CIMPA Summer School on Current Research on Finite Element Method

Lecture 2

IIT Bombay

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Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions

Stationary equations

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The type of stationary equation to be solved

$$-\nu\Delta w + (w\cdot\nabla)w + \nabla q = 0, \quad \text{in } \Omega,$$

$$\text{div } w = 0 \quad \text{in } \Omega, \quad w = u \quad \text{on } \Gamma_d, \quad \sigma(w,q)n = 0 \quad \text{on } \Gamma_n,$$

or

$$\lambda_0 z - \text{div}\sigma(z,p) + (w_s\cdot\nabla)z + (z\cdot\nabla)w_s + \nabla p = 0, \quad \text{in } \Omega,$$

$$\text{div } z = 0 \quad \text{in } \Omega, \quad z = u \quad \text{on } \Gamma_d, \quad \sigma(z,p)n = 0 \quad \text{on } \Gamma_n,$$

with

$$\sigma(z,p) = \nu(\nabla z + (\nabla z)^T) - p\, I,$$

and $\lambda_0 \geq 0$ is such that

$$\int_\Omega \frac{\nu}{2}((\nabla\phi + (\nabla\phi)^T) : (\nabla\phi + (\nabla\phi)^T))\,dx$$

$$+ \int_\Omega (\lambda_0\phi + (w_s \cdot \nabla)\phi + (\phi \cdot \nabla)w_s)\,dx$$

$$\geq \int_\Omega \frac{\nu}{4}((\nabla\phi + (\nabla\phi)^T) : (\nabla\phi + (\nabla\phi)^T))\,dx$$

for all $\phi \in \{v \in H^1(\Omega;\mathbb{R}^2) \mid v|_{\Gamma_d} = 0,\ \operatorname{div} v = 0\}$.

We assume that $w_s \in H^{3/2+\varepsilon_0}(\Omega;\mathbb{R}^2)$ for some $\varepsilon_0 > 0$.

We look for exact solutions and for solutions approximated by a FEM.

Outline of the lecture

- Mixed variational formulation in the case of H.B.C..
- Lifting of the N.H. B.C.
- Existence and regularity of solutions.
- Penalization of Dirichlet boundary conditions.
- FE approximation in the case of H.B.C..
- FE approximation in the case of N.H.B.C in strong form.
- FE approximation in the case of N.H.B.C in weak form.
- FE approximation in the case of N.H.B.C in penalized form.
- Practical aspects for solving nonlinear equations.

Variational formulations in the case of H.B.C..

We present the results for the Stokes equations

$$-\nu\Delta z + \nabla p = -\mathrm{div}\sigma(z,p) = f, \quad \text{in } \Omega,$$

$$\mathrm{div}\, z = 0 \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \Gamma_d, \quad \sigma(z,p)n = 0 \quad \text{on } \Gamma_n.$$

The variational formulation is stated either in

$$V^1_{\Gamma_d}(\Omega) = \{v \in H^1(\Omega;\mathbb{R}^2) \mid v|_{\Gamma_d} = 0,\ \mathrm{div}\, v = 0\}.$$

or in

$$H^1_{\Gamma_d}(\Omega;\mathbb{R}^2) = \{v \in H^1(\Omega;\mathbb{R}^2) \mid v|_{\Gamma_d} = 0\}.$$

Variational formulation in $V^1_{\Gamma_d}(\Omega)$

$$\text{Find } z \in V^1_{\Gamma_d}(\Omega;\mathbb{R}^2), \quad \text{such that}$$

$$a(z,\phi) + (f,\phi)_\Omega = 0, \quad \forall \phi \in V^1_{\Gamma_d}(\Omega),$$

where

$$a(z,\phi) = -\frac{\nu}{2}\int_\Omega (\nabla z + (\nabla z)^T) : (\nabla\phi + (\nabla\phi)^T)\, dx.$$

