Lectures on

Weak Solutions of Elliptic Boundary Value Problems

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1 Some abstract variational problems

Most partial differential equations in engineering and physics arise out of *variational principles*. We usually have a class of *admissible solutions* (say, displacements) and an energy functional associated to these admissible functions; we seek to minimize the energy to identify the solution of the problem. The corresponding *Euleri-Lagrange equation* gives the partial differential equation we started with. In the next section, we will see several examples of this general situation. In this section, we will discuss some abstract variational problems which will form the basis of our study of elliptic boundary value problems.

One of the classical results in functional analysis is the minimization of the norm (or distance) in a Hilbert space.

Theorem 1.1 *Let H be a real Hilbert space whose norm and inner product are denoted $\|.\|$ and $(.,.)$ respectively. Let $K \subset H$ be a closed convex subset. Let $x \in H$. Then there exists a unique $y \in K$ such that*

$$\|x-y\| \;=\; \min_{z\in K}\|x-z\|. \tag{1.1}$$

Further, y can be characterized as the unique vector in K such that, for all $z \in K$, we have

$$(x-y, z-y) \;\leq\; 0. \tag{1.2}$$

Proof: Let $d = \inf_{z\in K}\|x-z\| \geq 0$. Let $\{y_n\}$ be a minimizing sequence in K, *i.e.* $y_n \in K$ and $\|x-y_n\| \to d$. Then it is clear that $\{y_n\}$ is a bounded sequence in H and so, since H is reflexive, we can extract a weakly convergent subsequence $\{y_{n_k}\}$. Let $y_{n_k} \rightharpoonup y$ *weakly* in H. Then, since K is closed and convex, it is also weakly closed and so $y \in K$. Also, since the norm is a weakly lower semi-continuous functional, we have

$$\|x-y\| \;\leq\; \liminf_{k\to\infty}\|x-y_{n_k}\| \;=\; d.$$

Since $y \in K$, we also have $\|x-y\| \geq d$. Thus we have proved the existence of a vector $y \in K$ satisfying (1.1). If $y' \in K$ also satisfied (1.1), then by the convexity of K, we have $\frac{1}{2}(y+y') \in K$, and by the parallelogram identity we have

$$\left\|\frac{y+y'}{2}-x\right\|^2 \;=\; \frac{1}{2}\|y-x\|^2+\frac{1}{2}\|y'-x\|^2-\left\|\frac{y-y'}{2}\right\|^2 \;<\; d^2$$

which contradicts the definition of d. This proves the uniqueness of y.

We will now prove the characterization of y via (1.2). Let $x \in H$ and $y \in K$ satisfy (1.1). Let $z \in K$. If $0 < t < 1$, then, by the convexity of K, it follows that $tz + (1-t)y \in K$. Then, by virtue of (1.1), we have

$$\|x-y\| \;\leq\; \|x-(tz+(1-t)y)\| \;=\; \|(x-y)-t(z-y)\|.$$

Thus,
$$\|x-y\|^2 \;\leq\; \|x-y\|^2 - 2t(x-y,z-y) + t^2\|z-y\|^2$$
which yields
$$(x-y,z-y) \;\leq\; \frac{t}{2}\|z-y\|^2$$
and (1.2) follows on letting $t \to 0$. Conversely, if $x \in H$ and $y \in K$ satisfy (1.2), then, for $z \in K$, we have
$$\|x-y\|^2 - \|x-z\|^2 \;=\; 2(x-y,z-y) - \|z-y\|^2 \;\leq\; 0$$
from which it follows that x and y satisfy (1.1). ■

Remark 1.1 If $H = \mathbb{R}^2$ with the usual euclidean inner product and if K is a closed convex subset of the plane, the relation (1.2) can be geometrically interpreted as follows: the lines joining the point y with x and with any point $z \in K$ will always contain an obtuse angle. See Figure 1. ■

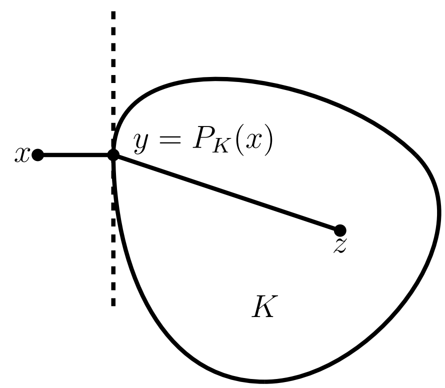

Figure 1

Notation Let H be a real Hilbert space and let $K \subset H$ be a closed convex subset. Let $x \in H$. We will denote the unique vector $y \in K$ satisfying (1.1) or, equivalently (1.2), whose existence is guaranteed by the preceding theorem, by $P_K x$. The mapping $P_K : H \to K$ is, in general, a *nonlinear* map and is usually called the **projection** of H onto K. ■

Corollary 1.1 *Let H be a Hilbert space and let $K \subset H$ be a closed convex subset. Then, for x_1 and x_2 in H, we have*
$$\|P_K x_1 - P_K x_2\| \;\leq\; \|x_1 - x_2\|. \tag{1.3}$$

Proof: It follows from (1.2) that
$$\begin{aligned}(x_1 - P_K x_1, P_K x_2 - P_K x_1) &\leq 0\\ (x_2 - P_K x_2, P_K x_1 - P_K x_2) &\leq 0.\end{aligned}$$

Adding these two relations, we get

$$((x_1 - x_2) - (P_K x_1 - P_K x_2), P_K x_2 - P_K x_1) \ \leq \ 0$$

which leads to

$$\|P_K x_2 - P_K x_1\|^2 \ \leq \ (x_2 - x_1, P_K x_2 - P_K x_1) \ \leq \ \|x_2 - x_1\| \ \|P_k x_2 - P_K x_1\|$$

by the Chauchy-Schwarz inequality, from which (1.3) follows immediately. ■

Definition 1.1 *Let H be a real Hilbert space. and let $a : H \times H \to \mathbb{R}$ be a bilinear form. The bilinear form is said to be* **continuous** *if there exists a constant $M > 0$ such that for all u and v in H, we have*

$$|a(u,v)| \ \leq \ M\|u\| \ \|v\|.$$

*It is said to be H-***elliptic** *if there exists a constant $\alpha > 0$ such that for every $v \in H$, we have*

$$a(v,v) \ \geq \ \alpha\|v\|^2.$$

The bilinear form is said to be **symmetric** *if for every u and v in H, we have*

$$a(u,v) \ = \ a(v,u). \ ■$$

Example 1.1 Let $H = \mathbb{R}^N$ with the usual euclidean inner product. Then every $N \times N$ matrix with real entries defines a continuous bilinear form. If $A = (a_{ij}), 1 \leq i,j \leq N$, and if we have $u = (u_1, \cdots, u_N)$ and $v = (v_1, \cdots, v_N)$, then the bilinear form is defined via the relation

$$a(u,v) \ \stackrel{\text{def}}{=} \ \sum_{i,j=1}^{N} a_{ij} u_j v_i \ = \ v^T A u$$

(if we consider u and v as column vectors and if v^T denotes the transpose of v). By the Cauchy-Schwarz inequality, we have

$$\begin{array}{rcccl} |a(u,v)| & = & |v^T A u| & = & |(v, Au)| \\ & \leq & \|v\| \ \|Au\| & \leq & \|A\| \ \|u\| \ \|v\|. \end{array}$$

If A is a symmetric and positive definite matrix, then the bilinear form is symmetric and H-elliptic since we know that

$$\sum_{ij=1}^{N} a_{ij} v_i v_j \ \geq \ \alpha\|v\|^2$$

where $\alpha > 0$ is the smallest eigenvalue of the matrix A. ■

We will see several examples of elliptic bilinear forms in the next section.

Theorem 1.2 *Let H be a real Hilbert space and let $a : H \times H \to \mathbb{R}$ be a continuous, symmetric and H-elliptic bilinear form. Let $K \subset H$ be a closed convex set. Let $f \in H$. Then, there exists a unique $u \in K$ such that*

$$a(u, v-u) \;\geq\; (f, v-u), \text{ for every } v \in K. \tag{1.4}$$

Further, $u \in K$ can be characterized as the minimizer of $J(.)$ over K, where

$$J(v) \;=\; \frac{1}{2}a(v,v) - (f,v), \;\; v \in H.$$

Proof: Define, for $u, v \in H$,

$$< u, v > \stackrel{\text{def}}{=} \; a(u,v).$$

Then, by the bilinearity, symmetry and ellipticity of $a(.,.)$, this defines an inner product on H. Let the corresponding norm be denoted by $\|.\|_a$ so that, for all $v \in H$, we have

$$\|v\|_a^2 \;=\; a(v,v).$$

The continuity and ellipticity of the bilinear form yield

$$\alpha\|v\|^2 \;\leq\; \|v\|_a^2 \;\leq\; M\|v\|^2$$

for every $v \in H$ and so the two norms are equivalent and H is a Hilbert space with the new norm as well. Now, by the Riesz representation theorem, there exists $\widetilde{f} \in H$ such that for every $v \in H$, we have

$$a(\widetilde{f}, v) \;= < \widetilde{f}, v > = \; (f, v)$$

since $v \mapsto (f,v)$ defines a continuous linear functional on H with the norm $\|.\|_a$. Then

$$\begin{array}{rcl} \frac{1}{2}\|v-\widetilde{f}\|_a^2 & = & \frac{1}{2}a(v-\widetilde{f}, v-\widetilde{f}) \\ & = & \frac{1}{2}a(v,v) - a(v,\widetilde{f}) + \frac{1}{2}a(\widetilde{f},\widetilde{f}) \\ & = & \frac{1}{2}a(v,v) - (f,v) + \frac{1}{2}\|\widetilde{f}\|_a^2 \\ & = & J(v) + \frac{1}{2}\|\widetilde{f}\|_a^2. \end{array}$$

Since $\frac{1}{2}\|\widetilde{f}\|_a^2$ is a constant, minimizing J over K is the same as minimizing $\|v - \widetilde{f}\|_a$ over K which yields, thanks to the characterization (1.2), the existence of a unique $u \in K$ such that

$$< \widetilde{f} - u, v - u > \leq \; 0$$

for all $v \in K$, which is the same as (1.4). ∎

We now show that we can relax the condition of symmetry of the bilinear form and still have a unique solution to (1.4). Of course, we can no longer identify the solution as the minimizer of a functional of the form J.

