Adaptive tree approximation with finite elements

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Outline

1. Basic notions in constructive approximation
2. Tree approximation
3. Mesh refinement with tree structure for PDEs
4. Approximation of gradients
5. Convergence rates for tree approximation of gradients

Literature

- Two-dimensional case: Section 1.3 and Chapter 6 in

  Nochetto/Veeser, *Primer of Adaptive FEM*, in: Lecture Notes in Math 2040, Springer 2012

- Multidimensional case: Chapter 4 in

  Nochetto/Siebert/Veeser, *Theory of Adaptive FEM: An introduction*, in: Multiscale, Nonlinear and Adaptive Approximation, DeVore/Kunoth (Eds.), Springer, 2009

Here we will consider only the 2d case.

Trianglar mesh refinement and PDEs

Assume we discretize a PDE with a triangular mesh. Then the following properties are often exploited:

- *Shape regularity:* The minimum angle of all triangles is uniformly bounded away from zero.
- *Conformity (edge-to-edge):* Triangles meet only in vertices or edges.

Notice:

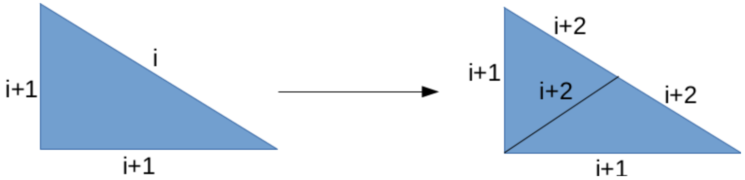
- Shape regularity puts constraints on iterated subdivisions of triangles.
- Conformity or limited non-conformity may propagate refinement (!).

Outline

1. Mesh refinement with tree structure for PDEs
   - Single bisections
   - Maintaining conformity
   - Cost of conformity

Definition

Given a triangle with a labeling $(0, 1, 1)$ of its edges, we prescribe



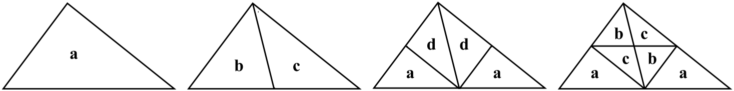
where the new vertex is the midpoint of the edge i.

This associates a tree with any triangle of an initial triangulation $\mathcal{M}_0$.

The generation $g(K)$ of a triangle K is its lowest edge label. Incrementing the generation halves the area.

Shape regularity, diameter and generation

The minimum angles of the triangles in a tree are bounded away from 0 in terms of the root triangle:

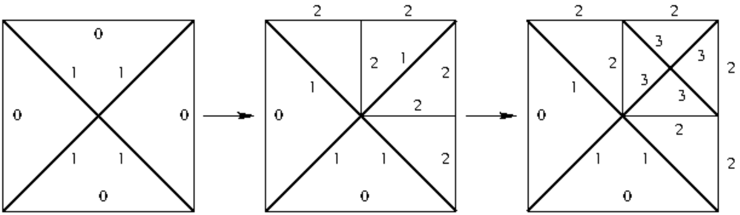


Thus, we have diam $K \approx |K|^{1/2}$ for all triangle in a tree, whence

$$\operatorname{diam} K \approx 2^{-g(K)/2}.$$

A few bisections in a triangulation . . .

The labeled bisections

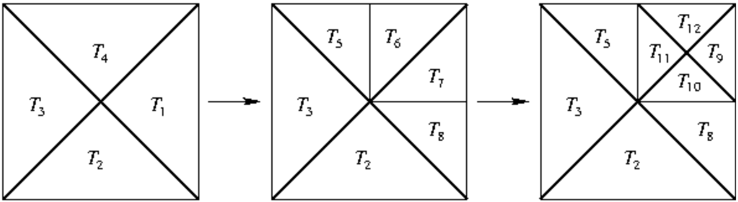


correspond to . . .

. . . and its trees

to the trees (or forest)

where



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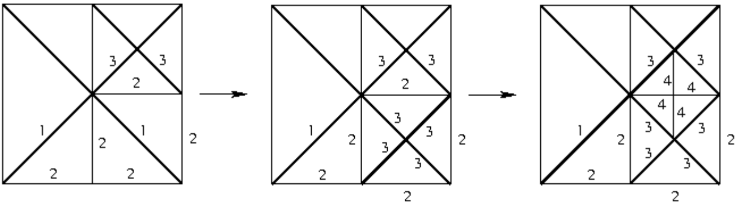
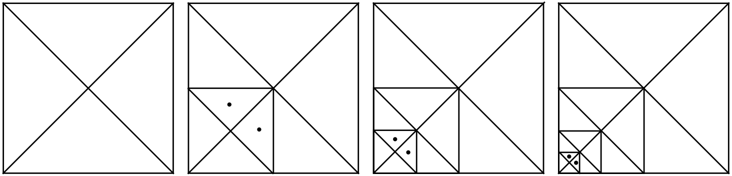
Refinement propagation due to conformity – 1

The bisection of one triangle may require the bisection of a triangle with *lower* generation:



Refinement propagation due to conformity – 2

. . . with *arbitrary* (!) lower generation (longest edge has always lowest label):



Recursive bisection – 1

Let $K \in \mathcal{M}$ be a triangle of a conforming mesh and denote by E its edge with the lowest label.

