Controllability, Observability, Stability and Stabilizability of Linear Systems

Raju K George, IIST, Thiruvananthapuram

email: rkg.iist@gmail.com, george@iist.ac.in

Consider the n-dimensional control system described by the vector differential equation :

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in (t_0, \infty) \tag{1}$$

$$x(t_0) = x_0$$

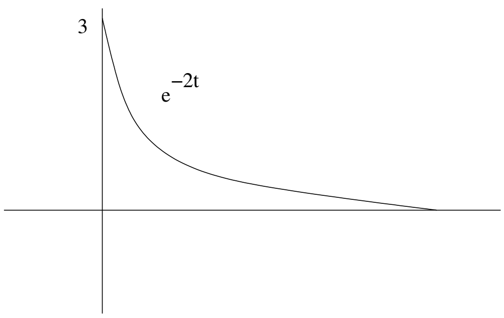
where, $A(t) = (a_{ij}(t))_{n\times n}$ is an $n \times n$ matrix with entries are continuous functions of t defined on $I = [t_0, t_1]$, $B(t) = (b_{ij}(t))_{n\times m}$ is an $n \times m$ matrix with entries are continuous function of t on I. The state $x(t)$ is an n-vector, control $u(t)$ is an m-vector. We first deal with controllability of one dimensional system which described by a scalar differential equation.

What is a control system ?:

Consider a 1-dimensional system

$$\frac{dx}{dt} = -2x, \quad x(0) = 3$$

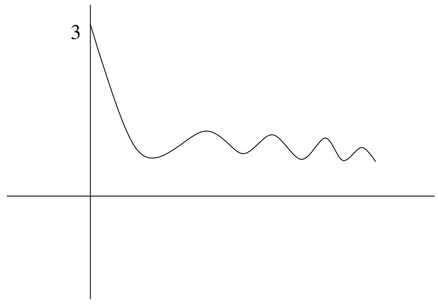
The solution of the system is $x(t) = 3e^{-2t}$ and its graph is shown in the following figure



If we add a nonhomogeneous term $\sin(t)$ called the forcing term or control term to it then the system is given by

$$\frac{dx}{dt} = -2x + sin(t)$$

$$x(0) = 3$$



Observe that the solution or the trajectory of the system is changed. That is, the evolution of the system is changed by adding the new forcing term to the system. Thus the system with a forcing term is called a control system.

Controllability Problem: The controllability problem is to check the existence of a forcing term or control function $u(t)$ such that the corresponding solution of the system will pass through a desired point $x(t_1) = x_1$.

We now show that the scalar control system

$$\dot{x} = ax + bu$$

$$x(t_0) = x_0$$

is controllable. We produce a control function $u(t)$ such that the corresponding solution starting with $x(t_0) = x_0$ also satisfies $x(t_1) = x_1$. Choose a differentiable function $z(t)$ satisfying $z(t_0) = x_0$ and $z(t_1) = x_1$. For example, by the method of linear interpolation, $z - x_0 = \frac{x_1 - x_0}{t_1 - t_0}(t - t_0)$. Thus the function

$$z(t) = x_0 + \frac{(x_1 - x_0)}{t_1 - t_0}(t - t_0)$$

satisfies

$$z(t_0) = x_0, z(t_1) = x_1$$

A Steering Control using $z(t)$: The form of the control system

$$\dot{x} = ax + bu$$

motivates a control of the form

$$u = \frac{1}{b}[\dot{x} - ax]$$

Thus we define a control using the funtion z by

$$u = \frac{1}{b}[\dot{z} - az]$$

$$\dot{x} = ax + b[\frac{1}{b}[\dot{z} - az]]$$

$$\dot{x} - \dot{z} = a(x - z)$$

$$\frac{d}{dt}(x - z) = a(x - z)$$

$$x(t_0) - z(t_0) = 0$$

Let $y = x - z$

$$\frac{dy}{dt} = ay$$

$$y(t_0) = 0.$$

The unique solution of the system is $y(t) = x(t) - z(t) = 0$. That is, $x(t) = z(t)$ is the solution of the controlled system satisfying the required end condition $x(t_0) = x_0$ and $x(t_1) = x_1$. Thus the control function

$$u(t) = \frac{1}{b}[\dot{z}(t) - az(t)] \text{ is a steering control.}$$

Remark : Here we have not only controllability but the control steers the system along the given trajectory z. This is a strong notion of controllability known as trajectory controllability. Trajectory controllability is possible for a time-dependent scalar system $\dot{x} = a(t)x + b(t)u \ : \ b(t) \neq 0 \quad \forall \quad t \in [t_0, t_1]$ In this case the steering control is

$$u = \frac{1}{b(t)}[\dot{z} - a(t)z]$$

n-dimensional system with m=n

Consider an n-dimensional system $x = Ax + Bu$, where $A$and $B$are $n \times n$ matrices and B is invertible matrix. Now consider a control function as in the case of scalar systen, given by

$$u = B^{-1}[\dot{z} - Az]$$

where, $z(t)$ is a n-vector valued and differentiable function satisfying $z(t_0) = x_0$ and $z(t_1) = x_1$. Using this control we have

$$\dot{x} = Ax + BB^{-1}[\dot{z} - Az]$$
$$\dot{x} - \dot{z} = A(x - z)$$
$$\dot{x}(t_0) - z(t_0) = 0$$
$$\implies x(t) = z(t)$$

Remark: If $BB^{-1} = I$, that is, if B has right inverse then also the system is trajectory controllable.

When $m < n$:

When $m < n$ we consider the system

$$\dot{x} \quad = \quad A(t)x + B(t)u \tag{2}$$

$$\begin{aligned} x(t_0) &= x_0 \\ A(t) &= (a_{ij}(t))_{n \times n}, \\ B(t) &= (b_{ij}(t))_{n \times m} \end{aligned}$$

Definition(Controllability) : The system (2) is controllable on an interval $[t_0, t_1]$ if $\forall \quad x_0, x_1 \in \mathbb{R}^n, \quad \exists$ controllable function $u \in L^2([t_0, t_1] : \mathbb{R}^m)$ such that the corresponding solution of (2) satisfying $x(t_0) = x_0$ also satisfies $x(t_1) = x_1$: Since x_0 and x_1 are arbitrary this notion is also known as exact controllability or complete controllability.

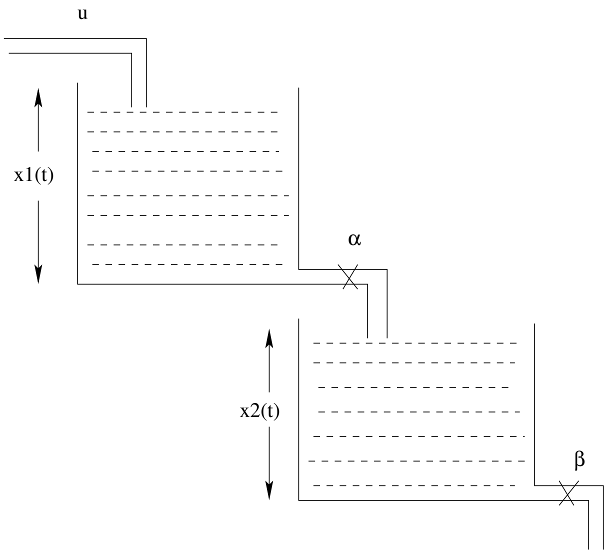
Subspace Controllability : Let $D \subset \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$ and if the system is controllable for all $x_0, x_1 \in D$ then we say that the system is controllable to the subspace D.

