Laboratory Sessions : FreeFem++
CIMPA Workshop
Navier Stokes Stabilization Code

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Abstract

This are a collection of comments on the code for Navier Stokes stabilization written by Dr Gislain. The objective is that you can quickly corelate the algorithm discussed in the lecture notes with the code.

There are a sequence of calculations, we point out the main steps at each stage and write out a few formuale.

We end with suggestions for a few simulations with the code and refer to corresponding page numbers in the Lecture notes (Lecture 1 to Lecture 5).

Contents

1 A simulation of Stabilization of Unsteady flow of the 2-D incompressible Navier Stokes flow

There are a sequence of program to be executed, we comment on each of them with some formulae to aid quick corelation of the code with the algorithm.

The sequence of programs to be executed are

1. *Param.pde*: The key parameters.
2. *MainStationaryNS.pde*: Computes the steady state flow of the NS equations at $Re = 100$ (which turns out to be unstable).
3. *MainLambda.pde*: Appropriate Eigen Value analysis.
4. *MainFeedback.pde*: Determination of the Feedback matrix to be applied on the Boundary for controlled time evolution of the flow.
5. *MainCont.pde*: Time evolution simulation of the controlled flow.

We now collect some comments to aid quick navigation of the code as per the algorithm.

1.1 Param.pde

Geometry of the flow is 2-D flow over a square obstacle, and the boundary conditions are Dirichlet at inlet and pipe boundaries. The boundary condition is Neumann pressure condition at outlet. Control is applied on the top and bottom surface of the square obstacle. Essentially, computed feedback control on the boundary of the square (top and bottom) will stabilize the unstable flow at $Re = 100$.

The parameters for simulations are,

- Reynolds number.
- and Prescribed decay rate (omega)(used in the Riccati solution during the contruction of the Feedback matrix).
- pert1: Perturbation of the inlet flow.

In this program, it appears that calculations are done on only ONE unstable eigenvector picked from the spectrum and this happens to be a complex conjugate complex eigenpair with positive real value. Also, this particular problem with $Re = 100$ has only one unstable complex conjugate eigenvalue (i.e., positive real part) in the spectrum considered (100 eigenvalues were considered)

Refer to page 4, Lecture 4. What this means is that Z_u is two dimensional.

1.2 MainStationaryNS.pde

Essentially a continuation of the Navier Stokes steady state solutions to obtain the steady state at $Re = 100$. There is Newton Raphson and the equation for the linearised step of the Newton Raphson is obtained from the linearised weak form of the Navier Stokes. (this is standard) .

Refer to Lecture 1, page 24 for the algorithm that has been implemented.

1.3 MainLambda.pde

- Solves

$$Au = \lambda Mu$$
$$A^T u = \lambda Mu$$

- Stores (nev=100) number of eigenvalues (real and imaginary part) close to zero and the corresponding eigenvectors. The notation is ev and iev for the eigenvalues of A and evt and $ievt$ for the eigenvalues of A^T. For the eigenvectors it is $Evu1$ and $XiVu1$ for A and A^T respectively.

  We repeat once again that this program appears to look for only one complex pair of eigenvalues (and the corresponding eigenvector, real and imaginary part) *with positive real part (and thus unstable) for A and the corresponding eigenvector of A^T (same eigenvalues). These eigen vectors for A are stored in a variable $E1$ and $iE1$ and for A^T are stored in $X1$ and $iX1$.*

$$M\begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix} = \int_\Omega u \cdot v$$

Where v is test function.

$$A\begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix} = -\int_\Omega (\nu\left(\nabla u_1 \cdot \nabla v_1 + \nabla u_2 \cdot \nabla v_2\right) + \left(U_{Re=100} \cdot \nabla\right) u \cdot v$$
$$+ \left(u \cdot \nabla\right) U_{Re=100} \cdot v - p\nabla \cdot v + q \times \nabla \cdot u)$$
$$+\text{Boundary Conditions}$$

where v_1, v_2 and q are test functions and $U_{Re=100}$ is computed unsteady solution at Reynolds number=100.

Note that all boundary conditions are homogenous Neumann.

- Store the unstable Eigenvector of A in $E1$ and $iE1$ (real and imaginary part of the eigenvector). (Real part of the eigenvalue is positive)

- Store the unstable Eigenvector of A^T in $X1$ and $iX1$ (real and imaginary part of the eigenvector). (the eigenvalues are same as above, i.e., the unstable value for A).

  (Note added by Prof JP Raymond: When the eigenvectors of A and A^T are computed, it is important to choose them such as the bi-orthogonality condition be satisfied.) See also page 6 and 15, 16 and 17 of Lecture 4.

- Refer to Lecture 1 page 15, for the linearised governing equation of the evolution of the perturbation of the flow from the unstable steady state (in this case, the steady flow at Reynolds number=100) .

- Refer to page 16 Lecture 2 for the discretised matrices that play a role in this eigenvalue analysis.

1.4 MainFeedback.pde

The feedback Matrix K is constructed in this routine.

- Select the *REAL PART OF THE UNSTABLE EIGENVECTOR OF* A^T. Call it $[W1,\ W2,\ WP]$
- Compute and save the stress (a vector function, and it corresponds with the unstable eigenvector mentioned above, i.e., two components of this vector $s = [s1,\ s2]$ are approximated on the boundary as piece linear ($P1$ FE space) and obtained by the solution to the following problem,

  Find $s = [s1,\ s2]$ on the boundary such that

  $$\int_{\text{Control Boundary}} s \cdot t - \left\{ \left[\nu \frac{\partial}{\partial n}(W1) - WP \times n_x \right] \times t1 + \left[\nu \frac{\partial}{\partial n}(W2) - WP \times n_y \right] \times t_2 = 0 \right.$$

  $\forall\, t = [t1,\ t2]$ in the test space defined as piecewise linear (i.e., P1) on the Control Boundary, (also, please note that the $W1, W2, WP$ in the above formula is the real part of the unstable eigenvetor of A^T).
- Store the two components of this stress vector in X_i1 and X_i2.
- Normalise this stress vector (all this does is ensures the stress vector at each node is 1.0).
- PLEASE NOTE: THIS IS THE CALCULATION THAT WAS DISCUSSED IN THE CLASS AS A CALCULATION THAT NEEDS A CAREFUL LOOK FOR INTERPRETATION THAT IS IS SYNC WITH THE LECTURE NOTES.

