CIMPA Summer School on Current Research on Finite Element Method

Lecture 5

IIT Bombay

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Stabilization of the Navier-Stokes equations

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Lyapunov equations.

If $-A \in \mathbb{R}^{n\times n}$ is stable ($\sigma(-A) < 0$), then the equation

$$W \in \mathbb{R}^{n\times n}, \quad W = W^* \geq 0,$$
$$W(-A^*) + (-A)W = -BB^*,$$

admits the unique solution

$$W = \int_0^\infty e^{-tA} BB^* e^{-tA^*} \, dt.$$

Proof.

$$W(-A^*) + (-A)W = \int_0^\infty [(-A)e^{-tA}BB^*e^{-tA^*} + e^{-tA}BB^*e^{-tA^*}(-A^*)]dt$$

$$= \int_0^\infty \frac{d}{dt}[e^{-tA}BB^*e^{-tA^*}]dt = \left[e^{-tA}BB^*e^{-tA^*}\right]_{t=0}^{t=\infty} = -BB^*$$

The underlying Riccati equation.

If $-A \in \mathbb{R}^{n\times n}$ is stable ($\sigma(-A) < 0$), if (A, B) is stabilizable, and if W is the solution to the Lyapunov equation

$$W \in \mathbb{R}^{n\times n}, \quad W = W^* \geq 0,$$
$$W(-A^*) + (-A)W = -BB^*,$$

then W is invertible and $P = W^{-1}$ is the solution to the Riccati equation

$$W \in \mathbb{R}^{n\times n}, \quad W = W^* \geq 0,$$
$$A^*P + PA - PBB^*P = 0.$$

Proof. For the invertibility of W when (A, B) is stabilizable see Lecture 1. By multiplying the Lyapunov equation on the left and on the right by W^{-1}, then

$$(-A^*)W^{-1} + W^{-1}(-A) = -W^{-1}BB^*W^{-1}.$$

Setting $P = W^{-1}$, we obtain the desired equation.

Slides form Lecture 1.

Characterization of the stabilizability of F.D.S.

When $0 \notin \sigma(A)$, the following conditions are equivalent

(a) (A, B) is stabilizable,

(b) $(A_u, B_u) = (\pi_u A, \pi_u B)$ is stabilizable,

(c) The Gramian

$$W^{\infty}_{-A_u, B_u} = \int_0^{\infty} e^{-tA_u} B_u B_u^* e^{-tA_u^*} \, dt$$

is invertible.

(d) There exists $\alpha > 0$ such that for all $\phi \in Z_u^*$,

$$(O.I.) \qquad (W^{\infty}_{-A_u, B_u} \phi, \phi)_Z = \int_0^{\infty} \left\| B_u^* e^{-tA_u^*} \phi \right\|_U^2 \, dt \ \geq \ \alpha \|\phi\|_Z^2.$$

Setting

$$P_u = (W^{\infty}_{-A_u, B_u})^{-1} \in \mathcal{L}(Z_u, Z_u^*), \quad P_u = P_u^* \geq 0,$$

we prove that $\qquad A_u - B_u B_u^* P_u \quad$ is exponentially stable.

Recall that

$$B_u = \pi_u B \quad \text{and} \quad B_u^* = B^* \pi_u^*,$$

where π_u^* is the projection onto Z_u^* along Z_s^*.

The operator P_u satisfies the following Algebraic Bernoulli equation (a homogeneous Algebraic Riccati equation)

$$P_u A_u + A_u^* P_u - P_u B_u B_u^* P_u = 0.$$

If we set

$$P = \pi_u^* P_u \pi_u.$$

Then $P \in \mathcal{L}(Z)$ is such that $P = P^* \geq 0$ and solves the following $(A.B.E.)$

