CIMPA Summer School on
Current Research on Finite Element Method

Lecture 4

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Stabilization of the Linearized Navier-Stokes equations

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The PDE model is

$$\frac{\partial z}{\partial t} + (w_s \cdot \nabla) z + (z \cdot \nabla) w_s - \operatorname{div} \sigma(z,p) = 0, \quad \operatorname{div} z = 0 \quad \text{in} \quad Q_\infty,$$

$$z = u \quad \text{on} \quad \Sigma_d^\infty,$$

$$\sigma(z,p)n = 0 \quad \text{on} \quad \Sigma_n^\infty,$$

$$z(0) = z_0 \quad \text{on} \quad \Omega.$$

The controlled system is

$$\Pi z' = A\Pi z + \mathcal{B}u, \qquad \Pi z(0) = z_0,$$

$$(I - \Pi) z = (I - \Pi) L u,$$

with

$$\mathcal{B} = (\lambda_0 I - A)\Pi L \in \mathcal{L}(L^2(\Gamma_c; \mathbb{R}^2), (\mathcal{D}(A^*))').$$

Complement to Lecture 3. Meaning of $\mathcal{B}^*$.

The controlled system for the Stokes equation is

$$\begin{aligned} &\Pi z' = A\Pi z + \mathcal{B}u, \qquad \Pi z(0) = z_0, \\ &(I - \Pi)z = (I - \Pi)Lu, \end{aligned}$$

is equivalent to

$$\begin{aligned} &\frac{\partial z}{\partial t} - \mathrm{div}\sigma(z,p) = 0, \qquad \mathrm{div} z = 0 \quad \text{in } \Omega \times (0,T), \\ &z = u \qquad \text{on } \Gamma_d \times (0,T), \quad \sigma(z,p)n = 0 \qquad \text{on } \Gamma_n \times (0,T), \\ &z(0) = z_0 \text{ in } \Omega. \end{aligned}$$

Choose (ϕ, ψ) an eigenvector of the Stokes operator.

Meaning of the controlled system

$$\frac{d}{dt}\int_\Omega \Pi z\,\phi\,dx = \frac{d}{dt}\int_\Omega z\,\phi\,dx = (\Pi z, \Pi\,\mathrm{div}\,\sigma(\phi,\psi))_\Omega + (u, \mathcal{B}^*\phi)_{\Gamma_c}$$

Meaning of the P.D.E.

$$\frac{d}{dt}\int_\Omega \Pi z\,\phi\,dx = \frac{d}{dt}\int_\Omega z\,\phi\,dx = \int_\Omega z\,\mathrm{div}\sigma(\phi,\psi)\,dx - \int_{\Gamma_c}\sigma(\phi,\psi)n\cdot u\,dx$$

$$= \int_\Omega \Pi z\,\Pi\mathrm{div}\sigma(\phi,\psi)\,dx - \int_{\Gamma_c}\sigma(\phi,\psi)n\cdot u\,dx.$$

Conclusion.

$$\mathcal{B}^*\phi = -\sigma(\phi,\psi)n.$$

We choose a control $u(t)$ of finite dimension

$$u(x,t) = \sum_{i=1}^{N_c} v_i(t)\, g_i(x).$$

The controlled system is

$$\Pi z' = A\Pi z + \mathcal{B}u = A\Pi z + \sum_{i=1}^{N_c} v_i\, \mathcal{B} g_i(x), \qquad \Pi z(0) = z_0.$$

We set

$$\sum_{i=1}^{N_c} v_i\, \mathcal{B} g_i(x) = Bv.$$

We have

$$\Pi z' = A\Pi z + Bv, \qquad \Pi z(0) = z_0,$$

$$(I - \Pi) z = \sum_{i=1}^{N_c} v_i (I - \Pi) L g_i.$$

The state variable Πz belongs to $V^0_{n,\Gamma_d}(\Omega)$. We look for a decomposition

$$Z = V^0_{n,\Gamma_d}(\Omega) = Z_u \oplus Z_s,$$

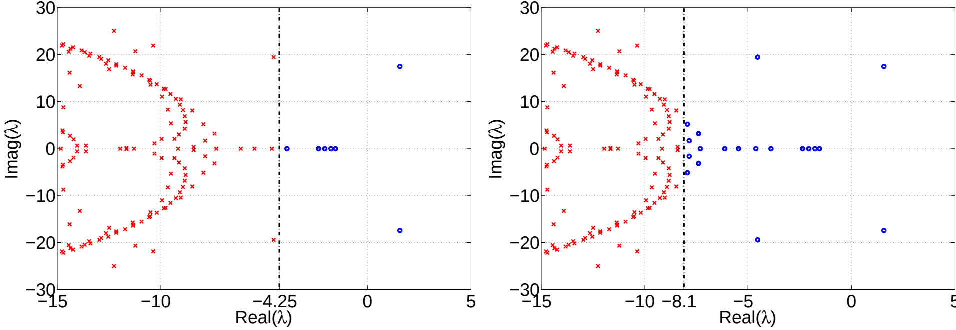
where Z_u and Z_s are invariant under A, $\dim Z_u < 30 < \infty$, and $A|_{Z_s}$ is stable

$$\|e^{tA|_{\mathbb{Z}_s}}\| \le Ce^{-(\alpha+\delta)t} \quad \text{for some } \delta > 0,$$

if we look for an exponential decay smaller than $-\alpha > 0$.
Thus

$$Z_u = \oplus_{j\in J_u} G_{\mathbb{R}}(\lambda_j) \quad \text{and} \qquad Z_u^* = \oplus_{j\in J_u} G^*_{\mathbb{R}}(\lambda_j).$$