Approximate Controllability: If D is dense in state space then the system is approximately controllable. But in $\mathbb{R}^n$, $\mathbb{R}^n$ is the only dense subspace of $\mathbb{R}^n$. Thus approximate controllability is equivalent to complete controllability in $\mathbb{R}^n$. For the subspace D we have

$$D \subseteq \mathbb{R}^n \text{ and } \quad \bar{\mathrm{D}} = \mathbb{R}^n \quad \text{implies} \quad \mathrm{D} = \mathbb{R}^n$$

Null Controllability : If every non - zero state $x_0 \in \mathbb{R}^n$ can be steered to the null state $0 \in \mathbb{R}^n$ by a steering control then the system is said to be null controllable.
We now see examples of controllable and uncontrollable systems.

Example: Tank Problem :



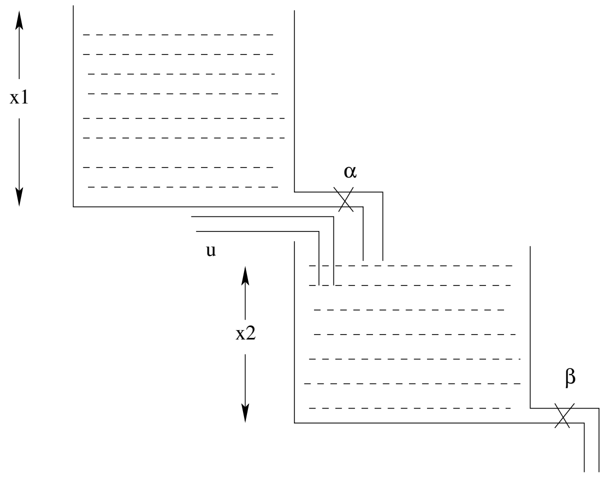
Let $x_1(t)$ be the water level in Tank 1 and $x_2(t)$ be the water level in Tank 2. Let α be the rate of outflow from Tank 1 and β be rate of outflow from Tank 2. Let u be the supply of water to the system. The system can be modelled into the following differential equations:

$$\frac{dx_1}{dt} = -\alpha x_1 + u$$
$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2$$

Model - 1 :

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ \alpha & -\beta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

Model - 2 :



$$\frac{dx_1}{dt} = -\alpha x_1$$
$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2 + u$$
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ \alpha & -\beta \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Obviously the second tank model is not controllable because supply can not change the water level in Tank 1. We will see later that the Model 1 is controllable whereas the model 2 is not controllable. Controllability analysis can be made in many real life problems like :

(i) Rocket launching Problem, Satellite control and control of aircraft

(ii) Biological System : Sugar Level in blood

(iii) Defence: Missiles & Anti-missiles problems.

(iv) Economy- regulating inflation rate

(v) Eology: Predator - Prey system

Solution of the Controlled System using Transition Matrix :
Consider the n-dimensional linear control system:

$$\dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0$$

Let $\Phi(t,t_0)$ be the transition matrix of the homogeneous system $\dot{x} = A(t)x$. The solution of the control system is given by (using variation of parameter method)

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau$$

The system is controllable iff for arbitrary initial and final states x_0, x_1 there exists a control function u such that

$$x_1 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

We first show that for linear systems complete controllability and null-controllability are equivalent.
Theorem :The linear system (1) is completely controllable iff it null-controllable.

Proof : It is obvious that complete controllability implies null-controllability.We now show that null-controllability implies complete controllability. Suppose that x_0 is to be steered to x_1.
Suppose that the system is null-controllable and let $w_0 = x_0 - \Phi(t_0, t_1)x_1$. Thus there exists a control u such that

$$\begin{aligned}
0 &= \Phi(t_1, t_0)w_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \\
&= \Phi(t_1, t_0)[x_0 - \Phi(t_0, t_1)x_1] + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \\
&= \Phi(t_1, t_0)x_0 - x_1 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \\
x_1 &= \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \\
&= x(t_1)
\end{aligned}$$

$\Longrightarrow \quad u$ steers x_0 to x_1 during $[t_0, t_1]$

Conditions for Controllability :
The system (1) is controllable iff $\exists \quad u \in L^2(I, \mathrm{I\!R}^{\mathrm{m}})$ such that

$$x_1 = \Phi(t_0, t_1)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

$$\text{ie,} x_1 - \Phi(t_0, t_1)x_0 = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

Define an operator $C : L^2(I, \mathrm{I\!R}^{\mathrm{m}}) \to \mathrm{I\!R}^{\mathrm{n}}$ by

$$Cu = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

Obviously, C is a bounded linear operator and Range of C is a subspace of $\mathrm{I\!R}^{\mathrm{n}}$. Since x_0, x_1 are arbitrary, the system is controllable iff C is onto.
Range(C) is called the Reachabie set of the system.

Theorem : The following statements are equivalent:

1. The linear system (1) is completely controllable.
2. C is onto
3. C^* is 1-1
4. CC^* is 1-1

In the above result, the operator C^* is the adjoint of the operator C. We now obtain the explicit form of C^*.

Adjoint Operator :The operator $C : L^2(I, \mathbb{R}^m) \to \mathbb{R}^n$ defines its adjoint $C^* : \mathbb{R}^n \to L^2(I, \mathbb{R}^m)$ in the following way:

$$
\begin{aligned}
< Cu, v >_{\mathbb{R}^n} &= < \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau, v >_{\mathbb{R}^n} d\tau \\
&= \int_{t_0}^{t_1} < \Phi(t_1,\tau)B(\tau)u(\tau)d\tau, v >_{\mathbb{R}^n} d\tau \\
&= \int_{t_0}^{t_1} < u(\tau), B^*(\tau)\Phi^*(t_1,\tau)v >_{\mathbb{R}^n} d\tau \\
&= < u, B^*()\Phi^*(t_1,\tau)v >_{L^2(I,\mathbb{R}^m)} \\
&= < u, C^*v >_{L^2(I,\mathbb{R}^m)} \\
(C^*v)(t) &= B^*(t)\Phi^*(t_1,t)v \\
\text{Using C* we get CC* in the form } CC^* &= \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^*(\tau)\phi^*(t_1,\tau)d\tau
\end{aligned}
$$

Observe that $CC^* : \quad \mathbb{R}^n \to \mathbb{R}^n$ is a bounded linear operator. Thus, CC^* is an n by n matrix.
Thus we have from the previous theorem that the system (1) is controllable $\iff$ C is onto
$\iff$ CC^* is 1-1
$\iff$ CC^* is an invertible matrix.
The matrix CC^* is known as the Controllability Grammian for the linear system and is given by
Controllability Grammian

$$W(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^*(\tau)\Phi^*(t_1,\tau)d\tau$$

By using inverse of the controllability Grammian we now define a steering control as given in the following theorem.

Theorem :The linear control system is controllable iff $W(t_0,t_1)$ is invertible and the steering control that move x_0 to x_1 is given by
$u(t) = B^*(t)\Phi^*(t_1,t)W^{-1}(t_0,t_1)[x_1 - \Phi(t_0,t_1)x_0]$
Proof : Controllability part is already proved earlier. We now show that the steering control defined above actually does the tranfer of states. The controlled state is given by

$$
\begin{aligned}
x(t) &= \Phi(t,t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau \\
x(t) &= \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)B^*(\tau)\Phi^*(t_1,\tau)W^{-1}(t_0,t_1)[x_1 - \Phi(t_0,t_1)x_0]d\tau \\
x(t_1) &= \Phi(t_1,t_0)x_0 + W(t_0,t_1)W^{-1}(t_0,t_1)[x_1 - \Phi(t_0,t_1)x_0] \\
x(t_1) &= x_1
\end{aligned}
$$

Remark :Among all controls steering x_0 to x_1, the control defined above is having minimum L^2-norm (energy). We will prove this fact later.
Define a matrix Q given by

$$Q = [B|AB|....A^{n-1}B]$$

It can be shown that Range of $W(t_0, t_1)$ = Range of Q

Controllability of the linear system and the rank of Q are related by the following Kalman's Rank Test.