Theorem 1.3 *(Stampacchia) Let H be a real Hilbert space and let $K \subset H$ be a closed convex set. Let $a(.,.)$ be a continuous and H-elliptic bilinear form on H. Given $f \in H$, there exists a unique $u \in K$ such that (1.4) is satisfied for every $v \in K$.*

Proof: Let $w \in H$ be a fixed element. By the continuity of $a(.,.)$, the map $v \mapsto a(w,v)$ is a continuous linear functional and so there exists (by the Riesz representation theorem) an element, denoted Aw, in H such that

$$(Aw, v) \ = \ a(w, v)$$

for every $v \in H$. The bilinearity of $a(.,.)$ implies the linearity of the map $w \mapsto Aw$ and, the continuity and H-ellipticity of $a(.,.)$ imply that

$$\left.\begin{array}{rcl} \|Aw\| & \leq & M\|w\|, \\ (Aw, w) & \geq & \alpha\|w\|^2 \end{array}\right\} \tag{1.5}$$

for all $w \in H$. Thus A is a bounded linear operator on H. Now, (1.4) can be writtten as

$$(Au, v-u) \ \geq \ (f, v-u), \ \text{for all } v \in K. \tag{1.6}$$

Let $\rho > 0$ be a positive constant, to be chosen presently. Then (1.6) is equivalent to finding $u \in K$ such that

$$(\rho f - \rho A u + u - u, v - u) \ \leq 0$$

for all $v \in K$. In other words (cf. (1.2)) we seek $u \in K$ such that

$$u \ = \ P_K(\rho f - \rho A u + u).$$

Thus we are looking for a fixed point of the map $F : H \to H$ (whose range lies in K) defined by

$$F(v) \ \stackrel{\text{def}}{=} \ P_K(\rho f - \rho A v + v).$$

Now, if $v, w \in H$, we have by Corollary 1.1, that

$$\|F(v) - F(w)\| \ \leq \ \|(v-w) - \rho A(v-w)\|.$$

Thus, by (1.5), we get

$$\begin{array}{rcl} \|F(v) - F(w)\|^2 & \leq & \|v-w\|^2 - 2\rho(A(v-w), v-w) + \rho^2\|A(v-w)\|^2 \\ & \leq & (1 - 2\rho\alpha + \rho^2 M^2)\|v-w\|^2. \end{array}$$

If we now choose ρ such that $0 < \rho < \frac{2\alpha}{M^2}$, then

$$1 - 2\rho\alpha + \rho^2 M^2 \ < \ 1$$

and so F is a contraction mapping and by the contraction mapping theorem, F has a unique fixed point u which must belong to K. This completes the proof. ■

If $K = V$, a closed *subspace* of H, then it is obviously convex. In that case, for any $w \in V$, set $v = w + u$ in (1.4). Thus, we get, for every $w \in V$,

$$a(u,w) \;\geq\; (f,w).$$

Since V is a subspace, this inequality also holds when $-w$ replaces w and so we have

$$a(u,w) \;=\; (f,w), \text{ for every } w \in V. \tag{1.7}$$

Notice that if we set $w = u$, then

$$\alpha\|u\|^2 \;\leq\; a(u,u) \;=\; (f,u) \;\leq\; \|f\|\,\|u\|,$$

which yields

$$\|u\| \;\leq\; \frac{1}{\alpha}\|f\|. \tag{1.8}$$

Thus, the mapping $f \mapsto u$ is a bounded linear operator from H into V. We have thus proved the following result.

Theorem 1.4 *(Lax-Milgram lemma) Let H be a Hilbert space and let $V \subset H$ be a closed subspace. Let $a(.,.)$ be a continuous and H-elliptic bilinear form on H. Let $f \in H$. Then, there exists a unique $u \in V$ such that (1.7) holds. In particular, this is true when $V = H$ itself. The mapping $G : H \to V$ defined by $Gf = u$ is a bounded linear operator and (1.8) holds. In addition, if $a(.,.)$ is symmetric, then u is the minimizer of the functional*

$$J(v) \;=\; \frac{1}{2}a(v,v) - (f,v)$$

over V. ■

Remark 1.2 In particular, if $a(u,v) = (u,v)$, the inner product on H, it follows that P_V satisfies the relation

$$(P_V f, w) \;=\; (f,w)$$

for every $w \in V$. In this case it is now clear that the projection P_V is a *linear operator* and is called the *orthogonal projection of H onto V*. ■

Remark 1.3 It is a simple exercise, based on the Riesz representation theorem, to see that in Theorems 1.2-1.4, we can have, on the right-hand side of (1.4), $\varphi(v-u)$ where $\varphi \in H'$, the dual of H, in place of $(f, v-u)$, $f \in H$. ■

Remark 1.4 When the bilinear form $a(.,.)$ is symmetric, and when $K = H$, the global minimum of the corresponding functional J defined in the Lax-Milgram lemma (Theorem 1.4) is attained at $u \in V$ which satisfies (1.7).

These can be considered as the *Euler-Lagrange equations* of the unconstrained optimization problem. When K is a closed convex proper subset, then we have a constrained optimization problem and we cannot expect the solution to satisfy equations but only the inequalities (1.4). These are called *variational inequalities.* ■

We will conclude this section with yet another abstract variational problem.

If Σ and V are Hilbert spaces, then a bilinear form $b : \Sigma \times V \to \mathbb{R}$ is said the be continuous if there exists a constant $M > 0$ such that

$$|b(\sigma, v)| \ \leq \ M\|\sigma\|_\Sigma \|v\|_V$$

for every $\sigma \in \Sigma$ and for every $v \in V$.

Theorem 1.5 *(Babuška-Brezzi) Let Σ and V be real Hilbert spaces. Let $a : \Sigma \times \Sigma \to \mathbb{R}$ and $b : \Sigma \times V \to \mathbb{R}$ be continuous bilinear forms. Let*

$$Z \ = \ \{\sigma \in \Sigma \mid b(\sigma, v) = 0 \text{ for every } v \in V\}.$$

Assume that $a(.,.)$ is Z-elliptic, i.e. there exists a constant $\alpha > 0$ such that for every $\sigma \in Z$, we have

$$a(\sigma, \sigma) \ \geq \ \alpha\|\sigma\|_\Sigma^2.$$

Assume further that the bilinear form $b(.,.)$ satisfies the Babuška-Brezzi *condition (also called the* inf-sup *condition): there exists $\beta > 0$ such that for every $v \in V$, we have*

$$\sup_{\substack{\tau \in \Sigma \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_\Sigma} \ \geq \ \beta\|v\|_V. \tag{1.9}$$

Let $\kappa \in \Sigma$ and let $\ell \in V$. Then, there exists a unique pair $(\sigma, u) \in \Sigma \times V$ such that

$$\left.\begin{array}{rcll} a(\sigma, \tau) + b(\tau, u) & = & (\kappa, \tau)_\Sigma & \text{for every } \tau \in \Sigma, \\ b(\sigma, v) & = & (\ell, v)_V & \text{for every } v \in V. \end{array}\right\} \tag{1.10}$$

where $(.,.)_\Sigma$ and $(.,.)_V$ denote the inner products in Σ and V respectively.

Proof: As in the case of Theorem 1.3, we can define continuous linear operators $A : \Sigma \to \Sigma$ and $B : \Sigma \to V$ such that for every $\sigma, \tau \in \Sigma$ and for every $v \in V$, we have

$$\begin{array}{rcl} (A\sigma, \tau)_\Sigma & = & a(\sigma, \tau) \\ (B\sigma, v)_V & = & b(\sigma, v). \end{array}$$

Then (1.10) is equivalent to solving the system of equations:

$$\begin{array}{rcl} A\sigma + B^*u & = & \kappa, \\ B\sigma & = & \ell, \end{array}$$

where $B^* : V \to \Sigma$ is the adjoint of B.

By virtue of (1.9), we deduce that for every $v \in V$ we have

$$\|B^* v\|_\Sigma \;\geq\; \beta \|v\|_V.$$

Thus B^* has closed range and is injective. Since we are dealing with continuous linear operators, it follows that B is surjective. Let us choose $\sigma_1 \in \Sigma$ such that $B\sigma_1 = \ell$. Now, $Z = \ker(B)$ is a closed subspace of Σ and so, by the Z-ellipticity of $a(.,.)$, it follows from the Lax-Milgram lemma (Theorem 3.1.4) that there exists a unique $\sigma_0 \in Z$ such that

$$a(\sigma_0, \tau) \;=\; (\kappa, \tau)_\Sigma - a(\sigma_1, \tau) \tag{1.11}$$

for every $\tau \in Z$. Since $Z = \ker(B)$, it follows that $B\sigma_0 = 0$ and so, if we set $\sigma = \sigma_1 + \sigma_0$, we still have $B\sigma = \ell$. It also follows from (1.11) that $\kappa - A\sigma \in Z^\perp$, the orthogonal complement of Z in Σ. But we know that

$$\text{Range}(B^*) \;=\; (\ker(B))^\perp \;=\; Z^\perp.$$

Thus, there exists $u \in V$ such that $B^* u = \kappa - A\sigma$ and so (σ, u) solves (1.10).

