Adaptive tree approximation with finite elements

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Outline

1. Basic notions in constructive approximation
2. Tree approximation
3. Mesh refinement with tree structure for PDEs
4. Approximation of gradients
5. Convergence rates for tree approximation of gradients

Literature

- A. Veeser, *Approximating gradients with piecewise polynomial functions*, Found. Comp. Math. (2015).
- For a "generalization", covering also the L^2-norm:

  F. Tantardini, A. Veeser, R. Verfürth, *Localization of the best error with finite elements in the reaction-diffusion norm*, Constr. Approx. (2015).

Outline

H^1_0-conformity and piecewise polynomials

Let $\mathcal{M}$ be any conforming triangulation of $\Omega \subset \mathbb{R}^d$ and ℓ the maximal polynomial degree. Recall

$$\mathbb{P}^\ell(\mathcal{M}) := \{ v \mid \forall K \in \mathcal{M} \; v_{|K} \in \mathbb{P}^\ell(K) \}$$

and take

$$S_0(\mathcal{M}) = \mathbb{P}^\ell(\mathcal{M}) \cap H^1_0(\Omega)$$

For any $v \in \mathbb{P}^\ell(\mathcal{M})$ holds

$$v \in H^1_0(\Omega) \iff v \in C^0(\bar{\Omega}) \text{ and } v_{|\partial\Omega} = 0$$

Global and local best errors

Write $\|\cdot\|_D$ for $\|\cdot\|_{L^2(D)}$. For fixed $v \in H_0^1(\Omega)$, consider

$$E\big(S_0(\mathcal{M})\big) := \inf\{\|\nabla(v-s)\|_\Omega \mid s \in S_0(\mathcal{M})\}$$

and, for each element $K \in \mathcal{M}$,

$$E\big(\mathbb{P}^\ell(K)\big) := \inf\Big\{\|\nabla(v-P)\|_K \mid P \in \mathbb{P}^\ell(K)\Big\}$$

$$E\big(\mathbb{P}^\ell(\mathcal{M})\big) := \left(\sum_{K\in\mathcal{M}} E\big(\mathbb{P}^\ell(K)\big)^2\right)^{1/2}$$

The latter do not take into account the 'conformity' constraints.

Shape coefficient

For any $K \in \mathcal{M}$, set

$$\rho(K) := \sup\left\{r > 0 \mid \exists x \; B_r(x) \subset K\right\}$$

and write

$$\sigma(\mathcal{M}) := \max_{K \in \mathcal{M}} \frac{\operatorname{diam}(K)}{\rho(K)},$$

for the shape coefficient of $\mathcal{M}$.

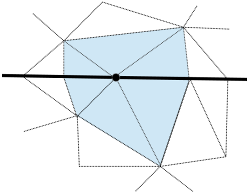
Face-connecteness

Assume that $z \in K \cap F$ where

- z is any node of $S(\mathcal{M}) := \mathbb{P}^\ell(\mathcal{M}) \cap H^1(\Omega)$,
- K is any element of $\mathcal{M}$,
- F is any $(d-1)$-face of $\mathcal{M}$

there is a path $K_0, \dots, K_m$ such that

$$K_0 = K, \quad K_m \subset F, \quad K_i \cap K_{i-1} \text{ is a } (d-1)\text{-face of } \mathcal{M}.$$



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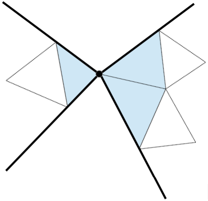

no

Localization of best H^1_0-error

There is a constant C_{de} depending on $\sigma(\mathcal{M})$ and ℓ such that

$$E\big(\mathbb{P}^\ell(\mathcal{M})\big) \leq E\big(S_0(\mathcal{M})\big) \leq C_{\mathrm{de}} E\big(\mathbb{P}^\ell(\mathcal{M})\big)$$

- does not hold for L^2-norm
- no apriori error estimate; no regularity involved
- ‘conformity’ constraints ok; beyond asymptotics
- approximately knowing the error of the Ritz projection is almost fully parallel

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   - Localization of the best H_0^1-error
   - About the proof of localization
   - Tree approximation for gradients
   - Approximation and gradient conformity

Explicit trace and Poincaré inequalities

Given an element K and a side F of K, there holds

$$\|w\|_F^2 \leq \frac{|F|}{|K|}\left(\|w\|_K^2 + \frac{2}{d}\,\|w\|_K \operatorname{diam} K\,\|\nabla w\|_K\right)$$

(cf. Ainsworth '07, for generalizatons cf. Veeser/Verfürth '09)

If $\int_K w = 0$, then

$$\|w\|_K \leq \frac{1}{\pi}\operatorname{diam} K\,\|\nabla w\|_K$$

(cf. Payne/Weinberger '60, ...)