Theorem (Kalman's Rank Condition) : If the matrices A and B are time - independent then linear system (1) is controllable iff

$$\text{Rank}(B|AB|...|A^{n-1}B) = n$$

Proof : Suppose that the system is controllable.

$$\text{Thus the operator} \quad C : L^2(I, \mathrm{I\!R}^{\mathrm{m}}) \to \mathrm{I\!R}^{\mathrm{n}} \quad \text{defined by}$$

$$Cu = \int_{t_o}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

is onto.We now prove that

$$\mathrm{I\!R}^{\mathrm{n}} = \mathrm{Range(C)} \subset \mathrm{Range(Q)}.$$

Let $x \in \mathrm{I\!R}^{\mathrm{n}}$ then $\exists u \in L^2(I, \mathrm{I\!R}^{\mathrm{n}})$ such that

$$\int_{t_o}^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau = x$$

Expand $e^{A(t_1-\tau)}$ by Cayley - Hamilton's Theorem.

$$\int_{t_o}^{t_1} [P_0(0) + P_1A + P_2A^2 + ... + P_{n-1}A^{n-1}]Bu(\tau)d\tau] = x$$

$$\Longrightarrow x \in \text{Range}[B|AB|A^2B|..........|A^{n-1}B]$$

Conversely, Suppose that condition holds but system is not controllable. ie,Rank of $W(t_0, t_0) \neq n$

$$\Longrightarrow \exists v \neq 0 \in \mathrm{I\!R}^{\mathrm{n}} \mathrm{such\ that W(t_0, t_1)v = 0}$$

$$\Longrightarrow v^T W(t_0, t_1)v = 0$$

$$\int_{t_o}^{t_1} v^T\Phi(t_1, \tau)BB^*\Phi^*(t_1, \tau)vd\tau = 0$$

$$\Longrightarrow \int_{t_o}^{t_1} ||B^*\Phi^*(t_1, \tau)v||^2 d\tau = 0$$

$$\Longrightarrow B^*\Phi^*(t_1, t)v = 0 \quad t \in [t_0, t_1$$

$$\Longrightarrow v^T\Phi(t_1, t)B = 0 \quad t \in [t_0, t_1]$$

$$v^Te^{A(t_1-t)}B = 0 \quad t \in [t_0, t_1]$$

Let $t = t_1, \quad v^TB = 0$

Differentiating $v^Te^{A(t_1-t)}B = 0$ with respect to t and putting $t = t_1$

$$-v^TAB = 0............v^TA^{n-1}B = 0$$

$$\Longrightarrow (v \perp \text{Range}[B|AB|....|A^{n-1}B])$$

Hence Rank of $Q \neq n$
Rank condition is violated and thus we get a contradiction and thus the system is controllable.

Examples : Tank Problem: Model I.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ \alpha & -\beta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$Q : [B : AB] = \begin{bmatrix} 1 & -\alpha \\ 0 & \alpha \end{bmatrix}$$

Rank $Q = 2 \quad \Longrightarrow$ System is controllable.
Model - 2 :

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ \alpha & -\beta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

$$Q = \begin{bmatrix} 0 & 0 \\ 1 & -\beta \end{bmatrix};$$

rank$(Q) = 1 \nless 2$
$\Longrightarrow$ System is not controllable.

Computation of Steering Control :

$$\begin{aligned} Cu &= w \\ CC^*v &= w \end{aligned}$$

where $u = C^*v$. The system is controllable iff

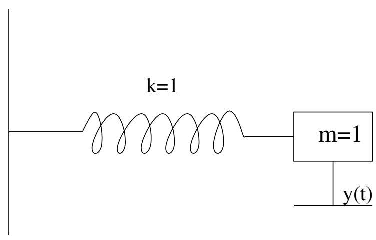
$$\begin{aligned} & C \text{ is onto.} \\ \Longleftrightarrow\ & C^* \text{is} 1-1. \\ \Longleftrightarrow\ & CC^* \text{is} 1-1. \\ \Longleftrightarrow\ & CC^* \text{is invertible.} \end{aligned}$$

If CC^* is invertible then

$$\begin{aligned} v &= (CC^*)^{-1}w \\ u &= C^*(CC^*)^{-1}w \end{aligned}$$

is the steering control.
Controllability Example :

Spring Mass System : Consider a spring mass system having unit mass and with spring constant 1. By Newton's law of motion we have the following differential equation.



$$y'' + y = 0$$

$$\text{Let} \quad x_1 = y$$

$$x_2 = y'$$

$$\dot{x_1} = x_2$$

$$\dot{x_2} = y'' = -y = -x_1$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Transition Matrix by Laplace Transform Method :

We know that $e^{At} = L^{-1}\{(sI - A)^{-1}\}$

$$(sI - A) = \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2 + 1}\begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}^T = \begin{pmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{pmatrix}$$

$$L^{-1}\{sI - A\}^{-1} = L^{-1}\begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \sin t \end{pmatrix}$$

Another Way - Matrix Expansion:

$$e^{At} = I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \dots\dots\dots$$

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -I$$

$$A^3 = -A$$

$$A^4 = I$$

$$A^5 = A$$

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix} + \begin{bmatrix} \frac{-t^2}{2!} & 0 \\ 0 & \frac{-t^2}{2!} \end{bmatrix} + \begin{bmatrix} 0 & \frac{-t^3}{3!} \\ \frac{-t^3}{3!} & 0 \end{bmatrix} + \begin{bmatrix} \frac{-t^4}{4!} & 0 \\ 0 & \frac{-t^4}{4!} \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \dots & t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots \\ -t + \frac{t^3}{3!} - \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \end{bmatrix} \\ &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \end{aligned}$$

Let the initial state and the desired final states be given by $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} x_1(T) \\ x_2(T) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

Transition Matrix is given by,

$$\Phi(T, t) = \begin{pmatrix} \cos(T - t) & \sin(T - t) \\ -\sin(T - t) & \cos(T - t) \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Controllability Grammian is given by,

$$\begin{aligned}
W(0,T) &= \int_0^T \begin{pmatrix} \sin(T-t) \\ \cos(T-t) \end{pmatrix} \begin{pmatrix} \sin(T-t) & \cos(T-t) \end{pmatrix} dt \\
&= \begin{pmatrix} \frac{1}{2}(T - \sin 2T) & \frac{1}{4}(1-\cos 2T) \\ \frac{1}{4}(1-\cos 2T & \frac{1}{2}(T + \frac{1}{2}\sin 2T) \end{pmatrix} \\
W^{-1}(0,T) &= \frac{4}{t^2 - \frac{1}{2}(1-\cos 2T)} \begin{pmatrix} T + \frac{1}{2}\sin 2T & \frac{1}{4}(\cos 2T - 1) \\ \frac{1}{4}(\cos 2T - 1) & \frac{1}{2}(T - \frac{1}{2}\sin 2T) \end{pmatrix}
\end{aligned}$$

The steering control is

$$\begin{aligned}
u(t) = & \frac{4}{T^2 - 1/2(1-\cos 2T)}(0,1) \begin{pmatrix} \cos(T-t) - \sin(T-t) & \\ \sin(T-t) & \cos(T-t) \end{pmatrix} W^{-1}(0,T) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\
= & \frac{1}{T^2 - \frac{1}{2}(1-\cos 2T)} \{[(T - \frac{1}{2})\cos(T-t) + \sin(T-t)] + \frac{1}{2}[\cos(T+t) - \sin(T+t)]\}
\end{aligned}$$

<u>Minimum norm control</u> :We now prove that the steering control defined in the above discussion is actually an optimal control.