To prove the uniqueness, let, if possible $(\widetilde{\sigma}, \widetilde{u})$ be another solution pair. Set $\sigma' = \sigma - \widetilde{\sigma}$ and $u' = u - \widetilde{u}$. Then

$$\left.\begin{array}{rcl} a(\sigma', \tau) + b(\tau, u') &=& 0, \quad \text{for every } \tau \in \Sigma, \\ b(\sigma', v) &=& 0, \quad \text{for every } v \in V. \end{array}\right\} \tag{1.12}$$

Thus, $\sigma' \in Z$ and setting $\tau = \sigma'$ in the first equation of (1.12), we get $a(\sigma', \sigma') = 0$ which yields $\sigma' = 0$. It follows from this that $B^* u' = 0$ which implies that $u' = 0$. This proves the uniqueness of the solution and completes the proof. ■

Remark 1.5 As mentioned in Remark 1.3, it is easy to see in the case of (1.10) the right-hand sides $(\kappa, \tau)_\Sigma$ and $(\ell, v)_V$ can be replaced by continuous linear functionals $\varphi(\tau)$ and $f(v)$ respectively. In this case we will have $A : \Sigma \to \Sigma'$, $B : \Sigma \to V'$ and $B^* : V \to \Sigma'$, where Σ' and V' are the duals of Σ and V respectively. ■

Remark 1.6 If the bilinear form $a(.,.)$ is symmetric, then we can consider the constrained optimization problem: find $\sigma \in \Sigma$ such that $B\sigma = \ell$ and

$$J(\sigma) \;=\; \min_{B\tau = \ell} J(\tau)$$

where

$$J(\tau) = \frac{1}{2} a(\tau, \tau) - (\kappa, \tau)_\Sigma.$$

Then we introduce the Lagrangian

$$\mathcal{L}(\tau, v) \;\stackrel{\text{def}}{=}\; \left[\frac{1}{2} a(\tau, \tau) - (\kappa, \tau)_\Sigma\right] + [b(\tau, v) - (\ell, v)_V]$$

and look for a saddle point. The corresponding set of equations satisfied by the saddle point will be preciosely the set (1.10).

Another way to look at (1.10) is from the point of view of elasticity. The first equation in (1.10) can be viewed as a *constitutive law* (for instance a stress-strain relationship like *Hooke's law*) and the second equation will correspond to the equlibrium equation or the equation of balance of forces. ■

In the next section, we will see numerous applications of the theorems of Lax-Milgram, Babuška-Brezzi and Stampacchia. While the Lax-Milgram formulation is the starting point of the so called *direct finite element methods*, the Babuška-Brezzi theorem will provide the basis for the *mixed finoite element methods*.

2 Examples of elliptic boundary value problems

In this section, we will present several examples of elliptic boundary value problems. In each case, we will state what we mean by a **weak solution** of the problem and study its existence and uniqueness using the results of the previous section.

Throughout this section, unless otherwise specified, Ω will denote a **bounded open set** in $\mathbb{R}^N$ and $\Gamma = \partial\Omega$ will denote its boundary.

2.1 The Dirichlet problem for second order elliptic operators

Consider the following problem:

$$\left.\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega, \\ u & = & 0 & \text{on } \Gamma, \end{array}\right\} \tag{2.1}$$

where $f : \Omega \to \mathbb{R}$ is a given function. A **classical solution** of (2.1) when $f \in \mathcal{C}(\overline{\Omega})$, is a function $u \in \mathcal{C}^2(\overline{\Omega})$ which satisfies this equation pointwise. If u is a classical solution, let us multiply the differential equation in (2.1) by $\varphi \in \mathcal{D}(\Omega)$ and integrate over Ω to get

$$-\int_\Omega (\Delta u)\varphi \, dx \;=\; \int_\Omega f\varphi \, dx.$$

Applying Green's theorem (integration by parts) and using the fact that φ vanishes on the boundary, we get

$$\int_\Omega \nabla u.\nabla\varphi \, dx \;=\; \int_\Omega f\varphi \, dx.$$

Since $u \in \mathcal{C}^2(\overline{\Omega})$ and $u = 0$ on Γ, we have that $u \in H_0^1(\Omega)$ and also $f \in L^2(\Omega)$. Further, $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$ and both sides of the above equation are continuous in φ with respect to the $H_0^1(\Omega)$-topology. Thus, it follows by density that

$$\int_\Omega \nabla u.\nabla v \, dx \;=\; \int_\Omega fv \, dx, \text{ for every } v \in H_0^1(\Omega). \tag{2.2}$$

Notice that in (2.2) *we do not need any information on the second derivatives of* u. Hence we say that if $u \in H_0^1(\Omega)$ satisfies (2.2), then it is a **weak solution** of (2.1). We have just seen that every classical solution is automatically a weak solution. We now show that a weak solution always exists uniquely for certain classes of functions f.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $\Gamma = \partial\Omega$. Let $f \in L^2(\Omega)$. Then there exists a unique weak solution $u \in H_0^1(\Omega)$ to (2.1). Further, u can be characterized as the minimizer in $H_0^1(\Omega)$ of the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by*

$$J(v) \;=\; \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx - \int_\Omega fv \, dx.$$

Proof: Set $V = H_0^1(\Omega)$ and let

$$a(u,v) \;=\; \int_\Omega \nabla u.\nabla v \, dx.$$

Since, by Poincaré's inequality, this is an inner product on $H_0^1(\Omega)$ whose norm $|.|_{1,\Omega}$ is equivalent to the usual norm $\|.\|_{1,\Omega}$, this is a symmetric, continuous and V-elliptic bilinear form on V. Further $v \mapsto \int_\Omega fv \, dx$ is a continuous linear functional on v since, again, by Poincaré's inequality, we have

$$\left|\int_\Omega fv \, dx\right| \;\leq\; |f|_{0,\Omega}|v|_{0,\Omega} \;\leq\; C|f|_{0,\Omega}|v|_{1,\Omega}.$$

The result now follows from the Lax-Milgram lemma (Theorem 1.4). ■

Remark 2.1. The same result is valid when $f \in H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$. We only need to replace $\int_\Omega fv \, dx$ by $< f, v >$, the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, *i.e.* the action of the linear functional f on v, in (2.2). ■

If $\varphi \in \mathcal{D}(\Omega)$, we get from (2.2) and Green's theorem that

$$-\int_\Omega (\Delta u)\varphi \, dx \;=\; \int_\Omega f\varphi \, dx.$$

Thus $-\Delta u = f$ in the sense of distributions. Further, if we know *a priori* that a weak solution u belongs to $\mathcal{C}^2(\overline{\Omega})$ and if $f \in \mathcal{C}(\overline{\Omega})$, then u has to be a classical solution of (2.1). For, in that case we have $u = 0$ on Γ and since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, we deduce that as functions in $L^2(\Omega)$ we have

$-\Delta u = f$ and so they are equal almost everywhere; but since they are continuous functions, they are therefore equal pointwise everywhere.

The question, therefore, is: *when is a weak solution smooth enough?* This is answered by a **regularity theorem**. We merely state for the record that if $f \in L^2(\Omega)$ and if Ω is a 'reasonably smooth domain', then the weak solution $u \in H^1_0(\Omega)$ is actually in $H^2(\Omega) \cap H^1_0(\Omega)$. If $f \in H^m(\Omega)$ and if Ω is of class $\mathcal{C}^{m+2}$, then $u \in H^{m+2}(\Omega) \cap H^1_0(\Omega)$. Thus, combining such information with the Sobolev imbedding theorems, we can decide whether a weak solution is classical or not.

We now turn to the inhomogeneous Dirichlet problem. Let $f : \Omega \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$ be given functions. Consider the problem:

$$\left.\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega, \\ u & = & g & \text{on } \Gamma. \end{array}\right\} \tag{2.3}$$

As before, would like to look for a weak solution inside $H^1(\Omega)$. If $u \in H^1(\Omega)$, then its trace belongs to $H^{\frac{1}{2}}(\Gamma)$. Thus, we assume that $g \in H^{\frac{1}{2}}(\Gamma)$ and that $f \in L^2(\Omega)$. Then, by the trace theorem, there exists $\widetilde{u} \in H^1(\Omega)$ such that

$$\widetilde{u}|_\Gamma \;=\; \gamma_0(\widetilde{u}) \;=\; g.$$

Now define

$$K \;=\; \widetilde{u} + H^1_0(\Omega) \;=\; \{v \in H^1(\Omega) \mid v - \widetilde{u} \in H^1_0(\Omega)\}.$$

Thus, K is a closed affine subspace of $H^1(\Omega)$ and we seek $u \in K$. We define a weak solution of (2.3) as an element $u \in K$ such that

$$\int_\Omega \nabla u.\nabla v \; dx \;=\; \int_\Omega fv \; dx \text{ for every } v \in H^1_0(\Omega). \tag{2.4}$$

We leave it to the reader to check that every classical solution is weak and that every weak solution in $\mathcal{C}^2(\overline{\Omega})$, with $f \in \mathcal{C}(\overline{\Omega})$ and $g \in \mathcal{C}(\Gamma)$, is classical. Choosing $v = \varphi \in \mathcal{D}(\Omega)$, we see that a weak solution satisfies $-\Delta u = f$ in the sense of distributions and we have imposed the condition that $u = g$ on Γ in the sense of trace. As for the existence of a weak solution u, set $u = \widetilde{u} + w$, where $w \in H^1_0(\Omega)$. Then, we seek $w \in H^1_0(\Omega)$ such that

$$\int_\Omega \nabla w.\nabla v \; dx \;=\; \int_\Omega fv \; dx - \int_\Omega \nabla\widetilde{u}.\nabla v \; dx \tag{2.5}$$

for every $v \in H^1_0(\Omega)$. Now the map

$$v \mapsto \int_\Omega fv \; dx - \int_\Omega \nabla\widetilde{u}.\nabla v \; dx$$

defines a continuous linear functional on $H^1_0(\Omega)$ and so can be written as $< F, v >$, with $F \in H^{-1}(\Omega)$. Thus, by the Lax-Milgram theorem or by

Theorem 2.1, we deduce the existence of a unique w satisfying the above equation which leads to the existence of a weak solution u to (2.3).