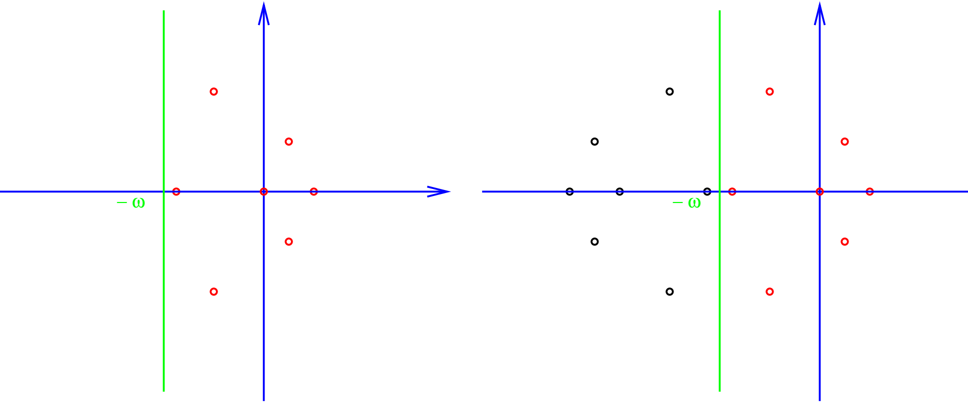
$$A.B.E. \qquad \begin{aligned} & P \in \mathcal{L}(Z), \quad P = P^* \geq 0, \\ & A^* P + PA - PBB^* P = 0, \\ & A - BB^* P \text{ generates} \\ & \text{an exponentially stable semigroup.} \end{aligned}$$

Moreover, the feedback $-B^*P$ provides the control of minimal norm in $L^2(0,\infty;U)$
$$u(t) = -B^*Pe^{t(A-BB^*P)}z_0,$$
and the spectrum of $A - BB^*P$ is

$$\sigma(A - BB^*P) = \{-\mathrm{Re}\lambda + i\,\mathrm{Im}\lambda \mid \lambda \in \sigma(A),\ \mathrm{Re}\lambda > 0\}$$
$$\cup\{\lambda \mid \lambda \in \sigma(A),\ \mathrm{Re}\lambda < 0\}.$$

To obtain a better exponential decay we can replace A by $A + \omega I$ and determine the corresponding feedback $-B^*P_\omega$. In that case the spectrum of $A - BB^*P_\omega$ is as follows

Spectrum of A and of $A - BB^* P_\omega$



Another way for finding a feedback operator

We consider the optimal control problem $(\mathcal{P})$

$$\text{Minimize} \;\; J(z,u) = \frac{1}{2}\int_0^\infty \left\| Cz(t) \right\|_Y^2 dt + \frac{1}{2}\int_0^\infty \big(Ru(t), u(t)\big)_U \, dt$$

$$z' = Az + Bu, \qquad z(0) = z_0,$$

where $C \in \mathcal{L}(Z,Y)$, $\quad R = R^* > 0$, $\quad R \in \mathcal{L}(U)$.

The solution to $(\mathcal{P})$ is obtained by

$$z' = Az + BKz, \quad z(0) = z_0, \quad K = -B^*P,$$

$$P \in \mathcal{L}(Z), \;\; P = P^* \geq 0, \quad A^*P + PA - PBR^{-1}B^*P + CC^* = 0.$$

The optimal cost is

$$J(z_{u_{opt}}, u_{opt}) = \frac{1}{2}\big(Pz_0, z_0\big)_Z.$$

The closed loop system is

$$\frac{\partial z}{\partial t} + (w_s \cdot \nabla) z + (z \cdot \nabla) w_s + (z \cdot \nabla) z - \operatorname{div} \sigma(z, p) = 0, \quad \operatorname{div} z = 0,$$
$$z = \mathcal{K} z \quad \text{on} \quad \Sigma_d^\infty, \quad \sigma(z, p) n = 0 \quad \text{on} \quad \Sigma_n^\infty,$$
$$\Pi z(0) = z_0 \quad \text{on} \quad \Omega.$$

