Optimal Control of Heat Equation

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contents

Optimal control problems

In many practical problems, a differential equation describes the evolution of the state of the system.
Sometimes the dynamics involves some control parameters also.

Our aim is to find the best control which maximizes a certain payoff criterion or cost functional.

Major Questions :

- Does an optimal control exist?
- How can we characterize such an optimal control?
- How to build an optimal control?

Optimal control problems

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Sometimes the dynamics involves some control parameters also.

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Moon Landing Problem

Question:
How to bring the spacecraft to land softly on the lunar surface using minimum amount of fuel?

The system can be modelled by

$$\dot{v}(t) = -g + \frac{\alpha(t)}{m(t)}$$

$$\dot{h}(t) = v(t)$$

$$\dot{m}(t) = -k\alpha(t)$$

Here $h(t)$ is the height ; $v(t)$ is the velocity
$m(t)$ is the mass of the spacecraft ; $\alpha(t)$ is the thrust

Moon Landing Problem

Problem is to minimize the fuel or to maximize the remaining amount when it lands.

Minimizing fuel is equivalent to maximising the mass when it lands.

Define

$$J(\alpha(.)) = m(T),$$

where T is the first time when $v(T) = 0$, that is it lands on the surface.

Then the problem is to maximize $J(\alpha(.))$.

Other constraints are

$$h(t) \geq 0, \quad m(t) \geq 0, \quad 0 \leq \alpha(t) \leq 1$$

Mathematical Framework

Control problem:

Solve the minimization problem

$$\text{Min} \int_0^T L(t, x(t), u(t))dt \; + \; g(x(T))$$

subject to

$$\frac{dx(t)}{dt} \; = \; f(t, x(t), u(t)),$$

$$\beta(x(0), x(T)) \; = \; 0 \; = \; \gamma(x(0), x(T)) \text{ in } \mathbb{R}^d$$

Control $u(t)$ is in $\Omega \; \subset \; \mathbb{R}^m$,

J is cost functional ; L is running cost; g is the terminal cost.

Augmented Functional

Question:
Can we use Lagrange Multiplier idea in optimal control problems too?
Assuming $g = 0$, define augmented functional, with multipliers $\lambda_0 \geq 0, \mu, \lambda$

$$\tilde{J}(x, \dot{x}, u, p, \lambda_0, \mu, \lambda) = \lambda_0 \; J(x, u)$$

$$+ \int_0^T p^T \; (f(x,u) - \dot{x}) \; + \; \mu^T \; \beta(x(0), x(T)) \; + \; \lambda^T \gamma(x(0), x(T))$$

At a minimum, vanishing of the derivative in each of the variable gives the optimality conditions.
Then the integrand is :

$$\tilde{L} \; = \; \lambda_0 \; L(x, u) + p^T \; (f(x,u) - \dot{x})$$

Introduce the Control Hamiltonian

$$H(x, p) \; = \; \lambda_0 \; L(x, u) + p^T \; (f(x, u)$$

Optimality Conditions

Formally, calculus of variations applied to $\tilde{J}$ to characterize the extremal gives :

$$\frac{d}{dt}\tilde{L}_{\dot{x}} \; - \; \tilde{L}_x = \; \dot{p} \; + \; H_x \; = \; 0$$

This gives the equation for the adjoint vector p :

$$\dot{p} \; = \; -p(t)^T \; Df(t, x^*(t), u^*(t)) \; - \; \nabla_x L(t, x^*(t), u^*(t))$$

with Transversality conditions, obtained by variations in the parameters

$$\tilde{L}_{\dot{x}}|_{t=0} \; = \; \nabla_1(\mu^T \beta \; + \; \lambda^T \gamma)$$

$$\tilde{L}_{\dot{x}}|_{t=T} \; = \; \nabla_2(\mu^T \beta \; + \; \lambda^T \gamma)$$

Partial gradient with respect to the first group of variables, $x(0)$ is ∇_1 ; for $x(T)$ is ∇_2.

