CIMPA Summer School on Current Research on Finite Element Method

Lecture 3

IIT Bombay

July 6 – July 17, 2015

Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions

Instationary equations

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1. Infinite Dimensional Systems – The Duhamel formula

1.1. The Duhamel formula for the heat equation with homogeneous B.C.

$$\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q = \Omega \times (0, \infty),$$
$$z = 0 \quad \text{in } \Sigma = \Gamma \times (0, \infty),$$
$$z(0) = z_0.$$

The spectrum of Δ with D.B.C. is constituted of isolated eigenvalues of finite multiplicity

$$(\lambda_j)_{1 \le j \le \infty}, \quad \lambda_j \in \mathbb{R}, \quad E(\lambda_j) = \mathrm{Ker}(\Delta - \lambda_j I) \subset H_0^1(\Omega),$$
$$L^2(\Omega) = \oplus_{j=1}^{\infty} E(\lambda_j), \quad \dim E(\lambda_j) = \ell_j.$$

Let

$$\{e_j^k,\ 1 \le k \le \ell_j\},$$

be an orthonormal basis of $E(\lambda_j)$, constituted of eigenfunctions of Δ in $H_0^1(\Omega)$.

Then

$$\{e_j^k \mid j \in \mathbb{N}^*,\ 1 \leq k \leq \ell_j\}$$

is a Hilbertian basis in $L^2(\Omega)$ and

$$e^{tA} z_0 = \sum_{j=1}^{\infty} \sum_{k=1}^{\ell_j} e^{\lambda_j t} (z_0, e_j^k)_{L^2(\Omega)} e_j^k .$$

The projection of z_0 onto $E(\lambda_{j_0})$ along $\oplus_{j=1,\, j\neq j_0}^{\infty} E(\lambda_j)$ is

$$\pi_{E(\lambda_{j_0})} z_0 = \sum_{k=1}^{\ell_{j_0}} (z_0, e_{j_0}^k)_{L^2(\Omega)} e_{j_0}^k .$$

Example. In the 1D case, $\Omega =]0, L[$, $\ell_j = 1$,

$$\lambda_j = -\frac{j^2\pi^2}{L^2}, \quad e_j(x) = \frac{\sqrt{2}}{\sqrt{L}} \sin\left(\sqrt{-\lambda_j}\, x\right) = \frac{\sqrt{2}}{\sqrt{L}} \sin\left(\frac{j\pi x}{L}\right),$$

$$e^{tA} z_0(x) = \sum_{j=1}^{\infty} e^{\lambda_j t} \left(\int_\Omega z_0(\xi) e_j(\xi) d\xi\right) e_j(x).$$

1.2. The Duhamel formula for the linearized Burgers equation

$$\frac{\partial z}{\partial t} - \Delta z + 2\,\partial_i w_s\, z + 2\,\partial_i z\, w_s = 0,$$

$$z = 0 \quad \text{on } \Sigma, \quad z(0) = z_0 \quad \text{in } \Omega.$$

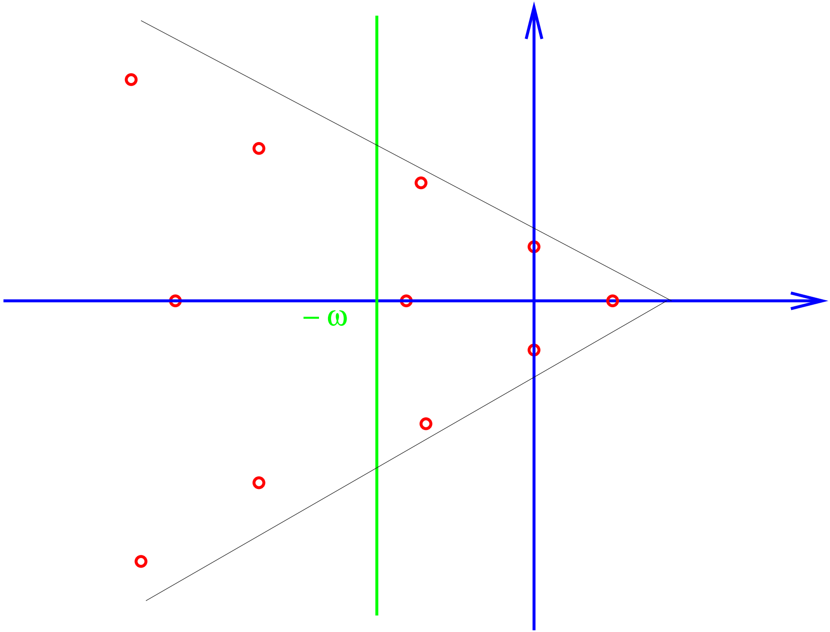
The spectrum of A, with $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$,
$Az = \Delta z - 2\,\partial_i w_s\, z - 2\,\partial_i z\, w_s$, is constituted of isolated eigenvalues of finite multiplicity

$$(\lambda_j)_{1 \le j \le \infty}, \quad \lambda_j \in \mathbb{C},$$

$$L^2_{\mathbb{R}}(\Omega) = \oplus_{j=1}^{\infty} G_{\mathbb{R}}(\lambda_j), \quad \dim G_{\mathbb{R}}(\lambda_j) = d(\lambda_j),$$

if λ_j is an eigenvalue $\overline{\lambda_j}$ is an eigenvalue too.

The spectrum of A^* the adjoint of A, with $A^* z = \Delta z + 2\,\partial_i z\, w_s$, is identical to the spectrum of A.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Since we are interested in real valued solutions, it is interesting to introduce the real generalized eigenspaces

$$\text{If } \mathrm{Im}(\lambda_j) = 0, \quad G_{\mathbb{R}}(\lambda_j) = \mathrm{vec}(\mathrm{Re}\, G_{\mathbb{C}}(\lambda_j)).$$

$$\text{If } \mathrm{Im}(\lambda_j) \neq 0, \quad G_{\mathbb{R}}(\lambda_j) = \mathrm{vec}(\mathrm{Re}\, G_{\mathbb{C}}(\lambda_j), \mathrm{Im}\, G_{\mathbb{C}}(\lambda_j)).$$

Let $\pi_{G_{\mathbb{R}}(\lambda_{j_0})}$ be the projection onto $G_{\mathbb{R}}(\lambda_{j_0})$ along $\oplus_{j=1,\, j\neq j_0}^{\infty} G_{\mathbb{R}}(\lambda_j)$. The system satisfied by $\pi_{G_{\mathbb{R}}(\lambda_j)} z_{z_0}$ is a finite dimensional system.

To define this projection, we introduce a basis of $G_{\mathbb{R}}(\lambda_{j_0})$

$$\left\{ e_{j_0}^k \mid k = 1, \cdots, N(\lambda_{j_0}) \right\},$$

and a basis of $G_{\mathbb{R}}^*(\bar{\lambda}_{j_0}) = G_{\mathbb{R}}^*(\lambda_{j_0})$ (the generalized eigenspace for A^* associated with $\bar{\lambda}_{j_0}$)

$$\left\{ \xi_{j_0}^k \mid k = 1, \cdots, d(\lambda_{j_0}) \right\}.$$

We can choose these bases in order that they satisfy the following bi-orthogonality property

$$\left(e_{j_1}^{k_1}, \xi_{j_2}^{k_2} \right)_{L^2(\Omega)} = \delta_{k_1}^{k_2}\, \delta_{j_1}^{j_2}.$$

In that case, we have

$$\pi_{G_{\mathbb{R}}(\lambda_{j_0})} f = \sum_{k=1}^{d(\lambda_{j_0})} \left(f, \xi_{j_0}^k\right)_{L^2(\Omega)} e_{j_0}^k.$$

and

$$\pi_{G_{\mathbb{R}}^*(\lambda_{j_0})} f = \sum_{k=1}^{d(\lambda_{j_0})} \left(f, e_{j_0}^k\right)_{L^2(\Omega)} \xi_{j_0}^k.$$

With the Jordan decomposition of the finite dimensional projected systems, we can determine the semigroup of the Linearized Burgers equation

$$e^{tA} z_0 = \sum_{j=1}^{\infty} \sum_{k=1}^{d(\lambda_j)} e^{\lambda_j t} (z_0, \xi_j^k)_{L^2(\Omega)} p_k(t) e_j^k,$$

where p_k is a polynomial.

