# Discrete Quasiinterpolation

**Theorem (approximation and stability).** $\exists 0 < C = C(\min \angle \mathbb{T}) < \infty$ $\forall \mathcal{T} \in \mathbb{T}\ \forall \hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}) \quad \hat{V} \in S^1_0(\hat{\mathcal{T}})\ \exists \forall V \in S^1_0(\mathcal{T})$

$V = \hat{V}$ on $\hat{\mathcal{T}} \cap \mathcal{T}$ and $||h_{\mathcal{T}}^{-1}(\hat{V} - V)||_{L^2(\Omega)} + |||V||| \leq C\,|||\hat{V}|||$.

Proof. Define $V \in S^1_0(\mathcal{T})$ by linear interpolation of nodal values

$$V(z) := \begin{cases} \hat{V}(z) & \text{if } z \in \mathcal{N}(\Omega) \cap \mathcal{N}(T) \text{ for some } T \in \mathcal{T} \cap \hat{\mathcal{T}} \\ \int_{\omega_z} \hat{V}\,dx/|\omega_z| & \text{if } z \in \mathcal{N}(\Omega) \text{ and } \mathcal{T}(z) \cap \hat{\mathcal{T}}(z) = \emptyset \\ 0 & \text{if } z \in \mathcal{N}(\partial\Omega) \end{cases}$$

Since $V$ and $\hat{V}$ are continuous at any vertex of any $T \in \mathcal{T} \cap \hat{\mathcal{T}}$, the first case applies in the definition of $V(z) = \hat{V}(z)$ for all $z \in \mathcal{N}(T)$. This proves $V = \hat{V}$ on $T \in \mathcal{T} \cap \hat{\mathcal{T}}$.

Given any node $z \in \mathcal{N}$ in the coarse triangulation, let $\omega_z = \text{int}(\cup \mathcal{T}(z))$ denotes its patch of all triangles $\mathcal{T}(z)$ in $\mathcal{T}$ with vertex $z$.

**Lemma A.** There exists $C(z) \approx \text{diam}(\omega_z)$ with

$$||\hat{V} - V(z)||_{L^2(\omega_z)} \leq C(z)\,||\nabla \hat{V}||_{L^2(\omega_z)}.$$

Proof in Case II: $z \in \mathcal{N}(\Omega)$ and $\mathcal{T}(z) \cap \hat{\mathcal{T}}(z) = \emptyset$ with $V(z) = \int_{\omega_z} \hat{V}\,dx/|\omega_z|$. Then, the assertion is a Poincare inequality with $C(z) = C_P(\omega_z)$. □

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Proof in Case III: $z \in \mathcal{N}(\partial\Omega)$ and $V(z) = 0$. Moreover, $\hat{V} - V$ vanishes at the two edges along $\partial\Omega$ of the open boundary patch $\omega_z$ with vertex $z$. Therefore, the assertion is indeed a Friedrichs inequality with $C(z) = C_F(\omega_z)$. □

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Proof in Case I: $\exists T \in \mathcal{T}(z) \cap \hat{\mathcal{T}}(z)$ for $z \in \mathcal{N}(\Omega)$ and $V = \hat{V}$ on $T$. This leads to homogenous Dirichlet boundary conditions on the two edges of the open patch $\omega_z \setminus T$ with vertex $z$ and $\hat{V} - V$ allows for a Friedrichs inequality (on the open patch as in Case III for a patch on the boundary)

$$||\hat{V} - V||_{L^2(\omega_z)} \leq C_F(\omega_z \setminus T)\, ||\nabla(\hat{V} - V)||_{L^2(\omega_z)}$$

However, this is not the claim! The idea is to realize that LHS$=||w||_{L^2(\omega_z)}$ for an $w := \hat{V} - \hat{V}(z)$, which is affine on $T$ and vanishes at vertex $z$. Hence (as an other inverse estimate or discrete Friedrichs inequality)

$$||w||^2_{L^2(T)} \leq C_{dF}(T)^2 \, ||\nabla w||^2_{L^2(T)} \leq C_{dF}(T)^2 \, ||\nabla w||^2_{L^2(\omega_z)}$$

E.g. the integral mean $w_T := \int_T w \, dx / |T|$ of $w := \hat{V} - \hat{V}(z)$ satisfies

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Compare with integral mean $\overline{w} := \int_{\omega_z} w \, dx / |\omega_z|$ and compute

$$\begin{aligned} |\overline{w} - w_T|^2 \, |T| = |T|^{-1} \, | \int_T (\overline{w} - w) dx|^2 &\leq ||w - \overline{w}||^2_{L^2(T)} \\ &\leq ||w - \overline{w}||^2_{L^2(\omega_z)} \leq C_P(\omega_z)^2 \, ||\nabla w||^2_{L^2(\omega_z)} \end{aligned}$$

