Applications of Optimal Control to Stabilization

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Contents

Stabilization

Consider the equation in Y

$$E \begin{cases} y'(t) = Ay(t) + Bu(t), \quad \forall\, t \geq 0, \\ y(0) = y_0, \\ y_0 \in Y, \quad u \in U, \end{cases}$$

where

- Y and U are two Hilbert spaces.
- The unbounded operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup on Y, denoted by $\{e^{tA}\}_{t\geq 0}$.
- The control operator $B \in \mathcal{L}(U; Y)$.

Suppose that $\{e^{tA}\}_{t\geq 0}$ is unstable.

Exponential stabilization

Aim Find a control $u \in L^2(0, \infty; U)$, in a feedback form

$$u(t) = Ky(t),$$

so that the closed loop system

$$y'(t) = (A + BK)y(t), \quad y(0) = y_0,$$

is exponentially stable on Y, i.e.

$$\|e^{t(A+BK)}\| \leq Me^{-\lambda t}, \quad \forall\, t \geq 0,$$

for some constants $M > 0$ and $\lambda > 0$.

Definition

The pair (A, B) is said to be stabilizable, if there exists $K \in \mathcal{L}(Y; U)$ such that $\{e^{t(A+BK)}\}_{t \geq 0}$ is exponentially stable on Y.

Remark If system E is null controllable, (A, B) is stabilizable.

Theorem (Datko theorem)

Let $\{S(t)\}_{t \geq 0}$ be a strongly continuous semigroup on Y. The semigroup $\{S(t)\}_{t \geq 0}$ is exponentially stable if and only if

$$\int_0^\infty \|S(t)y_0\|^2 < \infty, \quad \forall\, y_0 \in Y.$$

Connection with optimal control problem

In view of the above theorem, to solve the stabilization problem we look for the solution to the control problem P

$$\inf\{J(y,u), \quad (y,u) \quad \text{obeys} \quad E, \quad u \in L^2(0,\infty;U)\},$$
$$J(y,u) = \frac{1}{2}\int_0^\infty \|y(t)\|_Y^2\,dt + \frac{1}{2}\int_0^\infty \|u(t)\|_u^2\,dt.$$

Qn Does there exist $(\bar{y},\bar{u})$, the solution of the optimal problem P?

Qn Can the optimal control be found in the feedback form

$$\bar{u}(t) = K\bar{y}(t)?$$

Qn Is $\{e^{t(A+BK)}\}_{t\geq 0}$ exponentially stable?

Aim and outlines to prove it

- The optimal control problem P is called LQR problem with infinite horizon.
- First, we need to solve LQR problem with finite horizon and find the optimal control in a feedback form via the solution of a differential Riccati equation.
- Under a finite cost condition, we show that P admits unique solution and obtain the optimal control in a feedback form via the solution of a algebraic Riccati equation.
- Finally, the connection between the optimal control of P obtained in the feedback form and the stabilization of E is brought out.

Consider the problem $P(0,T,y_0)$

$$\inf\{J_T(y,u), \quad (y,u) \quad \text{obeys} \quad E, \quad u \in L^2(0,T;U)\},$$
$$J_T(y,u) = \frac{1}{2}\int_0^T \|y(t)\|_Y^2\,dt + \frac{1}{2}\int_0^T \|u(t)\|_U^2\,dt.$$

This problem $P(0,T,y_0)$ admits a unique solution $(\bar{y},\bar{u})$ characterized by

the system

$$\bar{y}'(t) = A\bar{y}(t) - BB^*p(t), \quad \bar{y}(0) = y_0,$$
$$-p'(t) = A^*p(t) + \bar{y}(t), \quad p(T) = 0,$$
$$\bar{u}(t) = -B^*p(t).$$

