Adaptive tree approximation with finite elements

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Outline

1. Basic notions in constructive approximation
2. Tree approximation
3. Mesh refinement with tree structure for PDEs
4. Approximation of gradients
5. Convergence rates for tree approximation of gradients

Literature – 1

about Besov spaces:

- G. G. Lorentz, R. A. DeVore, *Constructive approximation*, Springer, 1993.

about interpolation of spaces: chapter 14 in

- S. Brenner, R. Scott, *The mathematical theory of finite element methods*, Springer, 2008.

Literature – 2

What follows is the 'Besov' version of Theorem 3 in

- R. H. Nochetto, A. Veeser, *Primer of Adaptive FEM*, in: Lecture Notes in Math 2040, Springer 2012

and a partial alternative for:

- P. Binev, W. Dahmen, R. DeVore, P. Petrushev, *Approximation classes for adaptive methods*, Serdica Math. J. 28 (2002), 391–416.
- F. D. Gaspoz, P. Morin, *Approximation classes for adaptive higher order finite element approximation*, Math. Comp. 83 (2014), 2127–2160.

A simple example

We re-consider approximation of a continuous function on $[0,1]$ with piecewise constants in the maximum norm.

The inequalities

$$E(v, \mathbb{P}^0(I)) = \frac{1}{2}\left(\max_I v - \min_I v\right) \le \int_I |v'| \le |I| \sup_I |v'|$$

tell that the regularity ensuring the rate n^{-1} is, respectively, for

	uniform	free
	$\|v'\|_{L^\infty(0,1)}$	$\|v'\|_{L^1(0,1)}$

breakpoints.

Outline

1. Convergence rates for tree approximation of gradients
   - Measuring regularity with Besov spaces
   - Convergence rates

Shortcomings of weak derivatives and alternative?

Weak derivatives have the following shortcomings in measuring regularity:

- no fractional "number of derivatives"
- defined only for $p \geq 1$

Recall that, e.g., [Brezis '11, Prop. 8.5] shows, for $1 < p \leq \infty$,

$$f \in W^{1,p}(\mathbb{R}) \iff \exists C \geq 0 \; \forall h \in \mathbb{R} \;\; \|f(\cdot + h) - v\|_{L^p(\mathbb{R})} \leq C|h|$$

where $C = \|f'\|_{L^p(\mathbb{R})}$.

Difference operators

Let $\Omega \subset \mathbb{R}$ be a domain. Given $h \in \mathbb{R}^d$, set

$$\Omega_h := \{x \in \Omega \mid [x, x+h] \subset \Omega\}$$

and define

$$\Delta_h f(x) := \Delta_h(f, x, \Omega) := \begin{cases} f(x+h) - f(x), & x \in \Omega_h, \\ 0, & \text{otherwise.} \end{cases}$$

and, for $k \in \mathbb{N}$,

$$\Delta_h^k f = \Delta_h(\Delta_h^{k-1}) f \quad \text{in } \Omega.$$

Notice

$$P \in \mathbb{P}^\ell \implies \Delta_h^{\ell+1} P = 0.$$

Moduli of smoothness

Given $0 < p \leq \infty$ and $t > 0$, we define

$$\omega_k(f,t)_p := \omega_k(f,t,\Omega)_p := \sup_{|h|\leq t} \left\| \Delta_h^k f \right\|_{L^p(\Omega)}.$$

We have

- $\omega_k(f,\cdot)_p$ is increasing with $\omega_k(f,0)_p = 0$
- $\omega_k(\cdot,t)_p$ is a (quasi-)seminorm on $L^p(\Omega)$
- if

  $$\omega_k(f,t)_p = \begin{cases} o(t^k) & \text{if } p \geq 1, \\ o(t^{k-1+\frac{1}{p}}) & \text{if } 0 < p \leq 1, \end{cases}$$

  then $f \in \mathbb{P}^{k-1}(\Omega)$.

Besov spaces

Let $0 < p \leq \infty$ and $s > 0$, $0 < q \leq \infty$.

Choosing $k = [s] + 1$ and recalling $\int_0^1 t^\rho dt < \infty \iff \rho > -1$, define

$$|f|_{B^s_q(L^p(\Omega))} := \begin{cases} \sup_{t>0} t^{-s}\omega_k(f,t)_p, & q = \infty, \\ \left(\int_0^\infty \left[t^{-s}\omega_k(f,t)_p \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} & 0 < q < \infty, \end{cases}$$

and

$$B^s_q(L^p(\Omega)) := \{ f \in L^p(\Omega) \mid |f|_{B^s_q(L^p(\Omega))} < \infty \},$$
$$\|f\|_{B^s_q(L^p(\Omega))} := \|f\|_{L^p(\Omega)} + |f|_{B^s_q(L^p(\Omega))} .$$

Embeddings

Fix $0 < p \leq \infty$ and let $s_1, s_2, s > 0$ and $0 < q_1, q_2 \leq \infty$.