An interpolation operator

Let $\{\phi_z\}_{z\in\mathcal{N}}$ be the nodal basis functions of $S(\mathcal{M})$ and define a projection onto $S(\mathcal{M})$ by

$$\Pi u := \sum_{z\in\mathcal{N}} v_z \phi_z$$

with

$$v_z = \begin{cases} P_K(z) & \text{if } z \text{ interior } K \\ \int_{F_z} \phi_z^* v & \text{otherwise} \end{cases}$$

and $P_K \in \mathbb{P}^\ell(K)$ st $\|\nabla(v - P_K)\| = E(\mathbb{P}^\ell(K))$ and $\int_K P_K = \int_K v$,

F_z a face containing z and sharing its type,

$\{\phi_z^*\}_z$ as in Scott/Zhang '90

Sketch of proof – 1

Write

$$E\big(S_0(\mathcal{M})\big) \leq \|\nabla(v - \Pi v)\|_\Omega$$

and

$$\|\nabla(v - \Pi v)\|_K \leq \|\nabla(v - P_K)\|_K + \|\nabla(P_K - \Pi v)\|_K$$

and observe

$$\|\nabla(\Pi v - P_K)\|_K \leq \sum_{z \in \partial K} \left| \int_{F_z} \phi_z^*(v - P_K) \right| \|\nabla \phi_z\|_K$$

Sketch of proof – 2

If $F_z \subset K$, there holds

$$\left| \int_{F_z} \phi_z^*(v - P_K) \right| \leq \|\phi_z^*\|_{F_z} \, \|v - P_K\|_{F_z} \leq \frac{C}{|F_z|^{1/2}} \, \|v - P_K\|_{F_z}$$

and

$$\|v - P_K\|_{F_z} \leq \left(\frac{1}{\pi^2} + \frac{1}{d\pi} \right)^{1/2} \frac{|F_z|^{1/2}}{|K|^{1/2}} \operatorname{diam} K \, \|\nabla(v - P_K)\|_K$$

Otherwise use face-connectedness ...

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complete

Let $\mathcal{M}_0$ be an admissible (in 2d: with coinciding edge labels) initial mesh of Ω and let $\mathcal{T}$ be the corresponding forest of master trees.

Denote by $\mathbb{M}^{\mathcal{T}}$ the set of all meshes that can be generated from $\mathcal{M}_0$ by simplex bisection and by $\mathbb{M}^{\mathcal{T},\text{conf}}$ its subfamily of conforming (face-to-face) meshes.

Given any (possibly non-conforming) mesh $\mathcal{M}' \in \mathbb{M}^{\mathcal{T}}$, denote by $\texttt{complete}(\mathcal{M}')$ the smallest conforming refinement of $\mathcal{M}'$. Thanks to third part and Remark 7.1 of Binev/DeVore '04, we have

$$\#\texttt{complete}(\mathcal{M}') - \#\mathcal{M}_0 \preccurlyeq \#\mathcal{M}' - \#\mathcal{M}_0.$$

Mesh conformity in H_0^1-approximation

Denote by $\mathbb{M}_n^{\mathcal{T}}$ and $\mathbb{M}_n^{\mathcal{T},\text{conf}}$ the respective subfamilies with at most n simplex bisection.

Combining the localization of the best H_0^1-error with the inequality for `complete` gives:

There are $c_2 > 0$ depending on $\mathcal{M}_0$ and ℓ such that

$$E\big(S_0(\mathbb{M}_n^{\mathcal{T},\text{conf}})\big) \le C_{\text{de}} E\big(\mathbb{P}^\ell(\mathbb{M}_{c_2 n}^{\mathcal{T}})\big)$$

Setting for tree approximation

Let $\mathcal{M}_0$ be an initial mesh of Ω with coinciding edge labels and $\mathcal{T}$ the corresponding forest of master trees.