Indeed

$$\int_\Omega (\mathrm{div}\sigma(z,p) + f)\phi\, dx = \nu \int_\Omega (\nabla z + (\nabla z)^T) : \nabla\phi\, dx + \int_\Omega f\phi\, dx,$$

and

$$\int_\Omega (\nabla z + (\nabla z)^T) : (\nabla\phi - (\nabla\phi)^T)\, dx = 0.$$

Theorem. If $f \in L^2(\Omega;\mathbb{R}^2)$ (or $f \in (H^1_{\Gamma_d}(\Omega;\mathbb{R}^2))'$), the above var. problem admits a unique solution $z \in H^1_{\Gamma_d}(\Omega;\mathbb{R}^2)$.

Proof. Korn and Poincaré Inequalities, Lax Milgram Lemma.

Mixed variational formulation

Find $z \in H^1_{\Gamma_d}(\Omega;\mathbb{R}^2), \quad p \in L^2(\Omega),$

such that

$$a(z,\phi) + b(\phi,p) + (f,\phi)_\Omega = 0, \quad \forall \phi \in H^1_{\Gamma_d}(\Omega;\mathbb{R}^2),$$

$$b(z,\psi) = 0, \quad \forall \psi \in L^2(\Omega),$$

with

$$a(z,\phi) = -\frac{\nu}{2}\int_\Omega (\nabla z + (\nabla z)^T) : (\nabla\phi + (\nabla\phi)^T)\, dx,$$

$$b(\phi,p) = \int_\Omega \mathrm{div}\,\phi\, p\, dx.$$

Inf-Sup condition.

$$-a(z,z) \geq \alpha \|z\|^2_{H^1_{\Gamma_d}}, \quad \forall z \in H^1_{\Gamma_d}(\Omega;\mathbb{R}^2), \quad \alpha > 0,$$

$$\inf_{p \in L^2} \sup_{\phi \in H^1_{\Gamma_d}} \frac{b(\phi,p)}{\|\phi\|_{H^1_{\Gamma_d}} \|p\|_{L^2}} \geq \beta, \qquad \beta > 0.$$

Theorem. The inf-sup condition is satisfied. If $f \in L^2(\Omega;\mathbb{R}^2)$ (or $f \in (H^1_{\Gamma_d}(\Omega;\mathbb{R}^2))'$), the above mixed var. problem admits a unique solution $z \in H^1_{\Gamma_d}(\Omega;\mathbb{R}^2)$, $p \in L^2(\Omega)$.

Variational formulations in the case of N.H. B.C..

Lifting of the N.H.B.C.

$$
\begin{aligned}
&\text{Find } v \in H^1_{\Gamma_d}(\Omega;\mathbb{R}^2) \quad \text{such that}\\
&\operatorname{div} v = 0 \quad \text{in } \Omega,\\
&v = u \quad \text{on } \Gamma_d, \quad \varepsilon(v)n = 0 \quad \text{on } \Gamma_n,\\
&\operatorname{supp} u \subset \Gamma_c,
\end{aligned}
$$

where

$$
\varepsilon(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^T\right).
$$

Remark 1. We do not require that

$$
\int_\Gamma u \cdot n \, dx = 0.
$$

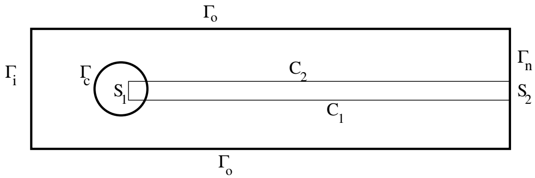
Remark 2. The problem is not a priori well posed.

