((1-t)u + tv, (1-t)u + tv)\big) - F\big((1-t)u + tv)\big)$$

has a minimum at $t = 0$.

Therefore $\Phi'(0) \geq 0$, which is equivalent to the variational inequality

$$a(u, v-u) \geq F(v-u)$$

that holds for all $v \in K$.

A Second Order Variational Inequality

$V = H_0^1(\Omega)$

$K = \{v \in V : v \geq \psi \text{ a.e.}\}$ ($\psi < 0$ on $\partial\Omega$)

$f \in L_2(\Omega)$

Find

$$u \in \underset{v \in K}{\text{argmin}} \left[\frac{1}{2}(\nabla v, \nabla v) - (f, v)\right]$$

obstacle problem for elastic membranes

A Second Order Variational Inequality

$V = H_0^1(\Omega)$

$K = \{v \in V : v \geq \psi \text{ a.e.}\}$ ($\psi < 0$ on $\partial\Omega$)

$f \in L_2(\Omega)$

Find

$$u \in \underset{v \in K}{\text{argmin}} \left[\frac{1}{2}(\nabla v, \nabla v) - (f, v)\right]$$

Since $(\nabla v, \nabla w)$ is an inner product on $H_0^1(\Omega)$ and K is a nonempty closed convex subset of $H_0^1(\Omega)$, there is a unique minimizer $u \in K$ characterized by the variational inequality

$$(\nabla u, \nabla(v - u)) \geq (f, v - u) \qquad \forall\, v \in K$$

A Second Order Variational Inequality

Brézis-Stampacchia 1968

The solution u of the second order problem belongs to $H^2(\Omega)$ under appropriate assumptions, i.e., the solutions of second order variational inequalities enjoy the same elliptic regularity as the solutions of second order boundary value problems.

A Second Order Variational Inequality

Brézis-Stampacchia 1968

The solution u of the second order problem belongs to $H^2(\Omega)$ under appropriate assumptions, i.e., the solutions of second order variational inequalities enjoy the same elliptic regularity as the solutions of second order boundary value problems.

$$\begin{aligned} & (\nabla u, \nabla(v-u)) \geq (f, v-u) && \forall\, v \in K \\ \Leftrightarrow \quad & (\Delta u + f, v-u) \leq 0 && \forall\, v \in K \end{aligned}$$

which is equivalent to

$$\Delta u + f \leq 0, \quad u - \psi \geq 0, \quad (\Delta u + f)(u - \psi) = 0$$

(Complementary Form of the variational inequality)

Finite Element Method

$\mathcal{T}_h$ is a triangulation of Ω.

$V_h \subset H_0^1(\Omega)$ is the P_1 finite element space associated with $\mathcal{T}_h$.

$\Pi_h : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow V_h$ is the nodal interpolation operator.

$K_h = \{v \in V_h : \ v \geq \Pi_h \psi\}$ $\qquad$ $(\Pi_h u \in K_h)$

$$\text{Find} \quad u_h = \underset{v \in K_h}{\operatorname{argmin}} \left[\frac{1}{2}(\nabla v, \nabla v) - (f, v)\right]$$

Finite Element Method

$\mathcal{T}_h$ is a triangulation of Ω.

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$K_h = \{v \in V_h : \ v \geq \Pi_h \psi\}$ $\qquad$ ($\Pi_h u \in K_h$)

$$\text{Find} \quad u_h = \underset{v \in K_h}{\text{argmin}} \left[\frac{1}{2} (\nabla v, \nabla v) - (f, v) \right]$$

There is a unique minimizer $u_h \in K_h$ characterized by the discrete variational inequality

$$(\nabla u_h, \nabla(v - u_h)) \geq (f, v - u_h) \qquad \forall\, v \in K_h$$

Error Estimate: $\|\nabla(u - u_h)\|_{L_2(\Omega)} \leq Ch$

Falk (1974)

Brezzi-Hager-Raviart (1977)

Error Estimate: $\|\nabla(u-u_h)\|_{L_2(\Omega)} \le Ch$

$$\|\nabla(u-u_h)\|^2_{L_2(\Omega)} = (\nabla(u-u_h), \nabla(u-u_h))$$

$$= (\nabla(u-u_h), \nabla(u-\Pi_h u)) + (\nabla(u-u_h), \nabla(\Pi_h u - u_h))$$

Error Estimate: $\|\nabla(u-u_h)\|_{L_2(\Omega)} \leq Ch$

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$$= (\nabla(u-u_h), \nabla(u-\Pi_h u)) + (\nabla(u-u_h), \nabla(\Pi_h u - u_h))$$

$$\leq \|\nabla(u-u_h)\|_{L_2(\Omega)} \|\nabla(u-\Pi_h u)\|_{L_2(\Omega)}$$

$$+ (\nabla u, \nabla(\Pi_h u - u_h)) - (\nabla u_h, \nabla(\Pi_h u - u_h))$$

Cauchy-Schwarz Inequality

Error Estimate: $\|\nabla(u-u_h)\|_{L_2(\Omega)} \leq Ch$

$$\|\nabla(u-u_h)\|^2_{L_2(\Omega)} = (\nabla(u-u_h), \nabla(u-u_h))$$

$$= (\nabla(u-u_h), \nabla(u-\Pi_h u)) + (\nabla(u-u_h), \nabla(\Pi_h u - u_h))$$

$$\leq \|\nabla(u-u_h)\|_{L_2(\Omega)} \|\nabla(u-\Pi_h u)\|_{L_2(\Omega)}$$
$$+ (\nabla u, \nabla(\Pi_h u - u_h)) - (\nabla u_h, \nabla(\Pi_h u - u_h))$$

$$\leq \|\nabla(u-u_h)\|_{L_2(\Omega)} \|\nabla(u-\Pi_h u)\|_{L_2(\Omega)}$$
$$+ (\nabla u, \nabla(\Pi_h u - u_h)) - (f, \Pi_h u - u_h)$$

Discrete Variational Inequality

$$(\nabla u_h, \nabla(v-u_h)) \geq (f, v-u_h) \quad \forall\, v \in K_h$$

$$\Pi_h u \in K_h$$

Error Estimate: $\|\nabla(u-u_h)\|_{L_2(\Omega)} \le Ch$

$$\|\nabla(u-u_h)\|^2_{L_2(\Omega)} = (\nabla(u-u_h), \nabla(u-u_h))$$

$$= (\nabla(u-u_h), \nabla(u-\Pi_h u)) + (\nabla(u-u_h), \nabla(\Pi_h u - u_h))$$

$$\le \|\nabla(u-u_h)\|_{L_2(\Omega)} \|\nabla(u-\Pi_h u)\|_{L_2(\Omega)}$$
$$+ (\nabla u, \nabla(\Pi_h u - u_h)) - (\nabla u_h, \nabla(\Pi_h u - u_h))$$

$$\le \|\nabla(u-u_h)\|_{L_2(\Omega)} \|\nabla(u-\Pi_h u)\|_{L_2(\Omega)}$$
$$+ (\nabla u, \nabla(\Pi_h u - u_h)) - (f, \Pi_h u - u_h)$$

$$= \|\nabla(u-u_h)\|_{L_2(\Omega)} \|\nabla(u-\Pi_h u)\|_{L_2(\Omega)}$$
$$- ((\Delta u + f), \Pi_h u - u_h)$$

