Stabilization of Heat Equation

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22 June - July 4, 2015

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One dimensional heat equation

The heat equation in $(0, L) \times (0, \infty)$ for $L > 0$

$$\begin{aligned} y_t - y_{xx} &= u\chi_O \quad \text{in} \quad (0, L) \times (0, \infty), \\ y(0, t) = y(L, t) &= 0 \quad \text{in} \quad (0, \infty), \\ y(x, 0) &= y_0(x) \quad \text{in} \quad (0, L), \end{aligned}$$

Questions : For a given $y_0 \in L^2(0, L)$, is there a control u with support in a subset O of $(0, L)$ such that the solution $y \in L^2(0, \infty; H_0^1(0, L)) \cap C([0, \infty); L^2(0, L))$ decays exponentially in time :

$$\|y(t)\| \leq Ke^{-\mu t}\|y_0\|, \quad \forall\, t \geq 0?$$

At what rate μ ?

Recall the eigenfunctions ϕ_k and eigenvalues λ_k of the homogeneous problem

$$\phi_k(x) = \sqrt{\frac{2}{L}} sin\left(\frac{k\pi x}{L}\right), \quad \forall\, k \in \mathbb{N}, \quad \forall\, x \in (0, L).$$

$$\lambda_k = -\frac{k^2\pi^2}{L^2}.$$

The family $(\phi_k)_{k\in\mathbb{N}}$ is a Hilbert basis of $L^2(0, L)$.

If $A = \frac{d^2}{dx^2}$ with domain $D(A) \; = \; H^2 \cap H_0^1$, then

$$\phi_k \in D(A), \quad A\phi_k = \lambda_k \phi_k.$$

For the homogeneous problem, the solution is

$$y(x, t) = \sum_{k=1}^{\infty} y_{0k} e^{-\frac{k^2\pi^2 t}{L^2}} \phi_k(x), \quad \forall\, x \in (0, L), \quad t \in (0, T),$$

The function $y \in L^2(0, T; H_0^1(0, L)) \cap C([0, T]; L^2(0, L))$.

Weak solution of heat equation

Definition (Weak solution of heat equation)

Let $y_0 \in L^2(0,L)$ and $f \in L^2(0,T;L^2(0,L))$. Then y belonging in

$W(0,T;H_0^1(0,L),H^{-1}(0,L)) = L^2(0,T;H_0^1(0,L) \cap H^1(0,T;H^{-1}(0,L))$,

is called a weak solution of the heat equation

$$\begin{aligned} y_t - y_{xx} = f \quad & \text{in} \quad (0,L)\times(0,T), \\ y(0,t) = y(L,t) = 0 \quad & \text{in} \quad (0,T), \\ y(x,0) = y_0(x) \quad & \text{in} \quad (0,L), \end{aligned}$$

if $y(\cdot,0) = y_0(\cdot)$ and for all $\phi \in H_0^1(0,L)$, y satisfies

$$\frac{d}{dt}\int_0^L y(t)\phi = -\int_0^L y_x(t)\phi_x + \int_0^L f(t)\phi.$$

Theorem (Existence and uniqueness of the weak solution)

Let $y_0 \in L^2(0, L)$ and $f \in L^2(0, T; L^2(0, L))$. The heat equation with forcing term f and initial condition y_0, admits a unique weak solution y and it satisfies

$$\|y\|_{L^\infty(0,T;L^2(0,L))} + \|y\|_{L^2(0,T;H_0^1(0,L))} \leq C(\|y_0\|_{L^2(0,L)} + \|f\|_{L^2(0,T;L^2(0,L))}),$$

for some constant $C > 0$.

- We can always choose μ where μ is strictly between two eigenvalues. Let us assume

$$0 \leq -\lambda_1 < \cdots < -\lambda_N < \mu < -\lambda_{N+1} < \cdots .$$

- Define

$$E^+ = \oplus_{k=1}^{N} E(\lambda_k), \quad E^- = \oplus_{k=N+1}^{\infty} E(\lambda_k),$$

where $E(\lambda_k) = Ker(\lambda_k I + A)$, the eigenspace associated to λ_k, for all $k \in \mathbb{N}$.