To find $\bar{u}$ in feedback form,

$$\bar{u} = K\bar{y}(t),$$

we study the family of problems $P(s, T, \zeta)$

$$\inf\{J_{s,T}(y,u), \quad (y,u) \quad \text{obeys} \quad (E_{s,\zeta}), \quad u \in L^2(0,T;U)\},$$
$$J_{s,T}(y,u) = \frac{1}{2}\int_s^T \|y(t)\|_Y^2\,dt + \frac{1}{2}\int_s^T \|u(t)\|_U^2\,dt,$$

and

$$(E_{s,\zeta}) \qquad y'(t) = Ay(t) + Bu(t), \quad y(s) = \zeta.$$

The solution (y^s_ζ, u^s_ζ) to $P(s,T,\zeta)$ is characterized by

$$\frac{dy^s_\zeta(t)}{dt} = Ay^s_\zeta(t) - BB^*p^s_\zeta(t), \quad y^s_\zeta(s) = \zeta,$$
$$-\frac{dp^s_\zeta(t)}{dt} = A^*p^s_\zeta(t) + y^s_\zeta(t), \quad p^s_\zeta(T) = 0,$$
$$u^s_\zeta(t) = -B^*p^s_\zeta(t).$$

Consider the mapping

$$P(s) : \zeta \longmapsto p^s_\zeta(s), \quad \forall\, s \in [0,T).$$

- The mapping is linear from Y into itself.
  We can show it using the linearity

$$(y^s_{\beta\zeta_1+\zeta_2}, p^s_{\beta\zeta_1+\zeta_2}, u^s_{\beta\zeta_1+\zeta_2}) = \beta(y^s_{\zeta_1}, p^s_{\zeta_1}, u^s_{\zeta_1}) + (y^s_{\zeta_2}, p^s_{\zeta_2}, u^s_{\zeta_2}).$$

- For all $t \in [0, T)$, $P(t) = P(t)^* \geq 0$.
  With an integration by parts formula between the solution p^s_ζ to

$$-p'(t) = A^* p(t) + y^s_\zeta(t), \quad p(T) = 0,$$

and the solution y^s_ξ to

$$y'(t) = Ay(t) - BB^* p^s_\xi(t), \quad y^s_\xi(s) = \xi,$$

we obtain

$$\Big(P(s)\zeta, \xi\Big)_Y = \int_s^T \Big(y^s_\zeta, y^s_\xi\Big)_Y + \int_s^T \Big(B^* p^s_\zeta, B^* p^s_\xi\Big)_U = \Big(\zeta, P(s)\xi\Big)_Y,$$

for all $\zeta \in Y$ and all $\xi \in Y$.

- For all $t \in [0,T)$, $P(t) \in \mathcal{L}(Y)$.
  - From the above identity, we write
$$\frac{1}{2}\Big(P(s)\zeta,\zeta\Big)_Y = J_{s,T}(y^s_\zeta, u^s_\zeta).$$
  - Consider the trajectory $z(t) = e^{(t-s)A}\zeta$ for all $t \geq s$ and $\zeta \in D(A)$. Then $z \in C^1([s,T];D(A)) \cap C([s,T];Y)$ and
$$J_{s,T}(z,0) = \|z\|^2_Y \leq M\|\zeta\|^2_Y, \quad \forall\, t \geq s,$$
for some constant $M > 0$.
  - Since y^s_ζ, u^s_ζ is the solution of the optimal problem $P(s,T,\zeta)$, we have
$$J_{s,T}(y^s_\zeta, u^s_\zeta) \leq J_{s,T}(e^{(t-s)A}\zeta, 0).$$
  - Combining all the above inequalities, we obtain for all $s \in [0,T)$,
$$\tfrac{1}{2}\Big(P(s)\zeta,\zeta\Big)_Y \leq M\|\zeta\|^2_Y.$$
  - Thus for all $t \in [0,T)$
$$\|P(t)^{\frac{1}{2}}\|_{\mathcal{L}(Y)} \leq M^{\frac{1}{2}}, \quad \|P(t)\|_{\mathcal{L}(Y)} \leq M.$$

- For any $t \in [s,T)$, $p^s_\zeta(t) = P(t)y^s_\zeta(t)$.
  - From the Dynamic programming principle (subsolutions of an optimal solution of a problem are themselves the optimal solutions for their corresponding subproblems), $u^s_\zeta(\cdot)$ and $u^t_{y^s_\zeta(t)}(\cdot)$, the optimal controls for the problems $P(s,T,\zeta)$ and $P(t,T,y^s_\zeta(t))$ satisfy

$$u^s_\zeta(\tau) = u^t_{y^s_\zeta(t)}(\tau), \quad \text{for almost} \quad \tau \in (t,T).$$

  - From the above relation, we derive

$$p^s_\zeta(t) = p^t_{y^s_\zeta(t)}(t), \quad t \in [s,T).$$

  - From the definition of map $P(\cdot)$, we obtain

$$p^t_{y^s_\zeta(t)}(t) = P(t)y^s_\zeta(t), \quad \forall\, t \in [0,T).$$

  - From the above two relations, the claim follows.