Preliminary results for the Stokes equations.

Trace of L^2 and divergence free vectorfields. If $v \in L^2(\Omega;\mathbb{R}^2)$, $\operatorname{div} v \in L^2(\Omega)$, then $v \cdot n|_\Gamma$ is defined by

$$\langle v \cdot n, \phi \rangle_\Gamma = \int_\Omega \operatorname{div} v \, L\phi \, dx + \int_\Omega v \cdot \nabla(L\phi) \, dx,$$

where $L\phi \in H^1(\Omega;\mathbb{R}^2)$ is a lifting of $\phi \in H^{1/2}(\Gamma;\mathbb{R}^2)$.

1.3. The Duhamel formula for the Stokes equation with Homogeneous Dirichlet B.C.

The Helmholtz decomposition

$$V_n^0(\Omega) = \Big\{ z \in L^2(\Omega;\mathbb{R}^d) \mid \text{ div } z = 0,\ z \cdot n = 0 \text{ on } \Gamma \Big\},$$

$$L^2(\Omega;\mathbb{R}^d) = V_n^0(\Omega) \oplus \text{grad}\, H^1(\Omega),$$

$$\Pi \ : \ L^2(\Omega;\mathbb{R}^d) \longmapsto V_n^0(\Omega).$$

$$\Pi z = z - \nabla p - \nabla q,$$

$$\Delta p = \mathrm{div}\, z \in H^{-1}(\Omega), \quad p \in H_0^1(\Omega),$$

$$\Delta q = 0, \quad \frac{\partial q}{\partial n} = (z - \nabla p) \cdot n \quad \text{on } \Gamma.$$

We set

$$V_0^1(\Omega) = H_0^1(\Omega;\mathbb{R}^d) \cap V_n^0(\Omega).$$

The Stokes operator with Dirichlet B.C.

$$A_0 = \Pi\Delta \quad \text{with} \quad \mathcal{D}(A_0) = H^2(\Omega;\mathbb{R}^d) \cap V_0^1(\Omega).$$

The spectrum of A_0 is real. We have a formula similar to the one for the heat equation

$$e^{tA}z_0 = \sum_{j=1}^{\infty}\sum_{k=1}^{\ell_j} e^{\lambda_j t}\big(z_0, e_j^k\big)_{L^2(\Omega)} e_j^k .$$

The Oseen operator with Dirichlet B.C.

$$\mathcal{D}(A) = H^2(\Omega;\mathbb{R}^d) \cap \cap V_0^1(\Omega),$$

$$Az = \nu\Pi\Delta z - \Pi((w_s\cdot\nabla)z) - \Pi((z\cdot\nabla)w_s),$$

$$\mathcal{D}(A^*) = H^2(\Omega;\mathbb{R}^d) \cap V_0^1(\Omega),$$

$$A^*\phi = \nu\Pi\Delta\phi + \Pi((w_s\cdot\nabla)\phi) - \Pi((\nabla w_s)^T\phi).$$

The spectrum of A is symmetric with respect to the real axis. The representation of the semigroup is similar to that of the Linearized Burgers equation.

The Stokes equation with Homogeneous mixed D/N B.C.

The Helmholtz decomposition

$$V^0_{n,\Gamma_d}(\Omega) = \Big\{ z \in L^2(\Omega;\mathbb{R}^d) \mid \ \mathrm{div}\, z = 0,\ z \cdot n = 0 \text{ on } \Gamma_d \Big\},$$

$$L^2(\Omega;\mathbb{R}^d) = V^0_{n,\Gamma_d}(\Omega) \oplus \mathrm{grad}\, H^1_{\Gamma_n}(\Omega),$$

$$\Pi \ : \ L^2(\Omega;\mathbb{R}^d) \longmapsto V^0_{n,\Gamma_d}(\Omega).$$

$$\Pi z = z - \nabla p - \nabla q,$$

$$\Delta p = \mathrm{div}\, z \in H^{-1}(\Omega), \quad p \in H^1_0(\Omega),$$

$$\Delta q = 0, \quad \frac{\partial q}{\partial n} = (z - \nabla p) \cdot n \quad \text{on } \Gamma_d, \quad q = 0 \quad \text{on } \Gamma_n.$$

We set

$$V^1_{\Gamma_d}(\Omega) = H^1_{\Gamma_d}(\Omega;\mathbb{R}^2) \cap V^0_n(\Omega),$$

$$H^1_{\Gamma_d}(\Omega;\mathbb{R}^2) = \{ z \in H^1(\Omega;\mathbb{R}^2) \mid z = 0 \text{ on } \Gamma_d \}.$$

Characterization of Stokes operator $(A_0, D(A_0))$ in the case of Mixed D/N B.C. with a right angle junction

$$\mathcal{D}(A_0) = \Big\{ z \in V^1_{\Gamma_d}(\Omega) \mid$$

$$\exists p \in L^2(\Omega) \text{ s. t. } \mathrm{div}\sigma(z,p) \in L^2(\Omega;\mathbb{R}^d)$$

$$\text{and } \sigma(z,p)\, n = 0 \quad \text{on} \quad \Gamma_n \Big\},$$

$$A_0 z = \Pi \mathrm{div}\sigma(z,p) \quad (\text{does not depend on } p).$$

In the case of mixed B.C.

$$A_0 z \neq \nu \Pi \Delta z.$$

Indeed, if (z,p) solves

$$-\Delta z + \nabla p = f,\ \operatorname{div} z = 0 \quad \text{in } \Omega, \qquad z = 0 \ \text{ on } \Gamma_d$$
$$\sigma(z,p)n = 0 \text{ on } \Gamma_n,$$

then

$$\Pi \nabla p \neq 0 \quad \text{because} \quad p|_{\Gamma_n} \neq 0,\ p|_{\Gamma_n} = \sigma(z,p)n \cdot n.$$

In the 2D and 3D cases with a right angle junction, we have

$$\mathcal{D}(A_0) \subset H^{3/2+\varepsilon_0}(\Omega;\mathbb{R}^d) \quad \text{for some} \quad \varepsilon_0 > 0.$$

(See Maz'ya and Rossmann, 2007.)