- We see that

$$L^2(0, L) = E^+ \oplus E^-.$$

- Let P_N be the orthogonal projection of $L^2(0, L)$ onto E^+, defined by

$$P_N f = \sum_{k=1}^{N} (f, \phi_k)_{L^2(0,L)} \phi_k, \quad \forall\, f \in L^2(0, L).$$

Theorem

Let y *be the solution of the heat equation with control* u*. Then* $P_N y$ *satisfies*

$$\partial_t P_N y - \partial_{xx} P_N y = P_N(u\chi_O) \quad \text{in} \quad (0,L)\times(0,\infty),$$
$$P_N y(0,t) = P_N y(L,t) = 0 \quad \text{in} \quad (0,\infty),$$
$$P_N y(\cdot,0) = P_N y_0(\cdot) \quad \text{in} \quad (0,L),$$

and $(I-P_N)y$ *satisfies*

$$\partial_t (I-P_N)y - \partial_{xx}(I-P_N)y = (I-P_N)(u\chi_O) \quad \text{in} \quad (0,L)\times(0,\infty),$$
$$(I-P_N)y(0,t) = (I-P_N)y(L,t) = 0 \quad \text{in} \quad (0,\infty),$$
$$(I-P_N)y(\cdot,0) = (I-P_N)y_0(\cdot) \quad \text{in} \quad (0,L).$$

Proof

- Let y be a solution of the heat equation.
- We show that $P_N y$ satisfies the weak formulation

$$\frac{d}{dt}\int_0^L P_N y(t)\phi = -\int_0^L P_N y_x(t)\phi_x + \int_0^L P_N(u\chi_O)\phi,$$

  for all $\phi \in E^+$ and so $\phi = \sum_{k=1}^N (\phi, \phi_k)_{L^2(0,L)}\phi_k$.
- It follows from the fact that $L^2(0, L)$ is the orthogonal sum Of E^+ and E^-, and definition of P_N.
- In fact, we show $\frac{d}{dt}\int_0^L y(t)\phi = \frac{d}{dt}\int_0^L P_N y(t)\phi$, $\int_0^L y_x(t)\phi_x = \int_0^L P_N y_x(t)\phi_x$ and $\int_0^L u\chi_O\phi = \int_0^L P_N(u\chi_O)\phi$, for all $\phi = \sum_{k=1}^N (\phi, \phi_k)_{L^2(0,L)}\phi_k \in E^+$.
- Similarly, we can show for $(I - P_N)y$.

- The system for $P_N y$ is in E^+ for all $t \geq 0$ and E^+ is the finite dimensional space.
- We study the controllability of the finite dimensional system for $P_N y$.
- Set finite dimensional control

$$u = \sum_{k=1}^{N} v_k \phi_k \in L^2(0,T;E^+),$$

for some time $T > 0$.

- Note that

$$P_N(u\chi_O) = P_N(\textstyle\sum_{k=1}^{N} v_k \phi_k \chi_O)$$
$$= \sum_{j=1}^{N} \sum_{k=1}^{N} v_j \Big(\int_O \phi_j \phi_k \Big) \phi_k.$$

- We introduce $B \in \mathcal{L}(\mathbb{R}^N, E^-)$ such as

$$Bv = \sum_{j=1}^{N} \sum_{k=1}^{N} v_j \Big(\int_O \phi_j \phi_k \Big) \phi_k,$$

for all $v = (v_1, \cdots, v_N)^T \in \mathbb{R}^N$.

- For this u, we write

$$\begin{aligned} &\partial_t P_N y - \partial_{xx} P_N y = Bv \quad \text{in} \quad (0, L) \times (0, \infty), \\ &P_N y(0, t) = P_N y(L, t) = 0 \quad \text{in} \quad (0, \infty), \\ &P_N y(\cdot, 0) = P_N y_0(\cdot) \quad \text{in} \quad (0, L), \end{aligned}$$

- Setting $P_N y = y_N$, $A^+ = \Delta P_N \in \mathcal{L}(E^+)$ and $P_N y_0 = y_{0,N}$, the above system can be written in E^+

$$y_N' = A^+ y_N + Bv, \quad y_N(0) = y_{0,N} \in E^+.$$

- Now we want to write an equivalent system in $\mathbb{R}^N$. Let $y_N = \sum_{k=1}^{N} y_k \phi_k \in E^+$ and $z = (y_1, \cdots, y_N)^T \in \mathbb{R}^N$.

- We write

$$z' = \Lambda z + Cv, \quad z(0) = z_0,$$

where

$$\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N) \in \mathbb{R}^N \times \mathbb{R}^N,$$

$$C = \Big(\int_O \phi_j \phi_k \Big)_{1 \le j \le N, 1 \le k \le N} \in \mathbb{R}^N \times \mathbb{R}^N,$$

$$z_0 = \Big((y_{0,N}, \phi_1)_{L^2(O)}, \cdots, (y_{0,N}, \phi_N)_{L^2(O)} \Big)^T \in \mathbb{R}^N.$$

Theorem

The matrix C is invertible if and only if the family $\{\phi_k|_O\}_{1\leq k\leq N}$ is linearly independent in $L^2(O)$.