- For all $\zeta \in Y$ and all $\xi \in Y$, $t \longmapsto \Big(P(t)\zeta, \xi\Big)_Y$ is continuous on $t \in [0, T)$.
  - From the Duhamel formula and relation in above slide, for all $t \in (s, T)$,

$$\|y_\zeta^s(t)\|_Y \leq \|e^{(t-s)A}\zeta\|_Y + \int_s^T \|e^{(t-\tau)A}BB^*P(\tau)y_\zeta^s(\tau)\|_Y \, d\tau.$$

  - Then, for some constant $C > 0$,

$$\|y_\zeta^s(t)\|_Y \leq C\|\zeta\|_Y, \quad \|p_\zeta^s(t)\|_Y \leq C\|\zeta\|_Y.$$

  - Using these estimates, we can show

$$lim_{h\to 0}\|y_\zeta^{s+h} - y_\zeta^s\|_{C([(s+h)\wedge s, T];Y)} = 0,$$

$$lim_{h\to 0}\|p_\zeta^{s+h} - p_\zeta^s\|_{C([(s+h)\wedge s, T];Y)} = 0,$$

  - From that, we deduce $t \longmapsto \Big(P(t)\zeta, \xi\Big)_Y$ is continuous on $[0, T)$ for all $\zeta \in Y$ and all $\xi \in Y$.

$P(\cdot)$, the solution to a differential Riccati equation

Definition ($C_s([0,T];\mathcal{L}(Y))$)

We denote by $C_s([0,T];\mathcal{L}(Y))$ the space of mapping P from $[s,T]$ to $\mathcal{L}(Y)$ such that $t \longmapsto P(t)\zeta$ belongs to $C([0,T];\mathcal{L}(Y))$ for all $\zeta \in Y$.

- We can prove that the map $P \in C_s([0,T];\mathcal{L}(Y))$.
- We show $P(\cdot)$ is the solution to the differential Riccati equation (DRE)

$$P(t) = P(t)^*, \quad P(t) \geq 0, \quad t \in [0,T],$$
$$P'(t) + A^*P(t) + P(t)A - P(t)BB^*P(t) + I = 0, \quad t \in [0,T],$$
$$P(T) = 0.$$

Definition (Solution to (DRE))

A function $P \in C_s([0,T];\mathcal{L}(Y))$ is a solution to (DRE) on $(0,T)$ if and only if, for every $(\zeta,\xi) \in D(A) \times D(A)$ the function $(P(\cdot)\zeta,\xi)_Y$ belongs to $W^{1,1}(0,T)$ and satisfies

$$
\begin{aligned}
&P(t) = P(t)^*, \quad P(t) \geq 0, \quad t \in [0,T],\\
&\tfrac{d}{dt}(P(t)\zeta,\xi)_Y + (P(t)\zeta, A\xi)_Y + (P(t)A\zeta,\xi)_Y\\
&-(P(t)BB^*\zeta,\xi)_Y + (\zeta,\xi)_Y = 0, \quad t \in [0,T],\\
&(P(T)\zeta,\xi)_Y = 0.
\end{aligned}
$$

Theorem

The map P defined above is a unique solution to (DRE) on $(0,T)$.

- Proof We shall calculate $\frac{d^+}{ds}\Big(P(s)\zeta,\xi\Big)_Y$, the right hand side derivative of the map

$$s \to \Big(P(s)\zeta,\xi\Big)_Y$$

on $[0,T)$ for all $(\zeta,\xi) \in D(A) \times D(A)$.

- For that, consider

$$\begin{aligned} y'(t) &= Ay(t) - BB^*p(t), \quad y(s) = \zeta, \\ -p'(t) &= A^*p(t) + y(t), \quad p(T) = 0, \end{aligned}$$

and

$$\begin{aligned} z'(t) &= Az(t) - BB^*q(t), \quad z(s) = \xi, \\ -q'(t) &= A^*q(t) + z(t), \quad q(T) = 0. \end{aligned}$$

As our previous notations, $(y,p) = (y^s_\zeta, p^s_\zeta)$ and $(z,q) = (y^s_\xi, p^s_\xi)$.