For any $K \in \mathcal{T}$, we set

$$\epsilon(K) = E\big(v, \mathbb{P}^\ell(K)\big)^2 = \inf_{P \in \mathbb{P}^\ell(K)} \|\nabla(v - P)\|_K^2$$

and, for any $\mathcal{M} \in \mathbb{M}^{\mathcal{T},\mathrm{conf}}$ that is conforming, we have

$$\mathcal{E}(\mathcal{M}) = \sum_{K \in \mathcal{M}} \epsilon(K) \approx E\big(v, S_0(\mathcal{M})\big)^2.$$

Thresholding algorithm

$\mathcal{M}'_t := \emptyset$
$\texttt{for all}\ K \in \mathcal{M}_0$
    $\texttt{if}\ \tilde{e}(K) > t\ \texttt{then grow}(K)$
$\mathcal{M}_t := \texttt{complete}(\mathcal{M}'_t)$

where $t > 0$ is given and $\texttt{grow}(K)$ is

$(K_1, K_2) := \texttt{bisect}(K)$
$\texttt{for}\ i = 1, 2$
    $\texttt{if}\ \tilde{e}(K_i) > t\ \texttt{then}$
        $\texttt{grow}(K_i)$
    else
        $\mathcal{M}'_t := \mathcal{M}'_t \cup \{K_i\}$

Instance optimality

If $\#\mathcal{M}_t \geq 3\#\mathcal{M}_0$, then

$$E\big(S_0(\mathcal{M}_t)\big) \leq 2C_{\mathrm{de}} E\big(\mathbb{P}^\ell(\mathcal{M}^{\mathcal{T}}_{c_2\#\mathcal{M}_t})\big)$$

for some $c_2 > 0$ depending on $\mathcal{M}_0$ and ℓ.

Comparing with Céa Lemma, the approximant here is

- determined without PDE,
- near best in a *nonlinear* approximation space that, for given n, is much bigger

Applications

The above thresholding algorithm or variants may be used

- to create benchmark for the adaptive solution of PDEs
- to coarsen in an adaptive algorithm of the form

$$\texttt{error reduction} \to \texttt{sparsity adjustment}$$

- to approximate data in the PDE, e.g., in connection with $-\Delta$ with

$$\epsilon(K) = \operatorname{diam}(K)^2 \, \|f\|_K^2 \qquad \text{(but \ldots)}$$

- to coarsen in the adaptive solution of evolutionary PDEs (rather reaction-diffusion or L^2-norm)

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Gradient conformity and best errors

Note

$$\{\nabla P \mid P \in \mathbb{P}^\ell\} = \left\{Q \in (\mathbb{P}^{\ell-1})^d \mid \partial_i Q_j = \partial_j Q_i,\ i,j = 1,\dots,d\right\}$$

Recalling

$$E\big(\mathbb{P}^\ell(K)\big) = \inf\left\{ \|\nabla(v-P)\|_K \mid P \in \mathbb{P}^\ell(K)\right\},$$

consider

$$E\big(\nabla v, \mathbb{P}^{\ell-1}(K)^d\big)_{L^2} := \inf\left\{\|\nabla v - Q\|_K \mid Q \in \mathbb{P}^{\ell-1}(K)^d\right\}.$$

Equivalence . . .

There holds

$$E\big(\nabla v, \mathbb{P}^{\ell-1}(K)^d\big)_{L^2} \leq E\big(\mathbb{P}^{\ell}(K)\big) \leq C_{\mathrm{de}} E\big(\nabla v, \mathbb{P}^{\ell-1}(K)^d\big)_{L^2}$$

where C_{de} depends only on ℓ and d

- coupling of partial derivatives ok
- approximately knowing the error of an H^1-seminorm-projection is almost fully parallel in terms of the error of the L^2-projections

. . . and about its proof

Denoting by $I_\ell v$ the averaged Taylor polynomial of Scott/Dupont '80,

$$
\begin{aligned}
E\big(\mathbb{P}^\ell(K)\big)^2 &\leq \sum_{i=1}^{d} \|\partial_i(v - I_\ell v)\|_K^2 = \sum_{i=1}^{d} \|\partial_i v - I_{\ell-1}(\partial_i v)\|_K^2 \\
&= \sum_{i=1}^{d} \|(\partial_i v - Q_i) - I_{\ell-1}(\partial_i v - Q_i)\|_K^2 \\
&\leq C_{\mathrm{de}} \, \|\nabla v - Q\|_K^2
\end{aligned}
$$

Independence on element shape by using an idea of Dekel/Leviatan '04.