Proof Since the matrix C is the Gram matrix in $L^2(O)$ of the family $\{\phi_k|_O\}_{k\in\mathbb{N}}$, the result follows.

Theorem

The family $\{\phi_k|_O\}_{1\leq k\leq N}$ is linearly independent in $L^2(O)$.

Aim To show if $\sum_{k=1}^{N}\alpha_k\phi_k|_O = 0$ in $L^2(O)$, then $\alpha_k = 0, \quad \forall\, 1\leq k\leq N$ and hence $\{\phi_k|_O\}_{1\leq k\leq N}$ is linearly independent in $L^2(O)$.

Proof

- Let $\sum_{k=1}^{N} \alpha_k \phi_k|_O = 0$ in $L^2(O)$.
- Denote $f = \sum_{k=1}^{N} \alpha_k \phi_k$ and we have $f|_O = 0$.
- Suppose f is in the form $f = \alpha_k \phi_k$ for some $k \in \{1, 2, \cdots, N\}$.
- Since $\phi_k(x) = \sqrt{\frac{2}{L}} sin\left(\frac{k\pi x}{L}\right)$ for all $x \in (0, L)$, using this expression we conclude that $f = 0$ on $(0, L)$, if it is given that $f|_O = 0$.

- Now suppose, $f = \sum_{k=1}^{N} \alpha_k \phi_k$.
- Then we reduce it to the first case as follows. We have

$$[\lambda_1 I + \partial_{xx}] f = \sum_{k=2}^{N} (\lambda_1 - \lambda_k) \alpha_k \phi_k,$$

using$[\lambda_1 I + \partial_{xx}]\phi_k = (\lambda_1 - \lambda_k)\phi_k$. Thus we eliminate ϕ_1 from the right hand side.

- Note that $(\lambda_j - \lambda_k) \neq 0$, for $j \neq k$.
- Similarly, to eliminate ϕ_1 and ϕ_2,

$$[\lambda_2 I + \partial_{xx}][\lambda_1 I + \partial_{xx}] f = \sum_{k=3}^{N} (\lambda_2 - \lambda_k)(\lambda_1 - \lambda_k) \alpha_k \phi_k.$$

- Repeating the above method finitely many times, we can reduce it in the first case and derive the result.

Theorem

The system

$$z' = \Lambda z + Cv, \quad z(0) = z_0,$$

is exactly controllable at time T, for all $T > 0$.

Proof

- By the above two theorems, we have C is invertible.
- By checking Kalman rank condition, we show the system is controllable.

From the above results, it follows

Theorem

For all $y_{0,N} \in E^+$, there exists a $v \in L^2(0, T; \mathbb{R}^N)$ such that y_N, the solution of the finite dimensional system satisfies $y_N(T) = 0$ and control v obeys

$$\|v(t)\|_{\mathbb{R}^N} \leq C\|y_{0,N}\|_{L^2(0,L)},$$

for some constant $C > 0$.

Theorem

For any $\mu > 0$, there exists a constant $K > 0$ such that for all initial condition $y_0 \in L^2(0, L)$, there exists a control $u \in L^2(0, \infty; L^2(0, L))$ such that $y_{y_0,u}$, the solution of the heat equation satisfies

$$\|y_{y_0,u}(t)\|_{L^2(0,L)} \leq Ke^{-\mu t}\|y_0\|_{L^2(0,L)}, \forall\, t \geq 0.$$

Proof

- Set control

$$u(t) = \begin{cases} \sum_{k=1}^{N} v_k(t)\phi_k, & 0 \leq t \leq T, \\ 0, & t > T, \end{cases}$$

where $v_{k1 \leq k \leq N}$ is obtained from the previous theorem such that $P_N y(T) = y_N(T) = 0$.