The choice for Z_u does not necessarily correspond to a spectral cut



Real Jordan decomposition of matrices

When $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, the solution to system

$$z' = Az + Bu, \quad z(0) = z_0,$$

is defined by

$$(E) \qquad z_{z_0,u}(t) = z(t) = e^{tA} z_0 + \int_0^t e^{(t-s)A} Bu(s)\, ds.$$

The same formula holds true in $\mathbb{C}^n$. If $(\lambda_i)_{1\le i\le r}$ are the complex eigenvalues of A, we can define

$$E(\lambda_j) = \mathrm{Ker}(A - \lambda_j I), \quad \dim E(\lambda_j) = \ell_j = \text{geometric multiplicity of } \lambda_j,$$

$$G(\lambda_j) = \mathrm{Ker}((\lambda_j I - A)^{m(\lambda_j)}), \quad \text{the generalized eigenspace ass. to } \lambda_j,$$

$$\dim G(\lambda_j) = d(\lambda_j) = \text{algebraic multiplicity of } \lambda_j.$$

We can decompose $\mathbb{R}^n$ into *real generalized eigenspaces*

$$\mathbb{R}^n = \oplus_{j=1}^r G_{\mathbb{R}}(\lambda_j), \quad G_{\mathbb{R}}(\lambda_j) = G_{\mathbb{R}}(\bar{\lambda}_j) = \text{vec}\{\text{Re}G(\lambda_j), \text{Im}G(\lambda_j)\},$$
$$AG_{\mathbb{R}}(\lambda_j) \subset G_{\mathbb{R}}(\lambda_j), \quad A = Q\Lambda Q^{-1},$$

$$\Lambda = \begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & 0 \\ & 0 & \ddots & \\ & & & \Lambda_r \end{pmatrix}, \quad \Lambda_j = \begin{pmatrix} J_j^1 & & & \\ & J_j^2 & & 0 \\ & 0 & \ddots & \\ & & & J_j^{\ell_j} \end{pmatrix},$$

and

$$e^{tA} = Qe^{t\Lambda}Q^{-1} = Q \begin{pmatrix} e^{t\Lambda_1} & & & \\ & e^{t\Lambda_2} & & 0 \\ & 0 & \ddots & \\ & & & e^{t\Lambda_r} \end{pmatrix} Q^{-1}.$$

Important identities

Identities in $\mathbb{C}$

$$A = Q_{\mathbb{C}}\Lambda_{\mathbb{C}}Q_{\mathbb{C}}^{-1}, \qquad AQ_{\mathbb{C}} = Q_{\mathbb{C}}\Lambda_{\mathbb{C}},$$

and

$$A^T Q_{\mathbb{C}}^{-T} = Q_{\mathbb{C}}^{-T}\Lambda_{\mathbb{C}}^T, \qquad (Q_{\mathbb{C}}^{-T})^{-T}Q = I_{\mathbb{C}^n}.$$

Identities in $\mathbb{R}$

$$A = Q\Lambda Q^{-1}, \qquad AQ = Q\Lambda,$$

and

$$A^T Q^{-T} = Q^{-T}\Lambda^T, \qquad (Q^{-T})^{-T}Q = I_{\mathbb{R}^n}.$$

In particular

$$e^{tA}G_{\mathbb{R}}(\lambda_j) \subset G_{\mathbb{R}}(\lambda_j).$$

We can rewrite the equation

$$z' = Az + Bu, \quad z(0) = z_0,$$

in the form

$$Q^{-1}z' = Q^{-1}AQQ^{-1}z + Q^{-1}Bu, \quad Q^{-1}z(0) = Q^{-1}z_0,$$

that is

$$\zeta' = \Lambda\zeta + \mathbb{B}u, \quad \zeta(0) = \zeta_0,$$

where $\zeta = Q^{-1}z$, $\mathbb{B} = Q^{-1}B$, $\zeta_0 = Q^{-1}z_0$.
The above system is equivalent to

$$\zeta_i' = \Lambda_i\zeta_i + \mathbb{B}_iu, \quad \zeta_i(0) = \zeta_{0,i}, \quad 1 \le i \le r.$$

The projectors π_u and π_u^*.

There exist a basis $\{e_1, \cdots, e_{d_u}\}$ of Z_u and a basis $\{\xi_1, \cdots, \xi_{d_u}\}$ of Z_u^*, bi-orthogonal. $d_u = \dim Z_u$.
The projections π_u onto Z_u parallel to Z_s and π_u^* onto Z_u^* parallel to Z_s^* are defined by

$$\boldsymbol{\pi}_u f = \sum_{i=1}^{d_u} (f, \xi_i) e_i \quad \text{and} \quad \boldsymbol{\pi}_u^* f = \sum_{i=1}^{d_u} (f, e_i) \xi_i, \quad \forall f \in Z.$$

Notice that the same formula holds for $f \in L^2(\Omega; \mathbb{R}^2)$

$$\boldsymbol{\pi}_u \Pi f = \sum_{i=1}^{d_u} (\Pi f, \xi_i) e_i = \sum_{i=1}^{d_u} (f, \xi_i) e_i = \boldsymbol{\pi}_u f$$

and

$$\boldsymbol{\pi}_u^* \Pi f = \sum_{i=1}^{d_u} (\Pi f, e_i) \xi_i = \sum_{i=1}^{d_u} (f, e_i) \xi_i = \boldsymbol{\pi}_u^* f.$$