Error Estimate: $\|\nabla(u-u_h)\|_{L_2(\Omega)} \le Ch$

$$
\begin{aligned}
\|\nabla(u-u_h)\|^2_{L_2(\Omega)} &= (\nabla(u-u_h), \nabla(u-u_h)) \\
&= (\nabla(u-u_h), \nabla(u-\Pi_h u)) + (\nabla(u-u_h), \nabla(\Pi_h u - u_h)) \\
&\le \|\nabla(u-u_h)\|_{L_2(\Omega)} \|\nabla(u-\Pi_h u)\|_{L_2(\Omega)} \\
&\qquad + (\nabla u, \nabla(\Pi_h u - u_h)) - (\nabla u_h, \nabla(\Pi_h u - u_h)) \\
&\le \|\nabla(u-u_h)\|_{L_2(\Omega)} \|\nabla(u-\Pi_h u)\|_{L_2(\Omega)} \\
&\qquad + (\nabla u, \nabla(\Pi_h u - u_h)) - (f, \Pi_h u - u_h) \\
&= \|\nabla(u-u_h)\|_{L_2(\Omega)} \|\nabla(u-\Pi_h u)\|_{L_2(\Omega)} \\
&\qquad - ((\Delta u + f), \Pi_h u - u_h) \\
&\le C_1 h \|\nabla(u-u_h)\|_{L_2(\Omega)} - ((\Delta u + f), \Pi_h u - u_h)
\end{aligned}
$$

Error Estimate: $\|\nabla(u-u_h)\|_{L_2(\Omega)} \le Ch$

$$\|\nabla(u-u_h)\|^2_{L_2(\Omega)} \le C_1 h \|\nabla(u-u_h)\|_{L_2(\Omega)} - ((\Delta u + f), \Pi_h u - u_h)$$

Error Estimate: $\|\nabla(u-u_h)\|_{L_2(\Omega)} \leq Ch$

$$\|\nabla(u-u_h)\|^2_{L_2(\Omega)} \leq C_1 h \|\nabla(u-u_h)\|_{L_2(\Omega)} - ((\Delta u + f), \Pi_h u - u_h)$$

$$-\,((\Delta u + f), \Pi_h u - u_h)$$

$$= -((\Delta u + f), (\Pi_h u - u) + (u - \psi) + (\psi - \Pi_h \psi) + (\Pi_h \psi - u_h))$$

Error Estimate: $\|\nabla(u-u_h)\|_{L_2(\Omega)} \leq Ch$

$$\|\nabla(u-u_h)\|^2_{L_2(\Omega)} \leq C_1 h\|\nabla(u-u_h)\|_{L_2(\Omega)} - ((\Delta u + f), \Pi_h u - u_h)$$

$$
\begin{aligned}
&-((\Delta u + f), \Pi_h u - u_h) \\
&\qquad = -((\Delta u + f), (\Pi_h u - u) + (u-\psi) + (\psi - \Pi_h\psi) + (\Pi_h\psi - u_h)) \\
&\qquad \leq -((\Delta u + f), (\Pi_h u - u) + (\psi - \Pi_h\psi))
\end{aligned}
$$

$-((\Delta u + f), u-\psi) = 0$ because $(\Delta u + f)(u-\psi) = 0$

$-((\Delta u + f), \Pi_h\psi - u_h) \leq 0$

because $(\Delta u + f) \leq 0$ and $u_h - \Pi_h\psi \geq 0$

Error Estimate: $\|\nabla(u-u_h)\|_{L_2(\Omega)} \leq Ch$

$$\|\nabla(u-u_h)\|^2_{L_2(\Omega)} \leq C_1 h\|\nabla(u-u_h)\|_{L_2(\Omega)} - ((\Delta u+f), \Pi_h u - u_h)$$

$$\begin{aligned}
&-((\Delta u+f), \Pi_h u - u_h) \\
&\quad = -((\Delta u+f), (\Pi_h u - u) + (u-\psi) + (\psi - \Pi_h\psi) + (\Pi_h\psi - u_h)) \\
&\quad \leq -((\Delta u+f), (\Pi_h u - u) + (\psi - \Pi_h\psi)) \\
&\quad \leq \|\Delta u+f\|_{L_2(\Omega)}(\|\Pi_h u - u\|_{L_2(\Omega)} + \|\psi - \Pi_h\psi\|_{L_2(\Omega)})
\end{aligned}$$

Cauchy-Schwarz Inequality

Error Estimate: $\|\nabla(u-u_h)\|_{L_2(\Omega)} \le Ch$

$$\|\nabla(u-u_h)\|^2_{L_2(\Omega)} \le C_1 h \|\nabla(u-u_h)\|_{L_2(\Omega)} - ((\Delta u + f), \Pi_h u - u_h)$$

$$\begin{aligned}
-\,((\Delta u + f), \Pi_h u - u_h) \\
&= -((\Delta u + f), (\Pi_h u - u) + (u - \psi) + (\psi - \Pi_h \psi) + (\Pi_h \psi - u_h)) \\
&\le -((\Delta u + f), (\Pi_h u - u) + (\psi - \Pi_h \psi)) \\
&\le \|\Delta u + f\|_{L_2(\Omega)} \big(\|\Pi_h u - u\|_{L_2(\Omega)} + \|\psi - \Pi_h \psi\|_{L_2(\Omega)}\big) \\
&\le C_2 h^2
\end{aligned}$$

Error Estimate: $\|\nabla(u-u_h)\|_{L_2(\Omega)} \leq Ch$

$$\|\nabla(u-u_h)\|^2_{L_2(\Omega)} \leq C_1 h\|\nabla(u-u_h)\|_{L_2(\Omega)} - ((\Delta u + f), \Pi_h u - u_h)$$

$$\begin{aligned}
-\,((\Delta u + f), \Pi_h u - u_h)& \\
&= -((\Delta u + f), (\Pi_h u - u) + (u - \psi) + (\psi - \Pi_h \psi) + (\Pi_h \psi - u_h)) \\
&\leq -((\Delta u + f), (\Pi_h u - u) + (\psi - \Pi_h \psi)) \\
&\leq \|\Delta u + f\|_{L_2(\Omega)} (\|\Pi_h u - u\|_{L_2(\Omega)} + \|\psi - \Pi_h \psi\|_{L_2(\Omega)}) \\
&\leq C_2 h^2
\end{aligned}$$

$$\begin{aligned}
\|\nabla(u-u_h)\|^2_{L_2(\Omega)} &\leq C_1 h\|\nabla(u-u_h)\|_{L_2(\Omega)} + C_2 h^2 \\
&\leq C_3 h^2 + \frac{1}{2}\|\nabla(u-u_h)\|^2_{L_2(\Omega)} + C_2 h^2
\end{aligned}$$

Obstacle Problem for the Clamped Kirchhoff Plate

Find $u = \underset{v \in K}{\operatorname{argmin}} \left[\frac{1}{2} a(v,v) - (f,v) \right]$

$$a(w,v) = \int_\Omega D^2 w : D^2 v \, dx \qquad D^2 w : D^2 v = \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}$$

$$(f,v) = \int_\Omega f v \, dx \qquad K = \{ v \in H_0^2(\Omega) : v \geq \psi \text{ on } \Omega \}$$

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$$(f,v) = \int_\Omega f v \, dx \qquad K = \{ v \in H_0^2(\Omega) : v \geq \psi \text{ on } \Omega \}$$

Since $a(\cdot,\cdot)$ is symmetric, bounded and coercive on $H_0^2(\Omega)$, it defines an inner product on $H_0^2(\Omega)$ and the norm $\|\cdot\|_a$ is equivalent to the Sobolev norm $\|\cdot\|_{H^2(\Omega)}$. Therefore $H_0^2(\Omega)$ is a Hilbert space under this inner product and K is a non-empty closed convex subset of this Hilbert space.