We have

$$s_1 > s_2 \implies B^{s_1}_{q_1}(L^p(\Omega)) \subset B^{s_2}_{q_2}(L^p(\Omega))$$

and

$$q_1 \leq q_2 \implies B^{s}_{q_1}(L^p(\Omega)) \subset B^{s}_{q_2}(L^p(\Omega))$$

Moreover, if Ω is Lipschitz and $0 < \tau \leq \infty$ then

$$s - \frac{d}{\tau} \geq -\frac{d}{p} \implies B^s_\tau(L^\tau(\Omega)) \subset L^p(\Omega).$$

Bramble-Hilbert lemma

Let $0 < p < \infty$ and $0 < s \leq \ell + 1$, $0 < \tau < \infty$ such that

$$\delta := s - \frac{d}{\tau} + \frac{d}{p} \geq 0.$$

Then, for any simplex K,

$$\inf_{P \in \mathbb{P}^\ell} \|v - P\|_{L^p(K)} \preccurlyeq \operatorname{diam}(K)^\delta \, |v|_{B^s_\tau(L^\tau(K))} \, ;$$

the hidden constant depends on p, s, τ, d and the shape coefficient of K.

See [Gaspoz/Morin '14, Lemma 4.15]

More smoothness moduli

Define

$$w_k(f,t,\Omega)_{p,q} := \left[\frac{1}{(2t)^d} \int_{[-t,t]^d} \left\| \Delta_h^k(f,\cdot,\Omega) \right\|_{L^p(\Omega)}^q dh \right]^{1/q}$$

and notice that if Ω_1, Ω_2 is a partition of Ω, then

$$w_k(f,t,\Omega_1)_{p,p} + w_k(f,t,\Omega_2)_{p,p} \leq w_k(f,t,\Omega_1 \cup \Omega_2)_{p,p}$$

Setadditivity of Besov seminorms

Given a simplicial mesh $\mathcal{M}$ of Ω, we have

$$w_k(f,t,K)_{p,q} \approx \omega_k(f,t,K)_p$$

for all $K \in \mathcal{M}$ and $t \preccurlyeq \operatorname{diam} K$; the hidden constants depend on k, p, q, and the shape coefficient $\sigma(\mathcal{M})$.

Therefore

$$\sum_{K \in \mathcal{M}} |v|^p_{B^s_p(L^p(K))} \preccurlyeq |v|^p_{B^s_p(L^p(\Omega))}$$

See [Gaspoz/Morin '14, Corollary 4.3 and Lemma 4.10].

Outline

Besov regularity and convergence rates

Let $\mathcal{M}_0$ be an admissible (in 2d: with coinciding edge labeling) initial mesh of Ω and $\mathbb{M}_n^{\mathcal{T},\mathrm{conf}}$ the corresponding set of all conforming refinements with at most n simplex bisections.

Then

$$v \in B^s_\tau\big(L^\tau(\Omega)\big) \quad \text{with} \quad s - \frac{d}{\tau} > 1 - \frac{d}{2}$$

implies the following decay rate for the best H^1_0-error:

$$E\big(v, \mathbb{P}^\ell(\mathbb{M}_n^{\mathcal{T},\mathrm{conf}})\big) = O\left(n^{-\frac{\min(s,r+1)-1}{d}}\right).$$

Generating a mesh with maximum strategy

Let $t > 0$ to be chosen later. Taking

$$\epsilon(K) = E\big(\mathbb{P}^\ell(K)\big)^2 = \inf_{P \in \mathbb{P}^\ell} \|\nabla(v - P)\|_K^2 \,,$$

run

$\mathcal{M} := \mathcal{M}_0$
$\texttt{while } \{K \in \mathcal{M} \mid \epsilon(K) > t\} \neq \emptyset$
    $\texttt{pick } K \in \mathcal{M} \texttt{ with } \epsilon(K) > t$
    $\mathcal{M} := \texttt{rec-bisect}(\mathcal{M}, K)$
$\mathcal{M}_t := \mathcal{M}$

Collect the picked elements in $\mathcal{M}_t^\star$; we have to bound $\#\mathcal{M}_t^\star$.