We set

$$z_u = \pi_u \Pi z = \pi_u z, \quad z_s = \pi_s \Pi z = \pi_s z, \quad A_u = \pi_u A, \quad B_u = \pi_u B, \quad A_s = \pi_s A,$$

We have $\Pi z = z_u + z_s$. The system satisfied by (z_u, z_s) is

$$z_u' = A_u z_u + B_u v, \qquad z_u(0) = z_{0,u} = \pi_u z_0,$$
$$z_s' = A_s z_s + B_s v, \qquad z_s(0) = z_{0,s} = \pi_s z_0.$$

We want to stabilize the equation for z_u with the exponential decay $e^{-\omega t}$. We set $z_{\omega,u}(t) = e^{\omega t} z_u(t)$, $v_\omega(t) = e^{\omega t} v(t)$:

$$z_{\omega,u}' = (A_u + \omega I) z_{\omega,u} + B_u v_\omega, \qquad z_{\omega,u}(0) = z_{0,u} = \pi_u z_0.$$

What is the meaning of the equation for z_u ?

$$z_u = \pi_u z = \sum_{i=1}^{d_u} (z, \xi_i) e_i = \sum_{i=1}^{d_u} \zeta_i e_i,$$

$$B_u v = \sum_{j=1}^{N_c} v_j \, \pi_u \mathcal{B} g_j = \sum_{j=1}^{d_u} \langle \mathcal{B} g_j, \xi_j \rangle e_j = \sum_{j=1}^{d_u} (g_j, \mathcal{B}^* \xi_j) e_j,$$

and

$$\mathcal{B}^* \xi_j = \sigma(\xi_j, p_{\xi_j}) n.$$

We also have

$$A_u z_u = \sum_{j=1}^{d_u} \zeta_j A_u e_j = \sum_{j=1}^{d_u} \zeta_j A e_j.$$

Thus, to find the differential equation satisfied by ζ_i, we take the inner product of

$$z_u' = A_u z_u + B_u v$$

with ξ_i and we obtain

$$\zeta_i' = \sum_{j=1}^{d_u} \zeta_j (Ae_j, \xi_i) + \sum_{k=1}^{N_c} v_k (g_k, \mathcal{B}^* \xi_i),$$

that is

$$\zeta_u' = \Lambda_u \zeta_u + \mathbb{B}_u v,$$

with $\zeta_u = (\zeta_1, \cdots, \zeta_{d_u})^T$,

$$[\Lambda_u]_{i,j} = (Ae_j, \xi_i)_\Omega \quad \text{for } 1 \leq i, j \leq d_u$$

and

$$[\mathbb{B}_u]_{i,k} = (g_k, \mathcal{B}^* \xi_i)_{\Gamma_c} \quad \text{for } 1 \leq i \leq d_u \quad \text{and } 1 \leq k \leq N_c.$$

We look for a feedback law of the form

$$v_\omega = -B_u^* \mathcal{P}_{\omega,u} z_{\omega,u} \quad \text{or} \quad v = -B_u^* \mathcal{P}_{\omega,u} z_u,$$

where $B_u^* \in \mathcal{L}(Z_u^*, \mathbb{R}^{N_c})$ is the adjoint of $B_u \in \mathcal{L}(\mathbb{R}^{N_c}, Z_u)$, and $\mathcal{P}_{\omega,u}$ is the solution to the algebraic Riccati equation

$$\begin{aligned}
&\mathcal{P}_{\omega,u} \in \mathcal{L}(Z_u, Z_u^*), \quad \mathcal{P}_{\omega,u} = \mathcal{P}_{\omega,u}^* \geq 0, \\
&\mathcal{P}_{\omega,u}(A_u + \omega I) + (A_u^* + \omega I)\mathcal{P}_{\omega,u} - \mathcal{P}_{\omega,u} B_u B_u^* \mathcal{P}_{\omega,u} \;=\; 0, \\
&\mathcal{P}_{\omega,u} \quad \text{is invertible.}
\end{aligned}$$

The spectrum of $A_u + \omega I - B_u B_u^* \mathcal{P}_{\omega,u}$ is determined in terms of the spectrum of $A_{\omega,u}$ by

$$\begin{aligned}
\operatorname{spec}(A_u + \omega I - B_u B_u^* \mathcal{P}_{\omega,u}) = \{\widetilde{\lambda}_j \in \operatorname{spec}(A_u + \omega I) \mid \\
\widetilde{\lambda}_j = -2\omega - \operatorname{Re}\lambda_j + i \operatorname{Im}\lambda_j, \quad j \in J_u\}.
\end{aligned}$$

We can also solve the equation

$$\begin{aligned} &\mathbb{P}_{\omega,u} \in \mathbb{R}^{d_u \times d_u}, \quad P_{\omega,u} = \mathbb{P}^*_{\omega,u} > 0, \\ &\mathbb{P}_{\omega,u}(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}) + (\Lambda_u^T + \omega I_{\mathbb{R}^{d_u}})\mathbb{P}_{\omega,u} - \mathbb{P}_{\omega,u}\mathbb{B}_u\mathbb{B}_u^*\mathbb{P}_{\omega,u} \;=\; 0. \end{aligned}$$