The Oseen operator $(A, D(A))$ is defined by

$$\mathcal{D}(A) = \mathcal{D}(A_0) \quad \text{and} \quad Az = A_0 z + \Pi\left((w_s \cdot \nabla)z + (z \cdot \nabla)w_s\right).$$

Theorem. The spectrum of A is symmetric with respect to the real axis. The representation of the semigroup is similar to that of the Linearized Burgers equation.
If w_s is regular enough and if $\text{div}\ w_s = 0$, we can verify that there exists $\lambda_0 > 0$ in the resolvent set of A satisfying

$$\big((\lambda_0 I - A)z, z\big)_{L^2(\Omega)} \geq \frac{\nu}{2}\|z\|^2_{H^1(\Omega)} \qquad \text{for all } z \in \mathcal{D}(A),$$

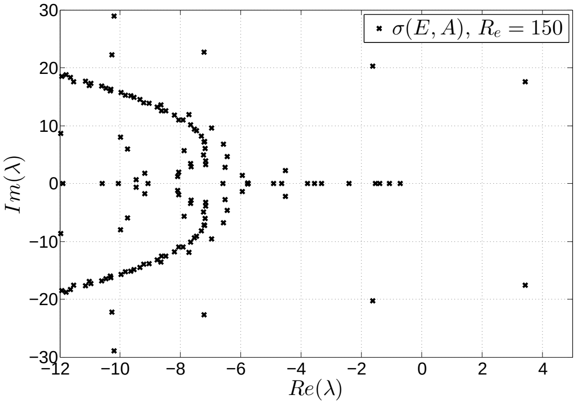
and

$$\big((\lambda_0 I - A^*)\phi, \phi\big)_{L^2(\Omega)} \geq \frac{\nu}{2}\|\phi\|^2_{H^1(\Omega)} \qquad \text{for all } \phi \in \mathcal{D}(A^*).$$

Moreover the resolvent of A, that is $(\lambda_0 I - A)^{-1}$, is a compact operator in $V^0_{n,\Gamma_d}(\Omega)$.

Consequence. The spectrum of A is contained in a sector. The eigenvalues are isolated, pairwise conjugate when they are not real, and of finite multiplicity.

Spectrum of A. $Re = \max(u_s)\,\text{Diam}/\nu = 150$ (Cylinder)



2. The Duhamel formula in the case of non homogeneous Dirichlet B.C.

2.1. The Stokes equation with homogeneous boundary conditions

$$\frac{\partial z}{\partial t} - \mathrm{div}\sigma(z,p) = f, \qquad \mathrm{div} z = 0 \quad \text{in } \Omega \times (0,T),$$

$$z = 0 \qquad \text{on } \Gamma_d \times (0,T),$$

$$\sigma(z,p)n = 0 \qquad \text{on } \Gamma_n \times (0,T),$$

$$z(0) = z_0 \text{ in } \Omega.$$

Applying the operator Π to the PDE, we obtain

$$\Pi z' = z' = Az + \Pi f, \quad z(0) = z_0,$$

$$\text{with} \quad Az = \Pi \mathrm{div}\sigma(z,p).$$

2.2. The Heat equation with non homogeneous B.C.

$$\frac{\partial z}{\partial t} - \Delta z = 0,$$

$$z = u \quad \text{on } \Sigma, \quad z(0) = z_0 \quad \text{in } \Omega\,.$$

The controlled system is of the form

$$z' = Az + Bu, \qquad z(0) = z_0.$$

But $B \notin \mathcal{L}(U, Z)$, and $B \in \mathcal{L}(U, (\mathcal{D}(A^*))')$.

Recall that $\mathcal{D}(A) = \mathcal{D}(A^*) = H^2(\Omega) \cap H_0^1(\Omega)$, $A = A^* = \Delta$.

The Dirichlet operator for the Laplace equation

$Du(t) = w(t)$ is the solution to

$$-\Delta w(t) = 0, \quad \text{in } \Omega, \qquad w(t) = u(t) \ \text{ on } \Gamma.$$

We set

$$z = y + w.$$

Equation satisfied by y

$$\frac{\partial y}{\partial t} - \Delta y = -w',$$
$$y = 0 \quad \text{on } \Sigma, \qquad y(0) = z_0 - w(0).$$

Evolution equation satisfied by y

$$y'(t) = Ay - w'(t), \qquad y(0) = z_0 - w(0).$$

With the Heat semigroup we obtain

$$y(t) = e^{tA}(z_0 - w(0)) - \int_0^t e^{(t-\tau)A}\, w'(\tau)\, d\tau.$$

Integrating by parts

$$y(t) = e^{tA} z_0 + \int_0^t (-A) e^{(t-\tau)A}\, w(\tau)\, d\tau - w(t).$$

Therefore

$$z(t) = y(t) + w(t) = e^{tA} z_0 + \int_0^t (-A) e^{(t-\tau)A} L u(\tau)\, d\tau.$$

This means that

$$z' = Az + (-A)Lu = Az + Bu, \qquad z(0) = z_0.$$

(We have to extend the semigroup to $(\mathcal{D}(A^*))'$). This equation is equivalent to

$$(z', \phi)_{L^2(\Omega)} = (z, A^*\phi)_{L^2(\Omega)} + (u, B^*\phi)_{L^2(\Gamma)}, \quad \forall \phi \in \mathcal{D}(A^*).$$

We say that z is a weak solution to this evolution equation when

$$\frac{d}{dt} \int_\Omega z(t)\, \phi = \int_\Omega z(t)\, A\phi - \int_\Gamma u(t)\, \frac{\partial \phi}{\partial n}, \qquad \forall \phi \in \mathcal{D}(A^*).$$

Here we have used that $A = A^*$. Thus

$$B^* \phi = -\frac{\partial \phi}{\partial n}.$$

We have

$$B^* \ : \ H^2(\Omega) \cap H_0^1(\Omega) \longmapsto L^2(\Gamma)$$

$$B \ : \ L^2(\Gamma) \longmapsto (H^2(\Omega) \cap H_0^1(\Omega))'$$

2.3. The Burgers equation with non homogeneous B.C.

The linearized control system is

$$\frac{\partial z}{\partial t} - \Delta z + 2\,\partial_i w_s\, z + 2\,\partial_i z\, w_s = 0,$$

$$z = u \quad \text{on } \Sigma, \quad z(0) = z_0 \quad \text{in } \Omega\,.$$

It is of the form

$$z' = Az + Bu, \qquad z(0) = z_0.$$

As before $B \notin \mathcal{L}(U, Z)$, but $B \in \mathcal{L}(U, (\mathcal{D}(A^*))')$.

$\mathcal{D}(A) = \mathcal{D}(A^*) = H^2(\Omega) \cap H^1_0(\Omega)$, $A^*\phi = \Delta\phi + 2\partial_i\phi\, w_s$.

The Dirichlet operator for the linearized Burgers equation

$Du(t) = w(t)$ is the solution to

$$\lambda_0 w(t) - \Delta w(t) + 2\,\partial_i w_s\, w(t) + 2\,\partial_i w(t)\, w_s = 0, \quad \text{in } \Omega,$$
$$w(t) = u(t) \ \text{ on } \Gamma.$$

We set

$$z = y + w.$$

Equation satisfied by y:

$$\frac{\partial y}{\partial t} - \Delta y + 2\,\partial_i w_s\, y + 2\,\partial_i y\, w_s = \lambda_0 w - w',$$
$$y = 0 \quad \text{on } \Sigma, \qquad y(0) = z_0 - w(0).$$

Evolution equation satisfied by y:

$$y'(t) = Ay + \lambda_0 w - w'(t), \qquad y(0) = z_0 - w(0).$$

With the Stokes semigroup we obtain

$$y(t) = e^{tA}(y_0 - w(0)) - \int_0^t e^{(t-\tau)A}\,(-\lambda_0 w(\tau) + w'(\tau))\,d\tau.$$

Integrating by parts

$$y(t) = e^{tA} z_0 + \int_0^t (\lambda_0 I - A) e^{(t-\tau)A}\, w(\tau)\,d\tau - w(t).$$

Therefore

$$z(t) = y(t) + w(t) = e^{tA} y_0 + \int_0^t (\lambda_0 I - A) e^{(t-\tau)A} Lu(\tau)\, d\tau.$$

This means that

$$z' = Az + (\lambda_0 I - A)Lu = Az + Bu, \qquad z(0) = z_0.$$

(we have to extend the semigroup to $(\mathcal{D}(A^*))'$)

We say that z is a weak solution to this evolution equation when

$$\frac{d}{dt}\int_\Omega z(t)\,\phi = \int_\Omega z(t)\,A^*\phi - \int_\Gamma u(t)\Big(\frac{\partial\phi}{\partial n} + 2w_s\,\phi\,n_i\Big), \qquad \forall\phi \in \mathcal{D}(A^*).$$

Here $B^*\phi = -\dfrac{\partial\phi}{\partial n} - 2w_s\,\phi\,n_i$.