- Using the expressions

$$y(t) = e^{(t-s)A}\zeta - \int_s^t e^{(t-\tau)A} BB^* p(\tau)\, d\tau,$$

$$z(t) = e^{(t-s)A}\xi - \int_s^t e^{(t-\tau)A} BB^* q(\tau)\, d\tau,$$

we can show

$$lim_{h\searrow 0}\|\frac{1}{h}\Big(y(s+h) - y(s)\Big) - A\zeta + BB^* p(s)\|_Y = 0,$$

$$lim_{h\searrow 0}\|\frac{1}{h}\Big(z(s+h) - z(s)\Big) - A\xi + BB^* q(s)\|_Y = 0.$$

- Using the equations satisfied by p and z and an integration by parts formula, we derive

$$\Big(P(s+h)y(s+h), z(s+h)\Big)_Y - \Big(P(s)y(s), z(s)\Big)_Y$$
$$= \int_{s+h}^{s} \Big(y(\tau), z(\tau)\Big)_Y + \Big(B^*p(\tau), B^*q(\tau)\Big)_U \, d\tau$$

- Using this identity and by a standard calculation, we can show that

$$\frac{d^+}{dt}(P(t)\zeta, \xi)_Y + (P(t)\zeta, A\xi)_Y + (P(t)A\zeta, \xi)_Y$$
$$-(P(t)BB^*\zeta, \xi)_Y + (\zeta, \xi)_Y = 0, \quad t \in [0, T],$$
$$(P(T)\zeta, \xi)_Y = 0.$$

- Since $s \to (P(s)\zeta, \xi)_Y$ is continuous on $[0, T)$ and $s \to \frac{d^+}{dt}(P(s)\zeta, \xi)_Y$ is bounded and continuous on $[0, , T)$, we can affirm that $s \to (P(s)\zeta, \xi)_Y$ is of class C^1 on $[0, T)$. Hence $P(\cdot)$ is a solution of *DRE*.
- For uniqueness, let $Q(\cdot)$ be a solution DRE.
- Claim $P(t) = Q(t), \quad \forall\, t \in [0, T]$.
- Consider for $s \in [0, T)$ and $\zeta \in D(A)$,

$$z'(t) = Az(t) - BB^*Q(t)z(t), \quad t \geq s, \quad z(s) = \zeta.$$

- Let $\{u_n\}_{n\in\mathbb{N}} \in C^1([0, T]; U)$ such that

$$u_n \to -B^*Qz, \quad \text{in} \quad L^2(0, T; U).$$

- $\{z_n\}_{n\in\mathbb{N}}$ satisfies

$$z'(t) = Az(t) - BB^*u_n(t), \quad t \geq s, \quad z(s) = \zeta.$$

- Set $q(t) = Q(t)z(t)$ and $q_n(t) = Q(t)z_n(t)$ for all $n \in \mathbb{N}$. Then using the terminal condition of DRE, $q(T) = Q(T) = 0$.
- Using DRE, we can show

$$-q_n'(t) = A^* q_n(t) + z_n(t) - Q(t)BB^*Q(t)z_n(t) - Q(t)Bu_n(t)$$

and taking $n \to \infty$,

$$-q'(t) = A^* q(t) + z.$$

- Thus (z, q) satisfies the problem $P(s, T, \zeta)$ and by the uniqueness of the solution of optimality problem $P(s, T, \zeta)$, we get $(z, q) = (y_\zeta^s, p_\zeta^s)$ and hence

$$Q(t) = P(t), \quad \forall\, t \in [0, T].$$

Remark Setting $Q(t) = P(T - t)$, where $P(\cdot)$ is the solution of DRE, we can show that $Q(\cdot)$ satisfies

$$
\begin{aligned}
&Q(t) = Q(t)^*, \quad Q(t) \geq 0, \quad t \in [0, T], \\
&Q'(t) = A^*Q(t) + Q(t)A - Q(t)BB^*Q(t) + I, \quad t \in [0, T], \\
&Q(0) = 0.
\end{aligned}
$$