<u>Theorem</u>: The control function defined by $u_0 = B^*(t)\Phi^*(t_1,t)W^{-1}(t_0,t_1)x_1$ is a minimum norm control among all other controls steering the system from state x_0 to state x_1.
That is, $||u_0|| \leq ||u||$ for all other steering controllers u in $L^2(I, \mathbb{R})$

<u>Proof</u>:
Let $u = u_0 + u - u_0$ and hence we have

$$\begin{aligned}
||u||^2 &= ||u_0 + (u - u_0)||^2 \\
&= < u_0 + (u-u_0), u_0 + (u-u_0) > \\
&= < u_0, u_0 > + < u_0, u - u_0 > + < u - u_0, u_0 > + < u - u_0, u - u_0 > \\
&= ||u_0||^2 + ||u - u_0||^2 + 2\mathrm{Re} < u_0, \quad u - u_0 >_{L^2}
\end{aligned}$$

Now,

$$\begin{aligned}
< u_0, u - u_0 >_{L^2} &= \int_{t_0}^{t_1} < u_0(t), u(t) - u_0(t) >_{\mathbb{R}^m} dt \\
&= \int_{t_0}^{t_1} < B^*(t)\Phi^*(t_1,t)W^{-1}(t_0,t_1)x_1, u(t) - u_0(t) > dt \\
&= < W^{-1}(t_0,t_1)x_1, \int_{t_0}^{t_1} \Phi(t_1,t)B(t)[u(t) - u_0(t)]dt > \\
&= < W^{-1}(t_0,t_1)x_1, \quad x_1 - x_1 > \\
&= 0
\end{aligned}$$

Since both u and u_0 are steering controllers.

Thus

$$||u||^2 = ||u_0||^2 + ||u - u_0||^2$$
$$\text{or} \quad ||u||^2 - ||u_0||^2 = ||u - u_0||^2 \geq 0$$
$$||u||^2 \geq ||u_0||^2$$

for all steering controllers u.

Adjoint Equation : An equation having solution x in some inner product space is said to be adjoint of an equation with solution p in the same inner product space if $< x(t), p(t) > =$ constant.

Theorem : The adjoint equation associated with $\dot{x} = A(t)x$ is $\dot{p}(t) = -A^*(t)p$

Proof :

$$\begin{aligned}
\frac{d}{dt} < x(t), p(t) > &= < \dot{x}(t), p(t) > + < x(t), \dot{p}(t) > \\
&= < A(t)x, p(t) > + < x(t), -A^*(t)p(t) > \\
&= < x(t), A^*(t)p(t) > + < x(t), -A^*(t)p(t) > \\
&= < x(t), 0 >= 0
\end{aligned}$$

$\therefore \quad < x(t), p(t) > =$ constant.

Theorem : If $\Phi(t, t_0)$ is the transition matrix of $\dot{x}(t) = A(t)x$ then $\Phi^*(t_0, t)$ is the transition matrix of its adjoint system $\dot{p} = -A^*(t)p$.

Proof :

$$\begin{aligned}
I &= \Phi^{-1}(t, t_0)\Phi(t, t_0) \\
0 = \frac{d}{dt}I &= \frac{d}{dt}[\Phi^{-1}(t, t_0)\Phi(t, t_0)] \\
&= \frac{d}{dt}[\Phi^{-1}(t, t_0)]\Phi(t, t_0) + \Phi^{-1}(t, t_0)\dot{\Phi}(t, t_0) \\
&= \dot{\Phi}(t_0, t)\Phi(t, t_0) + \Phi(t_0, t)A(t)\Phi(t, t_0) \\
0 &= [\dot{\Phi}(t_0, t) + \Phi(t_0, t)A(t)]\Phi(t, t_0) \\
\Longrightarrow \dot{\Phi}(t_0, t) &= -\Phi(t_0, t)A(t) \\
\dot{\Phi}^*(t_0, t) &= -A^*(t)\Phi^*(t_0, t) \\
\Longrightarrow \Phi^*(t_0, t) & \quad \text{is the transition matrix of the adjoint system.}
\end{aligned}$$

Remark : The system is self adjoint if $A(t) = -A^*(t)$ and in this case

$$\begin{aligned}
\Phi(t, t_0) &= \Phi^*(t_0, t) \\
&= \Phi^{-1}(t, t_0) \\
\Phi(t, t_0)\Phi^*(t, t_0) &= I
\end{aligned}$$

Observability

Problem of finding the state vector knowing only the output y over some interval of time$[t_0, t_1]$. Consider the input free system

$$\dot{x}(t) = A(t)x(t) \tag{3}$$

with the observation equation

$$y(t) = C(t)x(t),$$

where $C(t) = (c_{ij}(t))_{m\times n}$ matrix having entries as continuous functions of t.
Let $\Phi(t,t_0)$ be the transition matrix. The solution is $x(t) = \Phi(t,t_0)x_0$
Thus

$$y(t) = C(t)\Phi(t,t_0)x_0 \quad t_0 \le t \le t_1$$

Definition : System (3) is said to be observable over a time period $[t_0, t_1]$ if it is possible to determine uniquely the initial state $x(t_0) = x_0$ from the knowledge of the output $y(t)$ over $[t_0, t_1]$.
The complete state of the system is known if initial state x_0 is known.

Define a linear operator

$$L : \mathbb{R}^n \to L^2([t_0,t_1];\mathbb{R}^m)\text{by}$$
$$(Lx_0)(t) = C(t)\Phi(t,t_0)x_0$$

Thus,

$$y(t) = (Lx_0)(t) \quad t \in [t_0,t_1]$$

The system is observable iff L is invertible.

Theorem :The following statements are equivalent.

1. The linear system $\dot{x}(t) = A(t)x(t), \quad y(t) = C(t)x(t)$ is observable.
2. The operator L is 1-1.
3. The adjoint operator L^* is onto.
4. The operator L^*L is onto.

Remark : $L^*L : \mathbb{R}^n \to \mathbb{R}^n$ is an $n \times n$ matrix called Observability Grammian.

Finding $L^* : L^2 \to \mathbb{R}^n$:

$$\begin{aligned} < (Lx_0)(.), w(.) >_{L^2(I,\mathbb{R}^n)} &= \int_{t_o}^{t_1} < C(t)\Phi(t,t_0)x_0, w(t) >_{\mathbb{R}^n} dt \\ &= \int_{t_o}^{t_1} < x_0, \Phi^*(t,t_0)C^*(t)w(t) >_{\mathbb{R}^n} dt \\ &= < x_0, \int_{t_o}^{t_1} \Phi^*(t,t_0)C^*(t)w(t)dt >_{\mathbb{R}^n} \\ &= < x_0, L^*w(.) >_{\mathbb{R}^n} \end{aligned}$$

Thus,

$$L^*w = \int_{t_o}^{t_1} \Phi^*(t,t_0)C^*(t)w(t)dt$$

Observability Grammian The observability Grammian is given by

$$M(t_0,t_1) = L^*L = \int_{t_o}^{t_1} \Phi^*(t,t_0)C^*(t)C(t)\Phi(t,t_0)dt$$

The linear system is observable if and only if the observability Grammian is invertible.