2.4. The Stokes equation with non homogeneous boundary conditions

$$\frac{\partial z}{\partial t} - \mathrm{div}\sigma(z,p) = 0, \qquad \mathrm{div} z = 0 \quad \text{in } \Omega \times (0,T),$$

$$z = u \qquad \text{on } \Gamma_d \times (0,T),$$

$$\sigma(z,p)n = 0 \qquad \text{on } \Gamma_n \times (0,T),$$

$$z(0) = z_0 \text{ in } \Omega.$$

Before writing this system in the form

$$\Pi z' = A\Pi z + Bu, \quad \Pi z(0) = \Pi z_0,$$

$$(I - \Pi)z = (I - \Pi)Lu,$$

we notice that p is the solution to

$$\Delta p(t) = 0 \quad \text{in } \Omega, \quad \frac{\partial p(t)}{\partial n} = \frac{\partial z}{\partial t} \cdot n - \Delta z \cdot n \quad \text{on } \Gamma_d,$$

$$p(t) = \nu(\nabla z + (\nabla z)^T)n \cdot n \quad \text{on } \Gamma_n.$$

The Dirichlet operator for the Stokes equation

$Lu(t) = w(t)$ is the solution to

$$-\mathrm{div}\sigma(w(t),\rho(t)) = 0,\ \mathrm{div}\, w(t) = 0 \quad \text{in } \Omega, \quad w(t) = u(t) \ \text{on } \Gamma_d$$
$$\sigma(w,\rho)n = 0 \text{ on } \Gamma_n.$$

We set

$$z = y + w \quad \text{and} \quad p = q + \rho.$$

Equation satisfied by y:

$$\frac{\partial y}{\partial t} = \mathrm{div}\sigma(y,q) - w', \quad \mathrm{div}\ y = 0,$$
$$y = 0 \quad \text{on } \Sigma_d, \quad \sigma(y,q)n = 0 \quad \text{on } \Sigma_n, \quad y(0) = z_0 - w(0).$$

Evolution equation satisfied by y:

$$y'(t) = Ay - \Pi w'(t), \qquad y(0) = \Pi(z_0 - w(0)).$$

With the Stokes semigroup we obtain

$$y(t) = S(t)(z_0 - w(0)) - \int_0^t S(t-\tau)\,\Pi w'(\tau)\,d\tau.$$

Integrating by parts

$$y(t) = S(t)z_0 + \int_0^t (-A)S(t-\tau)\,\Pi w(\tau)\,d\tau - \Pi w(t).$$

Therefore

$$\Pi z(t) = y(t) + \Pi w(t) = S(t)z_0 + \int_0^t (-A)S(t-\tau)\,\Pi L u(\tau)\,d\tau.$$

This means that

$$\Pi z' = A\Pi z + (-A)\Pi L u, \qquad \Pi z(0) = z_0.$$

(we have to extend the semigroup to $(\mathcal{D}(A))'$)

What is the equation satisfied by $(I - \Pi)z$?

$$(I-\Pi)z(t) = (I-\Pi)w(t) = (I-\Pi)Lu(t).$$

The system satisfied by z is finally :

$$\Pi z' = A\Pi z + (-A)\Pi L u, \qquad \Pi z(0) = z_0,$$

$$(I-\Pi)z = (I-\Pi)Lu = (I-\Pi)L(u\cdot n\, n).$$

We have

$$(I - \Pi)z = (I - \Pi)Lu = (I - \Pi)L(u \cdot n\, n) = \nabla q,$$

where

$$\Delta q(t) = 0 \text{ in } \Omega, \quad \frac{\partial q(t)}{\partial n} = u(t) \cdot n \text{ in } \Gamma_d, \quad q(t) = 0 \text{ on } \Gamma_n.$$

3. Numerical approximation of the controlled system - Distributed control

Mixed variational formulation

Find $z \in L^2(0,\infty; H^1_{\Gamma_d}(\Omega;\mathbb{R}^2)) \cap H^1(0,\infty;(H^1_{\Gamma_d}(\Omega;\mathbb{R}^2))')$,

$p \in L^2(0,\infty; L^2(\Omega))$, such that

$$\frac{d}{dt}(z(t),\phi) = a(z(t),\phi) + b(\phi,p(t)) + \langle f(t),\phi\rangle, \quad \forall \phi \in H^1_{\Gamma_d}(\Omega;\mathbb{R}^2),$$

$$b(z(t),\psi) = 0, \quad \forall \psi \in L^2(\Omega),$$

$$z(0) = z_0.$$

and

$$a(z,\phi) = -\int_\Omega \Big(\frac{\nu}{2}(\nabla z + (\nabla z)^T) : (\nabla\phi + (\nabla\phi)^T) + ((w_s\cdot\nabla)z + (z\cdot\nabla)w_s)\phi\Big)\, dx,$$

$$b(\phi,p) = \int_\Omega \operatorname{div}\phi\, p\, dx.$$

Finite element approximation - Homogeneous B.C.

$X_h \subset H^1_{\Gamma_d}(\Omega; \mathbb{R}^d)$ – Approximation space for the velocity,

$M_h \subset L^2(\Omega)$ – Approximation space for the pressure,

We choose $\mathbb{P}_3$ for X_h, $\mathbb{P}_2$ for M_h.

The finite dimensional system is

$$\frac{d}{dt}(z(t), \phi) = a(z(t), \phi) + b(\phi, p(t)) + \langle f(t), \phi \rangle, \quad \forall \phi \in X_h,$$

$$b(z(t), \psi) = 0, \quad \forall \psi \in M_h,$$

$$z(0) = \sum_{i=1}^{N_y} z_{0,i}\, \phi_i.$$

Let us set

$$z = \sum_{i=1}^{N_y} z_i \phi_i, \quad p = \sum_{i=1}^{N_p} p_i \psi_i,$$

and introduce the coordinate vectors

$$\mathbf{z} = (z_1, \cdots, z_{N_y})^T, \quad \mathbf{p} = (p_1, \cdots, p_{N_p})^T,$$
$$\mathbf{f} = (f_1, \cdots, f_{N_y})^T.$$

The system satisfied by (z, p) is of the form

$$M_{yy}\mathbf{z}'(t) = A_{yy}\mathbf{z}(t) + A_{yp}\mathbf{p}(t) + M_{yy}\mathbf{f}(t), \quad \mathbf{z}(0) = \mathbf{z}_0,$$
$$A_{yp}^T \mathbf{z}(t) = 0.$$

The matrices M_{yy}, A_{yy} and A_{yp} are those introduced in Lecture 2.