A bound for the local best error

Using the Sobolev embedding, we may assume $s \leq \ell + 1$ with changing τ but without changing

$$\delta := s - \frac{d}{\tau} - \left(1 - \frac{d}{2}\right) > 0.$$

Using the Bramble-Hilbert lemma, we derive

$$\begin{aligned} \inf_{P \in \mathbb{P}^\ell} \|\nabla(v - P)\|_K &\preccurlyeq \inf_{Q \in [\mathbb{P}^{\ell-1}]^d} \|\nabla v - Q\|_K \\ &\preccurlyeq \operatorname{diam}(K)^\delta |\nabla v|_{B^s_\tau(L^\tau(K))} \end{aligned}$$

Counting requested bisections by generations – 1

Denote by $g(K)$ the generation of $K \in \mathcal{T}$. Given $j \in \mathbb{N}_0$, consider

$$L_j := \{K \in \mathcal{M}_t^\star \mid g(K) = j\},$$

whose elements are disjoint.

Therefore, with $\tilde{\delta} = \delta / d > 0$,

$$t^{\tau/2} \# L_j \leq \sum_{K \in L_j} \epsilon(K)^{\tau/2} \preccurlyeq 2^{-\tilde{\delta}\tau j} \sum_{K \in L_j} |v|^\tau_{B^s_\tau(L^\tau(K))} \preccurlyeq 2^{-\tilde{\delta}\tau j} |v|^\tau_{B^s_\tau(L^\tau(\Omega))},$$

ie

$$\# L_j \preccurlyeq |v|^\tau_{B^s_\tau(L^\tau(\Omega))} \, t^{-\tau/2} \, 2^{-\tilde{\delta}\tau j}.$$

Counting requested bisections by generations – 2

Independently of t, we have

$$2^{-j} \# L_j \preccurlyeq |\Omega|, \quad \text{ie} \quad \# L_j \preccurlyeq |\Omega| \, 2^j.$$

In summary,

$$\# \mathcal{M}_t^\star = \sum_{j=0}^{\infty} \# L_j \preccurlyeq \sum_{j=0}^{\infty} \min \left\{ 2^j, |v|^\tau_{B^s_\tau(L^\tau(\Omega))} \, t^{-\tau/2} 2^{-\tilde{\delta}\tau j} \right\}$$

$$\preccurlyeq |v|^\tau_{B^s_\tau(L^\tau(\Omega))} \, t^{-\tau/2} 2^{-\tilde{\delta}\tau k} \preccurlyeq \left(|v|_{B^s_\tau(L^\tau(\Omega))} \, t^{-1/2} \right)^{\frac{\tau}{1+\tilde{\delta}\tau}}$$

with k satisfying

$$2^k \approx |v|^\tau_{B^s_\tau(L^\tau(\Omega))} \, t^{-\tau/2} 2^{-\tilde{\delta}\tau k} \quad \text{ie} \quad 2^k \approx \left(|v|^\tau_{B^s_\tau(L^\tau(\Omega))} \, t^{-\tau/2} \right)^{\frac{1}{1+\tilde{\delta}\tau}}$$

Choosing t

Consequently, the above algorithm terminates and, in view of the cost of maintaining conformity, we have

$$\#\mathcal{M}_t - \#\mathcal{M}_0 \preccurlyeq \#\mathcal{M}_t^\star \preccurlyeq \left(|v|_{B^s_\tau(L^\tau(\Omega))} \, t^{-1/2} \right)^{\frac{\tau}{1+\tilde{\delta}\tau}}$$

Choosing

$$t = C \left[|v|_{B^s_\tau(L^\tau(\Omega))} \left(\frac{2}{n} \right)^{\frac{1+\tilde{\delta}\tau}{\tau}} \right]^2 ,$$

we have $\#\mathcal{M}_t - \#\mathcal{M}_0 \le n/2$ and therefore

$$\#\mathcal{M}_t \le (\#\mathcal{M}_t - \#\mathcal{M}_0) + \#\mathcal{M}_0 \le n$$

for all $n \ge 2\#\mathcal{M}_0$.

Error bound

The choice of t yields

$$E\big(S_0(\mathcal{M}_t)\big) \preccurlyeq E\big(\mathbb{P}^\ell(\mathcal{M}_t)\big) \preccurlyeq \#\mathcal{M}_t^{1/2} t^{1/2}$$
$$\preccurlyeq |v|_{B^s_\tau(L^\tau(\Omega))} \, n^{\left(\frac{1}{2} - \frac{1}{\tau} - \tilde{\delta}\right)} \preccurlyeq |v|_{B^s_\tau(L^\tau(\Omega))} \, n^{-\frac{s-1}{d}}$$

because

$$\frac{1}{2} - \frac{1}{\tau} - \tilde{\delta} = -\frac{s-1}{d}$$

thanks to

$$\tilde{\delta} = \frac{\delta}{d} = \frac{s}{d} - \frac{1}{\tau} - \frac{1}{d} + \frac{1}{2}.$$