We now prove the uniqueness of the weak solution (this is not obvious from the uniqueness of w, since $\widetilde{u}$ is not uniquely defined!). If u_1 and u_2 are two weak solutions to (2.3), then $u_1 - u_2 \in H_0^1(\Omega)$ and

$$\int_\Omega \nabla(u_1 - u_2).\nabla v \, dx \;=\; 0$$

for every $v \in H_0^1(\Omega)$. Choosing $v = u_1 - u_2$, we deduce that $|u_1 - u_2|_{1,\Omega} = 0$, which, by Poincaré's inequality, implies that $u_1 = u_2$.

We finally show that u depends continuously on the data f and g. Since the trace map γ_0 is surjective, it has a continuous right inverse. Thus there exists a constant $\widetilde{C} > 0$ and for every $g \in H^{\frac{1}{2}}(\Gamma)$, there exists $\widetilde{u} \in H^1(\Omega)$ such that $\gamma_0(\widetilde{u}) = g$ and

$$\|\widetilde{u}\|_{1,\Omega} \;\leq\; \widetilde{C}\|g\|_{\frac{1}{2},\Gamma}.$$

Setting $v = w$ in (2.5), we get

$$|w|_{1,\Omega}^2 \;\leq\; |f|_{0,\Omega}|w|_{0,\Omega} + |\widetilde{u}|_{1,\Omega}|w|_{1,\Omega} \;\leq\; (C_1|f|_{0,\Omega} + \|\widetilde{u}\|_{1,\Omega})|w|_{1,\Omega}$$

using Poincaré's inequality. Thus it follows that

$$|w|_{1,\Omega} \;\leq\; C_1|f|_{0,\Omega} + \|\widetilde{u}\|_{1,\Omega} \;\leq\; C_1|f|_{0,\Omega} + \widetilde{C}|g|_{\frac{1}{2},\Gamma}.$$

Again, by Poincaré's inequality, $\|w\|_{1,\Omega} \leq C_2|w|_{1,\Omega}$ and so since $u = \widetilde{u} + w$, we deduce that there exists a constant $C > 0$, depending only on Ω, such that

$$\|u\|_{1,\Omega} \;\leq\; C(|f|_{0,\Omega} + |g|_{\frac{1}{2},\Gamma})$$

which proves the continuous dependence of u on the data f and g.

Remark 2.2 Existence, uniqueness and continuous dependence on the data are the criteria for the **well-posedness** of a problem in the sense of Hadamard. ∎

We can generalize the preceding considerations to cover the case of a *second order elliptic operator*. Let $a_{ij} \in \mathcal{C}^1(\overline{\Omega}), 1 \leq i,j \leq N$ be functions satisfying the *ellipticity condition*

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \;\geq\; \alpha|\xi|^2 \tag{2.6}$$

for all $x \in \Omega$ and for all $\xi = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N$, where $\alpha > 0$ is a constant independent of x and ξ. Let $a_0 \in \mathcal{C}(\overline{\Omega})$. Consider the problem:

$$\left.\begin{array}{rcl} -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + a_0 u &=& f \;\text{ in } \Omega, \\ u &=& 0 \;\text{ on } \Gamma. \end{array}\right\} \tag{2.7}$$

The differential operator in the above equation is said to be a *uniformly elliptic operator in divergence form.*. If $f \in L^2(\Omega)$, a **weak solution** is defined as $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \; dx + \int_\Omega a_0 uv \; dx \;\; = \;\; \int_\Omega fv \; dx \tag{2.8}$$

for every $v \in H_0^1(\Omega)$. Again, if $f \in \mathcal{C}(\overline{\Omega})$, it is easy to check that every classical solution is a weak solution and also that a smooth weak solution is classical. If $a_0(x) \geq 0$ for all $x \in \Omega$, then the bilinear form

$$a(u,v) \;\; = \;\; \int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \; dx + \int_\Omega a_0 uv \; dx$$

is $H_0^1(\Omega)$-elliptic by virtue of the ellipticity condition (2.6) and Poincaré's inequality. Thus, by the Lax-Milgram lemma (Theorem 1.4), there exists a unique weak solution in this case. In addition if $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq N$, then $a(.,.)$ is symmetric and so the weak solution u minimizes the functional J on $H_0^1(\Omega)$, where

$$J(v) \;\; = \;\; \frac{1}{2} \int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_i} \; dx + \frac{1}{2} \int_\Omega a_0 v^2 \; dx - \int_\Omega fv \; dx.$$

Notice that to define and prove the existence of a weak solution via (2.8), it suffices that a_{ij}, a_0 are all in $L^\infty(\Omega)$ and $f \in H^{-1}(\Omega)$.

More generally, we can consider the following second order elliptic boundary value problem:

$$\left. \begin{array}{rcl} -\sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^N a_i \frac{\partial u}{\partial x_i} + a_0 u & = & f \quad \text{in } \Omega, \\ u & = & 0 \quad \text{on } \Gamma, \end{array} \right\} \tag{2.9}$$

Where the $a_{ij}, 1 \leq i, j \leq N$ satisfy the ellipticity condition (2.6) and $a_i \in C(\overline{\Omega})$ for $0 \leq i \leq N$. A **weak solution** is a function $u \in H_0^1(\Omega)$ such that

$$a(u,v) \;\; = \;\; \int_\Omega fv \; dx \tag{2.10}$$

for every $v \in H_0^1(\Omega)$, where

$$a(u,v) \;\; = \;\; \int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \; dx + \int_\Omega \sum_{i=1}^N a_i \frac{\partial u}{\partial x_i} v \; dx + \int_\Omega a_0 uv \; dx.$$

This bilinear form is not symmetric in general. **If** it is $H_0^1(\Omega)$-elliptic, then a unique weak solution exists by the Lax-Milgram lemma (Theorem 1.4). In general, we have the following result.

Theorem 2.2 *Assume that $a_{ij} \in L^\infty(\Omega)$ for $1 \le i,j \le N$ and satisfy the ellipticity condition (2.6). Assume that $a_i \in L^\infty(\Omega)$ for $0 \le i \le N$. Let $f \in L^2(\Omega)$. When $f = 0$, the set of weak solutions of (2.9) is a finite dimensional subspace of $H_0^1(\Omega)$. Assume that the dimension is d. Then, there exists a d-dimensional subspace $F \subset L^2(\Omega)$ such that (2.10) has a solution if, and only if, $f \in F^\perp$, the orthogonal complement of F in $L^2(\Omega)$.*

Proof: Choose $\lambda > 0$ such that

$$a_0(x) + \lambda \ge \gamma > 0$$

for all $x \in \Omega$. Let $|a_i|_{0,\infty,\Omega} \le \beta$ for $0 \le i \le N$. Then, for any $v \in H_0^1(\Omega)$, we have

$$\begin{aligned} a(v,v) + \lambda \textstyle\int_\Omega v^2 \, dx &\ge \alpha |v|_{1,\Omega}^2 - \beta |v|_{1,\Omega} |v|_{0,\Omega} + \gamma |v|_{0,\Omega}^2 \\ &= \alpha |v|_{1,\Omega}^2 + \left(\gamma^{\frac{1}{2}} |v|_{0,\Omega} - \tfrac{\beta \gamma^{-\frac{1}{2}}}{2} |v|_{1,\Omega} \right)^2 - \tfrac{\beta^2}{4\gamma} |v|_{1,\Omega}^2 \\ &\ge \left(\alpha - \tfrac{\beta^2}{4\gamma} \right) |v|_{1,\Omega}^2. \end{aligned}$$