We can look for a solution v of the form

$$v = v_0 + \alpha_1 v_1,$$

where

$$v_1 = u \quad \text{on } \Gamma_d, \quad \varepsilon(v)n = 0 \quad \text{on } \Gamma_n,$$
$$\text{supp } v_1 \subset \Omega \cup \Gamma_c.$$



For v_0 we consider the problem

$$\begin{aligned}
&\text{Find } v_0 \in H^1_{\Gamma_d}(\Omega;\mathbb{R}^2) \quad \text{such that} \\
&\operatorname{div} v_0 = -\alpha_1 \operatorname{div} v_1 \quad \text{in } \Omega, \\
&v_0 = u - \alpha_1 v_1 \quad \text{on } \Gamma_d, \\
&\operatorname{supp} v_0 \subset \Omega \cup \Gamma_c, \\
&\text{thus} \quad \varepsilon(v) n = 0 \quad \text{on } \Gamma_n.
\end{aligned}$$

Theorem. (Galdi's book) If $f \in H^m_0$ and Γ is of class C^{m+1}, the equation $\operatorname{div} v = f$ admits a solution v in H^{m+1}_0 for all $m \geq 1$.

Theorem. If $u \in H^{1/2}(\Gamma_d; \mathbb{R}^2)$ the above equation admits a solution (not unique) $v \in H^1_{\Gamma_d}(\Omega; \mathbb{R}^2)$.

If $u \in H^{3/2}(\Gamma_d; \mathbb{R}^2)$ the above equation admits a solution (not unique) $v \in H^2_{\Gamma_d}(\Omega; \mathbb{R}^2)$.

Stokes with N.H. Dirichlet B.C..

$$-\mathrm{div}\sigma(z,p) = f, \quad \text{in } \Omega,$$

$$\mathrm{div}\ z = 0 \quad \text{in } \Omega, \quad z = u \quad \text{on } \Gamma_d, \quad \sigma(z,p)n = 0 \quad \text{on } \Gamma_n.$$

We look for z of the form $z = v + \zeta$ with

$$-\mathrm{div}\sigma(\zeta,p) = f - 2\nu\,\mathrm{div}\,\varepsilon(v), \quad \text{in } \Omega,$$

$$\mathrm{div}\ \zeta = 0 \quad \text{in } \Omega, \quad \zeta = 0 \quad \text{on } \Gamma_d, \quad \sigma(\zeta,p)n = 0 \quad \text{on } \Gamma_n.$$

Theorem. If $u \in H^{1/2}(\Gamma_d;\mathbb{R}^2)$, the above equation admits a unique solution $z \in H^1(\Omega;\mathbb{R}^2)$, $p \in L^2(\Omega)$.

If $u \in H^{3/2}(\Gamma_d;\mathbb{R}^2)$ the above equation admits a unique solution $z \in H^{3/2+\varepsilon_0}(\Omega;\mathbb{R}^2)$, $p \in H^{1/2+\varepsilon_0}(\Omega)$.

Mixed variational formulation in the case of N.H. Dirichlet B.C. in strong form.

$$\text{Find } z \in H^1_{\Gamma_{o,i}}(\Omega;\mathbb{R}^2), \quad p \in L^2(\Omega),$$

such that

$$a(z,\phi) + b(\phi,p) + (f,\phi) = 0, \quad \forall \phi \in H^1_{\Gamma_d}(\Omega;\mathbb{R}^2),$$

$$b(z,\psi) = 0, \quad \forall \psi \in L^2(\Omega),$$

$$z = u \quad \text{on } \Gamma_c.$$

We notice that the solution z and the test functions ϕ do not belong to the same space.

Mixed variational formulation in the case of N.H.B.C. with two Lagrange multipliers.