Kalman's Rank Condition for Time Invariant System

If A and C are time-independent matrices,then we have the following Rank Condition for Observability.

Theorem : The linear system $\dot{x}(t) = Ax(t),\ y(t) = Cx(t)$ is observable iff the rank of the following Observability matrix O

$$O = \begin{pmatrix} C \\ CA \\ CA^2 \\ ... \\ CA^{n-1} \end{pmatrix}$$

is n.

Proof : The observation $y(t)$ and its time derivatives are given by,

$$\begin{aligned} y(t) &= Ce^{A(t)}x(0) \\ y^1(t) &= CAe^{At}x(0) \\ y^2(t) &= CA^2e^{At}x(0) \\ \ldots\ldots\ldots & \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ y^{n-1}(t) &= CA^{n-1}e^{At}x(0) \end{aligned}$$

At $t = 0$, we have the following relation.

$$\begin{aligned} y(0) &= Cx(0) \\ y^1(0) &= CAx(0) \\ y^2(0) &= CA^2x(0) \\ \ldots\ldots\ldots & \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ y^{n-1}(0) &= CA^{n-1}x(0) \end{aligned}$$

The initial condition $x(0)$ can be obtained from the equation.

$$\begin{pmatrix} C \\ CA \\ CA^2 \\ ... \\ CA^{n-1} \end{pmatrix} x(0) = \begin{pmatrix} y^0(0) \\ y^1(0) \\ y^2(0) \\ ... \\ y^{n-1}(0) \end{pmatrix}$$

The initial state $x(0)$ can be determined if the observability matrix on the left hand side has full rank n.

Hence the system is observable if the Kalman's Rank Condition holds true. Converse can be proved easily(exercise).

Reconstruction of initial state x_0 : We have

$$\begin{aligned} y &= Lx_0 \\ L^*y &= L^*Lx_0 \\ x_0 &= (L^*L)^{-1}L^*y \\ x_0 &= [M(t_0,t_1)]^{-1}\int_{t_o}^{t_1}\Phi^*(\tau,t_0)C^*(\tau)y(\tau)d\tau \end{aligned}$$

Duality Theorem :
The linear system

$$\dot{x} = A(t) + B(t)u \tag{4}$$

is controllable iff the adjoint system

$$\left.\begin{array}{l} \dot{x} = -A^*(t) \\ y = B^*(t)x \end{array}\right\} \tag{5}$$

is observable.

Proof : If $\Phi(t,t_0)$ is the transition matrix generated by $A(t)$ then $\Phi^*(t_0,t)$ is the transition matrix generated by $-A^*(t)$.
The system (5) is observable iff the observability Grammian

$$\begin{aligned} M(t_0,t_1) \quad &= \quad \int_{t_o}^{t_1}[\Phi^*(t_0,t)]^*(B^*)(t)^*B^*(t)\Phi^*(t_0,t)dt \quad \text{is non-singular} \\ &\Longleftrightarrow \int_{t_o}^{t_1}\Phi(t_0,t)B(t)B^*(t)\Phi^*(t_0,t)dt \quad \text{is non-singular} \\ &\Longleftrightarrow \int_{t_o}^{t_1}\Phi(t_1,t_0)\Phi(t_0,t)B(t)B^*(t)\Phi^*(t_1,t_0)\Phi^*(t_0,t)dt \quad \text{is non-singular} \\ &\Longleftrightarrow \int_{t_o}^{t_1}\Phi(t_1,t)B(t)B^*(t)\Phi^*(t_1,t)dt \quad \text{is non-singular} \\ &\Longleftrightarrow W(t_0,t_1) \quad \text{is non-singular} \\ &\Longleftrightarrow \text{The system (4) is controllable.} \end{aligned}$$

Example :

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

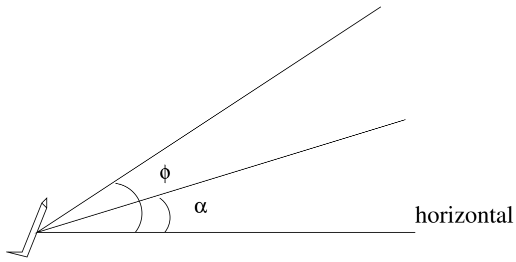
$$\dot{x} = Ax$$

$y(t) = [1,0,1]x(t)$. That is, $y = Cx$

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & -5 & 16 \\ 1 & -4 & 11 \end{bmatrix}$$

has rank 3.
$\Longrightarrow (A, C)$ is observable.

Airplane Model(linear Model) :



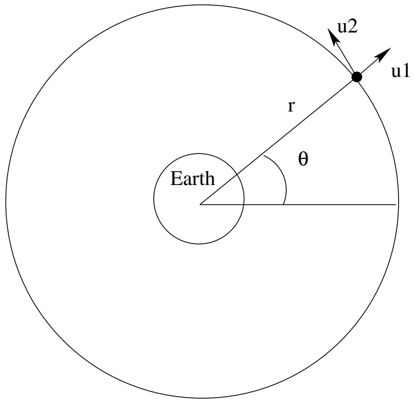
Let us define the following variables: $\phi(t)$:pitch angle $\equiv$ body of the plane inclined to an angle ϕ with the horizontal.
$\alpha(t)$:Flight Path Angle: The path of the flight is along a straight line and it is at an angle α with the horizontal.
$h(t)$: Altitude of the plane at time t.
c: Plane flies at a constant non-zero ground speed c.
w: Natural Oscillation frequency of the pitch angle.
a, b: the constants.
$u(t)$: The control input u is applied to the aircraft by the elevators at the tail of the flight.
$\alpha > 0$ for ascending $\alpha < 0$ for descending.
Now the mathematical model of the system for small ϕ and α is given by

$$\begin{aligned} \dot{\alpha} &= a(\phi - \alpha) \\ \ddot{\phi} &= -w^2(\phi - \alpha - bu) \\ \dot{h} &= c\alpha \end{aligned}$$

Consider the variables:
Let $x_1 = \alpha$,$x_2 = \phi$, $x_3 = \dot{\phi}$, $x_4 = a$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{pmatrix} = \begin{pmatrix} -a & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ w^2 & -w^2 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w^2 b \\ 0 \end{pmatrix} u$$

Show that the system is controllable.
Satellite Problem :



$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2 \end{pmatrix}$$

$u_1(t)$ - radial thrusters
u_2 - tangential thrusters

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t)$$

Show that the system is observable.
<u>Only radial distance measurements are available:</u> :

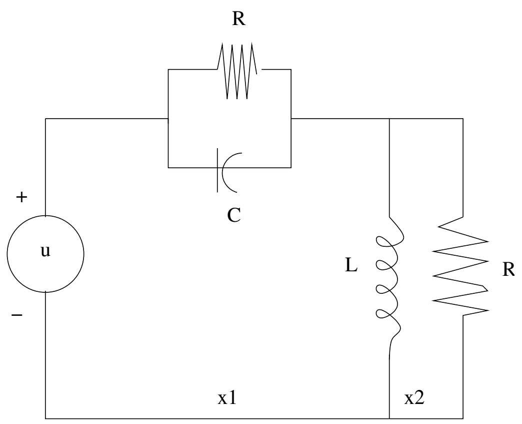
$$y_1(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x(t) = C_1 x(t)$$

$$\begin{bmatrix} C_1 \\ C_1A \\ C_1A^2 \\ C_1A^3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & -w^2 & 0 & 0 \end{pmatrix}$$

has rank 3.
Thus the system is not observable only with radistance measurements.
<u>Only measurements of angle are available</u> :

$$y_2 = [0, 0, 1, 0]x(t)$$
$$= C_2 x(t)$$
$$\text{rank}\begin{bmatrix} C_2 \\ C_2A \\ C_2A^2 \\ C_2A^3 \end{bmatrix} = 4$$