We can choose λ large enough so that $\alpha - \frac{\beta^2}{4\gamma} > 0$. Then the bilinear form

$$a(u,v) + \lambda \int_\Omega uv \, dx$$

will be $H_0^1(\Omega)$-elliptic and given $f \in L^2(\Omega)$, there will be a unique solution $u \stackrel{\text{def}}{=} Gf \in H_0^1(\Omega)$ such that

$$a(u,v) + \lambda \int_\Omega uv \, dx \; = \; \int_\Omega fv$$

for every $v \in H_0^1(\Omega)$. It is clear that the map $f \mapsto u = Gf$ is continuous from $L^2(\Omega)$ into $H_0^1(\Omega)$ (cf. Theorem 1.4) and since $H_0^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$ by the Rellich-Kondrosov theorem, it follows that G is a compact operator of $L^2(\Omega)$ into itself. Now, if u is a weak solution of (2.9), then clearly

$$u = G(f + \lambda u).$$

If we set $v = f + \lambda u$, then

$$v - \lambda G v \; = \; f.$$

Now, G is compact and $\lambda > 0$. So $I - \lambda G$ is invertible unless λ^{-1} is an eigenvalue of G. Thus, if λ^{-1} is not an eigenvalue, then a solution exists for all f (and $d = 0$). If λ^{-1} is an eigenvalue, then it has finite geometric multiplicity, since G is compact. So, if $f = 0$, the set of solutions is the corresponding eigenspace, which has finite dimension, say, d. Now, when

$f \neq 0$, by the Fredhölm alternative (cf. Kesavan [4]), solutions exist if, and only if, f satisfies the compatibility condition *viz.* $f \in \ker(I - \lambda G^*)^\perp$ and $\ker(I - \lambda G^*)$ has the same dimension d as $\ker(I - \lambda G)$. ■

Remark 2.3 If we can show that $d = 0$, then we saw that a weak solution exists uniquely for all $f \in L^2(\Omega)$. It can be shown, for instance, by means of a *maximum principle* that if $a_0 \geq 0$, then whatever may be $a_i \in L^\infty(\Omega)$ for $1 \leq i \leq N$, we have $d = 0$. Thus without further hypotheses on the $a_i, 1 \leq i \leq N$,we have the existence and uniqueness of a weak solution to (2.9) when $a_0 \geq 0$ (cf. Gilbarg and Trudinger [3]). ■

2.2 The Neumann problem

Let ν denote the unit outer normal on the boundary Γ of a bounded open set $\Omega \subset \mathbb{R}^N$. We denote by $\frac{\partial u}{\partial \nu}$ the exterior normal derivative on Γ of a smooth function u defined on Ω. We have $\frac{\partial u}{\partial \nu} = \nabla u.\nu$. If $f : \Omega \to \mathbb{R}$ is a given function, then consider the following boundary value problem:

$$\left.\begin{array}{rcll} -\Delta u + u & = & f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} & = & 0 & \text{on } \Gamma. \end{array}\right\} \tag{2.11}$$

If u is a classical (*i.e.* smooth enough) solution of (2.11), then, $u \in H^1(\Omega)$ and, by Green's formula, we have for every $v \in H^1(\Omega)$,

$$\int_\Omega \nabla u.\nabla v \; dx + \int_\Omega uv \;=\; \int_\Omega fv \; dx. \tag{2.12}$$

If $f \in L^2(\Omega)$, then we define a **weak solution** of (2.11) as a function $u \in H^1(\Omega)$ satisfying (2.12) for every $v \in H^1(\Omega)$. If we set

$$a(u, v) \;=\; \int_\Omega \nabla u.\nabla v \; dx + \int_\Omega uv \; dx,$$

then this is just the inner propduct in $H^1(\Omega)$ and so we trivially have the symmetry, continuity and $H^1(\Omega)$-ellipticity of the bilinear form and the existence of a unique weak solution follows immediately from the Lax-Milgram lemma (Theorem 1.4). Further, u will minimize the functional

$$J(v) \;=\; \frac{1}{2}\int_\Omega |\nabla v|^2 \; dx + \frac{1}{2}\int_\Omega |u|^2 \; dx - \int_\Omega fv \; dx$$

over all of $H^1(\Omega)$.

Now let u be a weak solution which also belongs to $H^2(\Omega)$. Then, by retracing the passage from (2.11) to (2.12), we get

$$\int_\Omega (-\Delta u)v \; dx + \int_\Omega uv \; dx + \int_\Gamma \frac{\partial u}{\partial \nu} v \; d\sigma \;=\; \int_\Omega fv \; dx,$$

where $\frac{\partial u}{\partial \nu}$ is nothing but the trace $\gamma_1(u)$ which is well defined since $u \in H^2(\Omega)$. If we choose $v \in \mathcal{D}(\Omega)$, then the integral on Γ will vanish and we deduce from this that $-\Delta u + u = f$ in the sense of distributions. Since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, and since all the integrals over Ω are continuous with respect to v in the $L^2(\Omega)$-topology, we also deduce that this equality holds in the sense of functions in $L^2(\Omega)$. Consequently the above relation be rewritten as

$$\int_\Omega (-\Delta u + u - f)v \; dx + \int_\gamma \frac{\partial u}{\partial \nu} v \; d\sigma \;\; = \;\; 0$$

for all $v \in H^1(\Omega)$ and this now reduces to

$$\int_\Gamma \frac{\partial u}{\partial \nu} v \; d\sigma \;\; = \;\; 0$$

for all $v \in H^1(\Omega)$. But $v|_\Gamma = \gamma_0(v) \in H^{\frac{1}{2}}(\Gamma)$. Since γ_0 is surjective onto this space which is dense in $L^2(\Gamma)$, we deduce from this that

$$\gamma_1(u) \;\; = \;\; \frac{\partial u}{\partial \nu} \;\; = \;\; 0$$

as an element of $L^2(\Gamma)$. Thus, if $u \in H^2(\Omega) \subset H^1(\Omega)$ is a weak solution, it satisfies the differential equation of (2.11) in the sense of distributions and the boundary condition in the sense of trace. If, further, $u \in \mathcal{C}^2(\overline{\Omega})$ (and $f \in \mathcal{C}(\overline{\Omega})$), then u will be a classical solution of (2.11).

Remark 2.4 We must emphasize here the important difference between the Dirichlet and Neumann problems. In case of the former, the boundary condition $u = 0$ (or $u = g$) had to be imposed *a priori* in the space of admissible functions where we seek a (weak) solution. On the other hand, for the Neumann problem, we imposed no such condition on the function space and it turns out that the boundary condition emerges as a natural consequence of the weak formulation. Dirichlet type boundary conditions are referred to as *essential boundary conditions* and have to be taken into account while formulating the problem in the weak sense while Neumann type boundary conditions are called *natural boundary conditions* and take care of themselves. ■

We can also study the Neumann problem for a general second order uniformly elliptic operator as in (2.7) or (2.9). In this case, the natural boundary condition will take the form

$$\sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_j} \nu_i \;\; = \;\; 0.$$

The quantity on the left-hand side of the above equation is called the **conormal derivative** associated to the differential operator and if $A = (a_{ij})$ is the matrix of the coefficients of the highest order terms, this is denoted $\frac{\partial u}{\partial \nu_A}$.

Let us now look at the inhomogeneous Neumann problem. Let $f : \Omega \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$ be given functions. Then we look for a function u such that

$$\left.\begin{array}{rcll} -\Delta u + u & = & f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} & = & g & \text{on } \Gamma. \end{array}\right\} \tag{2.13}$$

We will look for a weak solution in $H^1(\Omega)$ and so we need that $g \in H^{-\frac{1}{2}}(\Gamma)$. If $< .,. >_\Gamma$ denotes the duality pairing between $H^{-\frac{1}{2}}(\gamma)$ and $H^{\frac{1}{2}}(\Gamma)$, then a **weak solution** of the inhomogeneous Neumann problem (2.13) is a function $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u.\nabla v \; dx + \int_\Omega uv \; dx \;=\; \int_\Omega fv \; dx + < g, v >_\Gamma, \tag{2.14}$$

for every $v \in H^1(\Omega)$. If $g \in L^2(\Gamma)$, then we can replace $< g, v >_\Gamma$ by $\int_\Gamma gv \; d\sigma$. Again, it is easy to see, from the Lax-Milgram lemma, that a unique solution always exists to (2.14). If $u \in H^2(\Omega)$ is a weak solution, then as before we can deduce that $-\Delta u + u = f$ in the sense of distributions and that $\gamma_1(u) = \frac{\partial u}{\partial \nu} = g$. For a weak solution to belong to $H^2(\Omega)$, we must obviously have $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma)$.

If we consider the Neumann problem without the lower order term, *viz.*

$$\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} & = & g & \text{on } \Gamma, \end{array}$$

then its weak formulation would be to look for $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u.\nabla v \; dx \;=\; \int_\Omega fv \; dx + \int_\Gamma gv \; d\sigma,$$

assuming $g \in L^2(\Gamma)$. If $f \in L^2(\Omega)$, then a weak solution will exist if, and only if, f and g satisfy the following compatibility condition:

$$\int_\Omega f \; dx + \int_\gamma g \; d\sigma \;=\; 0.$$

That it is necessary comes from integrating both sides of the differential equation over Ω and using Green's theorem. Conversely, from the theory of the Fredhölm alternative (similar to the argument in the proof of Theorem 2.2), it follows that the above condition is sufficeint as well (since we can show that the solution space when $f = 0$ and $g = 0$ is one dimensional and consists of constant functions) and in this case we have an infinity of weak solutions, since adding a constant function to an arbitrary solution produces another one. Thus, there exists a unique solution which is orthogonal to the subspace of constant functions in $L^2(\Omega)$, *i.e.* satisfying

$$\int_\Omega u \; dx \;=\; 0.$$

In the same vein, we can consider weak formulations for other types of boundary value problems for second order elliptic operators. For instance, we can consider a *Robin condition* which is of the form $\frac{\partial u}{\partial \nu} + \alpha u = 0$ on Γ for some constant $\alpha > 0$. We can also consider an *oblique derivative problem i.e.* with $\alpha_1 \frac{\partial u}{\partial \nu} + \alpha_2 \frac{\partial u}{\partial \tau} = 0$ on Γ, where α_1 and α_2 are constants and $\frac{\partial u}{\partial \tau}$ denotes the tangential derivative along the boundary when $\Omega \subset \mathbb{R}^2$. Some of these will figure in the exercises at the end of this chapter. Another problem is the *mixed problem* of the following form:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_2, \end{aligned}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$. If the surface measure of Γ_1 is strictly positive, then Poincaré's inequality is still available for the space

$$V = \{v \in H^1(\Omega) \mid v|_{\Gamma_1} = 0\}.$$

Thus we can apply the Lax-Milgram lemma to the problem: find $u \in V$ such that for every $v \in V$,

$$\int_\Omega \nabla u.\nabla v \; dx = \int_\Omega fv \; dx.$$

This is the weak formulation of the mixed problem stated above (check!). Notice that the essential condition $v = 0$ on Γ_1 was imposed on the space V while nothing was done about the natural condition $\frac{\partial u}{\partial \nu} = 0$ on Γ_2.