Obstacle Problem for the Clamped Kirchhoff Plate

Find $u = \operatorname*{argmin}_{v \in K} \left[\frac{1}{2} a(v,v) - (f,v) \right]$

$$a(w,v) = \int_\Omega D^2 w : D^2 v \, dx \qquad D^2 w : D^2 v = \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}$$

$$(f,v) = \int_\Omega f v \, dx \qquad K = \{ v \in H_0^2(\Omega) : \ v \geq \psi \text{ on } \Omega \}$$

We can apply the abstract theory to conclude that the obstacle problem has a unique solution characterized by the variational inequality

$$a(u, v-u) \geq (f, v-u) \qquad \forall \, v \in K$$

Obstacle Problem for the Clamped Kirchhoff Plate

$$a(u, v-u) \ge (f, v-u) \qquad \forall\, v \in K$$

$$a(w,v) = \int_\Omega D^2 w : D^2 v\, dx$$

$$K = \{v \in H_0^2(\Omega) : v \ge \psi \text{ on } \Omega\}$$

If $u \in H^4(\Omega)$, then u satisfies the complementarity form of the variational inequality in a strong sense:

$$\Delta^2 u - f \ge 0, \quad u - \psi \ge 0, \quad (\Delta^2 u - f)(u - \psi) = 0$$

and the numerical analysis of the obstacle problem can proceed as in the second order case.

Regularity Results

$u \in H^3_{loc}(\Omega)$ under the assumption that $\psi \in C^2(\Omega)$.

Frehse (1971)

Since $u > \psi$ near $\partial\Omega$ and hence $\Delta^2 u = f$ near $\partial\Omega$, we have $u \in H^3(\Omega)$ by the convexity of Ω.

Kondrat'ev (1967)

Blum and Rannacher (1980)

$u \in C^2(\Omega)$ but in general $u \notin H^4_{loc}(\Omega)$ even if Ω, f and ψ are smooth.

Caffarelli and Friedman (1979)

The lack of H^4_{loc} regularity means that the complementarity form of the variational inequality only exists in a weak sense, which complicates the numerical analysis of the obstacle problem.

A Quadratic C^0 Interior Penalty Method

$\mathcal{T}_h$ = triangulation of Ω

V_h = quadratic finite element space $(\subset H_0^1(\Omega))$

$K_h = \{v \in V_h : v \geq \psi$ at all the vertices of $\mathcal{T}_h\}$

$$
\begin{aligned}
a_h(w,v) = \sum_{T \in \mathcal{T}_h} \int_T D^2 w : D^2 v \, dx + \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\partial^2 w/\partial n^2\}\!\} [\![\partial v/\partial n]\!] \, ds \\
+ \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\partial^2 v/\partial n^2\}\!\} [\![\partial w/\partial n]\!] \, ds \\
+ \sigma \sum_{e \in \mathcal{E}_h} |e|^{-1} \int_e [\![\partial w/\partial n]\!] [\![\partial v/\partial n]\!] \, ds
\end{aligned}
$$

A Quadratic C^0 Interior Penalty Method

$\mathcal{T}_h =$ triangulation of Ω

$V_h =$ quadratic finite element space $(\subset H_0^1(\Omega))$

$K_h = \{v \in V_h : v \geq \psi$ at all the vertices of $\mathcal{T}_h\}$

Discrete Problem

$$\text{Find } u_h = \underset{v \in K_h}{\text{argmin}} \left[\frac{1}{2} a_h(v,v) - (f,v)\right]$$

Since $a_h(\cdot,\cdot)$ is symmetric and coercive, the discrete obstacle problem has a unique solution characterized by a discrete variational inequality

$$a_h(u_h, v - u_h) \geq (f, v - u_h) \qquad \forall\, v \in K_h$$

Convergence Analysis

An Auxiliary Obstacle Problem

$$\text{Find} \quad \tilde{u}_h = \underset{v\in\tilde{K}_h}{\text{argmin}}\left[\frac{1}{2}a(v,v)-(f,v)\right]$$

where

$$\begin{aligned}\tilde{K}_h &= \{v \in H_0^2(\Omega) : v(p) \geq \psi(p) \quad \text{at all the vertices of } \mathcal{T}_h\} \\ &\supset K = \{v \in H_0^2(\Omega) : v \geq \psi \quad \text{on } \Omega\}\end{aligned}$$

Convergence Analysis

An Auxiliary Obstacle Problem

$$\text{Find} \quad \tilde{u}_h = \underset{v \in \tilde{K}_h}{\text{argmin}} \left[\frac{1}{2} a(v, v) - (f, v) \right]$$

where

$$\tilde{K}_h = \{v \in H_0^2(\Omega) : v(p) \geq \psi(p) \quad \text{at all the vertices of } \mathcal{T}_h\}$$
$$\supset K = \{v \in H_0^2(\Omega) : v \geq \psi \quad \text{on } \Omega\}$$

This is a minimization problem for the functional of the original obstacle problem but on a larger closed convex set.

Variational Inequality:

$$a(\tilde{u}_h, v - \tilde{u}_h) \geq (f, v - \tilde{u}_h) \qquad \forall\, v \in \tilde{K}_h$$

Convergence Analysis

An Auxiliary Obstacle Problem

$$\text{Find} \quad \tilde{u}_h = \underset{v \in \tilde{K}_h}{\text{argmin}} \left[\frac{1}{2} a(v,v) - (f,v) \right]$$

where

$$\begin{aligned} \tilde{K}_h &= \{ v \in H_0^2(\Omega) : \ v(p) \geq \psi(p) \quad \text{at all the vertices of } \mathcal{T}_h \} \\ &\supset K = \{ v \in H_0^2(\Omega) : \ v \geq \psi \quad \text{on } \Omega \} \end{aligned}$$

The auxiliary obstacle problem is an intermediate obstacle problem. On one hand it shares the space $H_0^2(\Omega)$ and the functional with the original obstacle problem. On the other hand it shares the same constraint as the discrete obstacle problem.

$$K_h \xrightarrow{\quad E_h \quad} \tilde{K}_h \xleftarrow{\quad \text{injection} \quad} K$$

Error Estimates

By using the auxiliary obstacle problem to connect the continuous and discrete obstacle problems, we can show that

Theorem $\|u - u_h\|_h \leq Ch$

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} |e| \|\{\!\{\partial^2 v/\partial n^2\}\!\}\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} |e|^{-1} \|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2$$

Error Estimates

By using the auxiliary obstacle problem to connect the continuous and discrete obstacle problems, we can show that

Theorem $\|u - u_h\|_h \leq Ch$

Key Ingredients

- the three variational inequalities

$$
\begin{aligned}
a(u, v-u) &\geq (f, v-u) && \forall\, v \in K \\
a_h(u_h, v-u_h) &\geq (f, v-u_h) && \forall\, v \in K_h \\
a(\tilde{u}_h, v-\tilde{u}_h) &\geq (f, v-\tilde{u}_h) && \forall\, v \in \tilde{K}_h
\end{aligned}
$$

- techniques from the calculus of variations
- properties of E_h

Error Estimates

By using the auxiliary obstacle problem to connect the continuous and discrete obstacle problems, we can show that

Theorem $\|u - u_h\|_h \leq Ch$

This error estimate is optimal since $u \in H^3(\Omega)$ and the norm $\|\cdot\|_h$ behaves like the $H^2(\Omega)$ norm.

Error Estimates

By using the auxiliary obstacle problem to connect the continuous and discrete obstacle problems, we can show that

Theorem $\|u - u_h\|_h \leq Ch$

It is not difficult to deduce an L_∞ error estimate from the energy error estimate since $\|\cdot\|_h$ behaves like $\|\cdot\|_{H^2(\Omega)}$ and

$$\|\zeta\|_{L_\infty(\Omega)} \leq C_\Omega \|\zeta\|_{H^2(\Omega)} \qquad \forall\, \zeta \in H^2(\Omega)$$

by the Sobolev inequality.

Theorem $\|u - u_h\|_{L_\infty(\Omega)} \leq Ch$

But numerical results indicate that this estimate is not sharp.