If E is a boundary edge, then we set

$$F_{\mathcal{M}}(K) := \emptyset,$$

else there exists a unique $K' \in \mathcal{M}$ with $E = K \cap K'$ and we set

$$F_{\mathcal{M}}(K) := K'.$$

Recursive bisection – 2

To construct the smallest conforming refinement of $\mathcal{M}$ in which K is bisected, we consider

```
rec-bisect(M, K)

if F_M(K) = ∅ then
    M =bisect(M, K)
else
    if F_M^2(K) ≠ K then
        rec-bisect(F_M(K))   % F_M(K) changes
    M =bisect(M, K, F_M(K))
return(M)
```

$\texttt{rec-bisect}(\mathcal{M}, K)$

$\texttt{if } F_{\mathcal{M}}(K) = \emptyset \texttt{ then}$
$\quad \mathcal{M} = \texttt{bisect}(\mathcal{M}, K)$
$\texttt{else}$
$\quad \texttt{if } F^2_{\mathcal{M}}(K) \neq K \texttt{ then}$
$\quad\quad \texttt{rec-bisect}(F_{\mathcal{M}}(K)) \quad \%\ F_{\mathcal{M}}(K)$ changes
$\quad \mathcal{M} = \texttt{bisect}(\mathcal{M}, K, F_{\mathcal{M}}(K))$
$\texttt{return}(\mathcal{M})$

Does it terminate and is the output conforming?

A sufficient condition for termination, . . .

If $\mathcal{M}$ is conforming and

each interelement edge receives coinciding labels,

then `rec-bisect`$(\mathcal{M}, K)$ has at most $g(K)$ recursive calls.

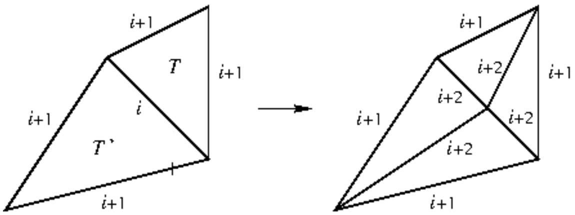
In fact, the edge labeling of K is $(i, i+1, i+1)$, where $E = K \cap F_{\mathcal{M}}(K)$ has label i also from $F_{\mathcal{M}}(K)$. Another call of `rec-bisect` with $F_{\mathcal{M}}(K)$ happens only if $\mathcal{M}$ is not changed and $F_{\mathcal{M}}(K)$ has the edge labeling $(i-1, i, i)$. therefore

$$g(F_{\mathcal{M}}(K)) = g(K) - 1.$$

for conservation of uniqueness of edge labeling, . . .

In the aforementioned case all intermediate meshes have a unique edge labeling

because, apart from the case $F_{\mathcal{M}}(K) = \emptyset$, only compatible bisections

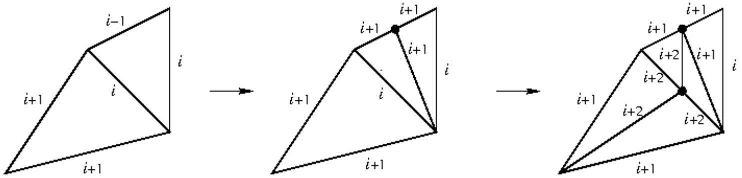


are performed.

and for conforming output

All intermediate mesh and the final one are conforming.

It remains to consider the situation after the return of a recursive call. Since `rec-bisect`$(\mathcal{M}, K)$ bisects K before returning, we obtain



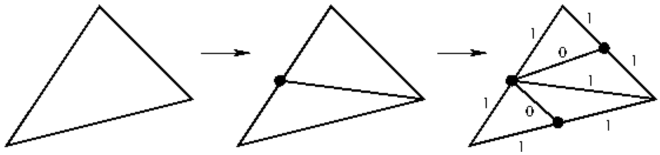
Assumption for initial labeling

Let $\mathcal{M}_0$ be a conforming initial triangulation and assume that the edge labeling $(0, 1, 1)$ for any triangle is such that

each interelement edge of $\mathcal{M}_0$ has coinciding labels.

Then iterative applications of `rec-bisect` are well-defined.