The projector Π_y^T in $\mathbb{R}^{N_y}$ onto $\mathrm{Ker} A_{yp}^T$ parallel to $\mathrm{Im}(M_{yy}^{-1} A_{yp})$ is

$$\Pi_y^T = I_{\mathbb{R}^{N_y}} - M_{yy}^{-1} A_{yp} (A_{yp}^T M_{yy}^{-1} A_{yp})^{-1} A_{yp}^T.$$

We multiply by Π_y to eliminate the pressure $\mathbf{p}$, and we obtain

$$\begin{aligned} & M_{yy} \Pi_y^T \mathbf{z}'(t) = \Pi_y A_{yy} \mathbf{z}(t) + M_{yy} \Pi_y^T \mathbf{f}(t), \quad \mathbf{z}(0) = \mathbf{z}_0, \\ & A_{yp}^T \mathbf{z}(t) = 0, \end{aligned}$$

that we rewrite in the form

$$\Pi_y^T \mathbf{z}'(t) = \Pi_y^T M_{yy}^{-1} A_{yy} \Pi_y^T \mathbf{z}(t) + \Pi_y^T \mathbf{f}(t), \quad \Pi_y^T \mathbf{z}(0) = \mathbf{z}_0.$$

or

$$\mathbf{z}'(t) = \Pi_y^T M_{yy}^{-1} A_{yy} \mathbf{z}(t) + \Pi_y^T \mathbf{f}(t), \quad \mathbf{z}(0) = \mathbf{z}_0.$$

4. Numerical approximation of the controlled system - Dirichlet control

Mixed variational formulation with 2 Lag. mult.

Find $z \in L^2(0,\infty; H^1_{\Gamma_{o,i}}(\Omega;\mathbb{R}^2)) \cap H^1(0,\infty; (H^1_{\Gamma_{o,i}}(\Omega;\mathbb{R}^2))')$,

$p \in L^2(0,\infty; L^2(\Omega))$, $\quad \tau \in L^2(0,\infty; H^{-1/2}(\Gamma_c;\mathbb{R}^2))$ such that

$$\frac{d}{dt}(z(t),\phi) = a(z(t),\phi) + b(\phi,p(t)) + \langle \tau(t),\phi\rangle_{\Gamma_c}, \quad \forall \phi \in H^1_{\Gamma_{o,i}}(\Omega;\mathbb{R}^2),$$

$$b(z(t),\psi) = 0, \quad \forall \psi \in L^2(\Omega),$$

$$\langle \zeta, z(t)\rangle_{\Gamma_c} = \langle \zeta, u(t)\rangle_{\Gamma_c}, \quad \forall \zeta \in H^{-1/2}(\Gamma_c;\mathbb{R}^2),$$

$$z(0) = z_0,$$

where $\langle \cdot,\cdot\rangle_\Gamma$ is the duality product between $H^{-1/2}(\Gamma_c;\mathbb{R}^2)$ and $H^{1/2}(\Gamma_c;\mathbb{R}^2)$,

$$\tau(t) = \sigma(z(t),p(t))n = \nu(\nabla z + (\nabla z)^T)n - pn,$$

and

$$a(z,\phi) = -\int_\Omega \Big(\frac{\nu}{2}(\nabla z + (\nabla z)^T) : (\nabla\phi + (\nabla\phi)^T) + ((w_s\cdot\nabla)z + (z\cdot\nabla)w_s)\phi\Big)\,dx,$$

$$b(\phi,p) = \int_\Omega \operatorname{div}\phi\, p\, dx.$$

Finite element approximation

$X_h \subset H^1_{\Gamma_{o,i}}(\Omega;\mathbb{R}^d)$ – Approximation space for the velocity,

$M_h \subset L^2(\Omega)$ – Approximation space for the pressure,

$S_h \subset H^{-1/2}(\Gamma_c;\mathbb{R}^d)$ – Approximation space for the normal trace of the stress tensor Γ_c.

We choose $\mathbb{P}_3$ for X_h, $\mathbb{P}_2$ for M_h, and $\mathbb{P}_2$ for S_h.

The finite dimensional system is

$$\frac{d}{dt}(z(t),\phi) = a(z(t),\phi) + b(\phi,p(t)) + \langle \tau(t),\phi\rangle, \quad \forall \phi \in X_h,$$

$$b(z(t),\psi) = 0, \quad \forall \psi \in M_h,$$

$$\langle \zeta, z(t)\rangle = \langle \zeta, u(t)\rangle, \quad \forall \zeta \in S_h,$$

$$z(0) = \sum_{i=1}^{N_z} z_{0,i}\,\phi_i.$$

Let us set

$$z = \sum_{i=1}^{N_z} z_i \phi_i, \quad p = \sum_{i=1}^{N_p} p_i \psi_i, \quad \tau = \sum_{i=1}^{N_\tau} \tau_i \zeta_i, \quad u = \sum_{k=1}^{N_b} u_k \zeta_k,$$

and introduce the coordinate vectors

$$\mathbf{z} = (z_1, \cdots, z_{N_z})^T, \quad \mathbf{p} = (p_1, \cdots, p_{N_p})^T, \quad \boldsymbol{\tau} = (\tau_1, \cdots, \tau_{N_\tau})^T,$$
$$\boldsymbol{\eta} = (p_1, \cdots, p_{N_p}, \tau_1, \cdots, \tau_{N_\tau})^T,$$
$$\mathbf{u} = (u_1, \cdots, u_{N_b})^T.$$

The system satisfied by $(z, p, \tau) = (z, \eta)$ is of the form

$$\begin{bmatrix} M_{zz} & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{z}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} = \begin{bmatrix} A_{zz} & A_{z\eta} \\ A_{z\eta}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} - \begin{bmatrix} 0 \\ M_{\eta\eta} \end{bmatrix} \mathbf{u},$$
$$\mathbf{z}(0) = \mathbf{z}_0.$$

or equivalently

$$\begin{bmatrix} M_{zz} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{p}(t) \\ \tau(t) \end{bmatrix} = \begin{bmatrix} A_{zz} & A_{zp} & A_{z\tau} \\ A_{zp}^T & 0 & 0 \\ A_{z\tau}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{p}(t) \\ \tau(t) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ M_{\tau\tau} \end{bmatrix} \mathbf{u},$$

$$\mathbf{z}(0) = \mathbf{z}_0.$$

Notice that

$$A_{zp}^T \mathbf{z} = 0 \quad \text{and} \quad A_{z\tau}^T \mathbf{z} = M_{\tau\tau} \mathbf{u}.$$

We can summarize by writing

$$A_{z\eta}^T \mathbf{z} = [0 \ M_{\eta\eta}]^T \mathbf{u}.$$

We also have

$$M_{zz}\mathbf{z}' = A_{zz}\mathbf{z} + A_{z\eta}\boldsymbol{\eta}, \quad \mathbf{z}(0) = \mathbf{z}_0,$$

$$A_{z\tau}^T \mathbf{z} = M_{\tau\tau}\mathbf{u}.$$

- We have an algebraic differential equation

- We can find a feedback by solving a Generalized Riccati equation

- But we have a problem of too high dimension.