General Polygons and General Boundary Conditions

Theorem $\|u - u_h\|_h + \|u - u_h\|_{L_\infty(\Omega)} \leq Ch^\alpha$

The index of elliptic regularity α is determined by the interior angles of Ω and the boundary conditions. For clamped plates we have $\frac{1}{2} < \alpha \leq 1$ and we can take α to be 1 if Ω is convex.

For the obstacle problem with the boundary conditions of simply supported plates, the value of α can be less than 1 even on convex polygonal domains. But we can improve the order of convergence using various techniques from the finite element arsenal.

Theorem With proper treatment we have

$$\|u - u_h\|_h + \|u - u_h\|_{L_\infty(\Omega)} \leq Ch$$

An Optimal Control Problem with Pointwise State Constraint

Ω is a bounded convex polygonal domain in $\mathbb{R}^2$.

Find $\quad y = \underset{v \in K}{\text{argmin}} \frac{1}{2}\Big[\|y - y_d\|^2_{L_2(\Omega)} + \beta\|\Delta y\|^2_{L_2(\Omega)}\Big]$

$$K = \{v \in H^2(\Omega) \cap H^1_0(\Omega) : \, v \leq \psi \quad \text{in } \Omega\}$$

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$$\text{Find} \quad y = \underset{v \in K}{\text{argmin}} \, \frac{1}{2} \Big[\|y - y_d\|^2_{L_2(\Omega)} + \beta \|\Delta y\|^2_{L_2(\Omega)} \Big]$$

$$K = \{ v \in H^2(\Omega) \cap H^1_0(\Omega) : \, v \le \psi \quad \text{in } \Omega \}$$

Equivalent Problem

$$\text{Find} \quad y = \underset{v \in K}{\text{argmin}} \Big(\frac{1}{2} \big[\beta a(v,v) + (v,v) \big] - (y_d, v) \Big)$$

$$a(w,v) = \int_\Omega D^2 w : D^2 v \, dx$$

$$K = \{ v \in H^2(\Omega) \cap H^1_0(\Omega) : \, v \le \psi \quad \text{in } \Omega \}$$

A Quadratic C^0 Interior Penalty Method

$\mathcal{T}_h$ = triangulation of Ω

V_h = quadratic finite element space $(\subset H_0^1(\Omega))$

$K_h = \{v \in V_h : v \leq \psi$ at all the vertices of $\mathcal{T}_h\}$

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\begin{aligned}
a_h(w,v) = \sum_{T\in\mathcal{T}_h} \int_T D^2 w : D^2 v\, dx &+ \sum_{e\in\mathcal{E}_h^i} \int_e \{\!\{\partial^2 w/\partial n^2\}\!\} [\![\partial v/\partial n]\!]\, ds \\
&+ \sum_{e\in\mathcal{E}_h^i} \int_e \{\!\{\partial^2 v/\partial n^2\}\!\} [\![\partial w/\partial n]\!]\, ds \\
&+ \sigma \sum_{e\in\mathcal{E}_h^i} |e|^{-1} \int_e [\![\partial w/\partial n]\!] [\![\partial v/\partial n]\!]\, ds
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$\mathcal{T}_h =$ triangulation of Ω

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Discrete Problem

$$\text{Find} \quad y = \underset{v \in K_h}{\operatorname{argmin}} \left(\frac{1}{2}\left[\beta a_h(v,v) + (v,v)\right] - (y_d, v) \right)$$

$$K_h = \{v \in V_h : v \le \psi \quad \text{at the vertices of } \mathcal{T}_h\}$$

Error Estimates

Theorem Without special treatment we have

$$\|y - y_h\|_h + \|y - y_h\|_{L_\infty(\Omega)} \leq Ch^\alpha$$

Theorem With proper treatment we have

$$\|y - y_h\|_h + \|y - y_h\|_{L_\infty(\Omega)} \leq Ch$$

$$\|v\|_h^2 = \beta\Big(\sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h^i} |e|^{-1}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2\Big) + \|v\|_{L_2(\Omega)}^2$$

Error Estimates

Theorem Without special treatment we have

$$\|y - y_h\|_h + \|y - y_h\|_{L_\infty(\Omega)} \leq Ch^\alpha$$

Theorem With proper treatment we have

$$\|y - y_h\|_h + \|y - y_h\|_{L_\infty(\Omega)} \leq Ch$$

We can compute an approximation u_h of the optimal control u from the discrete optimal state y_h through post-processing so that

$$\|u - u_h\|_{L_2(\Omega)} \approx \|y - y_h\|_h$$

Post-Processing

Procedure 1 Since $u = -\Delta y$, we take

$$u_h = -\Delta_h y_h$$

where Δ_h is the piecewise defined Laplacian.

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Procedure 2 Since

$$\int_\Omega \nabla y \cdot \nabla v \, dx = \int_\Omega uv \, dx \qquad \forall\, v \in H_0^1(\Omega)$$

we compute $u_h \in V_h$ by

$$\int_\Omega \nabla y_h \cdot \nabla v \, dx = \int_\Omega u_h v \, dx \qquad \forall\, v \in V_h$$

Post-Processing

Procedure 3 Since

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla v\, dx &= \int_\Omega \nabla(-\Delta y) \cdot \nabla v\, dx \\ &= \int_\Omega D^2 y : D^2 v\, dx \qquad \forall\, v \in H_0^1(\Omega) \cap H^2(\Omega)\end{aligned}$$

we compute $u_h \in V_h$ by

$$\int_\Omega \nabla u_h \cdot \nabla v\, dx = a_h(y_h, v) \qquad \forall\, v \in V_h$$

where $a_h(\cdot,\cdot)$ is the bilinear form for the quadratic C^0 interior penalty method that approximates the bilinear form

$$(w, v) \longrightarrow \int_\Omega D^2 w : D^2 v\, dx$$

Post-Processing

Procedure 1

numerical differentiation

Procedure 2

numerical differentiation followed by averaging

Procedure 3

numerical differentiation followed by smoothing

Post-Processing

Theorem For all three procedures we have

$$\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} \leq Ch^{\alpha}$$

without special treatments, and

$$\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} \leq Ch$$

with special treatments.

Post-Processing

Theorem For all three procedures we have

$$\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} \leq Ch^{\alpha}$$

without special treatments, and

$$\|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} \leq Ch$$

with special treatments.

Procedure 1, which only involves numerical differentiation, is the cheapest one. Procedure 2, which also requires solving a problem involving the mass matrix, is more expensive. Procedure 3, which requires solving a problem involving the stiffness matrix of a second order problem, is the most expensive one. But in practice Procedure 3 can outperform Procedure 2, which in turn can outperform Procedure 1.

Comparison

- There are finite element methods for this problem that are based on discrete versions of the optimal control problem (Deckelnick-Hinze 2007, Meyer 2008, Casas-Mateos-Vexler 2014) where

$$|y - y_h|_{H^1(\Omega)} + \|u - u_h\|_{L_2(\Omega)} = O(h)$$

for a smooth Ω or a rectangular Ω.

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  for a smooth Ω or a rectangular Ω.

- In our approach we have

$$\|y - y_h\|_h + \|u - u_h\|_{L_2(\Omega)} \leq O(h)$$

  for a smooth Ω or a rectangular Ω, where $\|\cdot\|_h$ is an H^2-like norm. Therefore the discrete state y_h in our approach provides more information for the optimal state y. Moreover the convergence of y_h to y in the H^1 norm or the L_∞ norm is of higher order in practice.

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$$\|y - y_h\|_h + \|u - u_h\|_{L_2(\Omega)} \leq O(h)$$

  for a smooth Ω or a rectangular Ω, where $\|\cdot\|_h$ is an H^2-like norm.

- We also observe high order convergence for the discrete optimal control u_h obtained by the second and third post-processing procedures.