The semi-discrete model is

$$\begin{aligned} &M_{zz}\mathbf{z}'(t) = A_{zz}\mathbf{z}(t) + A_{z\eta}\boldsymbol{\eta}(t), \quad \Pi^T\mathbf{z}(0) = \mathbf{z}_0, \\ &A_{z\eta}^T\mathbf{z}(t) = M_{\eta\eta}\,\mathbf{G}\,\mathbf{v}(t), \end{aligned}$$

with $\mathbf{z} = (z_1, \cdots, z_{N_z})^T$, $\mathbf{p} = (p_1, \cdots, p_{N_p})^T$, $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_{N_\tau})^T$,

$\boldsymbol{\eta} = (\eta_1, \cdots, \eta_{N_\tau})^T$ stands for $(p_1, \cdots, p_{N_p}, \tau_1, \cdots, \tau_{N_\eta})^T$,

$\mathbf{G} = [\mathbf{g}^1, \quad \mathbf{g}^{N_c}]$,

$\mathbf{g}^j = (g_1^j, \cdots, g_{N_b}^j)^T$.

We have proved that $(\mathbf{z}, \eta)$ is the solution to the above equation if and only if

$$\Pi^T \mathbf{z}'(t) = \mathbf{A}\,\Pi^T \mathbf{z}(t) + \mathbf{B}\,\mathbf{v}(t), \quad \Pi^T \mathbf{z}(0) = \mathbf{z}_0,$$
$$(I - \Pi^T)\mathbf{z}(t) = (I - \Pi^T)\mathbf{L}\,\mathbf{G}\,\mathbf{v}(t),$$
$$\eta(t) = (A_{z\eta}^T M_{zz}^{-1} A_{z\eta})^{-1} A_{z\eta}^T \mathbf{z}'(t) - (A_{z\eta}^T M_{zz}^{-1} A_{z\eta})^{-1} A_{z\eta}^T M_{zz}^{-1} A_{zz} \mathbf{z}(t),$$

where the infinitesimal generator **A** and the control operator **B** of the controlled system are

$$\mathbf{A} = \Pi^T M_{zz}^{-1} A_{zz} \quad \text{and} \quad \mathbf{B} = \Pi A_{zz} M_{zz}^{-1} A_{z\eta} (A_{z\eta}^T M_{zz}^{-1} A_{z\eta})^{-1} M_{\eta\eta}\, \mathbf{G}.$$

The projected system.

Eigenvalue problem for $\mathbf{A}\Pi^T$

$$\lambda \in \mathbb{C}^*, \quad \mathbf{f} \in \mathrm{Ker}(A_{z\eta}^T), \quad \mathbf{f} \neq 0_{\mathbb{C}^{N_z}}, \quad \mathbf{A}\Pi^T\mathbf{f} = \mathbf{A}\mathbf{f} = \lambda\mathbf{f},$$

is equivalent to

$$\lambda \in \mathbb{C}^*, \quad \mathbf{f} \in \mathbb{C}^{N_z}, \quad \mathbf{f} \neq 0_{\mathbb{C}^{N_z}}, \quad \mathbf{p_f} \in \mathbb{C}^{N_p}, \quad \boldsymbol{\tau}_\mathbf{f} \in \mathbb{C}^{N_\tau},$$

$$\begin{bmatrix} A_{zz} & A_{zp} & A_{z\tau} \\ A_{zp}^T & 0 & 0 \\ A_{z\tau}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{p_f} \\ \boldsymbol{\tau}_\mathbf{f} \end{bmatrix} = \lambda \begin{bmatrix} M_{zz} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{p_f} \\ \boldsymbol{\tau}_\mathbf{f} \end{bmatrix}.$$

We have the same equivalence for the generalized eigenvectors. If $\mathbf{F}$ contains all the eigenvectors and generalized eigenvectors of $\mathbf{A}\Pi^T$, we have

$$\Lambda_{\mathbb{C}} = \mathbf{F}^{-1}\,\mathbf{A}\Pi^T\,\mathbf{F}.$$

Adjoint Eigenvalue problem

Similarly, setting $\mathbf{A}^\sharp = \Pi^T M_{zz}^{-1} A_{zz}^T$,

$$\boldsymbol{\lambda} \in \mathbb{C}^*, \quad \phi \in \mathrm{Ker}(A_{z\eta}^T), \quad \phi \neq 0_{\mathbb{C}^{N_z}}, \quad \mathbf{A}^\sharp \Pi^T \phi = \mathbf{A}^\sharp \phi = \boldsymbol{\lambda} \phi,$$

is equivalent to

$$\boldsymbol{\lambda} \in \mathbb{C}^*, \quad \phi \in \mathbb{C}^{N_z}, \quad \phi \neq 0_{\mathbb{C}^{N_z}}, \quad \mathbf{p}_\phi \in \mathbb{C}^{N_p}, \quad \boldsymbol{\tau}_\phi \in \mathbb{C}^{N_\tau},$$

$$\begin{bmatrix} A_{zz}^T & A_{zp} & A_{z\tau} \\ A_{zp}^T & 0 & 0 \\ A_{z\tau}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \mathbf{p}_\phi \\ \boldsymbol{\tau}_\phi \end{bmatrix} = \boldsymbol{\lambda} \begin{bmatrix} M_{zz} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \mathbf{p}_\phi \\ \boldsymbol{\tau}_\phi \end{bmatrix}.$$

We have the same equivalence for the generalized eigenvectors. If $\mathbf{\Phi}$ contains all the eigenvectors and generalized eigenvectors of $\mathbf{A}^\sharp \Pi^T$, we have

$$\Lambda_{\mathbb{C}}^T = \mathbf{\Phi}^{-1} \, \mathbf{A}^\sharp \Pi^T \, \mathbf{\Phi}.$$

The two families **F** and **Φ** constituted of eigenvectors and generalized eigenvectors may be chosen bi-orthogonal.