Pontryagin Principle

Theorem

Under the usual smoothness assumptions, let $(\ x^*(.), u^*(.)\ )$ *be the optimal process.*
Then there exist $\lambda_0 \geq 0,\ \lambda,\ \mu\ \in\ \mathbb{R}^d$ *and a vector function* $p(.)$ *satisfying the adjoint equation together with boundary conditions given by transversality relations, such that for a.e.* $t \in [0, T]$,

$$\min_u\ H(t, x^*(t), p^*(t), u)\ =\ H(t, x^*(t), p^*(t), u^*(t))$$

Optimal control for elliptic equation : an example

- Ω is a domain with boundary Γ.
- $y =$ Electrical potential and $u =$ Current density,
  Or
- $y =$ Temperature distribution and $u =$ Thermal flux.

y satisfies

$$-\Delta y + y = f \quad \text{in} \quad \Omega, \quad \frac{\partial y}{\partial n} = u \quad \text{on} \quad \Gamma.$$

Problem: Minimize the distance between y and a given distribution y_d

$$\int_\Omega |y - y_d|^2.$$

The consumed energy is

$$\int_\Gamma |u|^2.$$

Control Problem:

$$\text{Minimize} \quad J(y,u) = \tfrac{1}{2}\int_\Omega |y - y_d|^2 + \tfrac{\beta}{2}\int_\Gamma |u|^2,$$

$$-\Delta y + y = f \quad \text{in} \quad \Omega, \quad \frac{\partial y}{\partial n} = u \quad \text{on} \quad \Gamma,$$

$$u \in L^2(\Gamma), \quad f \in L^2(\Omega),$$

$$y_d \in L^2(\Omega), \quad \beta > 0.$$

Optimal control for parabolic equation : an example

Cooling process in metallurgy:

- Ω is a domain with boundary Γ.
- T = Temperature in Ω, $c(T)$ = Specific heat capacity, $\rho(T)$ = Density,
- $K(T)$ = The conductivity of steel at the temperature T,
- R = a nonlinear function related to radiation law.

The problem is described by nonlinear heat equation of the form:

$$\rho(T)c(T)\frac{\partial T}{\partial t} = \mathrm{div}(K(T)\nabla T) \quad \text{in} \quad \Omega \times (0, t_f),$$

$$K(T)\frac{\partial T}{\partial n} = R(T, u) \quad \text{on} \quad \Gamma \times (0, t_f)$$

Here u is the control variable.

Control problem:

$$\text{Minimize} \quad J(y,u) = \beta_1 \int_\Omega |T(t_f) - \overline{T}|^2 + \beta_2 \int_0^{t_f} |u|^q,$$

$$\rho(T)c(T)\frac{\partial T}{\partial t} = \operatorname{div}(K(T)\nabla T) \quad \text{in} \quad \Omega \times (0, t_f),$$

$$K(T)\frac{\partial T}{\partial n} = R(T,u) \quad \text{on} \quad \Gamma \times (0, t_f),$$

$$\beta_1 > 0, \quad \beta_2 > 0,$$

Here t_f is the terminal time of the process ;
$\overline{T}$ is a desired profile of temperature;
exponent q is chosen in function of the radiation law R.

Sobolev spaces involving time

Search for solutions $u(x,t)$ in Sobolev spaces involving time.

Let Y be a Banach space.

Definition

The space

$$L^p(0,T;Y)$$

consists of all strongly measurable functions $u : [0,T] \to Y$ with

$i)$ $\|u\|_{L^p(0,T;Y)} := \left(\int_0^T \|u(t)\|_Y^p \, dt\right)^{\frac{1}{p}} < \infty$, for $1 \leq p < \infty$.

$ii)$ $\|u\|_{L^\infty(0,T;Y)} := \text{ess sup}_{t\in[0,T]} \|u(t)\|_Y < \infty$.

Definition

The space

$$C([0,T];Y)$$

consists of all continuous functions $u : [0,T] \to Y$ with

$$\|u\|_{C([0,T];Y)} := \max_{t \in [0,T]} \|u(t)\|_Y < \infty.$$

Differential equation in Banach space

Consider the equation

$$y' = Ay + f, \quad y(0) = y_0,$$

with $f \in C(\mathbb{R}; Y)$ and $y_0 \in Y$, Y is a Banach space.

