Adaptive tree approximation with finite elements

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Outline

1. Basic notions in constructive approximation
2. Tree approximation
3. Mesh refinement with tree structure for PDEs
4. Approximation of gradients
5. Convergence rates for tree approximation of gradients

Outline

1. Tree approximation
   - Abstract setting
   - An instance-optimal algorithm

Motivation

Tree approximation provides a solution to the practical problem for approximation with bisection.

Literature

- P. Binev, R. DeVore, *Fast computation in adaptive tree approximation*, Numer. Math. 97 (2004), 193–217.
  (ignore Section 7, apart from Remark 7.1)
- P. Binev, Tree approximation for *hp*-adaptivity, to appear.

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Master tree

Iterated interval bisection has the following abstract structure upon identifying intervals and nodes.

Let $\mathcal{T}$ be a *master tree*, ie an infinite binary tree such that

- each node has two children,
- each node has a parent, except one which is called the root (which corresponds to Ω).

(Sub)trees and partitions

A subset $T \subset \mathcal{T}$ is a *subtree* whenever

- T is finite,
- each node in T has its parent in T, except one.

A subtree T is *full* whenever each node has either two or non children. The nodes of a subtree T with no children are the *leaves* $\mathcal{L}(T)$.

A *tree* T is subtree with root Ω. The leaves $\mathcal{L}(T)$ of a full tree T form a partition of Ω and we write $T \in \mathbb{M}^{\mathcal{T}}$.

Subadditive error functionals

Given local error functionals $\epsilon : \mathcal{T} \to \mathbb{R}_0^+$, we define

$$\mathcal{E}(T) := \sum_{K \in \mathcal{L}(T)} \epsilon(K)$$

be the global error for any full tree T and assume *subadditivity*:

$$\epsilon(K_1) + \epsilon(K_2) \leq \epsilon(K)$$

whenever K_1 and K_2 are the children of $K \in \mathcal{T}$.

Set $\mathbb{M}_n^{\mathcal{T}} := \{T \in \mathbb{M}^{\mathcal{T}} \mid \#\mathcal{L}(T) \leq n\}$ and

$$\mathcal{E}(\mathbb{M}_n^{\mathcal{T}}) := \min_{T \in \mathbb{M}_n^{\mathcal{T}}} \mathcal{E}(T).$$

Example with interval bisection

Let $\mathbb{M}^{\mathcal{T}}$ the master tree associated with interval bisection $\mathbb{M}^{\text{bisect}}(\Omega)$.

For $0 < p < \infty$, the local 'errors'

$$\epsilon(K) = \inf_{P \in \mathbb{P}^\ell} \|v - P\|^p_{L^p(K)}$$

are subadditive. Moreover,

$$\mathcal{E}(T) = E\big(\mathbb{P}^\ell(\mathcal{L}(T))\big)^p$$

and

$$\mathcal{E}(\mathbb{M}^{\mathcal{T}}_n) = E\big(\mathbb{P}^\ell(\mathbb{M}^{\text{bisect}}_n)\big)^p = \min_{\mathcal{M} \in \mathbb{M}^{\text{bisect}}_n} E\big(\mathbb{P}^\ell(\mathcal{M})\big)^p.$$

Outline

Approximating with greedy (maximum) strategy

Starting from $T_0 = \{\Omega\}$, iterate

1. Compute $t_N := \max_{K \in \mathcal{L}(T_N)} \epsilon(K)$
2. Bisect all leaves $K \in T_N$ with $\epsilon(K) = t_N$ to obtain T_{N+1}
3. increment N

until, e.g., $\mathcal{E}(T_N)$ is sufficiently small.

(type of algorithm and first analysis goes back to Birman/Solomyak '67)

No instance optimality – 1

Consider approximation with piecewise constants in $L^2(0,1)$.

Let H be the Haar function given by

$$H(x) := \begin{cases} +1, & x \in [0, \frac{1}{2}], \\ -1, & x \in [\frac{1}{2}, 1], \\ 0, & \text{otherwise}, \end{cases}$$

and, for any $I = 2^{-m}[k, k+1]$ with $m \in \mathbb{N}_0$ and $k \in \{0, 2^m - 1\}$,

$$H_I(x) := 2^{m/2} H\big(2^m(x-k)\big)$$

so that

$$\int_0^1 |H_I|^2 = 1, \qquad \int_I H_I = 0.$$

No instance optimality – 2

Given $M, m \in \mathbb{N}$ and $\epsilon \in (0, \frac{1}{2})$, let $v \in L^2(0,1)$ be zero except for

$$v_{|[1-2^{-M},1]} = H_{[1-2^{-M},1]}, \qquad v_{|I} = \sqrt{1-\epsilon}\, H_I$$

for all $I = 2^{-m}[k, k+1]$ with $k = 0, \ldots, 2^{m-1} - 1$.