Numerical Results

The first set of numerical experiments involve the obstacle problem for clamped Kirchhoff plates on the unit square

$$(-.5, .5) \times (-.5, .5)$$

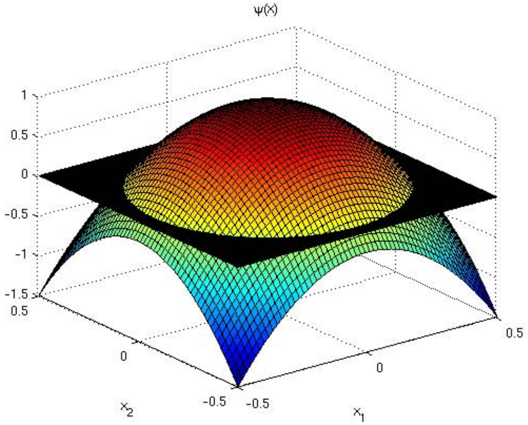
Numerical Results: Experiment 1

External Force

$$f = 0$$

Obstacle Function

$$\psi(x) = 1 - 5|x|^2 + |x|^4$$



Numerical Results: Experiment 1

h	relative error in $\|\cdot\|_h$	order	L_∞ error	order
2^{-1}	3.2401×10^{-1}		1.0000×10^{0}	
2^{-2}	4.5394×10^{-1}	-0.48	3.4417×10^{-1}	1.53
2^{-3}	4.9944×10^{-1}	-0.13	5.9705×10^{-2}	2.52
2^{-4}	3.8333×10^{-1}	0.38	2.6127×10^{-2}	1.19
2^{-5}	1.9609×10^{-1}	0.96	3.6557×10^{-3}	2.83
2^{-6}	9.2707×10^{-2}	1.08	1.2895×10^{-3}	1.50
2^{-7}	4.4712×10^{-2}	1.05	4.1668×10^{-4}	1.62
2^{-8}	2.1855×10^{-2}	1.03	1.0245×10^{-4}	2.02

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h} |e| \|\{\!\{\partial^2 v/\partial n^2\}\!\}\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h} |e|^{-1} \|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2$$

Numerical Results: Experiment 1

We can also visualize the approximation to the coincidence set

$$I = \{x : \ u(x) = \psi(x)\}$$

by plotting all the nodes p (vertices and midpoints) that satisfy the condition

$$|u_h(p) - \psi(p)| \leq L_\infty \text{ error}$$

where the L_∞ error is estimated by comparing solutions on consecutive levels.

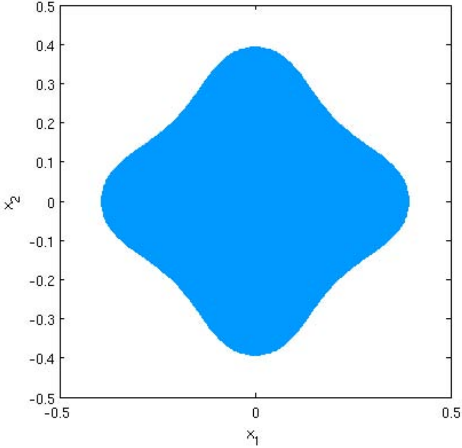
Since

$$\Delta^2 \psi = \Delta^2 (1 - 5|x|^2 + |x|^4) > 0$$

the non-coincidence set is connected.

Caffarelli and Friedman (1979)

Numerical Results: Experiment 1



Coincidence set I

$(h = 2^{-8})$

Numerical Results: Experiment 2

External Force

$$f = 0$$

Obstacle Function

$$\psi(x) = 1 - 5|x|^2 - |x|^4$$

This is similar to Experiment 1 except that

$$\Delta^2\psi = \Delta^2(1 - 5|x|^2 - |x|^4) < 0$$

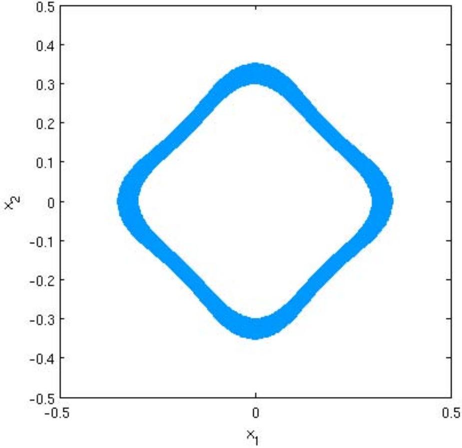
which implies that the interior of the coincidence set should be empty.

(Otherwise $\Delta^2 u = \Delta^2\psi < 0$ in an open set contradicting the fact that $\Delta^2 u$ is a nonnegative Borel measure.)

Numerical Results: Experiment 2

h	relative error in $\|\cdot\|_h$	order	L_∞ error	order
2^{-1}	3.4133×10^{-1}		1.0000×10^{0}	
2^{-2}	4.7596×10^{-1}	-0.47	3.3309×10^{-1}	1.58
2^{-3}	5.1117×10^{-1}	-0.10	7.2578×10^{-2}	2.19
2^{-4}	3.3897×10^{-1}	0.59	2.5308×10^{-2}	1.51
2^{-5}	1.6913×10^{-1}	1.00	7.6540×10^{-3}	1.72
2^{-6}	7.9146×10^{-2}	1.09	1.6226×10^{-3}	2.23
2^{-7}	3.8567×10^{-2}	1.03	5.8201×10^{-4}	1.47
2^{-8}	1.8889×10^{-2}	1.02	1.0995×10^{-4}	2.40

Numerical Results: Experiment 2



Coincidence set I

$(h = 2^{-8})$

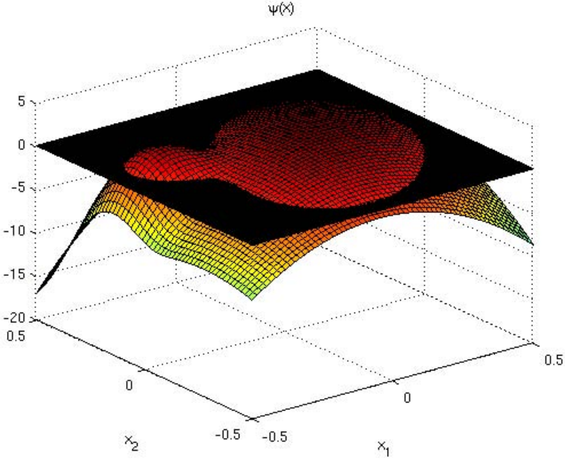
Numerical Results: Experiment 3

External Force

$$f(x) = \begin{cases} 1000e^{|x|^2} & \text{if } x_1 \leq 0 \text{ and } x_2 \geq 0 \\ 5000(3 + |x|^2) & \text{if } x_1 > 0 \text{ and } x_2 \geq 0 \\ -300\sin(\pi x_1)\sin(\pi x_2) & \text{if } x_1 \leq 0 \text{ and } x_2 < 0 \\ -200(1 - |x|^3) & \text{if } x_1 > 0 \text{ and } x_2 < 0 \end{cases}$$

Numerical Results: Experiment 3

A Nonsymmetric Obstacle Function ψ



Numerical Results: Experiment 3

h	relative error in $\|\cdot\|_h$	order	L_∞ error	order
2^{-1}	5.8955×10^{-1}		5.3265×10^{0}	
2^{-2}	4.1882×10^{-1}	0.49	9.1903×10^{-1}	2.53
2^{-3}	5.0682×10^{-1}	-0.28	9.8753×10^{-1}	-0.10
2^{-4}	3.6432×10^{-1}	0.48	4.4694×10^{-1}	1.14
2^{-5}	2.0144×10^{-1}	0.85	1.8191×10^{-1}	1.30
2^{-6}	9.8022×10^{-2}	1.04	7.9253×10^{-2}	1.20
2^{-7}	4.8507×10^{-2}	1.02	2.4371×10^{-2}	1.70

Numerical Results

The second set of numerical experiments involve the optimal control problem with pointwise state constraint.