Bi-orthogonality relationships

$$\mathbf{F}^T M_{zz} \mathbf{\Phi} = I_{\mathbb{C}^{N_z}} \quad \text{and} \quad \mathbf{\Phi}^T M_{zz} \mathbf{F} = I_{\mathbb{C}^{N_z}} .$$

Here $\mathbf{F}^T$ is the conjugate transposed of **F**.

We also have

$$\mathbf{F}\,\Lambda_{\mathbb{C}} = \mathbf{A}\,\Pi^T\,\mathbf{F} = I_{\mathbb{C}^{N_z}} , \quad \mathbf{\Phi}\,\Lambda_{\mathbb{C}}^T = \mathbf{A}^\sharp\,\Pi^T\,\mathbf{\Phi} = I_{\mathbb{C}^{N_z}} ,$$

$$\Lambda_{\mathbb{C}} = \mathbf{F}^{-1}\,\mathbf{A}\,\Pi^T\,\mathbf{F} \quad \text{and} \quad \Lambda_{\mathbb{C}}^T = \mathbf{\Phi}^{-1}\,\mathbf{A}^\sharp\,\Pi^T\,\mathbf{\Phi},$$

and

$$\Lambda_{\mathbb{C}} = \mathbf{\Phi}^T\,A_{zz}\,\Pi^T\,\mathbf{F} \quad \text{and} \quad \Lambda_{\mathbb{C}}^T = \mathbf{F}^T\,\Pi\,A_{zz}^T\,\mathbf{\Phi}.$$

Real bi-orthogonal families

If λ_j is real and if $\mathbf{f}^k$ is an eigenvector or a generalized eigenvector associated with λ_j, we can assume that $\mathbf{f}^k$ and ϕ^k are real vectors, and we set $(\mathbf{e}^k, \mathbf{p}_{\mathbf{e}^k}, \tau_{\mathbf{e}^k}) = (\mathbf{f}^k, \mathbf{p}_{\mathbf{f}^k}, \tau_{\mathbf{f}^k})$ and $(\xi^k, \mathbf{p}_{\xi^k}, \tau_{\xi^k}) = (\phi^k, \mathbf{p}_{\phi^k}, \tau_{\phi^k})$.

If λ_j is a complex eigenvalue with $\mathrm{Im}\,\lambda_j \neq 0$, then necessarily $\overline{\lambda_j}$ is also an eigenvalue, and if $\mathbf{f}^k$ and ϕ^k are eigenvectors or generalized eigenvectors associated with λ_j, then we may assume that $\overline{\mathbf{f}^k} = \mathbf{f}^m$ and $\overline{\phi^k} = \phi^m$ (for some m) are eigenvectors or generalized eigenvectors associated with $\overline{\lambda_j} = \lambda_m$. In that case we set

$(\mathbf{e}^k, \mathbf{p}_{\mathbf{e}^k}, \tau_{\mathbf{e}^k}) = \sqrt{2}\mathrm{Re}\,(\mathbf{f}^k, \mathbf{p}_{\mathbf{f}^k}, \tau_{\mathbf{f}^k})$,

$(\xi^k, \mathbf{p}_{\xi^k}, \tau_{\xi^k}) = \sqrt{2}\mathrm{Re}\,(\phi^k, \mathbf{p}_{\phi^k}, \tau_{\phi^k})$,

$(\mathbf{e}^m, \mathbf{p}_{\mathbf{e}^m}, \tau_{\mathbf{e}^m}) = \sqrt{2}\mathrm{Im}\,(\mathbf{f}^k, \mathbf{p}_{\mathbf{f}^k}, \tau_{\mathbf{f}^k})$, and

$(\xi^m, \mathbf{p}_{\xi^m}, \tau_{\xi^m}) = \sqrt{2}\mathrm{Im}\,(\phi^k, \mathbf{p}_{\phi^k}, \tau_{\phi^k})$.

If $\mathbf{E}$ contains all the vectors $(\mathbf{e}^k)_{1\leq k\leq N_z}$ and if $\mathbf{\Xi}$ contains the vectors $(\xi^k)_{1\leq k\leq N_z}$, the families may be ordered to have

a bi-orthogonality relationship

$$\mathbf{E}^T M_{zz} \mathbf{\Xi} = I_{\mathbb{R}^{N_z}} .$$

We also have

$$\Lambda = \mathbf{E}^{-1}\, \mathbf{A}\, \Pi^T\, \mathbf{E} \quad \text{and} \quad \Lambda^T = \mathbf{\Xi}^{-1}\, \mathbf{A}^\sharp\, \Pi^T\, \mathbf{\Xi},$$

and

$$\Lambda = \mathbf{\Xi}^T\, A_{zz}\, \Pi^T\, \mathbf{E} \quad \text{and} \quad \Lambda^T = \mathbf{E}^T\, \Pi\, A_{zz}^T\, \mathbf{\Xi}.$$