The controlled system

Method 1. We set $z = w + y$, $p = q + \rho$, $\tau = \xi + \vartheta$, with

$$-\lambda_0(w(t), \phi) + a(w(t), \phi) + b(\phi, q(t)) + \langle \xi, \phi \rangle = 0, \quad \forall \phi \in X_h,$$

$$b(w(t), \psi) = 0, \quad \forall \psi \in M_h,$$

$$\langle \zeta, w(t) \rangle = \langle \zeta, \mathbf{u} \rangle, \quad \forall \zeta \in S_h.$$

Thus, we have

$$-\lambda_0 M_{zz}\mathbf{w} + A_{zz}\mathbf{w} + A_{z\eta}(\mathbf{q}, \xi)^T = 0,$$

$$A_{z\eta}^T \mathbf{w} = [0 \ M_{\eta\eta}]^T \mathbf{u}.$$

Therefore

$$\mathbf{w} = -(-\lambda_0 M_{zz} + A_{zz})^{-1} A_{z\eta}(\mathbf{q}, \xi)^T,$$

$$-A_{z\eta}^T(-\lambda_0 M_{zz} + A_{zz})^{-1} A_{z\eta}\,(\mathbf{q}, \xi)^T = [0 \ M_{\eta\eta}]^T \mathbf{u},$$

that is

$$\mathbf{w} = (-\lambda_0 M_{zz} + A_{zz})^{-1} A_{z\eta}(A_{z\eta}^T(-\lambda_0 M_{zz} + A_{zz})^{-1} A_{z\eta})^{-1}[0 \ M_{\eta\eta}]^T \mathbf{u},$$

$$(\mathbf{q}, \xi)^T = -(A_{z\eta}^T(-\lambda_0 M_{zz} + A_{zz})^{-1} A_{z\eta})^{-1}[0 \ M_{\eta\eta}]^T \mathbf{u}.$$

The system for (y, ρ, ϑ) is

$$y(t) = \sum_{i=1}^{N_z} y_i(t)\,\phi_i, \quad \rho(t) \in M_h, \quad \vartheta(t) \in S_h,$$

$$\frac{d}{dt}(y(t), \phi)$$

$$= a(y(t), \phi) + b(\phi, \rho(t)) + \langle \vartheta(t), \phi \rangle - (w'(t), \phi) - \lambda_0(w(t), \phi), \quad \forall \phi \in X_h,$$

$$b(y(t), \psi) = 0, \quad \forall \psi \in M_h,$$

$$\langle \zeta, y(t) \rangle = 0, \quad \forall \zeta \in S_h,$$

$$y(0) = \textstyle\sum_{i=1}^{N_z} (z_{0,i} - w_i(0))\phi_i.$$

It can be writen in the form

$$\mathbf{y}(t) = (y_i(t))^T_{1\le i\le N_z}, \quad \mathbf{w}(t) = (w_i(t))^T_{1\le i\le N_z}, \quad \boldsymbol{\rho} = (\rho_j(t))^T_{1\le j\le N_p},$$

$$\boldsymbol{\vartheta}(t) = (\vartheta_k(t))^T_{1\le k\le N_\tau},$$

$$M_{zz}\,\mathbf{y}'(t) = A_{zz}\mathbf{y}(t) + A_{z\eta}(\boldsymbol{\rho}(t), \boldsymbol{\vartheta}(t))^T - M_{zz}(\mathbf{w}'(t) - \lambda_0\mathbf{w}(t)),$$

$$A^T_{z\eta}\mathbf{y}(t) = 0,$$

$$M_{zz}\mathbf{y}(0) = M_{zz}\mathbf{z}_0 - M_{zz}\mathbf{w}(0).$$

We look for $\mathbf{y}(t) \in \mathrm{Ker}A^T_{z\eta}$. Thus we look for the generator corresponding to the above evolution equation restricted to $\mathrm{Ker}A^T_{z\eta}$.

The discrete Helmholtz projector. We equip $Z = \mathbb{R}^{N_z}$ with the inner product

$$(\cdot,\cdot)_Z = (M_{zz}^{-1}\cdot,\cdot)_{\mathbb{R}^{N_z}}.$$

We have

$$Z = \mathrm{Ker} A_{z\eta}^{\sharp} \oplus \mathrm{Im}(A_{z\eta}),$$

where $A_{z\eta}^{\sharp}$ is the adjoint of $A_{z\eta}$ when $\mathbb{R}^{N_p+N_\tau}$ is equipped with the usual inner product and Z is equipped with $(\cdot,\cdot)_Z$. The projector Π onto $\mathrm{Ker} A_{z\eta}^{\sharp}$ parallel to $\mathrm{Im}(A_{z\eta})$ is

$$\Pi = I - A_{z\eta}(A_{z\eta}^{\sharp}A_{z\eta})^{-1}A_{z\eta}^{\sharp}.$$

We have

$$A_{z\eta}^{\sharp} = A_{z\eta}^{T}M_{zz}^{-1},$$

$$\Pi = I - A_{z\eta}(A_{z\eta}^{T}M_{zz}^{-1}A_{z\eta})^{-1}A_{z\eta}^{T}M_{zz}^{-1},$$

and the following decompositions

$$Z = \mathrm{Ker}(A_{z\eta}^{T}M_{zz}^{-1}) \oplus \mathrm{Im}(A_{z\eta}) \quad \text{and} \quad Z = \mathrm{Ker} A_{z\eta}^{T} \oplus \mathrm{Im}(M_{zz}^{-1}A_{z\eta}).$$

The projector onto $\mathrm{Ker} A_{z\eta}^{T}$ parallel to $\mathrm{Im}(M_{zz}^{-1}A_{z\eta})$ is

$$\Pi^{T} = I - M_{zz}^{-1}A_{z\eta}(A_{z\eta}^{T}M_{zz}^{-1}A_{z\eta})^{-1}A_{z\eta}^{T}.$$

Since $\Pi^T \mathbf{y} = \mathbf{y}$, if we apply the projector $\Pi = I - A_{z\eta}(A_{z\eta}^T M_{zz}^{-1} A_{z\eta})^{-1} A_{z\eta}^T M_{zz}^{-1}$ to the equation, we obtain

$$\begin{aligned}\Pi M_{zz} \Pi^T \mathbf{y}'(t) &= \Pi A_{zz} \Pi^T \mathbf{y}(t) - \Pi M_{zz}(\mathbf{w}'(t) - \lambda_0 \mathbf{w}(t)) \\ &= \Pi A_{zz} M_{zz}^{-1} \Pi M_{zz} \Pi^T \mathbf{y}(t) - \Pi M_{zz}(\mathbf{w}'(t) - \lambda_0 \mathbf{w}(t)), \\ \Pi M_{zz} \Pi^T \mathbf{y}(0) &= \Pi M_{zz} \mathbf{z}_0 - \Pi M_{zz} \mathbf{w}(0).\end{aligned}$$

We have used

$$\Pi A_{z\eta} = 0, \quad \Pi^2 = \Pi, \quad \Pi M_{zz} = M_{zz} \Pi^T, \quad \Pi M_{zz} = \Pi^2 M_{zz} = \Pi M_{zz} \Pi^T.$$

Thus

$$\Pi M_{zz} \Pi^T \mathbf{y}(t) = e^{t\mathbf{A}} \Pi M_{zz}(\mathbf{z}_0 - \mathbf{w}(0)) - \int_0^t e^{(t-s)\mathbf{A}} \Pi M_{zz}(\mathbf{w}'(s) - \lambda_0 \mathbf{w}(s))\, ds,$$

where

$$\mathbf{A} = \Pi^T M_{zz}^{-1} A_{zz}.$$

Integrating by parts, we arrive at

$$\Pi M_{zz}\Pi^T \mathbf{y}(t) = e^{t\mathbf{A}}\Pi M_{zz}\mathbf{z}_0 - \Pi M_{zz}\mathbf{w}(t) - \int_0^t (\mathbf{A}-\lambda_0 I)\, e^{(t-s)\mathbf{A}}\Pi M_{zz}\mathbf{w}(s)\, ds,$$

or

$$\begin{aligned}\Pi M_{zz}\mathbf{z}(t) &= \Pi M_{zz}\Pi^T \mathbf{y}(t) + \Pi M_{zz}\mathbf{w}(t)\\ &= e^{t\mathbf{A}}\Pi M_{zz}\mathbf{z}_0 - \int_0^t \Pi(\mathbf{A}-\lambda_0 I)\Pi^T\, e^{(t-s)\mathbf{A}}\Pi M_{zz}\mathbf{w}(s)\, ds.\end{aligned}$$