Consider the optimal control problem P

$$\inf\{J(y,u), \quad (y,u) \quad \text{obeys} \quad E, \quad u \in L^2(0,\infty;U)\},$$
$$J(y,u) = \frac{1}{2}\int_0^\infty \|y(t)\|_Y^2\,dt + \frac{1}{2}\int_0^\infty \|u(t)\|_U^2\,dt.$$

and

$$E \quad y'(t) = Ay(t) + Bu, \quad y(0) = y_0,$$

Main steps

- Give a suitable condition such that $J(y,u)$ finite for some y and u.
- Under this condition, problem P admits a unique solution.
- The optimal control of P can be found in a feedback form via solving an algebraic Riccati equation and it stabilizes E.
- For that, we need to go from finite horizon to infinite horizon problem taking $T \to \infty$ and using Banach-Steinhaus theorem.

- Finite cost condition(FCC) For every $y_0 \in Y$, there exists a $u_{y_0} \in L^2(0, \infty; U)$ such that

$$J(y(y_0, u_{y_0}), u_{y_0}) < \infty.$$

- If (A, B) is stabilizable, then the (FCC) is satisfied.
- Conversely, if (FCC) is satisfied, then (A, B) is stabilizable.

Lemma

Suppose that (FCC) is satisfied. Then P admits a unique solution. This solution $(\bar{y}, \bar{u})$ *obeys*

$$\bar{u}(t) = -B^* P \bar{y}(t), \quad \forall\, t \geq 0,$$

where P *is the minimal solution to ARE*

$$\begin{aligned} &P = P^* \geq 0, \\ &A^* P + PA^* - PBB^* P + I = 0. \end{aligned}$$

Moreover,

$$J(\bar{y}, \bar{u}) = \frac{1}{2} \Big(P y_0, y_0 \Big)_Y .$$

Definition (Solution to ARE)

An operator $P \in \mathcal{L}(Y)$ is a solution to ARE if and only if

$$P = P^* \geq 0,$$
$$(P\zeta, A\xi)_Y + (PA\zeta, \xi)_Y - (PBB^*\zeta, \xi)_Y + (\zeta, \xi)_Y = 0.$$

An operator P is a minimal solution to ARE, if it is a solution to ARE and obeys

$$P \leq Q,$$

for any solution Q to ARE.

Theorem

The ARE admits a unique minimal solution.

Proof

- Consider the problem $Q(s,T,\zeta)$

$$\inf\{I(s,T,\zeta,u),| \quad u \in L^2(0,T;U)\},$$

$$I(s,T,\zeta,u) = \frac{1}{2}\int_s^T \|y^s_{\zeta,u}(t)\|^2_Y \, dt + \frac{1}{2}\int_s^T \|u(t)\|^2_U \, dt,$$

and $y^s_{\zeta,u}$ is the solution to

$$y'(t) = Ay(t) + Bu(t), \quad t \geq s, \quad y(s) = \zeta.$$

- For each $\zeta \in Y$, let u_ζ be the solution of $Q(s, T, \zeta)$.
- Let P_{min} be the solution to the differential Riccati equation

$$P = P^* \geq 0, \quad P(0) = 0,$$
$$P' = A^*P + PA - PBB^*P + I.$$

- For every $\zeta \in Y$, $t \mapsto (P(t)\zeta, \zeta)_Y$ is nondecreasing.
  - Let $0 < T_1 < T_2$.
  - We have

$$\inf I(0, T_1, \zeta, u) = \frac{1}{2}(P(T_1)\zeta, \zeta)_Y,$$
$$\inf I(0, T_2, \zeta, u) = \frac{1}{2}(P(T_2)\zeta, \zeta)_Y.$$

- $$\begin{aligned} & \inf I(0,T_2,\zeta,u) \\ & = \inf\left\{I(0,T_2,\zeta,u) + \inf\{I(T_1,T_2,y^0_{\zeta,u},u)\}\right\} \\ & \geq \inf I(0,T_1,\zeta,u). \end{aligned}$$
- Hence $t \mapsto (P(t)\zeta,\zeta)_Y$ is nondecreasing.