Consequently, $$|\overline{w} - w_T|^2 \, |\omega_z| \leq \underbrace{|\omega_z|/|T|}_{\leq C_{sr}} \, C_P(\omega_z)^2 \, ||\nabla w||^2_{L^2(\omega_z)}$$

The orthogonality of $1$ and $w - \overline{w}$ in $L^2(\omega_z)$ is followed by Poincare's inequality to verify

$$\begin{aligned} ||w||^2_{L^2(\omega_z)} &= |\overline{w}|^2 \, |\omega_z| + ||w - \overline{w}||^2_{L^2(\omega_z)} \\ &\leq 2|\overline{w} - w_T|^2 \, |\omega_z| + 2|w_T|^2 \, |\omega_z| + C_P(\omega_z)^2 \, ||\nabla w||^2_{L^2(\omega_z)} \end{aligned}$$

The above estimates for $|w_T|^2 \, |T|$ and $|\overline{w} - w_T|^2 \, |T|$ lead to

$$||w||^2_{L^2(\omega_z)} \leq \underbrace{2|\omega_z|/|T| \left(C_{dF}(T) + 2C_P(\omega_z)^2\right)}_{=:C(z)^2} ||\nabla w||^2_{L^2(\omega_z)} \quad \square$$

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W.r.t. triangulation $\mathcal{T}$ and nodal basis functions $\varphi_1$, $\varphi_2$, $\varphi_3$ in $S^1(\mathcal{T})$, let $T = \text{conv}\{P_1, P_2, P_3\} \in \mathcal{T}$ and $\Omega_T := \omega_1 \cup \omega_2 \cup \omega_3$ for $\omega_j := \{\varphi_j > 0\}$

**Lemma B.** There exists $C(T) \approx h_T$ with

$$||\hat{V} - V||_{L^2(T)} \le C(T)\,||\nabla \hat{V}||_{L^2(\Omega_T)}.$$

Proof of Lemma B. N.B. $V = \sum_{j=1}^{3} V(P_j)\,\varphi_j$ and $1 = \sum_{j=1}^{3} \varphi_j$ on $T$
Hence

$$
\begin{aligned}
||\hat{V} - V||^2_{L^2(T)} &= \int_T |\sum_{j=1}^{3}(\hat{V} - V(P_j))\,\varphi_j|^2\,dx \\
&\leq \int_T (\sum_{j=1}^{3} |\hat{V} - V(P_j)|^2)\underbrace{(\sum_{k=1}^{3} \varphi_k^2)}_{\leq 1}\,dx \quad \text{(CS in } \mathbb{R}^3\text{)} \\
&\leq \sum_{j=1}^{3} ||\hat{V} - V(P_j)||^2_{L^2(T)} \\
&\leq \sum_{j=1}^{3} C(P_j)^2 ||\nabla\hat{V}||^2_{L^2(\omega_j)} \quad \text{(Lemma A)} \\
&\leq \underbrace{(\sum_{j=1}^{3} C(P_j)^2)}_{C^2(T)}\ ||\nabla\hat{V}||^2_{L^2(\Omega_T)} \qquad \square
\end{aligned}
$$

**Lemma C.** There exists $C > 0$ (which solely depends on $\min \angle \mathbb{T}$) with

$$||\nabla V||_{L^2(T)} \leq C\,||\nabla \hat{V}||_{L^2(\Omega_T)}.$$

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Proof. N.B. $\nabla V = \sum_{j=1}^{3} V(P_j) \, \nabla \varphi_j$ and $0 = \sum_{j=1}^{3} \nabla \varphi_j$ on $T$
Hence

$$\begin{aligned}
||\nabla V||_{L^2(T)}^2 &= \int_T |\sum_{j=1}^{3} (\hat{V} - V(P_j)) \nabla \varphi_j|^2 \, dx \\
&\leq \int_T (\sum_{j=1}^{3} |\hat{V} - V(P_j)|^2) \underbrace{(\sum_{k=1}^{3} |\nabla \varphi_k|^2)}_{\leq C_{sr}^2 / h_T^2} \, dx \quad \text{(CS in } \mathbb{R}^6\text{)} \\
&\leq C_{sr}^2 \, h_T^{-2} \sum_{j=1}^{3} \int_T |\hat{V} - V(P_j)|^2 \, dx \\
&\leq \ldots \text{(as before)} \ldots \\
&\leq \underbrace{C_{sr}^2 \, h_T^{-2} C^2(T)}_{=: C^2} ||\nabla \hat{V}||_{L^2(\Omega_T)}^2 \qquad \square
\end{aligned}$$

Finish of proof of theorem: $||h_{\mathcal{T}}^{-1}(\hat{V}-V)||_{L^2(\Omega)} + |||V||| \leq C\,|||\hat{V}|||$.