If $A \in \mathcal{L}(Y)$, then this equation admits a unique solution in $C^1(\mathbb{R}; Y)$ given by

$$y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} f(s)\, ds,$$

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \quad t \in \mathbb{R}.$$

When $A \in \mathcal{L}(Y)$, the family $(e^{tA})_{t\in\mathbb{R}}$ satisfies:

$i)$ $e^{0A} = I$,

$ii)$ $e^{tA} \in \mathcal{L}(Y), \quad \forall\, t \in \mathbb{R}$,

$iii)$ $e^{(s+t)A} = e^{sA} \circ e^{tA}, \quad \forall\, s \in \mathbb{R}, \forall\, t \in \mathbb{R}$,

$iv)$ $\lim_{t\to 0} \|e^{tA} - I\|_{\mathcal{L}(Y)} = 0$,

$v)$ $Ay = \lim_{t\to 0} \frac{(e^{tA}y - y)}{t}, \quad \forall\, y \in Y.$

Qn : What about unbounded operator A ?

One Dimensional Heat equation

The heat equation in $(0, L) \times (0, T)$

$$y \in L^2(0, T; H_0^1(0, L)) \cap C([0, T]; L^2(0, L)),$$
$$y_t - y_{xx} = 0 \quad \text{in} \quad (0, L) \times (0, T),$$
$$y(0, t) = y(L, t) = 0 \quad \text{in} \quad (0, T),$$
$$y(x, 0) = y_0(x) \quad \text{in} \quad (0, L),$$

where $T > 0$, $L > 0$ and $y_0 \in L^2(0, L)$.

Here $A = \frac{d^2}{dx^2}$ is defined on a suitable subspace, $D(A) = H^2 \cap H_0^1$ of $L^2(0, L))$ into $L^2(0, L))$ but is unbounded .

To find an expression for e^{tA}, introduce

$$\phi_k(x) = \sqrt{\frac{2}{L}} sin\left(\frac{k\pi x}{L}\right), \quad \forall\, k \in \mathbb{N}, \quad \forall\, x \in (0, L).$$

The family $(\phi_k)_{k\in\mathbb{N}}$ is an orthonormal basis of the Hilbert space $L^2(0, L)$.
ϕ_k is an eigenfunction of the operator $(A, D(A))$:

$$\phi_k \in D(A), \quad A\phi_k = \lambda_k \phi_k, \quad \lambda_k = -\frac{k^2\pi^2}{L^2}.$$

Search for a solution y in the form

$$y(x,t) = \sum_{k=1}^{\infty} g_k(t)\phi_k(x).$$

If

$$y_0(x) = y(x,0) = \sum_{k=1}^{\infty} g_k(0)\phi_k(x),$$

and if the PDE is satisfied in the sense of distributions in $(0,L)\times(0,T)$, then g_k obeys

$$g_k' + \frac{k^2\pi^2}{L^2} g_k = 0 \quad \text{in} \quad (0,T),$$

$$g_k(0) = y_{0k} = (y_0,\phi_k)_{L^2(0,L)}.$$

We have

$$g_k(t) = y_{0k} e^{-\frac{k^2\pi^2 t}{L^2}}, \quad \forall\, t \in (0,T).$$

The function $y \in L^2(0,T;H_0^1(0,L)) \cap C([0,T];L^2(0,L))$,

$$y(x,t) = \sum_{k=1}^{\infty} y_{0k} e^{-\frac{k^2\pi^2 t}{L^2}} \phi_k(x), \quad \forall\, x \in (0,L), \quad t \in (0,T),$$

is the solution of the heat equation.