- We also have

$$(P(t)\zeta,\zeta)_Y \leq 2I(0,t,\zeta,u_\zeta) \leq J(z(\zeta,u_\zeta),u_\zeta) < \infty.$$

- Thus $lim_{t\to\infty}(P(t)\zeta,\zeta)_Y$ exist and finite for every $\zeta \in Y$.
- Note that

$$(P(t)\zeta,\xi)_Y = \frac{1}{4}(P(t)(\zeta+\xi),(\zeta+\xi))_Y - \frac{1}{4}(P(t)(\zeta-\xi),(\zeta-\xi))_Y.$$

- Applying Banach-Steinhaus theorem to the family of operators $\{(P(t)\zeta, \cdot)_Y\}_{t\geq 0}$ for every $\zeta \in Y$, we obtain $sup_{t\geq 0}|(P(t)\zeta, \cdot)_Y| < \infty$.
- Again applying Banach-Steinhaus theorem to the family of operators $\{(P(t)\cdot, \cdot)_Y\}_{t\geq 0}$, we obtain $sup_{t\geq 0}|(P(t)\cdot, \cdot)_Y| < \infty$.
- Therefore, there exists $P^{\infty}_{min} \in \mathcal{L}(Y)$ such that

$$lim_{t\to\infty}(P(t)\zeta, \zeta)_Y = (P^{\infty}_{min}\zeta, \zeta)_Y.$$

- Since $P(t) = P(t)^* \geq 0$ for all $t \geq 0$, we have $P^{\infty}_{min} = (P^{\infty}_{min})^* \geq 0$.
- For every $\zeta \in D(A)$,

$$\frac{d}{dt}(P(t)\zeta, \zeta)_Y = (P(t)\zeta, A\zeta)_Y + (P(t)A\zeta, \zeta)_Y$$
$$-(P(t)BB^*\zeta, \zeta)_Y + (\zeta, \zeta)_Y, \quad t \in [0, T].$$

- Since the mapping, $t \mapsto (P(t)\zeta, \zeta)_Y$, is of class C^1, the right hand side of the above equation admits a limit when $t \to \infty$.
- Thus $lim_{t\to\infty} \dfrac{d}{dt}(P(t)\zeta, \zeta)_Y$ exists and we can show that this limit is necessarily zero.
- Hence P^{∞}_{min} satisfies ARE.
- To show P^{∞}_{min} is a minimal solution, we suppose that $\hat{P}$ is an another solution.
- Note that $\hat{P}$ is also a solution to the differential Riccati equation

$$P = P^* \geq 0, \quad P(0) = \hat{P},$$

$$P' = A^*P + PA - PBB^*P + I.$$

- Since $P_{min}(0) \leq \hat{P}(0)$, we have that $P_{min}(t) \leq \hat{P}(t) = \hat{P}$ for all $t \geq 0$.
- Taking $t \to \infty$, $P^{\infty}_{min} \leq \hat{P}$.
- Uniqueness of the minimal solution can be proved in a classical manner.

Theorem

The unique solution $(\bar{y}, \bar{u})$ *to* P *satisfies the feedback formula*

$$\bar{u} = -B^* P^{\infty}_{min} \bar{y}(t), \quad \forall\, t \geq 0,$$

where $\bar{y}$ *satisfies*

$$\bar{y}'(t) = A\bar{y}(t) - BB^* P^{\infty}_{min} \bar{y}(t), \quad \forall\, t \geq 0, \quad \bar{y}(0) = y_0.$$

Moreover,

$$J(\bar{y}, \bar{u}) = \frac{1}{2} \Big(P^{\infty}_{min} y_0, y_0 \Big)_Y.$$

Proof

- Let $\bar{y}$ be the solution to

$$\bar{y}'(t) = A\bar{y} - BB^* P^{\infty}_{min} \bar{y}, \quad \bar{y} = y_0.$$

- The solution to the problem

$$\Big\{\frac{1}{2}\int_0^\infty \|y_u(t)\|_Y^2\,dt + \frac{1}{2}\int_0^\infty \|u(t)\|_u^2\,dt \\ +\tfrac{1}{2}\Big(P_{min}^\infty y_u(T), y_u(T)\Big)_Y \mid u \in L^2(0,T;U)\Big\},$$

where y_u satisfies

$$E \quad y'(t) = Ay(t) + Bu, \quad y(0) = y_0,$$

is given by $(\hat{y}, \hat{u}) = (\hat{y}, -B^* P\hat{y})$, where P solves

$$P = P^* \geq 0, \quad P(T) = P_{min}^\infty,$$
$$P' + A^*P + PA^* - PBB^*P + I = 0.$$

and $\hat{y}$ solves

$$\hat{y}'(t) = A\hat{y}(t) - BP(t)\hat{y}(t), \quad \hat{y}(0) = y_0,$$

- P_{min}^∞ is the unique solution to the above DRE.