Numerical Results: Experiment 4

$$\Omega = (-0.5, 0, 5) \times (-0.5, 0.5)$$

$$\psi(x) = 0.1$$

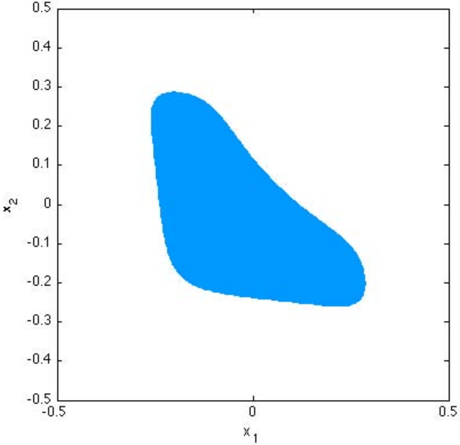
$$y_d = \sin\big(2\pi(x_1 + 0.5)(x_2 + 0.5)\big)$$

$$\beta = 10^{-3}$$

$$\alpha = 1$$

uniform meshes

Numerical Results: Experiment 4



Coincidence set

$(h = 2^{-8})$

Numerical Results: Experiment 4

h	relative error of state in $\|\cdot\|_h$	order	relative error of state in $|\cdot|_{H^1(\Omega)}$	order
2^{-1}	2.697×10^{0}		3.287×10^{0}	
2^{-2}	1.244×10^{0}	1.12	1.154×10^{0}	1.51
2^{-3}	6.764×10^{-1}	0.88	3.394×10^{-1}	1.77
2^{-4}	3.455×10^{-1}	0.97	9.206×10^{-2}	1.88
2^{-5}	1.749×10^{-1}	0.98	2.664×10^{-2}	1.79
2^{-6}	8.643×10^{-2}	1.01	7.282×10^{-3}	1.87
2^{-7}	4.267×10^{-2}	1.02	1.856×10^{-3}	1.97
2^{-8}	2.123×10^{-2}	1.01	4.671×10^{-4}	1.99

Relative errors of the state in $\|\cdot\|_h$ and $|\cdot|_{H^1(\Omega)}$

Numerical Results: Experiment 4

h	proc. 1	order	proc. 2	order	proc. 3	order
2^{-1}	1.736×10^{0}		2.675×10^{0}		1.729×10^{0}	
2^{-2}	1.057×10^{0}	0.72	1.563×10^{0}	0.78	1.515×10^{0}	0.19
2^{-3}	6.797×10^{-1}	0.64	9.448×10^{-1}	0.73	7.525×10^{-1}	1.01
2^{-4}	3.569×10^{-1}	0.93	3.529×10^{-1}	1.42	2.194×10^{-1}	1.78
2^{-5}	1.775×10^{-1}	1.01	1.217×10^{-1}	1.54	7.167×10^{-2}	1.61
2^{-6}	8.578×10^{-2}	1.05	4.198×10^{-2}	1.54	2.830×10^{-2}	1.34
2^{-7}	4.194×10^{-2}	1.03	1.313×10^{-2}	1.68	8.170×10^{-3}	1.79
2^{-8}	2.079×10^{-2}	1.01	4.442×10^{-3}	1.56	2.945×10^{-3}	1.47

Relative errors of the control in $\| \cdot \|_{L_2(\Omega)}$

Numerical Results: Experiment 4

h	proc. 2	order	proc. 3	order
2^{-1}	2.148×10^{0}		1.794×10^{0}	
2^{-2}	2.322×10^{0}	-0.11	1.585×10^{0}	0.18
2^{-3}	2.395×10^{0}	-0.04	1.427×10^{0}	0.15
2^{-4}	1.823×10^{0}	0.39	8.604×10^{-1}	0.73
2^{-5}	1.265×10^{0}	0.53	5.297×10^{-1}	0.70
2^{-6}	8.536×10^{-1}	0.57	3.845×10^{-1}	0.46
2^{-7}	5.484×10^{-1}	0.64	2.191×10^{-1}	0.81
2^{-8}	3.637×10^{-1}	0.59	1.522×10^{-1}	0.53

Relative errors of the control in $|\cdot|_{H^1(\Omega)}$

Numerical Results: Experiment 5

$$\Omega = (-0.5, 0.5) \times (-0.5, 0.5)$$

$$\psi(x) = \sin[2\pi(x_1 + 0.5)(x_2 + 0.5)] \sin[6\pi(x_1 - 0.5)(x_2 - 0.5)] + 0.2$$

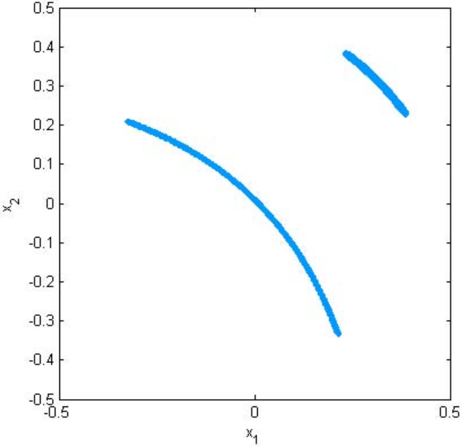
$$y_d = \begin{cases} \cos[\pi(x_1 - 0.25)] & x_1 > 0, x_2 > 0 \\ 5(x_1 + 0.25)^2 + (x_2 - 0.25)^2 & x_1 \le 0, x_2 > 0 \\ -\frac{1}{2}\exp[(x_1 + 0.25)^2 + (x_2 + 0.25)^2] & x_1 \le 0, x_2 \le 0 \\ \frac{1}{2} + [(x_1 - 0.25)^2 + (x_2 + 0.25)^2]^{3/2} & x_1 > 0, x_2 \le 0 \end{cases}$$

$$\beta = 10^{-3}$$

$$\alpha = 1$$

uniform meshes

Numerical Results: Experiment 5



Coincidence set

$(h = 2^{-8})$

Numerical Results: Experiment 5

h	relative error of state in $\|\cdot\|_h$	order	relative error of state in $\lvert\cdot\rvert_{H^1(\Omega)}$	order
2^{-2}	4.368×10^{-1}		2.911×10^{-1}	
2^{-3}	5.684×10^{-1}	-0.38	4.403×10^{-1}	-0.60
2^{-4}	3.350×10^{-1}	0.76	1.718×10^{-1}	1.36
2^{-5}	1.577×10^{-1}	1.09	4.984×10^{-2}	1.79
2^{-6}	7.552×10^{-2}	1.06	1.908×10^{-2}	1.39
2^{-7}	3.586×10^{-2}	1.07	3.954×10^{-3}	2.27
2^{-8}	1.727×10^{-2}	1.05	1.073×10^{-3}	1.88

Relative errors of the state in $\|\cdot\|_h$ and $\lvert\cdot\rvert_{H^1(\Omega)}$

Numerical Results: Experiment 5

h	proc. 1	order	proc. 2	order	proc. 3	order
2^{-2}	5.697×10^{-1}		6.204×10^{-1}		5.064×10^{-1}	
2^{-3}	5.555×10^{-1}	0.04	8.836×10^{-1}	-0.51	1.049×10^{0}	-1.05
2^{-4}	3.855×10^{-1}	0.53	5.062×10^{-1}	0.80	5.631×10^{-1}	0.90
2^{-5}	1.910×10^{-1}	1.01	2.035×10^{-1}	1.31	1.992×10^{-1}	1.50
2^{-6}	8.977×10^{-2}	1.09	9.077×10^{-2}	1.16	9.624×10^{-2}	1.05
2^{-7}	4.522×10^{-2}	0.99	3.558×10^{-2}	1.35	3.669×10^{-2}	1.39
2^{-8}	2.190×10^{-2}	1.05	1.475×10^{-2}	1.27	1.549×10^{-2}	1.24