This implies that even with the measurement of angle alone the system is observable.
<u>**Electrical Circuit Example**</u> :
Consider the following circuit:



The state space representation is given by,

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \frac{-2}{RC} & \frac{1}{C} \\ \frac{-1}{L} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{pmatrix} u(t)$$

Observation equation is given by

$$y(t) = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u(t)$$

$$Q = [B|AB] = \begin{bmatrix} \frac{1}{RC} & \frac{-2}{R^2C^2} + \frac{1}{LC} \\ \frac{1}{L} & \frac{-1}{RLC} \end{bmatrix}$$

The system is uncontrollable if $det = \frac{1}{R^2LC^2} - \frac{1}{L^2C} = 0$ iff $R = \sqrt{\frac{L}{C}}$
Observation Matrix is

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ \frac{2}{RC} & \frac{-1}{C} \end{bmatrix}$$

It has full rank implies the observability of the system.

Observability Example : Consider the spring mass system considered earlier. Let the observability equation be given by

$$y(t) = [0, 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$C = [0, 1]$$

$$\text{Observability Matrix} \quad O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rank is 2 $\Longrightarrow$ System is observable.
Computation of initial state x_0
Let $[t_0, t_1] = [-\pi, 0]$

$$\Phi(t, -\pi) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{pmatrix}$$

$$
\begin{aligned}
C\Phi(t,-\pi) &= [0,1]\begin{bmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{bmatrix} \\
&= [-\cos t - \sin t] \\
W(0,\pi) &= \int_{-\pi}^{0} \begin{bmatrix} -\cos t \\ -\sin t \end{bmatrix} \begin{bmatrix} -\cos t & -\sin t \end{bmatrix} dt \\
&= \int_{-\pi}^{0} \begin{bmatrix} \cos^2 t & \sin t\cos t \\ \cos t \sin t & \sin^2 t \end{bmatrix} dt \\
&= \frac{\pi}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Now using the reconstruction formula

$$
\begin{aligned}
\begin{pmatrix} x_1(-\pi) \\ x_2(-\pi) \end{pmatrix} &= \frac{2}{\pi}\int_{-\pi}^{0} \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix} \begin{pmatrix} \frac{1}{2}\cos t & \frac{1}{2}\sin t \end{pmatrix} dt \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \begin{pmatrix} -\cos^2 t & -\cos t\sin t \\ -\sin t\cos t & \sin^2 t \end{pmatrix} dt \\
&= \frac{1}{\pi}\begin{pmatrix} -\frac{\pi}{2} \\ -\frac{\pi}{2} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \\
&= x_0 \in \mathbb{R}^2
\end{aligned}
$$

STABILITY

Consider the dynamical system

$$\dot{x} = f(x,t) \tag{6}$$

Let $f(c,t) = 0$ for all t, where c is some constant vector. Then it follows that if $x(t_0) = c$ then $x(t) = c$, all $t \geq t_0$. Thus solutions starting at c remains there, and c is said to be an equilibrium or critical point (or state). Clearly, by introducing new variables $x_i^{\cdot} = x_i - c_i$, we can transform the equilibrium point to the origin.
We shall assume that $f(0,t) = 0$, $t \geq t_0$. We shall also assume that there is no other constant solution in the neighbourhood of the origin, so that the origin is an isolated equilibrium point.

Example : The equilibrium points of the system described by

$$\dot{x_1} = x_1 - 2x_1x_2$$

$$\dot{x_2} = -2x_2 + x_1x_2$$

are $(0,0)$ and $(2,1/2)$. The above equation is an example of a predator-pray population model due to Volterra and used in Biology.

Definition

An equilibrium state $x = 0$ is said to be: **Stable** if for any positive scalar ϵ there exist a positive scalar δ such that

$$||x(t_0)||_e < \delta \Rightarrow ||x(t)||_e < \epsilon, t \geq t_0$$

Asymptotically stable: if it is stable and if in addition $x(t) \to 0$ as $t \to \infty$.
Unstable: if it is not stable; that is, there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exist an $x(t_0)$ with $||x(t_0)|| < \delta$ and $t_1 > t_0$ such that $||x(t_1)|| \geq \epsilon$ for all $t > t_1$. If this holds for every $x(t_0)$ in $||x(t_0)||_e < \delta$ this equilibrium is completely unstable.

The above definitions are called 'stability in the sense of Lyapunov Regarded as a function of t in the n-dimensional state space, the solution $x(t)$ of (6) is called a trajectory or motion.

In two dimensions we can give the definitions a simple geometrical interpretation. If the origin O is stable, then give the outer circle C with radius ϵ, there exists an inner circle C_1 with radius δ_1 such that trajectories starting within C_1 never leaves C. If O is asymptotically stable then there is some circle C_2, radius δ_2 having the same property as C_1 but in addition trajectories starting inside C_2 tends to O as $t \to \infty$.

Linear System Stability

Consider the system

$$\dot{x} = Ax \tag{7}$$

Theorem

The system (7) is asymptotically stable at $X = 0$ if and only if A is stability matrix, i.e. all characteristic roots λ_k of A have negative real parts. System (7) is unstable at x=0 if any $\Re(\lambda_k) > 0$; and completely unstable if all $\Re(\lambda_k) > 0$.

Proof: The solution of (7) subject to $x(0) = x_0$ is

$$x(t) = exp(At)x_0 \tag{8}$$

with $f(\lambda) = exp(\lambda t)$ we have (using Sylvester's formula)

$$exp(At) = \sum_{k=1}^{q}(Z_{k1} + Z_{k2}t + Z_{k3}t^2 + ... + Z_{k\alpha_k}t^{\alpha_k - 1})exp(\lambda_k t)$$

where λ_k are the eigen values of A and α_k is the power of the factor $(\lambda - \lambda_k)$ in the minimal polynomial of A and Z_{kl} are constant matrices determined entirely by A. Using properties of norms we obtain

$$\begin{aligned} ||exp(At)|| &\leq \sum_{k=1}^{q}\sum_{l=1}^{\alpha_k} t^{l-1}||exp(\lambda_k t)||||Z_{kl}|| \\ &\leq \sum_{k=1}^{q}\sum_{l=1}^{\alpha_k} t^{l-1}exp[Re(\lambda_k)t]||Z_{kl}|| \to 0 \quad as \quad t \to \infty \end{aligned}$$

provided $\Re(\lambda_k) < 0$. Since the above is a finite sum of terms, each of which $\rightarrow 0$ as $t \rightarrow \infty$. Hence from (8) we get

$$x(t) \leq exp(At)x_0 \rightarrow 0$$

So the system is asymptotically stable. If any $\Re(\lambda_k)$ is positive then it is clear from the expression for $exp(At)$ that $||x(t)||$ tends to infinity as t tends to infinity, so the origin is unstable. ■

Illustration of stable and unstable trajectories

Time Varying Systems

Consider the non autonomous system

$$\dot{x} = A(t)x$$

where $A(t)$ is a continuous $n \times n$ matrix. For a nonlinear time-varying system it can be shown that even if all eigenvalues of $A(t)$ have negative real parts for all t the system may be unstable.