Find $z \in H^1_{\Gamma_{o,i}}(\Omega;\mathbb{R}^2), \quad p \in L^2(\Omega), \quad \tau \in H^{-1/2}(\Gamma_c;\mathbb{R}^2)$

such that

$$a(z,\phi) + b(\phi,p) + \langle \phi,\tau\rangle_{\Gamma_c} + (f,\phi)_\Omega = 0, \quad \forall \phi \in H^1_{\Gamma_{o,i}}(\Omega;\mathbb{R}^2),$$

$$b(z,\psi) = 0, \quad \forall \psi \in L^2(\Omega),$$

$$\langle z,\zeta\rangle_{\Gamma_c} = \langle \zeta,u\rangle_{\Gamma_c}, \quad \forall \zeta \in H^{-1/2}(\Gamma_c;\mathbb{R}^2),$$

where $\langle\cdot,\cdot\rangle_\Gamma$ is the duality product between $H^{-1/2}(\Gamma_c;\mathbb{R}^2)$ and $H^{1/2}(\Gamma_c;\mathbb{R}^2)$,

$$\tau = \sigma(z,p)n = \nu(\nabla z + (\nabla z)^T)n - pn.$$

We can also work with $H^1(\Omega;\mathbb{R}^2)$ in place of $H^1_{\Gamma_{o,i}}(\Omega;\mathbb{R}^2)$ if we use a Lagrange multiplier everywhere on Γ rather than only on Γ_c.

We look at the same time for z, p, and $\sigma(z,p)n$.

Inf-Sup condition.

$$-a(z,z) \geq \alpha \|z\|^2_{H^1_{\Gamma_{o,i}}}, \quad \forall z \in H^1_{\Gamma_{o,i}}(\Omega;\mathbb{R}^2), \quad \alpha > 0,$$

$$\inf_{(p,\tau)=\eta\in L^2\times H^{-1/2}} \sup_{\phi\in H^1_{\Gamma_{o,i}}} \frac{b(\phi,p)+\langle\tau,\phi\rangle_{\Gamma_c}}{\|\phi\|_{H^1_{\Gamma_{o,i}}}\|p\|_{L^2}} \geq \beta, \qquad \beta > 0.$$

Penalization of Dirichlet boundary conditions.

$$-\text{div}\sigma(z,p) = 0 \quad \text{and} \quad \text{div}\, z = 0 \quad \text{in } \Omega,$$

$$\varepsilon\sigma(z,p) + z = u \quad \text{on } \Gamma_d, \quad \sigma(z,p)n = 0 \quad \text{on } \Gamma_n.$$

The Dirichlet boundary condition on Γ_d is replaced by a Robin boundary condition.

Mixed variational formulation of the penalized problem.

Find $z \in H^1(\Omega;\mathbb{R}^2), \quad p \in L^2(\Omega),$

such that

$$a(z,\phi) - b(\phi,p) - \frac{1}{\varepsilon}\int_{\Gamma_d} z\,\phi + \frac{1}{\varepsilon}\int_{\Gamma_d} u\,\phi + (f,\phi) = 0, \quad \forall \phi \in H^1(\Omega;\mathbb{R}^2),$$

$$b(z,\psi) = 0, \quad \forall \psi \in L^2(\Omega).$$

Theorem. If $u \in H^{1/2}(\Gamma_d;\mathbb{R}^2)$, the above equation admits a unique solution $z_\varepsilon \in H^1(\Omega;\mathbb{R}^2)$, $p_\varepsilon \in L^2(\Omega)$.

If $u \in H^{3/2}(\Gamma_d;\mathbb{R}^2)$ the above equation admits a unique solution $z_\varepsilon \in H^{3/2+\varepsilon_0}(\Omega;\mathbb{R}^2)$, $p_\varepsilon \in H^{1/2+\varepsilon_0}(\Omega)$.

The sequence of solutions to the penalized problems converge to the solution of the exact problem.

FE approximation in the case of Homogeneous B.C..

Finite element approximation

$X_h \subset H^1_{\Gamma_d}(\Omega; \mathbb{R}^d)$ – Approximation space for the velocity,

$M_h \subset L^2(\Omega)$ – Approximation space for the pressure.

We choose $\mathbb{P}_3$ (or $\mathbb{P}_2$) for X_h and $\mathbb{P}_2$ (or $\mathbb{P}_1$) for M_h.