Non-constructive existence by Peterson's theorem about matching in cubic graphs. In alternative,



Two useful properties of `rec-bisect`

Let $\mathcal{M}$ be some mesh arising from a suitable $\mathcal{M}_0$ and iterative applications of `rec-bisect`. If

$$K \in \mathcal{M} \quad \text{and} \quad K' \in \texttt{rec-bisect}(\mathcal{M}, K) \setminus \mathcal{M},$$

then

$$g(K') \leq g(K) + 1$$

and

$$\operatorname{dist}(K', K) \leq C_{\text{dist}}\, 2^{-g(K')/2}$$

Proof of dist bound

Let $K_0 = K, K_1, \dots, K_m$ be the triangles in the recursive calls of `rec-bisect` and take j such that $K' \subset K_j$. Wlog $j > 1$. Then

$$\operatorname{dist}(K', K) \le \operatorname{diam}(K_{j-1}) + \operatorname{dist}(K_{j-1}, K_0) \le \sum_{k=0}^{j-1} \operatorname{diam}(K_k)$$

$$\precsim \sum_{k=0}^{j-1} 2^{-g(K_k)/2} = 2^{-g(K_{j-1})/2} \sum_{k=0}^{j-1} 2^{-k/2} \approx 2^{-g(K')/2}$$

`refine`

Given an 'admissible' initial mesh $\mathcal{M}_0$, let $\mathbb{M}^{\text{bisect}}(\mathcal{M}_0)$ denote the set of conforming, shape-regular meshes that can be generated by successive applications `rec-bisect`, starting from $\mathcal{M}_0$.

Given $\mathcal{M} \in \mathbb{M}^{\text{bisect}}(\mathcal{M}_0)$ and a set $\mathcal{M}^\star \subset \mathcal{M}$ of triangles to be bisected, we define

$\texttt{refine}(\mathcal{M}, \mathcal{M}^\star)$

$\texttt{while}\ \mathcal{M} \cap \mathcal{M}^\star \neq \emptyset$

$\quad \texttt{pick}\ K \in \mathcal{M} \cap \mathcal{M}^\star$

$\quad \mathcal{M} = \texttt{rec-bisect}(\mathcal{M}, K)$

$\texttt{return}(\mathcal{M})$

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Bound by total number of requested bisections

Let $\mathcal{M}_0$ be an initial triangulation with coinciding edge labels.

Moreover, given $k \in \mathbb{N}$, let $\mathcal{M}_0^\star, \dots, \mathcal{M}_{k-1}^\star$ and $\mathcal{M}_1, \dots, \mathcal{M}_k$ such that $\mathcal{M}_j^\star \subset \mathcal{M}_j$ and

$$\mathcal{M}_j = \mathtt{refine}(\mathcal{M}_{j-1}, \mathcal{M}_{j-1}^\star)$$

for $j = 0, \dots, k-1$.

Then

$$\#\mathcal{M}_k - \#\mathcal{M}_0 \preccurlyeq \sum_{j=0}^{k-1} \#\mathcal{M}_j^\star$$

Approach

Write $\mathcal{R} := \mathcal{M}_k \setminus \mathcal{M}_0$ and $\mathcal{M}_\cup^\star := \cup_{j=0}^{k-1} \mathcal{M}_j^\star$ and construct a function

$$\lambda : \mathcal{R} \times \mathcal{M}_\cup^\star \to \mathbb{R}_0^+$$

such that

$$\sum_{K^\star \in \mathcal{M}_\cup^\star} \lambda(K, K^\star) \geq C_1 \quad \text{and} \quad \sum_{K \in \mathcal{R}} \lambda(K, K^\star) \leq C_2$$

Then

$$C_1 \# \mathcal{R} \leq \sum_{K \in \mathcal{R}} \sum_{K^\star \in \mathcal{M}_\cup^\star} \lambda(K, K^\star) = \sum_{K^\star \in \mathcal{M}_\cup^\star} \sum_{K \in \mathcal{R}} \lambda(K, K^\star) \leq C_2 \# \mathcal{M}_\cup^\star.$$

Allocation function

Let $\big(a(\ell)\big)_{\ell \geq -1} \subset \mathbb{R}^+$ be such that

$$\sum_{\ell \geq -1} a(\ell) = A < \infty$$

and $B, C' > 0$ constants to be chosen later.

Given $K \in \mathcal{R}$ and $K^\star \in \mathcal{M}_\cup^\star$, we set

$$\lambda(K, K^\star) = a\big(g(K^\star) - g(K)\big)$$

whenever $g(K) \leq g(K^\star) + 1$ and $\mathrm{dist}(K, K^\star) \leq BC' 2^{-g(K)/2}$, as well as

$$\lambda(K, K^\star) = 0$$

otherwise.