Remark : y is not defined for $t < 0$.
Let us define

$$S(t)y_0 = \sum_{k=1}^{\infty} (y_0, \phi_k)_{L^2(0,L)} e^{-\frac{k^2\pi^2 t}{L^2}} \phi_k(x), \quad \forall\, x \in (0, L), \quad t \in (0, T),$$

Then we have

$i)$ $S(0) = I$,

$ii)$ $S(t) \in \mathcal{L}(L^2(0, L)), \quad \forall\, t > 0$,

$iii)$ $S(s + t)y_0 = S(s) \circ S(t)y_0, \quad \forall\, s \geq 0, \forall\, t \geq 0, \quad \forall\, y_0 \in L^2(0, L)$,

$iv)$ $\lim_{t \downarrow 0} \|S(t)y_0 - y_0\|_{L^2(0,L)} = 0, \quad \forall\, y_0 \in L^2(0, L)$.

Weak solutions to evolution equations

The differential equation is

$$y' = Ay + f, \quad y(0) = y_0,$$

with $f \in C(\mathbb{R}; Y)$ and $y_0 \in Y$, Y a Banach space.

Definition (Weak solution)

A weak solution to the equation in $L^p(0,T;Y)$, for $1 \le p < \infty$, is a function $y \in L^p(0,T;Y)$ such that, for all $z \in D(A^*)$, the mapping

$$t \to \langle y(t), z \rangle_{Y,Y'}$$

belongs to $W^{1,p}(0,T)$ and obeys

$$\frac{d}{dt}\langle y(t), z\rangle = \langle y(t), A^* z\rangle + \langle f(t), z\rangle, \quad \in (0,T),$$
$$\langle y(0), z\rangle = \langle y_0, z\rangle.$$

Theorem (Existence and Uniqueness of Weak solution)

If $y_0 \in Y$ and if $f \in L^p(0,T;Y)$, then the equation admits a unique weak solution in $L^p(0,T;Y)$.
Moreover, this solution belongs to $C([0,T];Y)$ and it satisfies

$$y(t) = S(t)y_0 + \int_0^t S(t-s)f(s), \quad \forall\, t \in [0,T].$$

From the expression for the solution

$$\|y(t)\|_{C([0,T];Y)} \leq C\left(\|y_0\|_Y + \|f\|_{L^1(0,T;Y)}\right),$$

for some positive constant C.

Optimal control problem for Heat equation

Let Ω be a bounded domain in R^n, with a boundary Γ of class C^2.
Let $T > 0$, set $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$.

The heat equation with a distributed control is

$$\frac{\partial y}{\partial t} - \Delta y = f + \chi_\omega u \quad \text{in} \quad Q,$$
$$y = 0 \quad \text{on} \quad \Sigma, \quad y(x,0) = y_0(x), \quad \text{in} \quad \Omega.$$

The optimal control problem is

$$\inf \left\{ J(y,u) \mid u \in L^2(\omega \times (0,T)) \right\},$$

for some suitable J.

Optimal control

Let us take

$$J(y,u) = \frac{1}{2}\int_Q |y - y_d|^2 + \frac{1}{2}\int_\Omega |y(T) - y_d(T)|^2 + \frac{\beta}{2}\int_{\omega\times(0,T)} |u|^2.$$

Here $\beta > 0$ and $y_d \in C([0,T];L^2(\Omega))$ and (y,u) solves the controlled heat equation .

Then we have estimate for the state variable:

$$\|y\|_{C([0,T];L^2(\Omega))} \leq C\left(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2(Q)} + \|u\|_{L^2(\omega\times(0,T))}\right).$$

Existence of a unique optimal control

- Set $F(u) = J(y(u), u)$.
- Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence in $L^2(\omega \times (0, T))$,

$$\lim_{n\to\infty} F(u_n) = \inf_{u\in L^2(\omega\times(0,T))} F(u).$$

- Let y_n be the solution corresponding to u_n. Suppose that $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\omega \times (0, T))$ and that

$$u_n \rightharpoonup \overline{u}, \quad \text{weakly in} \quad L^2(\omega \times (0, T)).$$

- Set $\overline{y} = y(\overline{u})$.