We also have similar identities with the pressures and the Lagrange multipliers

$$\begin{bmatrix} A_{zz} & A_{zp} & A_{z\tau} \\ A_{zp}^T & 0 & 0 \\ A_{z\tau}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{E}_p \\ \mathbf{E}_\tau \end{bmatrix} = \begin{bmatrix} M_{zz} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{E}_p \\ \mathbf{E}_\tau \end{bmatrix} \Lambda,$$

and

$$\begin{bmatrix} A_{zz}^T & A_{zp} & A_{z\tau} \\ A_{zp}^T & 0 & 0 \\ A_{z\tau}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Xi} \\ \boldsymbol{\Xi}_p \\ \boldsymbol{\Xi}_\tau \end{bmatrix} = \begin{bmatrix} M_{zz} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Xi} \\ \boldsymbol{\Xi}_p \\ \boldsymbol{\Xi}_\tau \end{bmatrix} \Lambda^T,$$

where the columns of $\mathbf{E}_p$ and $\mathbf{E}_\tau$ contains the vector coordinates of the pressure and the Lagrange multipliers associated with $\mathbf{E}$.

We shall use

$$A_{zz}^T \boldsymbol{\Xi} + A_{z\eta} \boldsymbol{\Xi} = M_{zz} \boldsymbol{\Xi} \Lambda.$$

With the real Jordan decomposition of $\mathbf{A}\Pi^T$ we have

$$\Lambda = \mathbf{E}^{-1}\mathbf{A}\Pi^T\mathbf{E} \quad \text{with} \quad \Lambda = \begin{pmatrix} \Lambda_r & \mathbf{0}_{\mathbb{R}^{N_z-d_\ell \times d_\ell}} \\ \mathbf{0}_{\mathbb{R}^{d_\ell \times N_z-d_\ell}} & \mathbf{0}_{\mathbb{R}^{d_\ell \times d_\ell}} \end{pmatrix},$$

where d_ℓ is the dimension of Ker$\mathbf{A}\Pi^T$ and $\Lambda_r \in \mathbb{R}^{N_z-d_\ell \times N_z-d_\ell}$ is invertible when $0 \notin \sigma(\mathbf{A})$.
The matrix $\mathbf{E} \in \mathbb{R}^{N_z \times N_z}$ may be decomposed in the form $\mathbf{E} = [\mathbf{E}_r\ \mathbf{E}_\ell]$ with $\mathbf{E}_r \in \mathbb{R}^{N_z \times N_z-d_\ell}$ and $\mathbf{E}_\ell \in \mathbb{R}^{N_z \times d_\ell}$, and the columns of $\mathbf{E}_\ell$ generate Ker$\mathbf{A}\Pi^T$.
Similarly

$$\Lambda^T = \mathbf{\Xi}^{-1}\mathbf{A}^\sharp\Pi^T\mathbf{\Xi} \quad \text{with} \quad \Lambda^T = \begin{pmatrix} \Lambda_r^T & \mathbf{0}_{\mathbb{R}^{N_z-d_\ell \times d_\ell}} \\ \mathbf{0}_{\mathbb{R}^{d_\ell \times N_z-d_\ell}} & \mathbf{0}_{\mathbb{R}^{d_\ell \times d_\ell}} \end{pmatrix},$$

where $\mathbf{A}^\sharp = \Pi^T M_{zz}^{-1} A_{zz}$.
We also have

$$\Lambda = \mathbf{\Xi}^T A_{zz}\Pi^T\mathbf{E} \quad \text{and} \quad \Lambda^T = \mathbf{E}^T\Pi A_{zz}^T\mathbf{\Xi}.$$

We introduce

$$\zeta = \mathbf{\Xi}^T M_{zz}\mathbf{z}.$$

If we multiply the first equation for $\mathbf{z}$ by $\mathbf{\Xi}^T$, we have

$$\mathbf{\Xi}^T M_{zz}\mathbf{z}'(t) = \mathbf{\Xi}^T A_{zz}\mathbf{z}(t) + \mathbf{\Xi}^T A_{z\eta}\boldsymbol{\eta}(t) = \mathbf{\Xi}^T A_{zz}\mathbf{z}(t).$$

Indeed $\mathbf{\Xi}^T A_{z\eta} = 0$, because $A_{z\eta}^T\mathbf{\Xi} = 0$. Moreover

$$\mathbf{\Xi}^T A_{zz} = \Lambda\,\mathbf{\Xi}^T M_{zz} - \mathbf{\Xi}_{\boldsymbol{\eta}}^T A_{z\eta}^T,$$

and

$$\mathbf{\Xi}^T M_{zz}\Pi^T = \mathbf{\Xi}^T \Pi\, M_{zz} = \mathbf{\Xi}^T M_{zz}.$$

Indeed $\Pi^T\mathbf{\Xi} = \mathbf{\Xi}$. Thus

$$\mathbf{\Xi}^T M_{zz}\,\mathbf{z}'(t) = \mathbf{\Xi}^T M_{zz}\,\Pi^T\mathbf{z}'(t) = \Lambda\,\mathbf{\Xi}^T M_{zz}\mathbf{z}(t) - \mathbf{\Xi}_{\boldsymbol{\eta}}^T A_{z\eta}^T\mathbf{z}(t)$$
$$= \Lambda\,\mathbf{\Xi}^T M_{zz}\,\Pi^T\mathbf{z}(t) - \mathbf{\Xi}_{\boldsymbol{\eta}}^T A_{z\eta}^T\mathbf{z}(t).$$