The finite dimensional system is

$$\text{Find } z \in X_h, \quad p \in M_h,$$

$$\text{such that}$$

$$a(z, \phi) + b(\phi, p) + (f, \phi) = 0, \quad \forall \phi \in X_h,$$

$$b(z, \psi) = 0, \quad \forall \psi \in M_h.$$

In this setting

$$z \in \mathbb{R}^{N_y}, \quad p \in \mathbb{R}^{N_p}.$$

Uniform discrete inf-sup condition.

$$-a(z,z) \geq \alpha \|z\|^2_{H^1_{\Gamma_d}}, \quad \forall z \in X_h, \quad \alpha > 0,$$

$$\inf_{p \in M_h} \sup_{\phi \in X_h} \frac{b(\phi, p)}{\|\phi\|_{H^1_{\Gamma_d}} \|p\|_{L^2}} \geq \beta, \qquad \beta > 0.$$

Let us set

$$z = \sum_{i=1}^{N_y} z_i \phi_i, \quad p = \sum_{i=1}^{N_p} p_i \psi_i, \quad f = \sum_{i=1}^{N_y} f_i \phi_i,$$

and introduce the coordinate vectors

$$\mathbf{z} = (z_1, \cdots, z_{N_y})^T, \quad \mathbf{p} = (p_1, \cdots, p_{N_p})^T,$$
$$\mathbf{f} = (f_1, \cdots, f_{N_y})^T.$$

The system satisfied by (z, p) is of the form

$$\begin{bmatrix} A_{yy} & A_{yp} \\ A_{yp}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} M_{yy} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix} = 0,$$

where

$$[A_{yy}]_{ij} = a(\phi_i, \phi_j), \quad 1 \le i, j \le N_y,$$
$$[A_{yp}]_{ij} = b(\phi_i, \phi_j), \quad 1 \le i \le N_y, \quad 1 \le j \le N_p,$$
$$[M_{yy}]_{ij} = (\phi_i, \phi_j)_\Omega, \quad 1 \le i, j \le N_y.$$

FE approximation in the case of N.H.B.C in strong form.

$$z = y + w, \quad \mathbf{z} = \mathbf{y} + \sum_{k=1}^{N_b} u_k \phi_k = \sum_{i=1}^{N_y} y_i \phi_i + \sum_{k=1}^{N_b} u_k \phi_k.$$

Since $y \in H^1_{\Gamma_d}(\Omega; \mathbb{R}^2)$, we choose

$$X_h \subset H^1_{\Gamma_d}(\Omega; \mathbb{R}^2).$$

The finite dimensional system is

Find $y \in X_h, \quad p \in M_h,$

such that

$$a(y + w, \phi) + b(\phi, p) = (f, \phi), \quad \forall \phi \in X_h,$$

$$b(y + w, \psi) = 0, \quad \forall \psi \in M_h.$$

In this setting

$$\mathbf{y} \in \mathbb{R}^{N_y}, \quad \mathbf{w} \in \mathbb{R}^{N_b}, \quad \mathbf{p} \in \mathbb{R}^{N_p}.$$

The system satisfied by $(z, p) = (y + w)$ is of the form

$$\begin{bmatrix} A_{yy} & A_{yp} \\ A_{yp}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} A_{yy} \\ A_{yp}^T \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ 0 \end{bmatrix} = 0,$$

where A_{yy}, A_{yp} and M_{yy} are defined as in the homogeneous case.