- From the previous part, we have

$$\Big(P(0)y_0, y_0\Big)_Y = \int_0^\infty \|\hat{y}(t)\|_Y^2\, dt + \int_0^\infty \|\hat{u}(t)\|_u^2\, dt + \Big(P_{min}^\infty \hat{y}(T), \hat{y}(T)\Big)_Y.$$

- Consequently, we have $(\bar{y}, \bar{u}) = (\hat{y}, \hat{u})$ and for every $T > 0$,

$$\Big(P_{min}^\infty y_0, y_0\Big)_Y = \int_0^\infty \|\bar{y}(t)\|_Y^2\, dt + \int_0^\infty \|\bar{u}(t)\|_u^2\, dt + \Big(P_{min}^\infty \bar{y}(T), \bar{y}(T)\Big)_Y.$$

- Taking $T \to \infty$, we obtain

$$2J(\bar{y}, \bar{u}) \le \Big(P_{min}^\infty y_0, y_0\Big)_Y.$$

- Considering the optimal problem

$$\Big\{ \frac{1}{2}\int_0^\infty \|y_u(t)\|_Y^2 + \|u(t)\|_u^2\, dt \mid u \in L^2(0,T;U), \Big\}$$

we have

$$\Big(P_{min}^\infty y_0, y_0\Big)_Y \leq \int_0^T \|\bar{y}(t)\|_Y^2 + \|\bar{u}(t)\|_u^2\, dt \leq 2J(\bar{y},\bar{u}).$$

and

$$\Big(P_{min}^\infty y_0, y_0\Big)_Y \leq \int_0^T \|y_u(t)\|_Y^2 + \|\bar{u}(t)\|_u^2\, dt \leq 2J(y_u,u),$$

for all $u \in L^2(0,\infty;U)$ and P_{min}^∞ is the unique solution to above DRE.

- Taking $T \to \infty$,

$$\left(P_{min}^{\infty} y_0, y_0\right)_Y \leq \int_0^{\infty} \|\bar{y}(t)\|_Y^2 + \|\bar{u}(t)\|_u^2 \, dt \leq 2J(\bar{y}, \bar{u}).$$

and

$$\left(P_{min}^{\infty} y_0, y_0\right)_Y \leq 2J(y_u, u), \quad \forall\, u \in L^2(0, \infty; U).$$

- Thus

$$\left(P_{min}^{\infty} y_0, y_0\right)_Y = 2J(\bar{y}, \bar{u}) = 2 \inf J(y, u)$$

and $(\bar{y}, \bar{u})$ is the unique solution to P.

Lemma

If P is a solution to the ARE, the operator $A - BB^*P$ with domain $D(A)$ is the generator of an exponentially stable semigroup on Y.

Proof

- Let $\zeta \in Y$ and y be the solution to

$$y' = Ay - BB^*Py, \quad y(0) = \zeta.$$

- First suppose that $\zeta \in D(A)$ and $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $C^1([0,\infty);U) \cap L^2(0,\infty;U)$ converging to $-B^*Py$ in $L^2(0,\infty;U)$. Let y_n be the solution to

$$y_n' = Ay_n + Bu_n, \quad y_n(0) = \zeta.$$

With the ARE, we get for all $s \geq 0$

$$\begin{aligned}
&\frac{d}{ds}(Py_n(s), y_n(s))_Y = 2(Ay_n(s) + Bu_n(s), Py_n(s))_Y \\
&= -(y_n(s), y_n(s))_Y + (B^*Py_n(s), B^*Py_n(s))_U \\
&\qquad\qquad +2(u_n(s), B^*Py_n(s))_U \\
&= -(y_n(s), y_n(s))_Y - (B^*Py_n(s), B^*Py_n(s))_U \\
&+2(B^*Py_n(s), B^*Py_n(s))_U + 2(u_n(s), B^*Py_n(s))_U .
\end{aligned}$$