2.3 The biharmonic operator

The biharmonic operator is given by Δ^2, where, as usual, Δ is the Laplace operator. This is a differential operator of the fourth order. The Dirichlet problem for the biharmonic operator is the following:

$$\left. \begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma. \end{aligned} \right\} \tag{2.15}$$

Let $f \in L^2(\Omega)$. If $\varphi \in \mathcal{D}(\Omega)$, then multiplying both sides of the differential equation by φ and repeatedly using Green's formula, we get

$$\int_\Omega \Delta u \Delta \varphi \; dx = \int_\Omega f\varphi \; dx. \tag{2.16}$$

For (2.16) to make sense, it is sufficient that $u \in H^2(\Omega)$ and in that case both the traces $u|_\Gamma = \gamma_0(u)$ and $\frac{\partial u}{\partial \nu}|_\Gamma = \gamma_1(u)$ are both well defined. Since we require them both to vanish, we have by the trace theorem that $u \in H^2_0(\Omega)$. Further, $\mathcal{D}(\Omega)$ is dense in $H^2_0(\Omega)$ and both sides of (2.16) are continuous with respect to φ in the $H^2_0(\Omega)$-topology. Thus (2.16) holds for all $\varphi \in H^2_0(\Omega)$.

Thus a **weak solution** of (2.15) is a function $u \in H_0^2(\Omega)$ which satisfies (2.16) for all $\varphi \in H_0^2(\Omega)$.

Consider the bilinear form on $H_0^2(\Omega)$ given by

$$a(u,v) \;=\; \int_\Omega \Delta u \Delta v \; dx.$$

This is clearly continuous, since by the Cauchy-Schwarz inequality, we have

$$|a(u,v)| \;\leq\; |\Delta u|_{0,\Omega}|\Delta v|_{0,\Omega} \;\leq\; C\|u\|_{2,\Omega}\|v\|_{2,\Omega}.$$

Further, we know that $u \mapsto |\Delta u|_{0,\Omega}$ defines a norm on $H_0^2(\Omega)$ equivalent to the usual norm (Exercise!). So

$$a(v,v) \;=\; |\Delta u|_{0,\Omega}^2 \;\geq\; \alpha\|u\|_{2,\Omega}^2$$

and so $a(.,.)$ is $H_0^2(\Omega)$-elliptic. Hence, by the Lax-Milgram lemma, there exists a unique weak solution which also minimizes the functional

$$J(v) \;=\; \frac{1}{2}\int_\Omega |\Delta v|^2 \; dx - \int_\Omega fv \; dx$$

over all of $H_0^2(\Omega)$. Again, we have continuous dependence on the data since

$$\alpha\|u\|_{2,\Omega}^2 \;\leq\; \int_\Omega |\Delta u|^2 \; dx \;=\; \int_\Omega fu \; dx \;\leq\; |f|_{0,\Omega}|u|_{0,\Omega} \;\leq\; |f|_{0,\Omega}\|u\|_{2,\Omega}$$

whence it follows that

$$\|u\|_{2,\Omega} \;\leq\; \frac{1}{\alpha}|f|_{0,\Omega}.$$

As usual, if u is a weak solution, we have $\Delta^2 u = f$ in the sense of distributions. If $u \in H^4(\Omega) \cap H_0^2(\Omega)$, then $\Delta^2 u = f$ as $L^2(\Omega)$ functions. If, in addition, we know that $u \in \mathcal{C}^4(\overline{\Omega})$, then $f \in \mathcal{C}(\overline{\Omega})$ and u will be a classical solution.

Remark 2.5 The weak formulation also makes sense if $f \in H^{-2}(\Omega)$, in which case we replace $\int_\Omega f\varphi$ by $< f, \varphi >$, the duality bracket between $H^{-2}(\Omega)$ and $H_0^2(\Omega)$. ■

Remark 2.6 While the Dirichlet problem for the Laplace operator could be used to model the behaviour of a membrane fixed along the boundary and acted upon by a vertical force when $\Omega \subset \mathbb{R}^2$. In the same way, the Dirichlet problem for the biharmonic operator describes the bending of a thin elastic plate which is *clamped* along the boundary and acted upon by a vertical force. The plate itself is a three-dimensional body and is approximated by its middle surface which occupies the region $\Omega \subset \mathbb{R}^2$. If we wish to consider a plate which is *simply supported* on the boundary and is fixed along it, then the boundary condition $u = 0$ will be retained but the condition $\frac{\partial u}{\partial \nu} = 0$

will be replaced by another boundary condition(which will correspond to the notion of a natural boundary condition for the biharmonic operator). ■

Let us now describe another weak formulation of the Dirichlet problem for the biharmonic operator when $f \in L^2(\Omega)$. Set $\sigma = -\Delta u \in L^2(\Omega)$. Then we can write

$$\int_\Omega \sigma\tau \; dx + \int_\Omega (\Delta u)\tau \; dx \;=\; 0 \tag{2.17}$$

for every $\tau \in L^2(\Omega)$. Then (2.16) becomes

$$-\int_\Omega \sigma\Delta v \; dx \;=\; \int_\Omega fv \; dx \tag{2.18}$$

for every $v \in H_0^2(\Omega)$. If we set $\Sigma = L^2(\Omega)$ and $V = H_0^2(\Omega)$ and define

$$\begin{array}{rcll} a(\sigma,\tau) & = & \int_\Omega \sigma\tau \; dx & \text{for every } \sigma,\tau \in \Sigma, \\ b(\tau,v) & = & \int_\Omega (\Delta v)\tau \; dx & \text{for every } \tau \in \Sigma \text{ and } v \in V, \end{array}$$

then the system (2.17)-(2.18) is as in the Babuška-Brezzi theorem (Theorem 1.5). Indeed, the bilinear form $a(.,.)$ is $L^2(\Omega)$-elliptic and so trivially Z-elliptic for any closed subspace Z of Σ. Also,

$$\sup_{\substack{\tau\in\Sigma \\ \tau\neq 0}} \frac{b(\tau,v)}{|\tau|_{0,\Omega}} \;\geq\; \frac{\int_\Omega \Delta v \Delta v \; dx}{|\Delta v|_{0,\Omega}} \;=\; |\Delta v|_{0,\Omega} \;\geq\; \alpha\|v\|_{2,\Omega}$$

for every $v \in V$ and so the Babuška-Brezzi condition is also verified. Thus there exists a unique pair $(\sigma, u) \in \Sigma \times V$ satisfying (2.17)-(2.18). Clearly, then $u \in H_0^2(\Omega)$ and $\sigma = -\Delta u$ and so u satisfies (2.16).

This formulation has no intrinsic value as we seem only to have increased the number of unknowns. Let us now describe another formulation of the same problem. Let $\widetilde{\Sigma} = H^1(\Omega)$ and let $\widetilde{V} = H_0^1(\Omega)$. Define

$$\begin{array}{rcll} a(\sigma,\tau) & = & \int_\Omega \sigma\tau \; dx & \text{for all } \sigma,\tau \in \widetilde{\Sigma}, \\ \widetilde{b}(\sigma,v) & = & -\int_\Omega \nabla\tau.\nabla v \; dx & \text{for all } \tau \in \widetilde{\Sigma}, v \in \widetilde{V}. \end{array}$$

Consider the problem: find $(\widetilde{\sigma}, \widetilde{u}) \in \widetilde{\Sigma} \times \widetilde{V}$ such that

$$\left.\begin{array}{rcll} a(\widetilde{\sigma},\tau) + \widetilde{b}(\tau,\widetilde{u}) & = & 0 & \text{for every } \tau \in \widetilde{\Sigma}, \\ -\widetilde{b}(\widetilde{\sigma},v) & = & \int_\Omega fv \; dx & \text{for every } v \in \widetilde{V}. \end{array}\right\} \tag{2.19}$$

In this case also $\widetilde{b}(.,.)$ satisfies the Babuška-Brezzi condition. Indeed,

$$\sup_{\substack{\tau\in\widetilde{\Sigma} \\ \tau\neq 0}} \frac{\widetilde{b}(\tau,v)}{\|\tau\|_{1,\Omega}} \;\geq\; \frac{\int_\Omega \nabla v.\nabla v \; dx}{\|v\|_{1,\Omega}} \;=\; \frac{|v|_{1,\Omega}^2}{\|v\|_{1,\Omega}} \;\geq\; \beta\|v\|_{1,\Omega}$$

by Poincaré's inequality. Unfortunately, the bilinear form $a(.,.)$ is not Z-elliptic since we cannot expect the $L^2(\Omega)$-norm to majorize the $H^1(\Omega)$-norm. Thus, we cannot directly apply Theorem 1.5 to prove the existence of a solution. However, it is easy to check that if a solution exists, then it is unique. We now prove the following result.