- We have
$$\|v(t)\|_{\mathbb{R}^N} \leq C\|y_{0,N}\|_{L^2(0,L)},$$
where $v = (v_1, \cdots, v_N)^T$.
- We also have $\|P_N y(T)\|_{L^2(0,L)} \leq C\|P_N y_0\|_{L^2(0,L)}$, for all $0 \leq t \leq T$ and $P_N y(t) = 0$ for all $t \geq T$.
- It yields
$$\|P_N y(t)\|_{L^2(0,L)} \leq Ce^{-\lambda_{N+1}t}e^{\lambda_{N+1}T}\|P_N y_0\|_{L^2(0,L)}, \quad \forall\, t \geq 0.$$

- Now to estimate the stable part for $(A^-, D(A^-))$ where $D(A^-) = E^- \cap D(A)$ and $A^- y = (I - P_N)y_x x$ for all $y \in D(A^-)$, we write

$$(I - P_N)y(t) = e^{A^- t}(I - P_N)y_0 +$$
$$\int_0^t e^{(t-s)A^-}(I - P_N)\Big(\sum_{k=1}^{N} v_k(s)\phi_k|_O\Big)\, ds.$$

- Using $\|e^{A^- t}(I - P_N)y_0\|_{L^2(0,L)} \leq e^{-\lambda_{N+1} t}\|P_N y_0\|_{L^2(0,L)}$ and the above estimates, we can show

$$\|(I - P_N)y(t)\|_{L^2(0,L)} \leq K e^{-\mu t}\|y_0\|_{L^2(0,L)}, \forall\, t \geq 0.$$

- Combining the estimates for $P_N y$ and $(I - P_N)y$, the theorem follows.

The heat equation in $(0,1)\times(0,\infty)$ with a boundary control

$$
\begin{aligned}
&y_t - y_{xx} = 0 \quad \text{in} \quad (0,1)\times(0,\infty),\\
&y(0,t) = 0, \quad y(1,t) = u(t) \quad \text{in} \quad (0,\infty),\\
&y(x,0) = y_0(x) \quad \text{in} \quad (0,1),
\end{aligned}
$$

where for all $t\in(0,\infty)$, $u(t)\in\mathbb{R}$, one dimensional control.

- We want to write the above equation in a Hilbert space Y as

$$y' = Ay + Bu, \quad y(0) = y_0 \in Y.$$

- B is the control operator and U is a Hilbert space for controls.
- Consider $Y = L^2(0,1)$ and $U = \mathbb{R}$, one dimensional space.
- Recall $A = \frac{d^2}{dx^2}$ with domain $D(A) = H^2(0,1)\cap H_0^1(0,1)$ and $D(A) = D(A^*)$ with $A = A^*$.
- ϕ_k is an eigenfunction of A for eigenvalue λ_k, where

$$\phi_k(x) = \sqrt{2}sin\,(k\pi x)\,, \quad \forall\, k\in\mathbb{N}, \quad \forall\, x\in(0,1).$$

$$\lambda_k = -k^2\pi^2.$$

Expression of B^*

Aim To determine B and B^* by the weak formulation.

- y is a weak solution of the heat equation with nonhomogeneous boundary condition if and only if for all $\psi \in D(A^*) = H^2(0,1) \cap H_0^1(0,1)$,

$$\frac{d}{dt}\int_0^1 y(x,t)\psi(x)dx = \int_0^1 y(x,t)\psi_{xx}(x)dx - u(t)\psi_x(1).$$

  y is a weak solution of the evolution equation if and only if for all $\psi \in D(A^*) = H^2(0,1) \cap H_0^1(0,1)$,

$$\frac{d}{dt}\int_0^1 y(x,t)\psi(x)dx = \int_0^1 y(x,t)A\psi(x)dx + (Bu(t),\psi)_{L^2(0,1)}.$$

- Comparing above two identities, we obtain

$$(Bu(t), \psi)_{L^2(0,1)} = -u(t)\psi_x(1), \quad t \in (0, \infty)$$

- Thus for all $t \in (0, \infty)$, $(u(t), B^*\psi)_{\mathbb{R}} = -u(t)\psi_x(1)$ and hence

$$B^*\psi = -\psi_x(1).$$

Hautus condition for stabilization:

- With a prescribed decay $\omega > 0$, $(A + \omega I, B)$ is stabilizable if and only if for all unstable eigenvalues $\lambda_k + \omega > 0$,

$$A^*\phi = \lambda_k\phi, \quad B^*\phi = 0$$

  implies $\phi = 0$.

- Let choose $\omega = 10$.
- The only unstable eigenvalue of $A + \omega I$ is $\lambda_1 + \omega = -\pi^2 + 10 > 0$.
- Unstable space spanned by ϕ_1, eigenfunction of $(A + \omega I)$ for eigenvalue $(\lambda_1 + \omega)$ and

$$E^+ = \mathbb{R}\phi_1, \quad E^- = \oplus_{k=2}^{\infty} \mathbb{R}\phi_k,$$

  where E^- is the stable eigenspace.
- Let us define the projectors associated with this decomposition

$$P_1 f = (f, \phi_1)_{L^2(0,1)}\phi_1, \quad (I - P_1)f = \sum_{k=2}^{\infty}(f, \phi_k)_{L^2(0,1)}\phi_k.$$