Let ε belong to $(0, 1]$. $\exists C_0 > 0$, $\exists \mu_0$, such that if $z_0 \in V^\varepsilon_{n,\Gamma_d}(\Omega)$, $\mu \in (0, \mu_0)$, and $\|z_0\|_{H^\varepsilon(\Omega;\mathbb{R}^2)} \le C_0 \mu$, then, the closed loop NSE admits a unique solution in the space

$$\Big\{ z \in L^2(0, \infty; \mathcal{D}((\lambda_0 I - A)^{1/2+\varepsilon/2})) \cap H^{1/2+\varepsilon/2}(0, \infty; V^0_{n,\Gamma_d}(\Omega)) + H^1(0, \infty; H^2(\Omega; \mathbb{R}^2)), \quad \|e^{\omega \cdot} z\|_{H^\varepsilon(\Omega;\mathbb{R}^2)} \le \mu \Big\}.$$

In particular the solution obeys

$$\|z(t)\|_{H^\varepsilon(\Omega;\mathbb{R}^2)} \le \mu e^{-\omega t}.$$

The closed loop nonlinear system

$$\Pi^T \mathbf{z}' = \mathbf{A}\,\Pi^T \mathbf{z} + \mathbf{B}\,\mathbb{K}\boldsymbol{\Xi}_u^T M_{zz}\mathbf{z} + F(\mathbf{z}),$$
$$\Pi^T \mathbf{z}(0) = \Pi^T \mathbf{z}_0,$$
$$(I - \Pi^T)\mathbf{z} = (I - \Pi^T)\mathbf{L}\mathbf{G}\,\mathbb{K}\boldsymbol{\Xi}_u^T M_{zz}\mathbf{z},$$

or equivalently

$$\begin{bmatrix} M_{zz} & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{z}(t) \\ \eta(t) \end{bmatrix}$$
$$= \begin{bmatrix} A_{zz} & A_{z\eta} \\ A_{z\eta}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ M_{\eta\eta} \end{bmatrix} \mathbf{G}\,\mathbb{K}\boldsymbol{\Xi}_u^T M_{zz}\mathbf{z} + F(\mathbf{z}),$$
$$\Pi^T \mathbf{z}(0) = \Pi^T \mathbf{z}_0,$$

is locally stable.

Theorem

The closed loop infinite dimensional nonlinear system

$$\frac{\partial z}{\partial t} + (w_s \cdot \nabla) z + (z \cdot \nabla) w_s + (z \cdot \nabla) z - \operatorname{div} \sigma(z,p) = 0, \quad \operatorname{div} z = 0,$$

$$z = G\,\mathbb{K}\zeta_u \quad \text{on} \quad \Sigma_d^\infty, \quad \sigma(z,p)n = 0 \quad \text{on} \quad \Sigma_n^\infty,$$

$$\Pi z(0) = z_0 \quad \text{on} \quad \Omega,$$

$$\zeta_u' = \Lambda_u \zeta_u + \mathbb{B}_u \mathbb{K}\zeta_u,$$

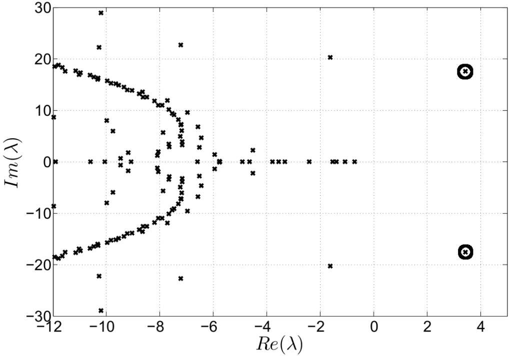
$$\zeta_u(0) = \zeta_{u,0} = \boldsymbol{\Xi}_u^T((z_0,\phi_1),\cdots,(z_0,\phi_{N_z}))^T.$$

where G stands for the vector function $(g_1,\cdots,g_{N_c})$,

is also locally stable.

Numerical results.

Spectrum of A – Re=150.



The first ten eigenvalues.

$\lambda_{1,2}$	λ_3	λ_4	λ_5	λ_6
$3.45 \pm 17.59i$	-0.71	-1.07	-1.38	$-1.6 \pm 20.22i$

$\lambda_{7,8}$	λ_9	λ_{10}
$-1.61 \pm 20.22i$	-2.42	-3.32

Table: Eigenvalues for $Re = 150$

Best control zone for a given Z_u.