Theorem 2.3 *Assume that the weak solution $u \in H_0^2(\Omega)$ of (2.16) satisfies the regularity condition $u \in H^3(\Omega)$. Then $(-\Delta u, u) = (\widetilde{\sigma}, \widetilde{u}) \in \widetilde{\Sigma} \times \widetilde{V}$ is the unique solution of (2.19).*

Proof: Clearly $(\sigma, u) = (-\Delta u, u)$ satisfies (2.17)-(2.18). But if $\tau \in H^1(\Omega)$, then

$$b(\tau, u) \ = \ \int_\Omega (\Delta u)\tau \ dx \ = \ -\int_\Omega \nabla u . \nabla \tau \ dx \ = \ \widetilde{b}(\tau, u).$$

Also, we have

$$-\int_\Omega \sigma \Delta v \ dx \ = \ \int_\Omega f v \ dx$$

for every $v \in H_0^2(\Omega)$ which yields

$$\int_\Omega \nabla \sigma . \nabla v \ dx \ = \ \int_\Omega f v \ dx.$$

In particular, this is true for every $v \in \mathcal{D}(\Omega)$. But $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$ and both sides of the above relation are continuous with respect to v for the $H_0^1(\Omega)$- topology. So the above relation also holds for all $v \in H_0^1(\Omega)$. Thus, $\widetilde{\sigma} = \sigma = -\Delta u$ and $\widetilde{u} = u$ satiusfy (2.19). This completes the proof. ■

Remark 3.2.7 This latter formulation is very useful from the computational point of view, as we shall see later. It is due to Ciarlet and Raviart [1]. It depends on the regularity result that a weak solution of (2.15) belongs to $H^3(\Omega)$. If Ω is of class $\mathcal{C}^\infty$, we in fact have that $u \in H^4(\Omega)$ when $f \in L^2(\Omega)$. If Ω is a polygon, or if it is a Lipschitz domain, then we do have $u \in H^3(\Omega)$ by a result of Kondrat'ev [5]. ■

3 Eigenvalue problems

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with boundary Γ. In this section we will consider the eigenvalue problem: find $\lambda \in \mathbb{R}$ and function(s) $u \not\equiv 0$ such that

$$\left.\begin{array}{rcll} -\Delta u & = & \lambda u & \text{in } \Omega, \\ u & = & 0 & \text{on } \Gamma. \end{array}\right\} \tag{3.1}$$

Given a solution pair (λ, u) of the above problem, we say that λ is an **eigenvalue** and u is an **eigenfunction** of the Laplace operator (with homogeneous Dirichlet boundary conditions). Notice that if u_1 and u_2 are eigenfunctions for an eigenvalue λ, then so is $\alpha u_1 + \beta u_2$ for any constants α and β. Thus, the set of all eigenfunctions corresponding to an eigenvalue, together with the null function, forms a vector space and this space is called the **eigenspace** corresponding to the eigenvalue λ.

Remark 3.1 In a manner analogous to (3.1), we can pose eigenvalue problems for other (homogeneous) boundary conditions. We can also pose the problem for other elliptic operators. The results in this section give a flavour of the kind of properties that eigenvalues and eigenfunctions of elliptic operators generally enjoy. Of course, depending on the operator and the boundary conditions, the validity of the various properties will differ. ■

Theorem 3.1 *There exists an orthonormal basis* $\{w_n\}$ *of* $L^2(\Omega)$ *and a sequence of positive real numbers* $\{\lambda_n\}$, *with* $\lambda_n \to \infty$ *as* $n \to \infty$, *such that* $w_n \in H_0^1(\Omega) \cap \mathcal{C}^\infty(\Omega)$,

$$0 \;<\; \lambda_1 \;\leq\; \lambda_2 \;\leq\; \cdots \;\leq\; \lambda_n \;\leq\; \lambda_{n+1} \;\leq\; \cdots,$$

and

$$-\Delta w_n \;=\; \lambda_n w_n \text{ in } \Omega.$$

Further, the dimension of the eigenspace of each λ_n *is finite.*

Proof: Given $f \in L^2(\Omega)$, define $Gf \in H_0^1(\Omega)$ as the weak solution of the problem: $-\Delta u = f$, in Ω and $u = 0$ on Γ. Thus, for every $v \in H_0^1(\Omega)$, we have

$$\int_\Omega \nabla(Gf).\nabla v \; dx \;=\; \int_\Omega fv \; dx.$$

Then $G : L^2(\Omega) \to H_0^1(\Omega)$ is a continuous linear map. Since Ω is bounded, $H_0^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$ by the Rellich-Kondrosov theorem and so after composition by this inclusion, we can consider G as a compact map from $L^2(\Omega)$ into itself (with range contained in $H_0^1(\Omega)$). This map is self-adjoint since, for arbitrary f and g in $L^2(\Omega)$, we have

$$\int_\Omega (Gf)g \; dx \;=\; \int_\Omega \nabla(Gf).\nabla(Gg) \; dx \;=\; \int_\Omega f(Gg) \; dx.$$

Further,

$$\int_\Omega (Gf)f \; dx = = \; \int_\Omega |\nabla(Gf)|^2 \; dx \;=\; |Gf|_{1,\Omega}^2$$

which is strictly positive if $f \neq 0$.

It now follows from the theory of compact self-adjoint and positive definite operators on a separable Hilbert space that there exists an orthonormal basis $\{w_n\}$ of eigenfunctions and a family $\{\mu_n\}$ of eigenvalues, with finite geometric multiplicity, decreasing to zero as $n \to \infty$ such that $Gw_n = \mu_n w_n$ (see, for instance, Kesavan [4]). Thus, clearly $w_n \in H_0^1(\Omega)$. Since G is positive definite, we have $\mu_n \neq 0$ for all n. Setting $\lambda_n = \mu_n^{-1}$, we get

$$w_n \;=\; G(\lambda_n w_n)$$

and so for every $v \in H_0^1(\Omega)$, we have

$$\int_\Omega \nabla w_n.\nabla v \; dx \;=\; \lambda_n \int_\Omega w_n v \; dx$$

which is the weak formulation of (3.1). In particular, $w_n = 0$ on Γ and satisfies $-\Delta w_n = \lambda_n w_n$ in the sense of distributions.

Finally, let $x \in \Omega$ and let $r > 0$ be such that the ball $B(x;r) \subset \Omega$. By an interior regularity result, we get that $w_n \in H^2(B(x;r))$ since $w_n \in L^2(B(x;r))$ and $-\Delta w_n = \lambda_n w_n$ in this ball. This, in turn, now implies that $w_n \in H^4(B(x;r))$ and so on. Thus $w_n \in H^k(B(x;r))$ for all positive integers k and, by the Sobolev imbedding theorem, it follows that $w_n \in \mathcal{C}^\infty(B(x;r))$. Since $x \in \Omega$ was arbitrarily chosen, it follows that $w_n \in \mathcal{C}^\infty(\Omega)$. This completes the proof. ∎

Remark 3.2 If Ω is of class $\mathcal{C}^\infty$, then $w_n \in \mathcal{C}^\infty(\overline{\Omega})$ for all positive integers n. ∎

Remark 3.3 If $H_0^1(\Omega)$ is provided with the inner-product

$$(u, v) \;=\; \int_\Omega \nabla u.\nabla v \; dx,$$

then $\{\lambda_n^{-\frac{1}{2}} w_n\}$ is an orthonormal basis for $H_0^1(\Omega)$. Indeed,

$$\frac{1}{\sqrt{\lambda_k \lambda_m}} \int_\Omega \nabla w_k.\nabla w_m \; dx \;=\; \left(\frac{\lambda_k}{\lambda_m}\right)^{\frac{1}{2}} \int_\Omega w_k w_m \; dx \;=\; \delta_{km}.$$

Further, if $u \in H_0^1(\Omega)$ is such that $(u, w_m) = 0$ for all m, then

$$0 \;=\; \int_\Omega \nabla u.\nabla w_m \; dx \;=\; \lambda_m \int_\Omega u w_m \; dx$$

and so $\int_\Omega u w_m \; dx = 0$ for all m. Since $\{w_m\}$ is an orthonormal basis for $L^2(\Omega)$, it follows that $u = 0$. This proves the claim.