- Thus

$$P_1 Bu = (Bu, \phi_1)_{L^2(0,1)}\phi_1 = (u, B^*\phi_1)_{\mathbb{R}}\phi_1 = -u\phi_1'(1)\phi_1,$$

$$(I - P_1)Bu = \sum_{k=2}^{\infty}(Bu, \phi_k)_{L^2(0,1)}\phi_k = -u\sum_{k=2}^{\infty}\phi_k'(1)\phi_k.$$

- Using the expression of ϕ_k, we have

$$\phi_k'(1) = \sqrt{2}\pi k(-1)^k.$$

- The series $\sum_{k=2}^{\infty} \phi_k'(1)\phi_k$ converges in $(D(A^*))'$, but not in $L^2(0,1)$.

Checking of Hautus condition

- Recall that only $\lambda_1 + \omega > 0$. Let $A^*\phi = \lambda_1\phi$ and $B^*\phi = 0$.
- Using $A^* = A$, $\phi = c\phi_1$, for all $c \in \mathbb{R}$.
- Since $B^*\phi = 0$, we get $c\phi'(1) = -\sqrt{2}c\pi = 0$.
- Hence $c = 0$ and so $\phi = 0$.
- Hautus condition is satisfied and $(A + \omega I, B)$ is stabilizable.

Control of minimal norm

- To determine control of minimal norm, we project the equation $y' = (A+\omega)y + Bu$ onto E^+.
- Let $P_1 y = y_1\phi_1$. The equation for y_1 is

$$y_1' = (10-\pi^2)y_1 - u(t)\phi'(1) = (10-\pi^2)y_1 - u(t)\sqrt{2}\pi,$$
$$y_1(0) = (y_0, \phi_1)_{L^2(0,1)}.$$

- The Bernoulli equation for the system is

$$p > 0, \quad 2(10-\pi^2)p - (-\pi\sqrt{2})^2 p^2 = 0.$$

- Thus

$$p = \frac{2(10-\pi^2)}{2\pi^2}.$$

- The control of minimal norm obeys the feedback law

$$u(t) = -(-\pi\sqrt{2})\frac{2(10-\pi^2)}{2\pi^2}y_1(t).$$

- $y_1(\cdot)$ satisfies the closed loop system

$$y_1' = -(10-\pi^2)y_1, \quad y_1(0) = (y_0, \phi_1)_{L^2(0,1)}.$$

- Hence $y_1(t) = (y_0, \phi_1)_{L^2(0,1)}e^{-(10-\pi^2)t}$, for all $t \geq 0$ and so this system is stable.

- The closed loop system for $\hat{y} = e^{10t} y$ is

$$\hat{y}_t - \hat{y}_{xx} = 0 \quad \text{in} \quad (0,1) \times (0,\infty),$$
$$\hat{y}(0,t) = 0, \quad \hat{y}(1,t) = (\pi\sqrt{2})\frac{2(10-\pi^2)}{2\pi^2}(\hat{y}(t),\phi_1)_{L^2(0,1)} \quad \text{in} \quad (0,\infty),$$
$$\hat{y}(x,0) = y_0(x) \quad \text{in} \quad (0,1).$$

- Since the above system is stabilizable, the closed loop system for y is

$$y_t - y_{xx} = 0 \quad \text{in} \quad (0,1) \times (0,\infty),$$
$$y(0,t) = 0, \quad y(1,t) = (\pi\sqrt{2})\frac{2(10-\pi^2)}{2\pi^2}(y(t),\phi_1)_{L^2(0,1)} \quad \text{in} \quad (0,\infty),$$
$$y(x,0) = y_0(x) \quad \text{in} \quad (0,1),$$

and y satisfies

$$\|y(t)\|_{L^2(0,1)} \leq Ce^{-10t}\|y_0\|_{L^2(0,1)},$$

for some constant $C > 0$.

- A. Bensoussan, G. Da Prato, M. Delfour, S. K. Mitter, Representation and control of infinite dimensional systems. Second edition. Systems and Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 2007.
- Lasiecka and Triggiani, Differential and Algebraic Riccati Equations with Applications to Boundary/Point control Problems, Springer-Verlag, 1991
- J-L Lions, Optimal Control of systems governed by Partial Differential Equations, Springer, 1971.
- Jean-Pierre Raymond, Optimal Control of PDEs, FICUS Course Notes, 2001.
- Jean-Pierre Raymond, Optimal control and stabilisation of flow related models, FICUS Course Notes, 2001.