Since $\zeta = \mathbf{\Xi}^T M_{zz}\mathbf{z}$ and $A_{z\eta}^T\mathbf{z} = M_{\eta\eta}\mathbf{G}\,\mathbf{v}$, we have

$$\zeta'(t) = \Lambda\,\zeta(t) - \mathbf{\Xi}_{\boldsymbol{\eta}}^T M_{\eta\eta}\mathbf{G}\,\mathbf{v}(t) = \Lambda\,\zeta(t) + \mathbb{B}\,\mathbf{v}(t).$$

Choice of invariant subspaces. We have

$$\mathbb{R}^{N_z} = \mathrm{Ker} A_{z\eta}^T \oplus \mathrm{Ker} \mathbf{A}\Pi^T.$$

We look for a decomposition of $\mathrm{Ker} A_{z\eta}^T$ of the form

$$\mathrm{Ker} A_{z\eta}^T = \mathbb{Z}_u \oplus \mathbb{Z}_s \quad \text{with} \quad \mathbf{A}\mathbb{Z}_u \subset \mathbb{Z}_u \quad \text{and} \quad \mathbf{A}\mathbb{Z}_s \subset \mathbb{Z}_s,$$

and such that

$$\| e^{t\mathbf{A}|_{\mathbb{Z}_s}} \| \le C e^{-(\alpha+\delta)t} \quad \text{for some } \delta > 0.$$

To each decomposition corresponds a decomposition of $\zeta = \zeta_u + \zeta_s$, with $\zeta_u = \mathbf{\Xi}_u^T M_{zz} \mathbf{z}$, and

$$\zeta_u' = \Lambda_u \zeta_u + \mathbb{B}_u \mathbf{v}, \quad \zeta_u(0) = \zeta_{u,0},$$
$$\zeta_s' = \Lambda_s \zeta_s + \mathbb{B}_s \mathbf{v}, \quad \zeta_s(0) = \zeta_{s,0}.$$

Our choice for $\mathbb{Z}_u$ is based on the analysis of the *degree of stabilizability* of the projected systems onto the different generalized invariant eigenspaces.

Approximation procedure

Approximation of Z_u and π_u

$$Z_u = \oplus_{j=1}^{N_u} G_{\mathbb{R}}(\lambda_j) = \text{vect}\{e_1, \cdots, e_{d_u}\}, \quad Z_u^* = \oplus_{j=1}^{N_u} G_{\mathbb{R}}^*(\lambda_j) = \text{vect}\{\xi_1, \cdots, \xi_{d_u}$$

Let $G_{\mathbb{R}}(\boldsymbol{\lambda}_j)$ and $G_{\mathbb{R}}^*(\boldsymbol{\lambda}_j)$ be the real generalized eigenspaces associated with $\boldsymbol{\lambda}_j$, that is

$$G_{\mathbb{R}}(\boldsymbol{\lambda}_j) = \text{vect}\{\text{Re}\,\mathbf{f}^k, \text{Im}\,\mathbf{f}^k \mid \mathbf{f}^k \in G_{\mathbb{C}}(\boldsymbol{\lambda}_j)\} \quad \text{and} \quad G_{\mathbb{R}}^*(\boldsymbol{\lambda}_j) = \text{vect}\{\text{Re}\,\phi^k, \text{Im}\,\phi$$

We set

$$\mathbb{Z}_u = \oplus_{j=1}^{N_u} G_{\mathbb{R}}(\boldsymbol{\lambda}_j) = \text{vect}\{\mathbf{e}^1, \cdots, \mathbf{e}^{d_u}\},$$

$$\mathbb{Z}_u^* = \oplus_{j=1}^{N_u} G_{\mathbb{R}}^*(\boldsymbol{\lambda}_j) = \text{vect}\{\boldsymbol{\xi}^1, \cdots, \boldsymbol{\xi}^{d_u}\}$$

$$\mathbb{Z}_s = \text{vect}\{\mathbf{e}^{d_u+1}, \cdots, \mathbf{e}^{N_z-d_\ell}\},$$

$$\mathbb{Z}_s^* = \text{vect}\{\boldsymbol{\xi}^{d_u+1}, \cdots, \boldsymbol{\xi}^{N_z-d_\ell}\}.$$

$$\mathbb{K} = \text{vect}\{\mathbf{e}^{N_z-d_\ell+1}, \cdots, \mathbf{e}^{N_z}\} = \text{vect}\{\boldsymbol{\xi}^{N_z-d_\ell+1}, \cdots, \boldsymbol{\xi}^{N_z}\} = \text{Ker}(\mathbf{A}\,\Pi^T)$$

$$\mathbf{e}^i = (e_k^i)_{1\le k\le N_z}, \quad \text{with} \quad \mathbf{e}_i = \sum_{k=1}^{N_z} e_k^i\,\phi_k.$$

We have

$$\mathbb{R}^{N_z} = \mathbb{Z}_u \oplus \mathbb{Z}_s \oplus \mathbb{K} = \mathbb{Z}_u^* \oplus \mathbb{Z}_s^* \oplus \mathbb{K}.$$

We define the functions $\mathbf{e}_i \in X_h$ and $\boldsymbol{\xi}_i \in X_h$ by

$$\mathbf{e}_i = \sum_{k=1}^{N_z} e_k^i \, \phi_k \quad \text{and} \quad \boldsymbol{\xi}_i = \sum_{k=1}^{N_z} \xi_k^i \, \phi_k,$$

where

$$(e_k^i)_{1 \le k \le N_z} = \mathbf{e}^i \quad \text{and} \quad (\xi_k^i)_{1 \le k \le N_z} = \boldsymbol{\xi}^i.$$