Lemma B and C show for some generic constant $C > 0$ and any $T \in \mathcal{T}$ that

$$||h_T^{-1}(\hat{V}-V)||^2_{L^2(T)} + ||\nabla V||^2_{L^2(T)} \leq C\,||\nabla \hat{V}||^2_{L^2(\Omega_T)}$$

The sum over all those inequalities for $T \in \mathcal{T}$ concludes the proof because the overlap of $(\Omega_T)_{T\in\mathcal{T}}$ is bounded by generic constant $C(\min\angle\mathbb{T})$. □

**Theorem.** (A3) holds in CFEM4PMP

Proof. Given discrete solution $U$ (resp. $\hat{U}$) of CFEM in PMP w.r.t. $\mathcal{T}$ (resp. refinement $\hat{\mathcal{T}}$), set $\hat{e} := \hat{U} - U \in S_0^1(\hat{\mathcal{T}})$ with quasiinterpolant $e \in S_0^1(\mathcal{T})$ as above. Then, $v := \hat{e} - e$ satisfies

$$\delta^2(\mathcal{T}, \hat{\mathcal{T}}) = |||\hat{e}|||^2 = a(\hat{e}, v) = F(v) - a(U, v)$$

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$$\delta^2(\mathcal{T}, \hat{\mathcal{T}}) = |||\hat{e}|||^2 = a(\hat{e}, v) = F(v) - a(U, v)$$

A piecewise integration by parts with a careful algebra with the jump terms for appropriate signs shows

$$\begin{aligned} -a(U,v) &= -\sum_{E\in\mathcal{E}(\Omega)} \int_E v\,[\partial U/\partial \nu_E]_E\,ds \\ &\leq \sqrt{\sum_{E\in\mathcal{E}(\Omega)} |E|^{-1} ||v||^2_{L^2(E)}} \sqrt{\sum_{E\in\mathcal{E}(\Omega)} |E|\, ||[\partial U/\partial \nu_E]_E||^2_{L^2(E)}} \end{aligned}$$

Recall trace inequality

$$|E|^{-1}||v||^2_{L^2(E)} \leq C_{tr}(h^{-2}_{\omega_E}||v||^2_{L^2(\omega_E)} + ||\nabla v||^2_{L^2(\omega_E)})$$

to estimate

$$\sum_{E\in\mathcal{E}(\Omega)} |E|^{-1}||v||^2_{L^2(E)} \leq C_{tr} \sum_{E\in\mathcal{E}(\Omega)} (h_{\omega_E}^{-2}||v||^2_{L^2(\omega_E)} + ||\nabla v||^2_{L^2(\omega_E)})$$
$$\lesssim ||h_{\mathcal{T}}^{-1} v||^2_{L^2(\Omega)} + |||e|||^2 \leq C^2\, |||\hat{e}|||^2$$

with the approximation and stability of the quasiinterpolation to estimate

$$\sum_{E\in\mathcal{E}(\Omega)} |E|^{-1}||v||^2_{L^2(E)} \leq C_{tr} \sum_{E\in\mathcal{E}(\Omega)} (h^{-2}_{\omega_E}||v||^2_{L^2(\omega_E)} + ||\nabla v||^2_{L^2(\omega_E)})$$

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Weighted Cauchy inequality followed by approximation of quasiinterpolation show

$$F(v) \leq ||h_{\mathcal{T}} f||_{L^2(\Omega)}\, ||h_{\mathcal{T}}^{-1} v||_{L^2(\Omega)} \leq C||h_{\mathcal{T}} f||_{L^2(\Omega)}\, |||\hat{e}|||$$

All this plus shape-regularity leads to reliability

$$\delta^2(\mathcal{T},\hat{\mathcal{T}}) = |||\hat{e}|||^2 \leq \Lambda_3\, \eta(\mathcal{T})|||\hat{e}|||.$$

The extra fact $v = 0$ on $\mathcal{T}\cap\hat{\mathcal{T}}$ and a careful inspection on disappearing integrals in the revisited analysis lead to the asserted upper bound in (A3),

$$\delta^2(\mathcal{T},\hat{\mathcal{T}}) \leq \Lambda_3\, \eta(\mathcal{T},\mathcal{T}\setminus\hat{\mathcal{T}}).$$

□