Disturbance in the initial condition The actuator, that is the control operator B depends on the 'suction-blowing zone'. Thus the control zone depends on θ. For a given θ, we denote by B_u^θ the control operator.
For a given initial condition z_0, the control $v_{z_0}^\theta$ of minimal norm stabilizing the system

$$z_u' = A_{\omega,u} z_u + B_u^\theta v_\theta, \quad z_u(0) = z_{0,u} = \pi_u\, z_0,$$

is obtained by solving

$$z_u' = A_{\omega,u} z_u - B_u^\theta B_u^{\theta\,*} P_{\omega,u}^\theta z_u, \quad z_u(0) = \pi_u\, z_0,$$

and by setting

$$v_{z_0}^\theta(t) = -B_u^{\theta\,*} P_{\omega,u}^\theta z_u(t),$$

where P^θ is the solution to the Riccati equation

$$P_{\omega,u}^\theta \in \mathcal{L}(Z_u, Z_u^*), \quad P_{\omega,u}^\theta = (P_{\omega,u}^\theta)^* > 0,$$
$$P_{\omega,u}^\theta(A_u + \omega I) + (A_u^* + \omega I)P_{\omega,u}^\theta - P_{\omega,u}^\theta\, B_u^\theta (B_u^\theta)^* P_{\omega,u}^\theta \;=\; 0.$$

Actually we solve

$$\mathbb{P}^{\theta}_{\omega,u} \in \mathbb{R}^{d_u \times d_u}, \quad \mathbb{P}^{\theta}_{\omega,u} = (\mathbb{P}^{\theta}_{\omega,u})^* > 0,$$

$$\mathbb{P}^{\theta}_{\omega,u}(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}) + (\Lambda_u^T + \omega I_{\mathbb{R}^{d_u}})\mathbb{P}^{\theta}_{\omega,u} - \mathbb{P}^{\theta}_{\omega,u}\,\mathbb{B}^{\theta}_u(\mathbb{B}^{\theta}_u)^*\mathbb{P}^{\theta}_{\omega,u} \;=\; 0.$$

Choose the zone minimizing the control norm in the case of the worst normalized perturbation in the initial condition and stabilizing the system for a given exponential decay rate.

For a given initial condition z_0, the norm of the stabilizing control of minimal norm is

$$\|v^\theta_{z_0}\|^2_{L^2(0,\infty;\mathbb{R}^2)} = (P^\theta_{\omega,u} z_0, z_0).$$

The worst case is

$$\max_{\|z_0\|_{L^2}=1} \|v^\theta_{z_0}\|^2_{L^2(0,\infty;\mathbb{R}^2)} = (P^\theta_{\omega,u} z_0, z_0) = \lambda_{\max}(P^\theta_{\omega,u}).$$

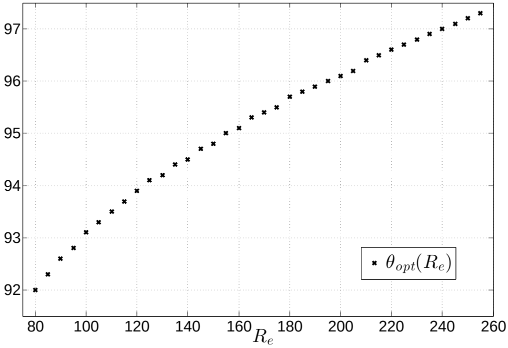
If we look for the optimal control zone characterized by θ_{opt} on the cylinder – i.e. the control zone minimizing the norm of the control in the case of the worst perturbation – we have to solve the following Min-Max problem

$$\theta_{\text{opt}} = \arg\min_\theta \max \sigma\left(P^\theta_{\omega,u}\right).$$

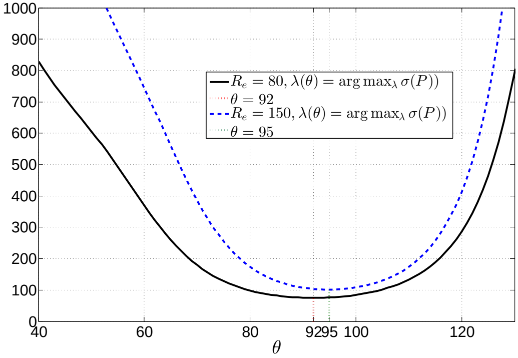
The approximate Min-Max problem is

$$\theta_{\text{opt}} = \arg\min_\theta \max \sigma\left(\mathbb{P}^\theta_{\omega,u}\right).$$

The best control zone in terms of *Re*.