We have

$$\Pi M_{zz}\mathbf{z}'(t) = \mathbf{A}\,\Pi M_{zz}\mathbf{z}(t) - (\mathbf{A}-\lambda_0 I)\,\Pi M_{zz}\,\mathbf{Lu}, \quad \Pi M_{zz}\mathbf{z}(0) = \Pi M_{zz}\mathbf{z}_0,$$

with

$$\mathbf{Lu} = \mathbf{w}$$

We also have

$$(I-\Pi)M_{zz}\mathbf{z} = (I-\Pi)M_{zz}\mathbf{w} = (I-\Pi)M_{zz}\mathbf{Lu}.$$

We can also write the equations as follows

$$\Pi^T \mathbf{z}'(t) = \Pi\, \mathbf{A}\, \Pi^T \mathbf{z}(t) - \Pi\, \mathbf{A}\, \Pi^T\, \mathbf{L}\mathbf{u}, \quad \Pi^T \mathbf{z}(0) = \Pi^T \mathbf{z}_0,$$

and

$$(I - \Pi^T)\mathbf{z} = (I - \Pi^T)\mathbf{w} = (I - \Pi^T)\mathbf{L}\mathbf{u}.$$

Method 2. A simplified approach. Useful identities

$$\Pi A_{z\eta} = 0,\ \Pi^2 = \Pi,\ \Pi M_{zz} = M_{zz}\Pi^T,\ \Pi M_{zz} = \Pi M_{zz}\Pi^T,\ M_{zz}^{-1}\Pi = \Pi^T M_{zz}^{-1}.$$

We start with

$$M_{zz}\mathbf{z}' = A_{zz}\mathbf{z} + A_{z\eta}\boldsymbol{\eta},$$

$$\Pi M_{zz}\mathbf{z}' = \Pi A_{zz}\mathbf{z} + \Pi A_{z\eta}\boldsymbol{\eta} = \Pi A_{zz}\mathbf{z},$$

$$\Pi M_{zz}\mathbf{z}' = M_{zz}\Pi^T\mathbf{z}' = \Pi A_{zz}\Pi^T\mathbf{z} + \Pi A_{zz}(I - \Pi^T)\mathbf{z},$$

and

$$(I - \Pi^T)\mathbf{z} = M_{zz}^{-1}A_{z\eta}(A_{z\eta}^T M_{zz}^{-1} A_{z\eta})^{-1} A_{z\eta}^T\mathbf{z},$$

$$A_{z\eta}^T\mathbf{z} = M_{\eta\eta}\mathbf{u}.$$

Thus

$$\Pi M_{zz}\mathbf{z}' = M_{zz}\Pi^T\mathbf{z}' = \Pi A_{zz}\Pi^T\mathbf{z} + \Pi A_{zz}M_{zz}^{-1}A_{z\eta}(A_{z\eta}^T M_{zz}^{-1}A_{z\eta})^{-1}M_{\eta\eta}\mathbf{u}$$

$$\Pi^T\mathbf{z}' = \mathbf{A}\Pi^T\mathbf{z} + \mathbf{B}\mathbf{u},$$

$$\mathbf{A} = \Pi^T M_{zz}^{-1}A_{zz}, \quad \mathbf{B} = -(\mathbf{A} - \lambda_0\Pi)\,\Pi\,\mathbf{L}.$$

This formula is equivalent to the previous one.

Lifting operator

Set $\mathbf{Lu} = \mathbf{w}$, where $(\mathbf{w}, \eta_{\mathbf{w}})$ is the solution to

$$-\lambda_0 M_{zz}\mathbf{w} + A_{zz}\mathbf{w} + A_{z\eta}\boldsymbol{\eta}_{\mathbf{w}} = 0,$$
$$A_{z\eta}^T\mathbf{w} = M_{\eta\eta}\mathbf{u},$$

that is

$$\mathbf{w} = (-\lambda_0 M_{zz} + A_{zz})^{-1}A_{z\eta}(A_{z\eta}^T(-\lambda_0 M_{zz} + A_{zz})^{-1}A_{z\eta})^{-1}M_{\eta\eta}\mathbf{u},$$
$$\boldsymbol{\eta}_{\mathbf{w}} = -(A_{z\eta}^T(-\lambda_0 M_{zz} + A_{zz})^{-1}A_{z\eta})^{-1}M_{\eta\eta}\mathbf{u}.$$

When we can take $\lambda_0 = 0$, we have

$$A_{zz}\mathbf{w} + A_{z\eta}\boldsymbol{\eta}_{\mathbf{w}} = 0,$$
$$A_{z\eta}^T\mathbf{w} = M_{\eta\eta}\mathbf{u},$$

that is

$$\mathbf{w} = A_{zz}^{-1}A_{z\eta}(A_{z\eta}^T(A_{zz})^{-1}A_{z\eta})^{-1}M_{\eta\eta}\mathbf{u},$$
$$\boldsymbol{\eta}_{\mathbf{w}} = -(A_{z\eta}^T A_{zz}^{-1}A_{z\eta})^{-1}M_{\eta\eta}\mathbf{u}.$$

We show that $(\mathbf{z}, \eta)$ is the solution of the descriptor system if and only if $\mathbf{z}$ is the solution to

$$\Pi^T \mathbf{z}' = \mathbf{A}\,\Pi^T \mathbf{z} + \mathbf{B}\mathbf{u}, \quad \Pi \mathbf{z}(0) = \Pi \mathbf{z}_0,$$

$$\mathbf{B} = -(\mathbf{A} - \lambda_0 \Pi)\,\Pi\,\mathbf{L}, \quad \mathbf{A} = \Pi^T M_{zz}^{-1} A_{zz},$$

$$(I - \Pi^T)\mathbf{z} = (I - \Pi^T)\mathbf{w} = (I - \Pi^T)\mathbf{L}\mathbf{u},$$

and η is

$$\eta = (A_{z\eta}^T M_{zz}^{-1} A_{z\eta})^{-1} A_{z\eta}^T (I - \Pi^T)\mathbf{L}\mathbf{u}' - (A_{z\eta}^T M_{zz}^{-1} A_{z\eta})^{-1} A_{z\eta}^T M_{zz}^{-1} A_{zz} \Pi \mathbf{z}$$
$$- (A_{z\eta}^T M_{zz}^{-1} A_{z\eta})^{-1} A_{z\eta}^T M_{zz}^{-1} A_{zz} (I - \Pi^T)\mathbf{z}.$$

The pressure $\mathbf{p}$ depends on $\Pi^T \mathbf{z}$, $\mathbf{u}$ and $\mathbf{u}'$.