- Show

$$y_n \rightharpoonup \overline{y}, \quad \text{weakly in} \quad L^2(Q),$$
$$y_n(T) \rightharpoonup \overline{y}(T), \quad \text{weakly in} \quad L^2(\Omega).$$

- Using the weakly lower semicontinuity of $\|\cdot\|^2_{L^2(Q)}$, $\|\cdot\|^2_{L^2(\Omega)}$, $\|\cdot\|^2_{L^2(\omega\times(0,T))}$, show that $\overline{u}$ is a solution of the minimizing problem.

- Uniqueness of the solution follows from the strict convexity of F.

Derivative of the state variable

Introduce
Equations satisfied by $z_\lambda = y(u+\lambda v) - y(u)$ is

$$\frac{\partial z_\lambda}{\partial t} - \Delta z_\lambda = \lambda \chi_\omega v \quad \text{in} \quad Q,$$
$$z_\lambda = 0 \quad \text{on} \quad \Sigma, \quad z_\lambda(x,0) = 0, \quad \text{in} \quad \Omega.$$

From the estimate it follows that

$$\|z_\lambda\|_{C([0,T];L^2(\Omega))} \leq C|\lambda|\|v\|_{L^2(\omega\times(0,T))}.$$

This yields that as λ tends to zero,

$$y(u+\lambda v) \to y(u) \quad \text{in} \quad C([0,T];L^2(\Omega)).$$

Derivative of F

Recall that

$$F'(u) = \lim_{\lambda \searrow 0} \frac{F(u+\lambda v) - F(u)}{\lambda}.$$

By a classical calculation, we have

$$F'(u)v = \int_Q (y(u) - y_d) z(v)$$
$$+ \int_\Omega (y(u)(T) - y_d(T)) z(v)(T) + \beta \int_{\omega \times (0,T)} uv,$$

where z satisfies

$$\frac{\partial z}{\partial t} - \Delta z = \chi_\omega v \quad \text{in} \quad Q,$$
$$z = 0 \quad \text{on} \quad \Sigma, \quad z(x,0) = 0, \quad \text{in} \quad \Omega.$$

Identification of $F'(u)$

Our aim: To find q such that

$$\int_Q (y(u) - y_d)z(v) + \int_\Omega (y(u)(T) - y_d(T))z(v)(T) = \int_{\omega\times(0,T)} qv$$

Let p be any regular function defined in $\overline{Q}$.

Using an integration by parts between $z(v)$ and p we have

$$\int_{\omega\times(0,T)} pv = \int_Q (z_t - \Delta z)p$$
$$= \int_Q z(-p_t - \Delta p) + \int_\Omega z(T)p(T) - \int_\Sigma \frac{\partial z}{\partial n} p$$

Consider the adjoint problem

$$-\frac{\partial p}{\partial t} - \Delta p = y(u) - y_d \quad \text{in} \quad Q,$$
$$p = 0 \quad \text{on} \quad \Sigma, \quad p(x,T) = (y(u) - y_d)(T), \quad \text{in} \quad \Omega.$$

Then we have

$$F'(u)v = \int_{\omega\times(0,T)} (p + \beta u)v,$$

if the above equations are justified.

For that we need to study the existence and the regularity of the solution of the adjoint problem.

Adjoint problem

Theorem

Let $g \in L^2(Q)$ and $p_T \in L^2(\Omega)$.
The terminal boundary value problem

$$-\frac{\partial p}{\partial t} - \Delta p = g \quad \text{in} \quad Q,$$
$$p = 0 \quad \text{on} \quad \Sigma, \quad p(x,T) = p_T, \quad \text{in} \quad \Omega$$

is wellposed and p satisfies

$$\|p\|_{C([0,T];L^2(\Omega))} \leq C \left(\|p_T\|_{L^2(\Omega)} + \|g\|_{L^2(Q)} \right).$$