The smallest eigenvalue of $\mathbb{P}_{\omega,u}$.



Disturbance in the inflow boundary condition

Here, we consider the equation

$$\frac{\partial z}{\partial t} + (w_s \cdot \nabla) z + (z \cdot \nabla) w_s + (z \cdot \nabla) z - \operatorname{div} \sigma(z, p) = 0, \quad \operatorname{div} z = 0 \quad \text{in} \quad Q$$

$$z = \sum_{i=1}^{N_c} v_i \, g_i + \mu \, h \quad \text{on} \quad \Sigma_d^\infty = \Gamma_d \times (0, \infty),$$

$$\sigma(z, p) n = 0 \quad \text{on} \quad \Sigma_n^\infty,$$

$$\Pi z(0) = 0 \quad \text{on} \quad \Omega.$$

The above equation is equivalent to the following system

$$\Pi z' = A \Pi z + \sum_{i=1}^{N_c} v_i \mathcal{B} g_i + \mu \, h = A \Pi z + B v + \mu \, \mathcal{B} h, \qquad \Pi z(0) = 0,$$

$$(I - \Pi) z = (I - \Pi) \left(\sum_{i=1}^{N_c} v_i \, L g_i + \mu \, L h \right) = (I - \Pi) \left(L g \cdot v + \mu \, L h \right).$$

We consider the system

$$z_u' = A_{\omega,u} z_u - B_u^\theta B_u^{\theta\,*} P_{\omega,u}^\theta z_u + \mu\, \pi_u \mathcal{B} h, \quad z_u(0) = 0.$$

We assume that μ is a regularization of a Dirac distribution δ_{t_0} with $t_0 > 0$. If $\mu = \delta_{t_0}$, the above equation reduces to

$$z_u' = A_{\omega,u} z_u - B_u^\theta B_u^{\theta\,*} P_{\omega,u}^\theta z_u, \quad z_u(t_0) = \pi_u \mathcal{B} h.$$

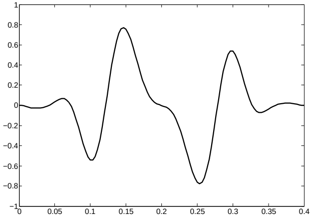
The norm of the control corresponding to the worst normalized boundary condition h is

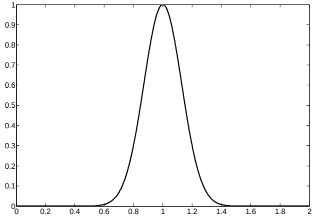
$$\max_{\|h\|_{L^2(\Gamma_i)}=1} \left(P_{\omega,u}^\theta \pi_u \mathcal{B} h, \pi_u \mathcal{B} h\right)_{L^2(\Omega)} = \max \sigma\left(\mathcal{B}^* \pi_u^* P_{\omega,u}^\theta \pi_u \mathcal{B}\right).$$

The boundary perturbation in the inflow boundary $\Gamma_i \times (0, \infty)$

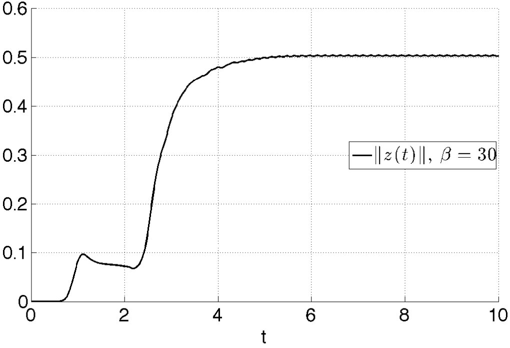
$$f(t, x_2, \beta) = \beta\, e^{-30(t-1)^2} \left(\sigma(\boldsymbol{\xi}_1, \mathbf{p}_{\boldsymbol{\xi}^1})\mathbf{n}.\mathbf{n}, 0\right)^T,$$

with $\beta > 0$.