Relative errors of the control in $\|\cdot\|_{L_2(\Omega)}$

Numerical Results: Experiment 5

h	proc. 2	order	proc. 3	order
2^{-2}	9.991×10^{-1}		6.770×10^{-1}	
2^{-3}	1.760×10^{0}	-0.82	1.534×10^{0}	-1.18
2^{-4}	1.763×10^{0}	-0.00	1.382×10^{-1}	0.15
2^{-5}	1.520×10^{0}	0.21	1.038×10^{-1}	0.41
2^{-6}	1.226×10^{0}	0.31	8.742×10^{-1}	0.25
2^{-7}	8.800×10^{-1}	0.48	5.743×10^{-1}	0.61
2^{-8}	6.804×10^{-1}	0.37	4.549×10^{-1}	0.34

Relative errors of the control in $|\cdot|_{H^1(\Omega)}$

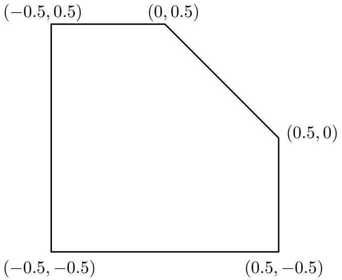
Numerical Results: Experiment 6

$\Omega =$ a pentagon

$$\psi(x) = 0.1$$

$$y_d = \sin\big(2\pi(x_1 + 0.5)(x_2 + 0.5)\big)$$

$$\beta = 10^{-3}$$

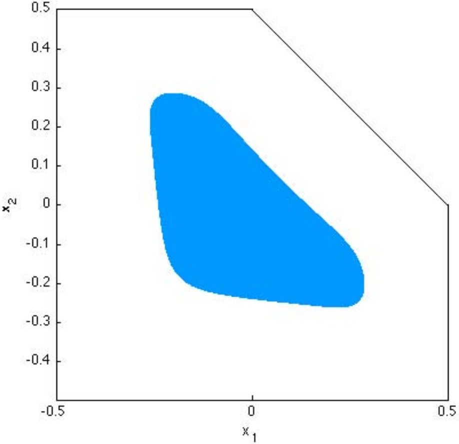


$$\alpha = \frac{1}{3}$$

uniform meshes

$$\|y - y_h\|_h + \|u - u_h\|_{L_2(\Omega)} \le Ch^{\frac{1}{3}}$$

Numerical Results: Experiment 6



Coincidence set

$(h = 2^{-9})$

Numerical Results: Experiment 6

h	relative error of state in $\|\cdot\|_h$	order	relative error of state in $|\cdot|_{H^1(\Omega)}$	order
2^{-2}	1.275×10^{0}		1.156×10^{0}	
2^{-3}	7.305×10^{-1}	0.80	3.484×10^{-1}	1.73
2^{-4}	3.607×10^{-1}	1.02	8.279×10^{-2}	2.07
2^{-5}	1.958×10^{-1}	0.88	2.217×10^{-2}	1.90
2^{-6}	1.176×10^{-1}	0.73	6.526×10^{-3}	1.76
2^{-7}	7.897×10^{-2}	0.57	2.131×10^{-3}	1.61
2^{-8}	5.772×10^{-2}	0.45	8.760×10^{-4}	1.28
2^{-9}	4.416×10^{-2}	0.39	4.570×10^{-4}	0.94

Relative errors of the state in $\|\cdot\|_h$ and $|\cdot|_{H^1(\Omega)}$

Numerical Results: Experiment 6

h	proc. 1	order	proc. 2	order	proc. 3	order
2^{-2}	2.244×10^{0}		1.729×10^{0}		1.716×10^{0}	
2^{-3}	1.556×10^{0}	0.53	1.084×10^{0}	0.67	8.585×10^{-1}	1.00
2^{-4}	7.624×10^{-1}	1.03	3.937×10^{-1}	1.46	2.461×10^{-1}	1.80
2^{-5}	3.944×10^{-1}	0.95	1.365×10^{-1}	1.53	8.015×10^{-2}	1.62
2^{-6}	2.257×10^{-1}	0.81	5.331×10^{-2}	1.36	3.234×10^{-2}	1.31
2^{-7}	1.472×10^{-1}	0.62	2.539×10^{-2}	1.07	9.605×10^{-3}	1.75
2^{-8}	1.061×10^{-1}	0.47	1.716×10^{-2}	0.56	3.625×10^{-3}	1.41
2^{-9}	8.064×10^{-2}	0.40	1.312×10^{-2}	0.39	1.416×10^{-3}	1.36

Relative errors of the control in $\|\cdot\|_{L_2(\Omega)}$

Numerical Results: Experiment 6

h	proc. 2	order	proc. 3	order
2^{-2}	1.074×10^{0}		1.809×10^{0}	
2^{-3}	1.123×10^{0}	-0.06	1.685×10^{0}	0.10
2^{-4}	8.438×10^{-1}	0.41	9.972×10^{-1}	0.76
2^{-5}	6.000×10^{-1}	0.49	1.038×10^{-1}	0.71
2^{-6}	4.728×10^{-1}	0.34	4.498×10^{-1}	0.44
2^{-7}	4.8583×10^{-1}	-0.04	2.583×10^{-1}	0.80
2^{-8}	6.782×10^{-1}	-0.48	1.787×10^{-1}	0.53
2^{-9}	1.047×10^{-1}	-0.63	1.254×10^{-1}	0.51

Relative errors of the control in $|\cdot|_{H^1(\Omega)}$

Numerical Results: Experiment 7

$\Omega =$ a pentagon

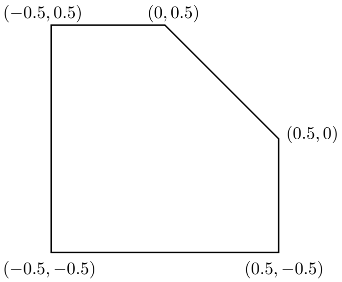
$$\psi(x) = 0.1$$

$$y_d = \sin\big(2\pi(x_1 + 0.5)(x_2 + 0.5)\big)$$

$$\beta = 10^{-3}$$

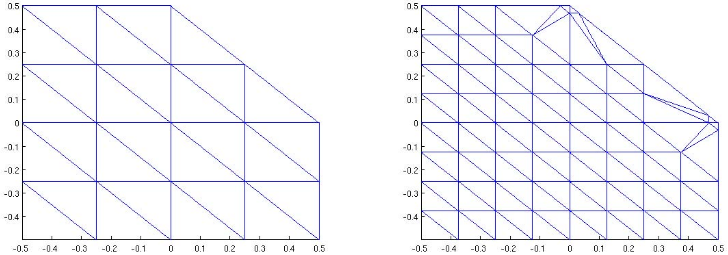
$$\alpha = \frac{1}{3}$$

graded meshes



$$\|y - y_h\|_h + \|u - u_h\|_{L_2(\Omega)} \leq Ch$$

Numerical Results: Experiment 7



$\mathcal{T}_0$ (left) and $\mathcal{T}_1$ (right)

Numerical Results: Experiment 7

h	relative error of state in $\|\cdot\|_h$	order	relative error of state in $|\cdot|_{H^1(\Omega)}$	order
2^{-3}	5.153×10^{-1}		2.246×10^{-1}	
2^{-4}	3.433×10^{-1}	0.59	1.099×10^{-1}	1.03
2^{-5}	1.984×10^{-1}	0.79	4.501×10^{-2}	1.29
2^{-6}	1.096×10^{-1}	0.86	1.501×10^{-2}	1.58
2^{-7}	6.013×10^{-2}	0.87	5.728×10^{-3}	1.39
2^{-8}	3.284×10^{-2}	0.87	1.949×10^{-4}	1.56
2^{-9}	1.780×10^{-2}	0.88	6.608×10^{-4}	1.56