Integration by parts between z and p

Theorem

Suppose that $\phi \in L^2(Q)$, $g \in L^2(Q)$, $p_T \in L^2(\Omega)$.
Then the solution z of the equation

$$\frac{\partial z}{\partial t} - \Delta z = \phi \quad \text{in} \quad Q,$$
$$z = 0 \quad \text{on} \quad \Sigma, \quad z(x,0) = 0, \quad \text{in} \quad \Omega.$$

and the solution p of the adjoint equation satisfy the following formula

$$\int_Q \phi p = \int_Q zg + \int_\Omega z(T) p_T.$$

Outline of the proof

- If $p_T \in H_0^1(\Omega)$, then z and p belong to

$$L^2(0,T;D(A)) \cap H^1(0,T;L^2(\Omega)).$$

Green's formula gives

$$\int_\Omega -\Delta z(t)p(t)dx = \int_\Omega -\Delta p(t)z(t)dx,$$

for almost every $t \in [0,T]$, and

$$\int_0^T \int_\Omega \frac{\partial z}{\partial t} p = -\int_0^T \int_\Omega \frac{\partial p}{\partial t} z + \int_\Omega z(T)p_T.$$

Thus IBP formula is established for $p_T \in H_0^1(\Omega)$.

- Then by density argument and using the estimate, we obtain the equation for any $p_T \in L^2(Q)$.

Theorem

If $(\overline{y}, \overline{u})$ *is the solution, then*

$$\overline{u} = -\frac{1}{\beta} p|_{\omega \times (0,T)},$$

where p *is the solution of the adjoint problem corresponding to* $\overline{y}$*:*

$$-\frac{\partial p}{\partial t} - \Delta p = \overline{y} - y_d \quad \text{in} \quad Q,$$
$$p = 0 \quad \text{on} \quad \Sigma, \quad p(x,T) = \overline{y}(T) - y_d(T), \quad \text{in} \quad \Omega.$$

Next theorem is the converse of the above one.

Theorem

If a pair $(\widetilde{y}, \widetilde{p}) \in C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega))$ *obeys the system*

$$\begin{aligned}
&\frac{\partial \widetilde{y}}{\partial t} - \Delta \widetilde{y} = f - \frac{1}{\beta}\chi_\omega \widetilde{p} \quad \text{in} \quad Q, \\
&\widetilde{y} = 0 \quad \text{on} \quad \Sigma, \quad \widetilde{y}(x,0) = y_0(x), \quad \text{in} \quad \Omega, \\
&-\frac{\partial \widetilde{p}}{\partial t} - \Delta \widetilde{p} = \widetilde{y} - y_d \quad \text{in} \quad Q, \\
&\widetilde{p} = 0 \quad \text{on} \quad \Sigma, \quad \widetilde{p}(x,T) = \widetilde{y}(T) - y_d(T), \quad \text{in} \quad \Omega,
\end{aligned}$$

Then the pair $(\widetilde{y}, -\frac{1}{\beta}\widetilde{p})$ *is the optimal solution to the problem.*

Abstract result

Consider the problem $P(0, T, y_0)$

$$\inf\{J_T(y,u), \quad (y,u) \mid u \in L^2(0,T;U)\},$$
$$J_T(y,u) = \frac{1}{2}\int_0^T \|y(t)\|_Y^2\,dt + \frac{1}{2}\int_0^T \|u(t)\|_u^2\,dt.$$

and (y, u) satisfies

$$y'(t) = Ay(t) + Bu(t), \quad \forall\, t \geq 0,$$
$$y(0) = y_0,$$
$$y_0 \in Y, \quad u \in U,$$

where

- Y and U are two Hilbert spaces.
- The unbounded operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup on Y.
- The control operator $B \in \mathcal{L}(U; Y)$.

Theorem

This problem $P(0, T, y_0)$ *admits a unique solution* $(\bar{y}, \bar{u})$ *and* $(\bar{y}, \bar{u})$ *satisfies the system*

$$
\begin{aligned}
&\bar{y}'(t) = A\bar{y}(t) - BB^*p(t), \quad \bar{y}(0) = y_0, \\
&-p'(t) = A^*p(t) + \bar{y}(t), \quad p(T) = 0, \\
&\bar{u}(t) = -B^*p(t).
\end{aligned}
$$

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