The norm of the uncontrolled solution for $\beta = 30$



Feedback gain

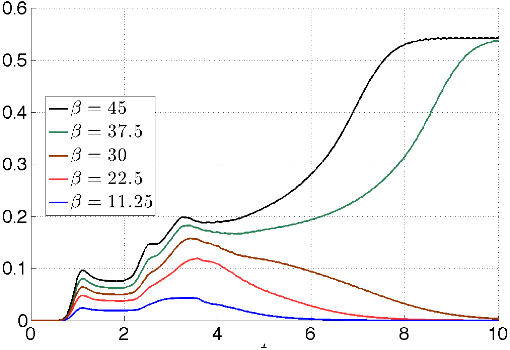
Reduced model

$$\zeta_u' = \boldsymbol{\Lambda}_u \zeta_u + \mathbb{B}_u \mathbf{v}, \quad \zeta_u(0) = \zeta_{u,0}.$$

The closed loop system is

$$\begin{aligned} M_{zz}\mathbf{z}' &= A_{zz}\mathbf{z} + A_{z\eta}\boldsymbol{\eta}, \quad \Pi^T \mathbf{z}(0) = \mathbf{z}_0, \\ A_{z\eta}^T \mathbf{z} &= M_{\eta\eta}\, \mathbf{G}\, \mathbb{K}_u \zeta_u, \\ \zeta_u' &= \Lambda_u\, \zeta_u + \mathbb{B}_u\, \mathbb{K}_u \zeta_u, \quad \zeta_u(0) = \zeta_{u,0}. \end{aligned}$$

Efficiency of the control law for dim$Z_u = 2$ and $\omega = 0$



Choice of $\mathbb{Z}_u$

Degree of stabilizability – Re=150

λ_i	$\lambda_{1,2}$	λ_3	λ_4	λ_5	λ_6
$10^4 d_i$	4.21	$4.24\,10^{-2}$	$1.52\,10^{-6}$	$8.06\,10^{-3}$	$3.06\,10^{-3}$

λ_i	$\lambda_{7,8}$	λ_9	λ_{10}
$10^4 d_i$	1.30	$1.82\,10^{-2}$	$8.17\,10^{-4}$

Table: Stabilizability of different eigenspaces for $Re = 150$, $d_i = 1/\max\sigma(\mathbb{P}_{\omega,i})$.

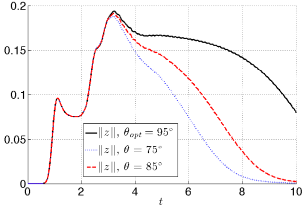
$\mathbb{P}_{\omega,i}$ is the solution to

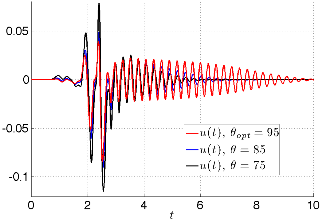
$$\mathbb{P}_{\omega,i} \in \mathbb{R} \quad \text{or} \quad \mathbb{P}_{\omega,i} \in \mathbb{R}^{2\times 2}, \quad \mathbb{P}_{\omega,i} = \mathbb{P}^*_{\omega,i} > 0,$$

$$\mathbb{P}_{\omega,i}(\Lambda_i + \omega I) + (\Lambda_i^T + \omega I)\mathbb{P}_{\omega,i} - \mathbb{P}_{\omega,i}\,\mathbb{B}_i\mathbb{B}_i^*\mathbb{P}_{\omega,i} \;=\; 0.$$