Integrating between 0 to t, for all $t > 0$, we obtain

$$\begin{aligned}
&(Py_n(t), y_n(t))_Y + \int_0^t \left(\|y_n(s)\|_Y^2 + \|B^*Py_n(s)\|_U^2 \right) ds \\
&= (P\zeta, \zeta) + 2 \int_0^t \left(\|B^*Py_n(s)\|_U^2 + (u_n(s), B^*Py_n(s))_U \right) ds.
\end{aligned}$$

Now passing through the limit $n \to \infty$, we get

$$(Py(t), y(t))_Y + \int_0^t \left(\|y(s)\|_Y^2 + \|B^*Py(s)\|_U^2 \right) ds = (P\zeta, \zeta).$$

- By a density argument, the above identity is also true for any $\zeta \in Y$.
- Since P is a nonnegative operator, from above inequality we obtain

$$\int_0^t \left(\|y(s)\|_Y^2 + \|B^*Py(s)\|_U^2 \right) ds \leq$$
$$(Py(t), y(t))_Y + \int_0^t \left(\|y(s)\|_Y^2 + \|B^*Py(s)\|_U^2 \right) ds$$
$$= (P\zeta, \zeta).$$

- Hence, taking $t \to \infty$, we get

$$\int_0^\infty \left(\|y(s)\|_Y^2 + \|B^*Py(s)\|_U^2 \right) ds \leq (P\zeta, \zeta),$$

and by Datko theorem, the lemma follows.

Lemma

Let P and Q be two solutions to ARE. Suppose that the operator $A - BB^*P$ with domain $D(A)$, is the generator of an exponentially semigroup in Y. Then $P \geq Q$.

Proof

- Since P and Q are solutions of ARE, we have

$$(P-Q)(A-BB^*P) + (A-BB^*P)^*(P-Q)$$
$$+(P-Q)BB^*(P-Q) = 0.$$

- From this identity, for $\zeta \in Y$, we deduce

$$\frac{d}{dt}\Big((P-Q)e^{t(A-BB^*P)}\zeta, e^{t(A-BB^*P)}\zeta\Big)_Y$$
$$= -\|B^*(P-Q)e^{t(A-BB^*P)}\zeta\|_U^2.$$

- Integrating from 0 to T, for all $T > 0$, we obtain

$$\begin{aligned}\Big((P-Q)\zeta,\zeta\Big)_Y &= \Big((P-Q)e^{T(A-BB^*P)}\zeta, e^{T(A-BB^*P)}\zeta\Big)_Y \\ &\quad + \int_0^T \|B^*(P-Q)e^{t(A-BB^*P)}\zeta\|_U^2 dt \\ &\geq \Big((P-Q)e^{T(A-BB^*P)}\zeta, e^{T(A-BB^*P)}\zeta\Big)_Y .\end{aligned}$$

- Taking $T \to \infty$ and using that $A - BB^*P$ generates exponentially stable semigroup in Y, we get

$$\Big((P-Q)\zeta,\zeta\Big)_Y \geq 0, \quad \forall\, \zeta \in D(A),$$

that is $P \geq Q$.

Algorithm to solve Ricatti Equation

Numerical resolution of the finite dimensional Riccati equation

$$P = P^* \geq 0, \quad A^*P + PA - PBB^*P + C^*C = 0.$$

Hypothesis (A, B) is stabilizable and (A, C) is detectable.
Methods based on computation of eigenvalues of the matrix

$$H = \begin{pmatrix} A & -BB^* \\ CC^* & -A^* \end{pmatrix}$$

The spectrum of H is symmetric with respect to the origin and has no eigenvalues with a zero real part.

Algorithm

Algorithm

1. Calculate the eigenvalues and eigenvectors of H by QR- method (writing the matrix as a product of an orthogonal matrix and an upper triangular matrix).
2. Select the eigenvectors corresponding to the eigenvalues with with a negative real part. Let V_1 be the matrix whose columns correspond to these vectors:
$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix}.$$
3. Solve $V_{11}^* P = V_{21}^*$, to calculate P.

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