Example: If $A(t) = \begin{bmatrix} a\cos^2 t - 1 & 1 - \frac{a\sin 2t}{2} \\ -1 - \frac{a\sin 2t}{2} & a\sin^2 t - 1 \end{bmatrix}$

the solution $x(t) = \begin{bmatrix} e^{(a-1)t}\cos t & e^{-t}\sin t \\ e^{(a-1)t}\sin t & e^{-t}\cos t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

with $x(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

It can be seen that $x(t) \rightarrow \infty$ as $t \rightarrow \infty$, if $1 < a < 2$; even though the eigenvalues of $A(t)$ are $\frac{a-2}{2} \pm \sqrt{(\frac{2-a}{2})^2 - (2-a)}$ which have negative real part. ■

We give an example to show that instability criteria of the linear time invarying systems do not apply to linear time varying systems.

Consider the system $\dot{x}(t) = A(t)x(t)$ with

$$A(t) = \begin{bmatrix} (-\frac{11}{2}) + (\frac{15}{2})\sin 12t & (\frac{15}{2})\cos 12t \\ & \\ (\frac{15}{2})\cos 12t & (-\frac{11}{2}) - (\frac{15}{2})\sin 12t \end{bmatrix}. \qquad (9)$$

The eigenvalues of $A(t)$ are 2 and -13 for all t. The eigenvalue 2 has a positive real part. However the state transition matrix of $A(t)$ in (9) is [?]

$$X(t,0) = \begin{bmatrix} \frac{1}{2}e^{-t}(\cos 6t + 3\sin 6t) & \frac{1}{6}e^{-t}(\cos 6t + 3\sin 6t) \\ +\frac{1}{2}e^{-10t}(\cos 6t - 3\sin 6t) & +\frac{1}{6}e^{-10t}(\cos 6t - 3\sin 6t) \\ & \\ \frac{1}{2}e^{-t}(3\cos 6t - \sin 6t) & \frac{1}{6}e^{-t}(3\cos 6t - \sin 6t) \\ -\frac{1}{2}e^{-10t}(3\cos 6t + \sin 6t) & +\frac{1}{6}e^{-10t}(3\cos 6t + \sin 6t) \end{bmatrix}$$

Clearly $\|X(t,0)\| < \infty$ for all t and $\|X(t,0)\| \rightarrow 0$ as $t \rightarrow \infty$. So the system is asymptotically stable.

We conclude from above examples that stability and instability of linear time varying systems cannot be determined from the eigenvalues of their system matrix $A(t)$.

Perturbed Linear Systems

Consider the perturbations of the systems $\dot{x}(t) = A(t)x(t)$ in the following form

$$\dot{x} \quad = \quad A(t)x + B(t)x \ ; \quad x(t_0) = x_0 \tag{10}$$

where $B(t)$ is an $n \times n$ continuous matrix defined on $[0, \infty)$ and satisfies the condition

$$\lim_{t\to\infty} \|B(t)\| = 0. \tag{11}$$

Theorem

If $\lim\limits_{t\to\infty} A(t) = A$, a constant matrix and if all the characteristic roots of A have negative real parts and $B(t)$ satisfies the condition (11) then all the solutions of the system (10) tend to zero as $t \to \infty$.

Proof

The solution of equation (10) is given by

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)B(s)x(s)\mathrm{d}s.$$

Since all the characteristic roots of A have negative real parts there exits positive constants M and α such that $\|\Phi(t, t_0)\| \le Me^{-\alpha(t-t_0)}$, $t \ge t_0$. Further the condition (11) holds and $B(t)$ is continuous. Hence there exists a constant b such that $\|B(t)\| \le b$. Hence

$$\begin{aligned} \|x(t)\| \quad &\le \quad \|\Phi(t, t_0)x_0\| + \int_{t_0}^{t} \|\Phi(t, s)B(s)x(s)\|\mathrm{d}s \\ &\le \quad M\|x_0\|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}b\|x(s)\|\mathrm{d}s. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x(t)\|e^{\alpha(t-t_0)} \quad &\le \quad M\|x_0\| + Mb\int_{t_0}^{t} e^{\alpha(s-t_0)}\|x(s)\|\mathrm{d}s \\ &\le \quad M\|x_0\|e^{Mb(t-t_0)}. \end{aligned}$$

By Gronwall's inequality

$$\|x(t)\| \le M\|x_0\|e^{(Mb-\alpha)(t-t_0)}, \ t \ge t_0$$

and $\lim\limits_{t\to\infty} \|x(t)\| = 0$ when we take $Mb < \alpha$, which is always possible when t_0 is sufficiently large. Hence the theorem. ∎

Nonlinear Perturbation

Consider the equation of the form

$$\dot{x} = Ax + g(x) \tag{12}$$

where $g(x)$ is very small compared to x and it is continuous. If $g(0) = 0$ then $x(t) = 0$ is an equilibrium solution of (12), we would like to determine whether it is stable or unstable. This is stated in the following theorem.

Theorem

Suppose that the function $g(x)/\|x\|$ is a continuous function of x which tends to zero as $x \to 0$. Then the solution $x(t)$ of (12) is asymptotically stable if the solution $x(t)$ of the linearized equation $\dot{x} = Ax$ is asymptotically stable.

Proof
We know that any solution $x(t)$ of (12) can be written in the form

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}g(x(s))\mathrm{d}s. \tag{13}$$

We wish to show that $\|x(t)\|$ tends to zero as t tends to infinity. Since $\dot{x} = Ax$ is asymptotically stable all the eigenvalues of A have negative real part. Then we can find positive constants K and α such that $\|e^{At}x(0)\| \leq Ke^{-\alpha t}\|x(0)\|$ and $\|e^{A(t-s)}g(x(s))\| \leq Ke^{-\alpha(t-s)}\|g(x(s))\|$. Moreover, from our assumption that $g(x)/\|x\|$ is continuous and vanishes at $x = 0$, we can find a positive constant σ such that $\|g(x)\| \leq \alpha\|x\|/2K$ if $\|x\| \leq \sigma$. Consequently, eqution (13) implies that

$$\begin{aligned}
\|x(t)\| &\leq \|e^{At}x(0)\| + \int_0^t \|e^{A(t-s)}g(x(s))\|\mathrm{d}s \\
\|x(t)\| &\leq Ke^{-\alpha t}\|x(0)\| + \frac{\alpha}{2}\int_0^t e^{-\alpha(t-s)}\|x(s)\|\mathrm{d}s
\end{aligned}$$

as long as $\|x(s)\| \leq \sigma$, $0 \leq s \leq t$. Multiplying both sides by $e^{\alpha t}$ gives

$$e^{\alpha t}\|x(t)\| \leq K\|x(0)\| + \frac{\alpha}{2}\int_0^t e^{\alpha s}\|x(s)\|\mathrm{d}s.$$

Applying Gronwall's inequality we get

$$\|x(t)\| \leq K\|x(0)\|e^{-\frac{\alpha t}{2}} \tag{14}$$

as long as $\|x(s)\| \leq \sigma$, $0 \leq s \leq t$. Now, if $\|x(0)\| \leq \sigma/K$, then the inequality (14) guarantees that $\|x(t)\| \leq \sigma$ for all t. Consequently the inequality (14) is true for all $t \geq 0$ if $\|x(0)\| \leq \sigma/K$. Finally we observe that from (14), $\|x(t)\| \leq K\|x(0)\|$ and $\|x(t)\|$ approaches zero as t approaches infinity. Therefore the equation $x(t) = 0$ is asymptotically stable. ∎

The motion of a simple pendulum with damping is given by

$$\ddot{\theta} + \frac{k}{m}\dot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{15}$$

We know that

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots = \theta + f(\theta)$$

where $\frac{f(\theta)}{\theta} \to 0$ as $\theta \to 0$. This gives

$$\ddot{\theta} + \frac{k}{m}\dot{\theta} + \frac{g}{l}\theta + \frac{g}{l}f(\theta) = 0.$$

Equation (15) is equivalent to (12) with $x = (\theta, \dot{\theta})^*$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & \frac{k}{m} \end{bmatrix},$$

and $g(x) = (0, -\frac{g}{l}f(\theta))^*$. The matrix A has eigenvalues $-\frac{k}{m} \pm (\frac{k^2}{4m^2} - \frac{g}{l})^{\frac{1}{2}}$ which have negative real parts if k, m, g, l are positive and $\frac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$. Hence the pendulum problem (15) is asymptotically stable.