For $M > 2^{m-1}$, we have

$$\frac{E(T_N)}{E(T_n^*)} = 2^{m-1}(1-\epsilon) \quad \text{for} \quad \frac{n}{N} = \frac{2^m + 1}{2^{m-1} + M}$$

where T_n^* has the leaves $2^{m+1}[k, k+1]$, $k = 0, \ldots, 2^m - 1$ and $[\frac{1}{2}, 1]$.

Modified error functionals

Define $\tilde{\epsilon} : \mathcal{T} \to \mathbb{R}_0^+$ by

$$\tilde{\epsilon}(\Omega) = \epsilon(\Omega)$$

and, if K^* is the parent of K,

$$\frac{1}{\tilde{\epsilon}(K)} = \frac{1}{\epsilon(K)} + \frac{1}{\tilde{\epsilon}(K^*)}$$

with obvious conventions.

Instance optimality with modified functionals

Then the greedy strategy on $\tilde{\epsilon}$ instead of ϵ yields, for each $n \leq N$,

$$\mathcal{E}(\tilde{T}_N) \leq \frac{N}{N-n+1}\mathcal{E}(\mathbb{M}_n^{\mathcal{T}}).$$

The number of operations is $O(N)$, apart from computing ϵ and sorting.

Sorting can be avoided with the help of dyadic bins, at the cost of a multiplicative factor 2.

Bounding $E(\tilde{T}_N)$ in terms of $\tilde{t}_N - 1$

Given $K \in \mathcal{T}$, denote by $\mathcal{A}(K)$ its ancestors.

For any $K \in \mathcal{T}$, an induction yields

$$\frac{1}{\tilde{\epsilon}(K)} = \frac{1}{\epsilon(K)} + \sum_{K' \in \mathcal{A}(K)} \frac{1}{\epsilon(K')} \qquad (\star)$$

In particular, the modified error functionals $\tilde{\epsilon}$ reduce even if the original ones ϵ do not.

Bounding $\mathcal{E}(\tilde{T}_N)$ in terms of $\tilde{t}_N$

Multiplying $(\star)$ with $\tilde{\epsilon}(K)\epsilon(K)$ gives

$$\begin{aligned}\mathcal{E}(\tilde{T}_N) &\leq \sum_{K\in\mathcal{L}(T_N)} \tilde{\epsilon}(K)\left(1+\sum_{K'\in\mathcal{A}(K)}\frac{\epsilon(K)}{\epsilon(K')}\right)\\ &\leq t_N\left(\#\mathcal{L}(T_N)+\sum_{K'\in T_N\setminus\mathcal{L}(T_N)}\ \sum_{K\in\mathcal{L}(T_N):K\subset K'}\frac{\epsilon(K)}{\epsilon(K')}\right)\\ &\leq t_N\#T_N \leq 2N\, t_N\end{aligned}$$

where in the last but one step we have used subadditivity.

Bounding $\tilde{t}_N$ in terms of $\mathcal{E}(\mathbb{M}_n^{\mathcal{T}})$ – 1

Let T be a subtree with root $K^* \in \mathcal{T}$ and such that $\tilde{\epsilon}(K) \geq t_N$ for all nodes $K \in T$. Then

$$\epsilon(K^*) \geq \# T\, t_N.$$

by induction. Key step: if K is the parent of K_1 and K_2, then multiplying

$$\epsilon(K_1) + \epsilon(K_2) \geq t_N \left[k_1 + k_2 + \frac{\epsilon(K_1) + \epsilon(K_2)}{\tilde{\epsilon}(K)} \right]$$

with $\frac{\epsilon(K)}{\epsilon(K_1)+\epsilon(K_2)} \geq 1$ yields

$$\epsilon(K) \geq t_N \left[k_1 + k_2 + \frac{\epsilon(K)}{\tilde{\epsilon}(K)} \right] = t_N \left[k_1 + k_2 + \frac{\epsilon(K)}{\epsilon(K)} + \frac{\epsilon(K)}{\tilde{\epsilon}(K')} \right]$$

where = is only possible if K has a parent, say K'.

Bounding $\tilde{t}_N$ in terms of $\mathcal{E}(\mathbb{M}_n^{\mathcal{T}})$ – 2

Let $T^* \in \mathbb{M}_n^{\mathcal{T}}$ such that $\mathcal{E}(T^*) = \mathcal{E}(\mathbb{M}_n^{\mathcal{T}})$.

For any $K^* \in \mathcal{L}(T^*)$, consider

$$T_{K^*} = \{K \in T_N \setminus \mathcal{L}(T_N) \mid K \subset K^*\}$$

and derive

$$E(T^*) = \sum_{K^* \in \mathcal{L}(T^*)} \epsilon(K^*) \geq t_N \sum_{K^* \in \mathcal{L}(T^*)} \# T_{K^*} \geq t_N(N-n).$$

For the claimed constant, one has to reason more carefully, but along these steps.

PDE-motivated tree approximation?

PDE-motivated approximation settings differ from the preceding examples by

- ‘Natural’ error norms involve derivatives, e.g.,

$$\|\nabla v\|_{L^2(\Omega)}$$

- Often approximants have additional structure, reflecting in
  - their boundary values,
  - their global regularity and
  - the underlying meshes.