Relative errors of the state in $\|\cdot\|_h$ and $|\cdot|_{H^1(\Omega)}$

Numerical Results: Experiment 7

h	proc. 1	order	proc. 2	order	proc. 3	order
2^{-3}	5.568×10^{-1}		4.717×10^{-1}		1.587×10^{0}	
2^{-4}	3.776×10^{-1}	0.56	2.953×10^{-1}	0.68	5.001×10^{-1}	1.67
2^{-5}	2.148×10^{-1}	0.81	1.532×10^{-1}	0.95	1.428×10^{-1}	1.81
2^{-6}	1.151×10^{-1}	0.90	7.764×10^{-2}	0.98	5.409×10^{-2}	1.40
2^{-7}	6.221×10^{-2}	0.89	4.084×10^{-2}	0.93	1.554×10^{-2}	1.80
2^{-8}	3.355×10^{-2}	0.89	2.165×10^{-2}	0.92	5.675×10^{-3}	1.45
2^{-9}	1.795×10^{-2}	0.90	1.135×10^{-2}	0.93	1.906×10^{-3}	1.57

Relative errors of the control in $\|\cdot\|_{L_2(\Omega)}$

Numerical Results: Experiment 7

h	proc. 3	order
2^{-3}	2.6689×10^{0}	
2^{-4}	1.5051×10^{0}	0.83
2^{-5}	8.0510×10^{-1}	0.90
2^{-6}	5.3368×10^{-1}	0.59
2^{-7}	2.8460×10^{-1}	0.91
2^{-8}	2.0225×10^{-1}	0.49
2^{-9}	1.3918×10^{-1}	0.54

Relative errors of the control in $|\cdot|_{H^1(\Omega)}$

Adaptive C^0 Interior Penalty Methods

A Residual Based Error Estimator for Clamped Plates

$$\eta_h = \Big(\sum_{e \in \mathcal{E}_h} \eta_{e,1}^2 + \sum_{e \in \mathcal{E}_h^i} (\eta_{e,2}^2 + \eta_{e,3}^2) + \sum_{T \in \mathcal{T}_h} \eta_T^2 \Big)^{\frac{1}{2}},$$

where

$$\eta_{e,1} = \frac{\sigma}{|e|^{\frac{1}{2}}} \| [\![\partial u_h / \partial n]\!] \|_{L_2(e)}$$

$$\eta_{e,2} = |e|^{\frac{1}{2}} \| [\![\partial^2 u_h / \partial n^2]\!] \|_{L_2(e)}$$

$$\eta_{e,3} = |e|^{\frac{3}{2}} \| [\![\partial^3 u_h / \partial n^3]\!] \|_{L_2(e)}$$

$$\eta_T = h_T^2 \| f - \Delta^2 u_h \|_{L_2(T)}$$

Brenner-Gudi-S (2010)

Adaptive C^0 Interior Penalty Methods

Theorem η_h is a reliable and (locally) efficient estimator for $\|\tilde{u}_h - u_h\|_h$.

Since the functions u and $\tilde{u}_h$ have singularities at the same locations, we can use η_h as an error indicator in adaptive computations for u.

S.C. Brenner, J. Gedicke, S. and Y. Zhang

An *a posteriori* error analysis for a quadratic C^0 interior penalty method for the obstacle problem of clamped Kirchhoff plates

(in preparation)

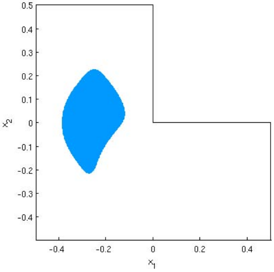
Numerical Results for an L-Shaped Domain

Domain $\Omega = (-0.5, 0.5)^2 \setminus [0, 0.5]^2$

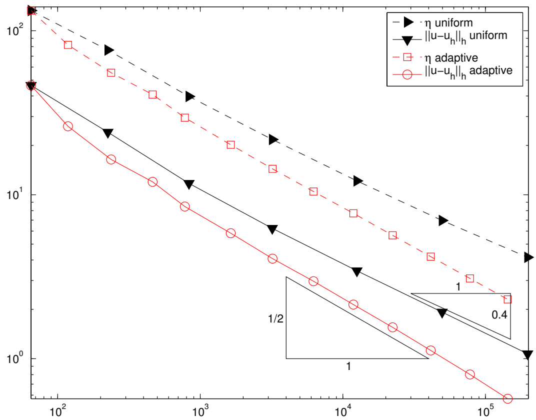
External Force $f = 0$

Obstacle Function $\psi(x) = 1 - \left[\frac{(x_1+0.25)^2}{0.2^2} + \frac{x_2^2}{0.35^2}\right]$

Coincidence Set

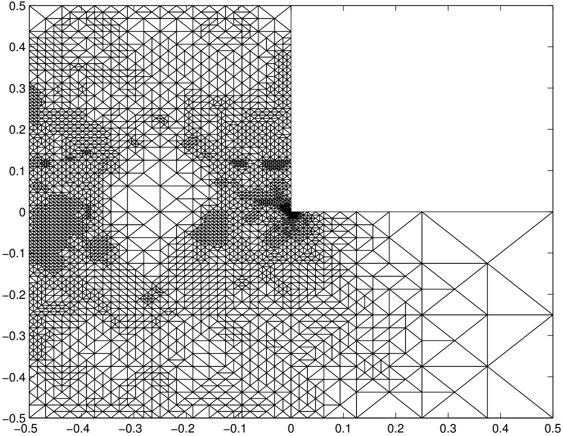


Numerical Results for an L-Shaped Domain



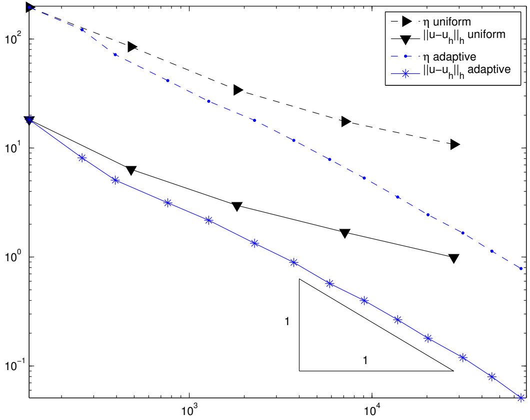
Quadratic C^0-IPM on uniform and adaptive meshes

Numerical Results for an L-Shaped Domain



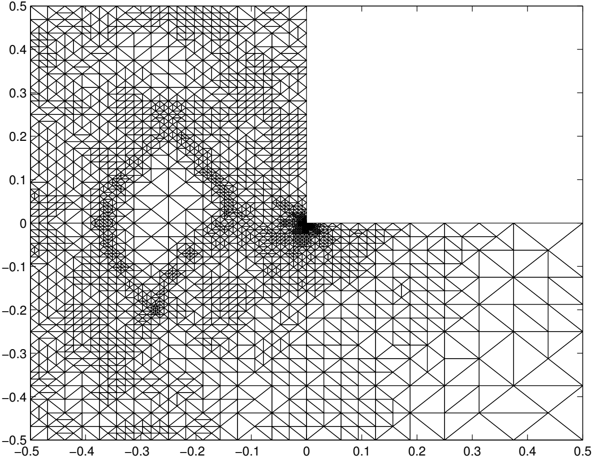
Adaptive mesh with about 3000 nodes

Numerical Results for an L-Shaped Domain



Cubic C^0-IPM on uniform and adaptive meshes

Numerical Results for an L-Shaped Domain



Adaptive mesh with about 2000 nodes

Open Problems

- Sharp error estimates in $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_{L_\infty(\Omega)}$
- Convergence in $\|\cdot\|_{H^1(\Omega)}$ for optimal control generated by the post-processing procedure 3
- Convergence and optimality of adaptive methods
- Fast solvers