Approximation of B_u. Recall that $B_u \in \mathcal{L}(\mathbb{R}^{N_c}, Z)$ is defined by

$$B_u v = \sum_{j=1}^{d_u} \sum_{i=1}^{N_c} v_i \left\langle g_i, \sigma(\xi_j, p_{\xi_j}) n \right\rangle_{\Gamma_c} e_j.$$

Thus, we define the approximation $\mathbf{B}_u \in \mathcal{L}(\mathbb{R}^{N_c}, X_h)$ of B_u by setting

$$\mathbf{B}_u \mathbf{v} = \sum_{j=1}^{d_u} \sum_{i=1}^{N_c} v_i \left\langle g_i, \tau_{\boldsymbol{\xi}_j} \right\rangle_{\Gamma_c} \mathbf{e}_j,$$

where $\tau_{\boldsymbol{\xi}_j} = \sum_{i=1}^{N_\tau} \tau_i^j \zeta_i$ is the Lagrange multiplier in the mixed formulation associated with the Dirichlet condition satisfied by $\boldsymbol{\xi}_j$.

Besides, $\left\langle g_i, \tau_{\boldsymbol{\xi}_j} \right\rangle_{\Gamma_c}$ is calculated with an integration formula on Γ_c.

More precisely, we set

$$\left\langle g_i, \tau_{\boldsymbol{\xi}_j} \right\rangle_{\Gamma_c} = (\tau^j)^T M_{\tau\tau} \mathbf{g}^i,$$

where $\mathbf{g}_i = \sum_{j=1}^{N_\tau} \mathbf{g}_j^i \zeta_j$ is the finite element approximation of g_i and $\mathbf{g}^i = (\mathbf{g}_1^i, \cdots, \mathbf{g}_{N_\tau}^i)^T$ is the coordinate vector of $\mathbf{g}_i$ in the basis $(\zeta_j)_{1 \le j \le N_\tau}$. Therefore, if we denote by $\mathbb{B}_u$ the matrix of $\mathcal{B}_u$ in the canonical basis of $\mathbb{R}^{N_c}$ and in the basis $(\mathbf{e}_j)_{1 \le j \le d_u}$ of X_h, we have

$$\mathbb{B}_u \mathbf{v} = \sum_{i=1}^{N_c} v_i \, \boldsymbol{\Xi}_{\tau,u}^T M_{\tau\tau} \mathbf{g}^i = \boldsymbol{\Xi}_{\tau,u}^T M_{\tau\tau} \mathbf{G} \, \mathbf{v} \quad \text{and} \quad \mathbb{B}_u = \boldsymbol{\Xi}_{\tau,u}^T M_{\tau\tau} \, \mathbf{G}.$$

Stabilization of the projected system. We can find a feedback law by solving the matrix Riccati equation

$$\mathbb{P}_{\omega,u} \in \mathbb{R}^{d_u \times d_u}, \quad P_{\omega,u} = \mathbb{P}^*_{\omega,u} > 0,$$
$$\mathbb{P}_{\omega,u}(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}) + (\Lambda_u^T + \omega I_{\mathbb{R}^{d_u}})\mathbb{P}_{\omega,u} - \mathbb{P}_{\omega,u}\mathbb{B}_u\mathbb{B}_u^*\mathbb{P}_{\omega,u} = 0.$$

Now, we set

$$\mathbb{K} = -\mathbb{B}_u^* \mathbb{P}_{\omega,u} = [K_{i,j}]_{1\leq i \leq N_c,\, 1\leq j\leq d_u},$$

and

$$\mathcal{K} z = \sum_{i=1}^{N_c} \sum_{j=1}^{d_u} K_{i,j} \left(z, \xi_j\right)_{L^2(\Omega;\mathbb{R}^2)} g_i.$$

When we apply the feedback control law $\mathcal{K}$ to the semi-discrete NSE, we obtain the following nonlinear closed loop system:

$$\frac{\partial z}{\partial t} + (w_s \cdot \nabla) z + (z \cdot \nabla) w_s + (z \cdot \nabla) z - \operatorname{div} \sigma(z,p) = 0, \quad \operatorname{div} z = 0 \quad \text{in} \quad Q$$

$$z = \mathcal{K} z \quad \text{on} \quad \Sigma_d^\infty, \quad \sigma(z,p)n = 0 \quad \text{on} \quad \Sigma_n^\infty,$$

$$\Pi z(0) = z_0 \quad \text{on} \quad \Omega.$$

We determine a feedback control law $\mathbb{K}$ able to stabilize the semi-discrete NSE with an exponential decay rate strictly smaller than $-\alpha$. From our analysis, it follows that the following linear closed loop system

$$M_{zz}\mathbf{z}'(t) = A_{zz}\mathbf{z}(t) + A_{z\eta}\boldsymbol{\eta}(t), \quad \Pi^T\mathbf{z}(0) = \mathbf{z}_0,$$
$$A_{z\eta}^T\mathbf{z}(t) = M_{\eta\eta}\,\mathbf{G}\,\mathbb{K}\zeta_u(t),$$
$$\zeta_u'(t) = \Lambda_u\,\zeta_u(t) + \mathbb{B}_u\,\mathbb{K}\zeta_u(t), \quad \zeta_u(0) = \zeta_{u,0},$$