Strong formulation of the Dirichlet B.C.

$$z = \sum_{i=1}^{N_y} z_i \phi_i + \sum_{k=1}^{N_b} \sum_{j=1}^{N_c} u_k \phi_k.$$

We introduce the mass matrices

$$[M_{yy}]_{ij} = (\phi_i, \phi_j), \quad [M_{yb}]_{ik} = (\phi_i, \phi_k), \quad [M_{bb}]_{k\ell} = (\phi_k, \phi_\ell) \quad \text{for } 1 \leq i, j \leq N_y,$$

and

$$[A_{yy}]_{ij} = -a(\phi_i, \phi_j), \quad [A_{yb}]_{ik} = -a(\phi_i, \phi_k), \quad \text{for } 1 \leq i, j \leq N_y, \ 1 \leq k \leq N_b,$$
$$[A_{yp}]_{ij} = -b(\phi_i, \psi_j), \quad \text{for } 1 \leq i \leq N_y, \ 1 \leq j \leq N_p,$$
$$\mathbf{u}_k, \quad \text{for } 1 \leq k \leq N_b.$$

The projector Π_y^T in $\mathbb{R}^{N_y}$ onto $\mathrm{Ker} A_{yp}^T$ parallel to $\mathrm{Im}(M_{yy}^{-1} A_{yp})$ is

$$\Pi_y^T = I_{\mathbb{R}^{N_y}} - M_{yy}^{-1} A_{yp} (A_{yp}^T M_{yy}^{-1} A_{yp})^{-1} A_{yp}^T.$$

The semi-discrete system is now

$$M_{yy}\mathbf{z}'(t) + M_{yb}\mathbf{u}'(t) = A_{yy}\mathbf{z}(t) + A_{yb}\mathbf{u} + A_{yp}\mathbf{p}, \quad \Pi_y^T\mathbf{z}(0) = \mathbf{z}_0,$$
$$A_{yp}^T\mathbf{z}(t) + A_{yb}^T\mathbf{u}(t) = 0.$$

We introduce the solution $(\mathbf{w}, \mathbf{p}_\mathbf{w})$ to the equation

$$A_{yy}\mathbf{w} + A_{yb}\mathbf{u} + A_{yp}\mathbf{p}_\mathbf{w} = 0, \quad A_{yp}^T\mathbf{w} + A_{yb}^T\mathbf{u} = 0.$$

We look for $(\mathbf{z}, \mathbf{p})$ in the form $(\mathbf{z}, \mathbf{p}) = (\mathbf{y} + \mathbf{w}, \mathbf{q} + \mathbf{p}_\mathbf{w})$. The equation for $(\mathbf{y}, \mathbf{q})$ is

$$M_{yy}\mathbf{y}'(t) = A_{yy}\mathbf{y}(t) - \big(M_{yy}\mathbf{w}'(t) + M_{yb}\mathbf{u}'(t)\big) + A_{yp}\mathbf{q}, \quad \Pi_y^T\mathbf{y}(0) = \mathbf{z}_0 - \Pi_y^T\mathbf{w}_0$$
$$A_{yp}^T\mathbf{y}(t) = 0.$$

We multiply by Π_y to eliminate the pressure $\mathbf{q}$, and we obtain

$$\begin{aligned}
&\Pi_y M_{yy}\mathbf{y}'(t) = M_{yy}\Pi_y^T\mathbf{y}'(t) = \Pi_y A_{yy}\mathbf{y}(t) - \Pi_y\big(M_{yy}\mathbf{w}'(t) + M_{yb}\mathbf{u}'(t)\big),\\
&\Pi_y^T\mathbf{y}(0) = \mathbf{z}_0 - \Pi_y^T\mathbf{w}_0,\\
&A_{yp}^T\mathbf{y}(t) = 0,
\end{aligned}$$

that we rewrite in the form

$$\begin{aligned}
&\Pi_y^T\mathbf{y}'(t) = \Pi_y^T M_{yy}^{-1}A_{yy}\Pi_y^T\mathbf{y}(t) - \Pi_y^T\big(\mathbf{w}'(t) + M_{yy}^{-1}M_{yb}\mathbf{u}'(t)\big),\\
&\Pi_y^T\mathbf{y}(0) = \mathbf{z}_0 - \Pi_y^T\mathbf{w}_0.
\end{aligned}$$

Thus, setting $\mathbf{A} = \Pi_y^T M_{yy}^{-1}A_{yy}$, we have

$$\begin{aligned}
\Pi_y^T\mathbf{y}(t) &= e^{t\mathbf{A}}(\mathbf{z}_0 - \Pi_y^T\mathbf{w}_0) - \int_0^t e^{(t-s)\mathbf{A}}\Pi_y^T(\mathbf{w}'(s) + M_{yy}^{-1}M_{yb}\mathbf{u}'(s))ds\\
&= e^{t\mathbf{A}}\big(\mathbf{z}_0 - \Pi_y^T\mathbf{w}_0\big) + e^{t\mathbf{A}}\Pi_y^T\big(\mathbf{w}(0) + M_{yy}^{-1}M_{yb}\mathbf{u}(0)\big)\\
&\quad -\Pi_y^T\big(\mathbf{w}(t) + M_{yy}^{-1}M_{yb}\mathbf{u}(t)\big) - \int_0^t e^{(t-s)\mathbf{A}}\mathbf{A}\Pi_y^T\big(\mathbf{w}(s) + M_{yy}^{-1}M_{yb}\mathbf{u}(s)\big)ds,
\end{aligned}$$

from which we deduce

$$\Pi_y^T\big(\mathbf{y}(t)+\mathbf{w}(t)+M_{yy}^{-1}M_{yb}\mathbf{u}(t)\big) = e^{t\mathbf{A}}(\mathbf{z}_0+\Pi_y^T M_{yy}^{-1}M_{yb}\mathbf{u}(0)) - \int_0^t e^{(t-s)\mathbf{A}}\mathbf{A}\Pi_y^T\big(\mathbf{w}(s)+M_{yy}^{-1}M_{yb}\mathbf{u}(s)\big)\,ds.$$

We finally obtain

$$\begin{aligned}
&\Pi_y^T\big(\mathbf{y}'(t)+\mathbf{w}'(t)+M_{yy}^{-1}M_{yb}\mathbf{u}'(t)\big)\\
&= \mathbf{A}\Pi_y^T\big(\mathbf{y}(t)+\mathbf{w}(t)+M_{yy}^{-1}M_{yb}\mathbf{u}(t)\big) + \mathbf{A}\Pi_y^T\big(\mathbf{w}(t)+M_{yy}^{-1}M_{yb}\mathbf{u}(t)\big),\\
&\mathbf{y}(0)+\Pi_y^T\big(\mathbf{w}(0)+M_{yy}^{-1}M_{yb}\mathbf{u}(0)\big) = \mathbf{z}_0+\Pi_y^T M_{yy}^{-1}M_{yb}\mathbf{u}(0).
\end{aligned}$$

If we use a mass lumping method, we make the approximation $M_{yb}=0$.
In this controlled system, the state variable is $\Pi_y^T\big(\mathbf{y}+\mathbf{w}+M_{yy}^{-1}M_{yb}\mathbf{u}\big)$, the infinitesimal generator is $\mathbf{A}=\Pi_y^T M_{yy}^{-1}A_{yy}$, and the control operator is

$$\mathbf{u}\longmapsto \mathbf{A}\Pi_y^T\big(\mathbf{w}+M_{yy}^{-1}M_{yb}\mathbf{u}\big),$$

since **w** can be expressed in terms of **u**.

Penalization of the Dirichlet B.C.

The system is

$$\begin{bmatrix} M_{zz} & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{z} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} A_{zz} & A_{zp} \\ A_{zp}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{p} \end{bmatrix} - \frac{1}{\varepsilon} \begin{bmatrix} M_{\tau\tau} \\ 0 \end{bmatrix} (\mathbf{z}_{|\Gamma_d} - \mathbf{u}).$$

or

$$\begin{bmatrix} M_{zz} & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{z} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} A_{zz} - \frac{1}{\varepsilon} M_{\tau\tau}\gamma_d & A_{zp} \\ A_{zp}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{p} \end{bmatrix} + \frac{1}{\varepsilon} \begin{bmatrix} M_{\tau\tau} \\ 0 \end{bmatrix} \mathbf{u}.$$

We can set

$$A_{zz,\varepsilon} = A_{zz} - \frac{1}{\varepsilon} M_{\tau\tau}\gamma_d.$$

In the above equation $A_{zp}^T \mathbf{z} = 0$. Thus, we can directly apply the projector Π^T, onto Ker(A_{zp}^T) along Im($M_{zz}^{-1} A_{zp}$), to the above equation to eliminate the pressure.