Upper bound

Fix $K^\star \in \mathcal{M}_\cup^\star$ and, for $0 \leq g \leq g(K^\star) + 1$, consider

$$\mathcal{M}(g) := \{K \in \mathcal{R} \mid \operatorname{dist}(K, K^\star) \leq BC' 2^{-g(K)/2} \text{ and } g(K) = g\}$$

Since all these triangles lie in a ball centered in the midpoint of $K^\star$ and with radius $\preccurlyeq 2^{-g}$, we have $\#\mathcal{M}(g) \preccurlyeq 1$

Therefore

$$\sum_{K \in \mathcal{R}} \lambda(K, K^\star) \preccurlyeq \sum_{g=0}^{g(K^\star)+1} a(g(K^\star) - g) \preccurlyeq \sum_{\ell \geq -1} a(\ell) \preccurlyeq A.$$

Lower bound – going back some way

Fix $K_0 \in \mathcal{R}$ and define $K_0, \dots, K_J$ with $J \le k-1$ by

$$K_j \text{ arises from } \texttt{rec-bisect}(\cdot, K_{j+1}) \text{ and } K_{j+1} \in \mathcal{M}_\cup^\star.$$

We have

- $g(K_J) = 0$
- $g(K_{j+1}) \ge g(K_j) - 1$

Let s be the smallest value such that $g(K_s) = g(K_0) - 1$. Then $K_1, \dots, K_s$ are candidates for contributing to the lower bound:

$$g(K_0) \le g(K_j) + 1 \quad \text{and} \quad K_j \in \mathcal{M}_\cup^\star \qquad (j = 1, \dots, s).$$

Lower bound – definition of C'

Fix any $j \in \{1, \ldots, s\}$. Similar to before, we derive

$$\begin{aligned}\operatorname{dist}(K_0, K_j) &\leq \sum_{i=1}^{j} \operatorname{dist}(K_{i-1}, K_i) + \sum_{i=1}^{j-1} \operatorname{diam}(K_i)\\ &\leq C' \sum_{i=0}^{j-1} 2^{-g(K_i)/2},\end{aligned}$$

which defines C'.

Lower bound – $m(\ell, j)$

To estimate further, we introduce, for $\ell \geq 0$ and $j = 1, \ldots, s$,

$$m(\ell, j) := \#\{K \in \{K_0, \ldots, K_{j-1}\} \mid g(K) = g(K_0) + \ell\}$$

Observe that $m(\ell, \cdot)$ is increasing and

$$m(\ell, 1) = \begin{cases} 1, & \text{if } \ell = 0, \\ 0, & \text{if } \ell > 0. \end{cases}$$

Then

$$\operatorname{dist}(K_0, K_j) \leq C' 2^{-g(K_0)/2} \sum_{\ell=0}^{\infty} m(\ell, j) 2^{-\ell/2}$$

Lower bound – definition of B

In order to compare with $m(\ell, j)$, we choose $\big(b(\ell)\big)_{\ell \geq 0} \subset \mathbb{R}^+$ such that

$$B := \sum_{\ell=0}^{\infty} b(\ell) 2^{-\ell/2} < \infty$$

and

$$\inf_{\ell \geq 0} b(\ell) \geq 1, \qquad \inf_{\ell \geq 1} a(\ell) b(\ell) \geq c_{ab} > 0$$

e.g., $b(\ell) = 2^{\ell/3}$ if $a(\ell) = (\ell + 2)^{-2}$.

According to the relationship of m and b, we consider several cases.

Lower bound – case $m(\cdot, s) \leq b$

If

$$m(\ell, s) \leq b(\ell) \quad \text{for all } \ell \geq 0,$$

then

$$\text{dist}(K_0, K_s) \leq BC' 2^{-g(K_0)/2}$$

and so

$$\sum_{K^\star \in \mathcal{M}^\star_\cup} \lambda(K_0, K^\star) \geq \lambda(K_0, K_s) = a(-1) > 0$$

Lower bound – case $m(\cdot, s) \not\leq b$

Assume there is $\ell \geq 0$ with $m(\ell, s) > b(\ell)$. Choose $\ell^* \geq 0$ and $j^* \in \{1, \dots, s\}$ such that

$$m(\ell^*, j^*) > b(\ell^*)$$

and

$$\forall \ell, j \quad m(\ell, j) > b(\ell) \implies j \geq j^*.$$

Since $m(\ell^*, 1) \leq 1$ and $b(\ell^*) \geq 1$, we have $j^* \geq 2$. Hence, for all $i = 1, \dots, j^* - 1$, we have $m(\cdot, i) \leq b$ and so

$$\text{dist}(K_0, K_i) \leq BC' 2^{-g(K_0)/2}.$$

Therefore

$$\sum_{K^\star \in \mathcal{M}^\star_\cup} \lambda(K_0, K^\star) \geq m(\ell^*, j^*) a(\ell^*) \geq b(\ell^*) a(\ell^*) \geq c_{ab} > 0.$$