It can also be written in the form

$$\begin{bmatrix} A_{yy} & A_{yp} \\ A_{yp}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} A_{yb} \\ A_{yp}^T \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ 0 \end{bmatrix} = 0.$$

Approximation in the case of N.H.B.C in a weak form with a Lagrange multiplier

Find $z \in H^1_{\Gamma_{o,i}}(\Omega;\mathbb{R}^2), \quad p \in L^2(\Omega), \quad \tau \in H^{-1/2}(\Gamma_c;\mathbb{R}^2)$

such that

$$a(z,\phi) + b(\phi,p) + \langle \phi,\tau\rangle_{\Gamma_c} + (f,\phi) = 0, \quad \forall \phi \in H^1_{\Gamma_{o,i}}(\Omega;\mathbb{R}^2),$$

$$b(z,\psi) = 0, \quad \forall \psi \in L^2(\Omega)$$

$$\langle z,\zeta\rangle_{\Gamma_c} = \langle u,\zeta\rangle_{\Gamma_c}, \quad \forall \zeta \in H^{-1/2}(\Gamma_c;\mathbb{R}^2),$$

where $\langle\cdot,\cdot\rangle_\Gamma$ is the duality product between $H^{-1/2}(\Gamma_c;\mathbb{R}^2)$ and $H^{1/2}(\Gamma_c;\mathbb{R}^2)$,

$$\tau = \sigma(z,p)n = \nu(\nabla z + (\nabla z)^T)n - pn.$$

Finite element approximation

$X_h \subset H^1_{\Gamma_{o,i}}(\Omega; \mathbb{R}^2)$ – Approximation space for the velocity,

$M_h \subset L^2(\Omega)$ – Approximation space for the pressure,

$S_h \subset H^{-1/2}(\Gamma_c; \mathbb{R}^2)$ – Approximation space for the normal trace of the stress tensor Γ_c.

We choose $\mathbb{P}_3$ for X_h, $\mathbb{P}_2$ for M_h, and $\mathbb{P}_2$ for S_h.

The finite dimensional system is

$$\text{Find } z \in X_h, \quad p \in M_h,$$

such that

$$a(z,\phi) + b(\phi,p) + \langle \phi, \tau \rangle_{\Gamma_c} = 0, \quad \forall \phi \in X_h,$$

$$b(z,\psi) = 0, \quad \forall \psi \in M_h,$$

$$\langle z, \zeta \rangle_{\Gamma_c} = \langle u, \zeta \rangle_{\Gamma_c}, \quad \forall \zeta \in S_h,$$

In this setting

$$z \in \mathbb{R}^{N_z}, \quad p \in \mathbb{R}^{N_p}, \quad p \in \mathbb{R}^{N_\tau}.$$

Let us set

$$z = \sum_{i=1}^{N_z} z_i \phi_i, \quad p = \sum_{i=1}^{N_p} p_i \psi_i, \quad \tau = \sum_{i=1}^{N_\tau} \tau_i \zeta_i, \quad u = \sum_{k=1}^{N_b} u_k \zeta_k,$$

and introduce the coordinate vectors

$$\mathbf{z} = (z_1, \cdots, z_{N_z})^T, \quad \mathbf{p} = (p_1, \cdots, p_{N_p})^T, \quad \boldsymbol{\tau} = (\tau_1, \cdots, \tau_{N_\tau})^T,$$
$$\boldsymbol{\eta} = (p_1, \cdots, p_{N_p}, \tau_1, \cdots, \tau_{N_\tau})^T,$$
$$\mathbf{u} = (u_1, \cdots, u_{N_b})^T, \quad 1 \le i \le N_c.$$

The system satisfied by $(z, p, \tau) = (z, \eta)$ is of the form

$$\begin{bmatrix} A_{zz} & A_{z\eta} \\ A_{z\eta}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \boldsymbol{\eta} \end{bmatrix} - \begin{bmatrix} 0 \\ M_{\eta\eta} \end{bmatrix} \mathbf{u} = 0,$$
$$A_{z\eta} = [A_{zp} \; A_{z\tau}].$$

or equivalently

$$\begin{bmatrix} A_{zz} & A_{zp} & A_{z\tau} \\ A_{zp}^T & 0 & 0 \\ A_{z\tau}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{p} \\ \tau \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ M_{\tau\tau} \end{bmatrix} \mathbf{u} = 0.$$