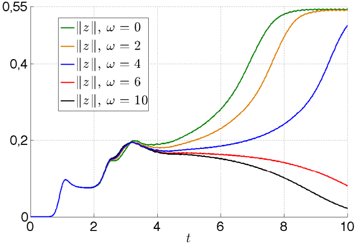
Influence of the control zone

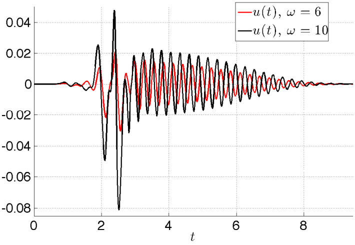
$$f(t, x_2, \beta) = \beta\, e^{-30(t-1)^2} \left(\sigma(\xi_1, \mathbf{p}_{\xi^1})\mathbf{n}.\mathbf{n}, 0\right)^T, \quad \beta = 45.$$





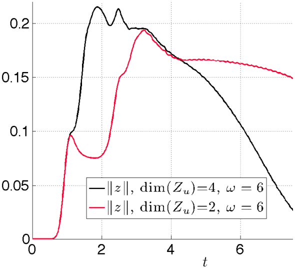
Influence of ω on the feedback gains

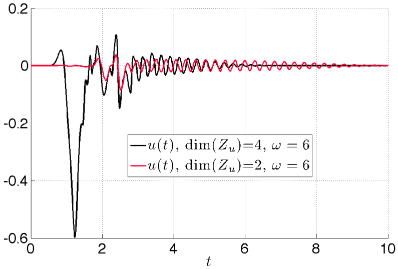




Influence of Z_u

We compare $\mathbb{Z}_u = G(\boldsymbol{\lambda}_1)$ with $\mathbb{Z}_u = G(\boldsymbol{\lambda}_1) \oplus G(\boldsymbol{\lambda}_7)$.





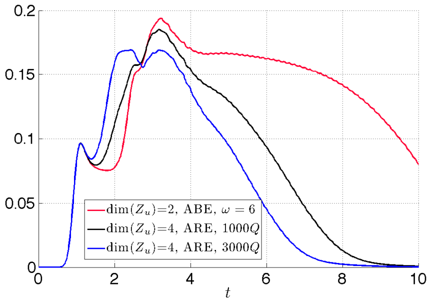
Influence of Q in the Riccati equation

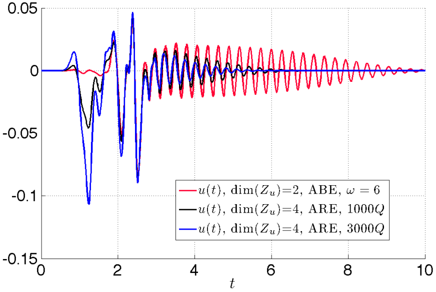
We can improve the efficiency of the feedback control law by choosing

$$\mathbb{P}_{\omega,u} \in \mathcal{L}(\mathbb{R}^{d_u}), \quad \mathbb{P}_{\omega,u} = \mathbb{P}^*_{\omega,u} \geq 0,$$
$$\mathbb{P}_{\omega,u}(\Lambda_u + \omega\Delta) + (\Lambda_u^T + \omega\Delta)\mathbb{P}_{\omega,u} - \mathbb{P}_{\omega,u}\,\mathbb{B}_u\mathbb{B}_u^*\mathbb{P}_{\omega,u} + r\,Q \;=\; 0,$$
$$\mathbb{P}_{\omega,u} \quad \text{is invertible and } \Lambda_u + \omega I_{\mathbb{R}^{d_u}} - \mathbb{B}_u\mathbb{B}_u^*\mathbb{P}_{\omega,u} \quad \text{is stable},$$

where

$$\Delta = \begin{pmatrix} I_{\mathbb{R}^2} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathbb{R}^2} \end{pmatrix}.$$


0.2
0.15
0.1
0.05
0
0
2
4
6
8
10
t
$\dim(Z_u)$=2, ABE, $\omega = 6$
$\dim(Z_u)$=4, ARE, $1000Q$
$\dim(Z_u)$=4, ARE, $3000Q$


0.05
0
-0.05
-0.1
-0.15
0
2
4
6
8
10
t
$u(t)$, $\dim(Z_u)$=2, ABE, $\omega = 6$
$u(t)$, $\dim(Z_u)$=4, ARE, $1000Q$
$u(t)$, $\dim(Z_u)$=4, ARE, $3000Q$

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