  I JUST NOTE THE FORMULA HERE AND HOPE THAT WE CAN EVENTUALLY ADD NOTES THAT WILL BRING OUT THE FORMULA WITH THE REQUISITE CLARITY.

  SO TO REPEAT: WE ARE UNABLE TO GIVE A CLEAR EXPLANATION OF WHAT IS EXACTLY GOING ON IN THIS ONE STEP

  Contruct the Control Matrix G.

  The Control matrix G is contructed as follows,

$$
\begin{aligned}
G\begin{bmatrix} g_1 \\ g_2 \\ gp \end{bmatrix} &= \int_\Omega (\nu\,(\nabla g_1 \cdot \nabla v_1 + \nabla g_2 \cdot \nabla v_2) \\
&\quad -gp\nabla \cdot v + q \times \nabla \cdot g) \\
&\quad +\text{Boundary Conditions}
\end{aligned}
$$

(again, v_1, v_2 and q are test functions), where the boundary conditions on the *Control Boundary* are smoothened *control stress vector functions* calculated in the previous step i.e., X_i1 and X_i2.

- The next part of the code (visualization of control jets) is purely for the sake of visualization and does not play a part in the sequence of calculations.

- Contruct the feedback matrix K. This consists of a series of computations,

  1. Notation for the unstable eigenvector of A is Eu

     $$Eu = \begin{bmatrix} E1 & iE1 \\ \vdots & \vdots \end{bmatrix}_{\text{total degrees of freedom} \times 2}$$

     for the unstable eigenvector of A^T is Xu

     $$Xu = \begin{bmatrix} X1 & iX1 \\ \vdots & \vdots \end{bmatrix}_{\text{total degrees of freedom} \times 2}$$

  2. Now we calculate a 2×2 matrix R as follows

     $$R = Xu^T G Eu Eu^T G^T Xu$$

     and also let us retrieve the real and imaginary part of the unstable eigenvalue. Notation in the program uses $L(1)$ has the real part of the unstable eigenvalue. $L(2)$ has the imaginary part of the unstable eigenvalue.

  3. Now, given R (computed above), L (the values of the unstable eigenvalues) and the decay rate (named omega=(10 default) in the program, a parameter of the simulation) we can calculate the symmetric matrix 2×2 matrix Pi, via an analytic solution of the Riccati equations (obtained previously from MAPLE and coded in the program).

     Refer to page 26, Lecture 4 for the ARE (algebraic Riccati equation) that needs to be solved. In this code all the matrices are 2×2 and a system of three nonlinear algebraic equations for the three components of the symmetric matrix $\mathbb{P}_{\omega,u} \in \mathbb{R}^{2\times 2}$ are analytical solved by the symbolic package MAPLE (this is clearly commented in the FreeFem++ code). We have attached this simple MAPLE calculation for your reference.

4. Finally, we calculate the feedback matrix K

$$\begin{aligned} B &= Xu^T GEu \\ K &= B^T PiXu^T M \end{aligned}$$

i.e,

$$K = Eu^T G^T XuPiXu^T M$$

1.5 MainCont.pde

This is time simulation of the Controlled flow over the obstacle and easy to follow from the codes. We just mention the main point, *the time simulation is controlled on the boundary by the stress vector $[X_i 1, X_i 2]$ on the boundary weighted by the action of the Feedback matrix K on the previous time solution $u_{previous}$.*

As mentioned in Param.pde, this is the implementation of BDF2 scheme for the Unsteady Navier Stokes equation.

2 Discussion on the simulations possible with the Codes

Suggestions for parameter variations in Simulations and possibilities of generalization of the code.

2.1 Reynolds number

The default value is 100. The lectures present results for a cylinder with Reynolds number 150. We can repeat the entire calculations with Reynolds number = 150.

2.2 Omega: decay parameter

Refer to page 20 Lecture 5 where the results of variation of ω were discussed. These results can be obtained by variation of the parameter omega (see Param.pde, the default value is 10). This parameter plays a role in the solution of the Riccatti equation (See page 26 of Lecture 4).

2.3 Perturbation of the flow at the inlet

The results presented on page 19 of Lecture 5 can be obtained by variation of the parameter (see variable pert1 in Param.pde)

2.4 Geometry of the obstacle

The obstacle is a square in the code. The results presented in the Lecture notes are for a flow of Reynolds number 150 over a cylinder. The students can attempt to change the geometry of a cylinder and reproduce the results presented in the Lecture notes.

2.4.1 Discussion of *best control zone*

In the case of a cylinder, one can study the *best control zone* discussed pn pages 7-10 of Lecture 5 of the lecture notes.

2.5 Choice of the unstable subspace Z_u

As we understand, in this program we can choose ONLY one pair of unstable eigenvectors (thus $Z_u = 2$) corresponding to the subspace associated with an unstable complex eigenvalue for the simulation of the control.

See page 18 and also page 6 of Lecture 5 for remarks on the choice of Z_u. As discussed, this will involve more careful solution of the ARE (algebraic Riccati equation), a simplified MAPLE analytical solution that can deal with 2×2 matrices is used in the present code.