Notice that

$$A_{zp}^T \mathbf{z} = 0 \quad \text{and} \quad A_{z\tau}^T \mathbf{z} = M_{\tau\tau} \mathbf{u}.$$

We can summarize by writing

$$A_{z\eta}^T \mathbf{z} = M_{\eta\eta} \mathbf{u}.$$

The matrices are defined by

$$[A_{zz}]_{ij} = a(\phi_i, \phi_j), \quad 1 \le i, j \le N_z,$$

$$[A_{zp}]_{ij} = b(\phi_i, \phi_j) \quad 1 \le i \le N_z, \quad 1 \le j \le N_p,$$

$$[M_{zz}]_{ij} = (\phi_i, \phi_j)_\Omega, \quad 1 \le i, j \le N_z.$$

$$[M_{\tau\tau}]_{ij} = (\phi_i, \phi_j)_{\Gamma_c}, \quad 1 \le i, j \le N_b.$$

FE approximation in the case of N.H.B.C in penalized form.

$$\begin{bmatrix} A_{zz} & A_{zp} \\ A_{zp}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{p} \end{bmatrix} - \frac{1}{\varepsilon} \begin{bmatrix} M_{\tau\tau} \\ 0 \end{bmatrix} \mathbf{z}_{|\Gamma_d} + \frac{1}{\varepsilon} \begin{bmatrix} M_{\tau\tau} \\ 0 \end{bmatrix} \mathbf{u} = 0.$$

Practical aspects for solving nonlinear equations.

Step 1. Solve the Stokes eq. with $Re = 1$. Solve Navier-Stokes eq. by the Newton methode with $Re = 1$, initialized with the solution to the Stokes eq..

Step 2. For $k = 1, \cdots$, solve the Navier-Stokes eq. with $Re = 1 + k\delta$, initialized with the solution to the Navier-Stokes eq. for $Re = 1 + (k-1)\delta$.

Exercises.

1. Optimal control of the Stokes or L.N.S. equations with a distributed control

$$\lambda_0 z - \mathrm{div}\sigma(z,p) + (w_s\cdot\nabla)z + (z\cdot\nabla)w_s + \nabla p = u\chi_{\mathcal{O}}, \qquad \text{in } \Omega,$$

$$\mathrm{div}\, z = 0 \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \Gamma_d, \quad \sigma(z,p)n = 0 \quad \text{on } \Gamma_n.$$

Prove the existence and give the optimality conditions for the problem

$$\min\{J(z,u) \mid u \in L^2(\Omega;\mathbb{R}^2),\ (z,u) \text{ is a solution to the P.D.E.}\},$$

$$J(z,u) = \frac{\alpha}{2}\int_\Omega |z - z_d|^2 + \frac{1}{2}\int_\Omega |u|^2.$$

2. Optimal control of the Stokes or L.N.S. equations with a boundary control

$$\lambda_0 z - \mathrm{div}\sigma(z,p) + (w_s\cdot\nabla)z + (z\cdot\nabla)w_s + \nabla p = u\chi_{\mathcal{O}}, \quad \text{in } \Omega,$$

$$\mathrm{div}\ z = 0 \quad \text{in } \Omega, \quad z = u\chi_{\Gamma_c} \quad \text{on } \Gamma_d, \quad \sigma(z,p)n = 0 \quad \text{on } \Gamma_n.$$

Prove the existence and give the optimality conditions for the problem

$$\min\{J(z,u) \mid u \in L^2(\Gamma_c;\mathbb{R}^2),\ (z,u) \text{ is a solution to the P.D.E.}\},$$

$$J(z,u) = \frac{\alpha}{2}\int_\Omega |z - z_d|^2 + \frac{1}{2}\int_{\Gamma_c} |u|^2.$$

3. Write the discrete versions of the two previous problems.