Nonlinear Systems

Lyapunov function: We define a Lyapunov function $V(x)$ as follows:

1. $V(x)$ and all its partial derivatives $\frac{\partial V}{\partial x_i}$ are continuous.
2. $V(x)$ is positive definite, i.e. and $V(0) > 0$ for $x \neq 0$ in some neighborhood $||x|| \leq k$ of the origin.
3. the derivative of V

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + \ldots.. + \frac{\partial V}{\partial x_n}\dot{x_n} \\ &= \frac{\partial V}{\partial x_1}f_1 + \frac{\partial V}{\partial x_2}f_2 + \ldots.. + \frac{\partial V}{\partial x_n}f_n \end{aligned}$$

   is negative semidefinite i.e. $(\dot{V}(0)) = 0$, and for all x in $||x|| \leq k$, $\dot{V}(x) \leq 0$.

Theorem

If for the differential equations (6) we can find a definite function V such that by virtue of the given equations its derivative is either identically equal to zero or is semidefinite with the opposite sign of V, then the unperturbed motion is stable.

Proof: Let us choose an arbitrary and sufficiently small positive number $\epsilon > 0$ and construct the sphere $\sum x_j^2 = \epsilon$. Next, inside this sphere we construct the surface V. This is always possible because V is a continuous function that is equal to zero at the origin. Now we choose a small enough δ so that the sphere $\sum x_j^2 = \delta$ lies inside the surface $V = c$ with no points in common. Let us show that an image point$M$ set in motion from the sphere δ never reaches the sphere ϵ. This will prove the stability of the motion.

Without loss of generality we may assume that the function V is positive definite (if $V < 0$ we can consider the function $-V$). According to the hypothesis of the theorem, the derivative of V, by virtue of the equations of the perturbed motion, is either negative or identically equal to zero i.e. $\dot{V} \leq 0$.

Then from the obvious identity $V - V_0 = \int_0^t \dot{V} dt$, where V_0 is value of the function at the initial point M_0, we get $V - V_0 \leq 0$ which implies $V \leq V_0$.

From this inequality it follows that for $t \geq t_0$ the image point M is located either on the surface $V = V_0 = c_1$ for ($\dot{V} \equiv 0$) or inside this surface. Thus an image point M set in motion from the position M_0 located inside or on the surface of the sphere δ never moves outside the surface $V = c_1$, moreover, it can not reach the surface of the sphere ϵ. This proves the theorem. ■

Applications of Lyapunov theory to linear systems

Consider the real linear time invariant system

$$\dot{x} = Ax \tag{16}$$

We now show how Lyapunov theory can be used to deal directly with (16) by taking as a potential Lyapunov function the quadratic form

$$V = x^T P x \tag{17}$$

where P is a real symmetric matrix. The time derivative of V with respect to (16) is

$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x \\ &= -x^T Q x, \end{aligned}$$

where

$$A^T P + PA = -Q, \tag{18}$$

and it is easy to see that Q is also symmetric. If P and Q are both positive definite then the system (16) is asymptotically stable. If Q is positive definite and P is negative definite or indefinite then in both cases V can take negative values in the neighbourhood of the origin so (16) is unstable.

Theorem

The real matrix A is a stability matrix if and only if for any given real symmetric positive definite (r.s.p.d.) matrix Q the solution P of the continuous Lyapunov matrix equation (18) is also positive definite. ■

Notice that if would be no use choosing P to be positive definite and calculating Q from (18). For unless Q turned out to be definite or semidefinite nothing could be inferred about asymptotic stability from the Lyapunov theorems. If A has complex elements then the above theorem still holds but with P and Q in (18) Hermitian, A^T replaced by A^*.

STABILIZABILITY

Stabilization

In this section we will discuss the linear time invariant system

$$\dot{x} = Ax + Bu,\ x \in \mathbf{R}^n,\ u \in \mathbf{R}^m. \tag{19}$$

If this represents the linearization of some operating plant about a desired equilibrium state then it is represented by $x = 0,\ u = 0$. Now it is well known that the uncontrolled ($u = 0$) homogeneous system

$$\dot{x} = Ax$$

fails to be asymptotically stable. One of the tasks of the control analyst is to use the control u in such a way as to remedy this situation. Because of simplicity for both implementation and analysis, the traditionally favoured means for accomplishing this objective is the use of a linear feedback relation

$$u = Kx \tag{20}$$

where the control $u(t)$ is determined as a linear function of the current state $x(t)$. The problem now becomes that of choosing the $m \times n$ feedback matrix K, in such a way that modified homogenous system realized by substituting (20) into (19), that is,

$$\dot{x} = (A + BK)x \tag{21}$$

is such that $A + BK$ has only eigenvalues with negative real parts.

The system (19) is called an open loop system while the modified system (21) is called a closed loop system.

One of the basic results in the control theory of constant coefficient linear system is that controllability implies stabilizability, the latter being the property of (19) which admits the possibility of selecting K so that $A + BK$ is stability matrix.

Definition

The linear time invariant control system (19) is stabilizable if there exists an $m \times n$ matrix K such that $A + BK$ is a stability matrix.

Theorem

If the system (19) is controllable, then it is stabilizable.

Proof. Assume that the system (19) is controllable. Then the controllability grammian matrix

$$W_T = W(0, T) = \int_0^T e^{-At} BB^* e^{-A^*t} \mathrm{d}t \tag{22}$$

is positive definite for $T > 0$. Define the linear feedback control law

$$u = -B^* W_T^{-1} x = K_T x \tag{23}$$

it can be shown that u stabilizes (19). Now we compute

$$\begin{aligned} AW_T + W_T A^* &= \int_0^T [Ae^{-At}BB^*e^{-A^*t} + e^{-At}BB^*e^{-A^*t}A^*]\mathrm{d}t \\ &= -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}[e^{-At}BB^*e^{-A^*t}]\mathrm{d}t \\ &= -e^{-AT}BB^*e^{-A^*T} + BB^*. \end{aligned} \tag{24}$$

Since W_T is positive and symmetric for $T > 0$, we can modify (24) as follows

$$\begin{aligned} (A - BB^*W_T^{-1})W_T + W_T(A - BB^*W_T^{-1})^* + BB^* & \\ +e^{-AT}BB^*e^{-A^*T} &= 0. \end{aligned}$$

Now

$$BB^* + e^{-AT}BB^*e^{-A^*T} \geq BB^*$$

and, since W_T is positive, we see that $(A - BB^*W_T^{-1})^*$, and hence $A - BB^*W_T^{-1}$ itself is a stability matrix. ■

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