It is now easy to see that the Fourier expansion of a function $u \in H_0^1(\Omega)$ is given by

$$u \;=\; \sum_{n=1}^\infty \left(\int_\Omega u w_n \; dx\right) w_n$$

in *both* the spaces $L^2(\Omega)$ and $H_0^1(\Omega)$. ∎

Example 3.1 let $N = 1$ and let $\Omega = (0, 1)$. Then the problem (3.1) reads as:

$$\begin{aligned} -u'' &= \lambda u \text{ in } (0,1), \\ u(0) &= u(1) \;=\; 0. \end{aligned}$$

Multiplying the differential equation by u and integrating by parts, using the boundary condition, we get

$$\int_0^1 |u'|^2 \, dx \;=\; \lambda \int_0^1 |u|^2 \, dx.$$

Thus, it follows that $\lambda \geq 0$. But if $\lambda = 0$, $u' = 0$ and so u is a constant and must vanish since it vanishes at the end points. Thus $\lambda > 0$. Now, a general solution of the differential equation $u'' + \lambda u = 0$, for positive λ is given by

$$u(x) \;=\; A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x.$$

Since $u(0) = 0$, we get $A = 0$. Thus $B \neq 0$ and since $u(1) = 0$, it follows that $\sqrt{\lambda}$ is an integer multiple of π. Thus the eigenvalues are $\{n^2\pi^2\}_{n=1}^\infty$ and the corresponding eigenfunctions are constant multiples of $w_n(x) = \sin n\pi x$. We know from the theory of Fourier series that $\{\sqrt{2} \sin n\pi x\}_{n=1}^\infty$ forms an orthonormal basis for $L^2(0,1)$. ■

Example 3.2 Let $N = 2$ and let $\Omega = (0,1) \times (0,1)$. It is easy to see that if $\Lambda_{nm} = (n^2 + m^2)\pi^2$ and if $W_{nm} = \sin n\pi x \sin m\pi y$, then

$$-\Delta W_{nm} \;=\; \Lambda_{nm} W_{nm}$$

in Ω and that W_{nm} vanishes on the boundary of Ω. Thus Λ_{nm} are all eigenvalues with corresponding eigenfunctions W_{nm}. Are there any other eigenvalues? It can be shown that $\{2W_{nm}\}_{n,m=1}^\infty$ is an orthonormal basis for $L^2(\Omega)$ and so there can be no further eigenfunctions. Thus these are all the eigenvalues and eigenfunctions in this case. ■

Remark 3.4 When we numbered the eigenvalues, we wrote the sign '$\leq$' between them. In the case of Example 3.1, all these eigenvalues were distinct. However, in the case of Example 3.2, we have $\lambda_1 = \Lambda_{11} = 2\pi^2$, while $\lambda_2 = \lambda_3 = \Lambda_{12} = \Lambda_{21} = 5\pi^2$. In the latter case, the eigenspace is two-dimensional and is spanned by the functions $\sin \pi x \sin 2\pi y$ and $\sin 2\pi x \sin \pi y$. Thus when we number the eigenvalues in increasing order, we repeat the same eigenvalue as many times as the dimension of the eigenspace, and this number is called its *geometric multiplicity*. ■

We now give a variational characterization of the eigenvalues and eigenfunctions obtained in Theorem 3.1. We define, for $v \in H_0^1(\Omega), v \neq 0$, the *Rayleigh quotient* $R(v)$ by

$$R(v) \;\stackrel{\text{def}}{=}\; \frac{\int_\Omega \nabla v . \nabla v \, dx}{\int_\Omega v^2 \, dx}.$$

We denote by V_m the space spanned by the first m eigenfunctions $\{w_1, \cdots, w_m\}$. By the orthonormality of the eigenfunctions, they are linearly independent and so $\dim(V_m) = m$.

Theorem 3.2 *Let m be an aribtrary positive integer. Then $\lambda_m = R(w_m)$. Further,*

$$\lambda_m = \max_{\substack{v\in V_m\\ v\neq 0}} R(v) \tag{3.2}$$

$$= \min_{\substack{v\perp V_{m-1}\\ v\neq 0}} R(v) \tag{3.3}$$

$$= \min_{\substack{W\subset H_0^1(\Omega)\\ \dim(W)=m}} \max_{\substack{v\in W\\ v\neq 0}} R(v). \tag{3.4}$$

In particular,

$$\lambda_1 = \min_{\substack{v\in H_0^1(\Omega)\\ v\neq 0}} R(v) \tag{3.5}$$

Proof: If (λ, u) is an eigen pair, then for every $v \in V$, we have

$$\int_\Omega \nabla u.\nabla v \; dx = \lambda \int_\Omega uv \; dx. \tag{3.6}$$

It is now obvious from this that $\lambda_m = R(w_m)$ for any positive integer m. Let $v \in V_m$. Then

$$v = \sum_{k=1}^m \alpha_k w_k,$$

where $\alpha_k = \int_\Omega v w_k \; dx$. Using the orthonormality of the w_k, we get

$$R(v) = \frac{\sum_{k=1}^m \lambda_k \alpha_k^2}{\sum_{k=1}^m \alpha_k^2} \leq \lambda_m$$

since the λ_k are increasing with k. Thus the maximum value of $R(v)$ as v varies over V_m is less than, or equal to, λ_m. Since $w_m \in V_m$ and $R(w_m) = \lambda_m$, we get (3.2).

Now, let $v \perp V_{m-1}$ (this orthogonality holds in $L^2(\Omega)$ as well as in $H_0^1(\Omega)$ simultaneously). Then

$$v = \sum_{k=m}^\infty \alpha_k w_k = \lim_{\ell\to\infty} \sum_{k=m}^\ell \alpha_k w_k.$$

If we set $v_\ell = \sum_{k=m}^\ell \alpha_k w_k$, then $v_\ell \to v$ as $\ell \to \infty$ both in $H_0^1(\Omega)$ and $L^2(\Omega)$. Thus, it follows that $R(v_\ell) \to R(v)$. Now,

$$R(v_\ell) = \frac{\sum_{k=m}^\ell \lambda_k \alpha_k^2}{\sum_{k=m}^\ell \alpha_k^2} \geq \lambda_m$$

and so $R(v) \geq \lambda_m$ as well. Again $w_m \perp V_{m-1}$ and this establishes (3.3). In particular $m = 1$ gives (3.5).

Finally, let W be an m-dimensional subspace of $H_0^1(\Omega)$. Then obviously $W \cap V_{m-1}^{\perp} \neq \{0\}$ (why?). Thus, there exists $w \neq 0, w \in W \cap V_{m-1}^{\perp}$. Then by (3.3), $R(w) \geq \lambda_m$ and so

$$\max_{\substack{v \in W \\ v \neq 0}} R(v) \;\geq\; R(w) \;\geq\; \lambda_m.$$

Again (3.4) follows since $\dim(V_m) = m$ and on this subspace, the maximum value of the Rayleigh quotient is indeed λ_m. This completes the proof. ■

Remark 3.5 The characterization (3.4) is intrinsic in the sense that it does not depend on the choice of eigenfunctions. ■

We will now prove an important property of the eigenpair (λ_1, w_1).

Lemma 3.1 *Let $w \in H_0^1(\Omega), w \neq 0$ be such that $R(w) = \lambda_1$. Then w is an eigenfunction corresponding to λ_1.*

Proof: Let $v \in H_0^1(\Omega)$ be arbitrarily chosen. Let $t > 0$. Then $w + tv \in H_0^1(\Omega)$ and by virtue of (3.5) we have that $R(w+tv) \geq \lambda_1 = R(w)$. By normalizing, we can assume that $|w|_{0,\Omega}^2 = 1$. Then

$$\frac{\int_\Omega \nabla(w+tv).\nabla(w+tv)\; dx}{\int_\Omega (w+tv)^2\; dx} \;\geq\; \int_\Omega \nabla w.\nabla w\; dx \;=\; \lambda_1.$$

Cross multiplying and simplifying, we get

$$t^2 \int_\Omega \nabla v.\nabla v\; dx + 2t \int_\Omega \nabla w.\nabla v\; dx \;\geq\; \lambda_1 \left(2t \int_\Omega wv\; dx + t^2 \int_\Omega v^2\; dx \right).$$

Dividing throughout by 2t and letting $t \to 0$, we get

$$\int_\Omega \nabla w.\nabla v\; dx \;\geq\; \lambda_1 \int_\Omega wv\; dx.$$

We can replace v by $-v$ and get the reverse inequality as well and so w satisfies (3.6) and this completes the proof. ■

Theorem 3.3 *Let $\Omega \subset \mathbb{R}^N$ be an open and connected set. Then λ_1 is a simple eigenvalue (i.e. $\dim(V_1) = 1$) and any eigenfunction corresponding to it does not change sign in Ω. In particular, we can always choose w_1 to be strictly positive in Ω.*

Proof: Let w be any eigenfunction corresponding to λ_1. Then we know that w^+ and w^- also belong to $H_0^1(\Omega)$. Setting $v = w^+$ and $v = w^-$ in the weak form

$$\int_\Omega \nabla w.\nabla v\; dx \;=\; \lambda_1 \int_\Omega wv\; dx,$$

we easily deduce that

$$R(w^+) \;=\; R(w^-) \;=\; \lambda_1.$$

Consequently, by the preceding lemma, both w^+ and w^- will be eigenfunctions, if they are non-zero and hence will also be in $\mathcal{C}^\infty(\Omega)$. But since $-\Delta w^+ = \lambda_1 w^+ \geq 0$, it follows from the strong maximum principle that $w^+ \equiv 0$ or $w^+ > 0$ in Ω. The same conclusion holds for w^- as well. But both w^+ and w^- cannot be simultaneously non-zero. Thus $w = w^+$ or $w = w^-$ and so w cannot change sign in Ω. This then implies that we cannot have two mutually orthogonal eigenfunctions for λ_1 and so λ_1 is simple and we can always choose an eigenfunction which is strictly positive in Ω. This completes the proof. ■

Remark 3.6 This theorem can also be proved as a consequence of the Krein-Rutman theorem. In finite dimensional spaces, we have an analogous result known as the Perron-Fröbenius theorem which states that for a non-negative irreducible matrix, the spectral radius is a simple eigenvalue and that there exists a strictly positive eigenvector. Here, the operator G defined in Theorem 3.1 can be shown to have analogous properties in infinite dimensions and so its spectral radius μ_1 is a simple eigenvalue with eigenfucntions of constant sign. The advantage of the above proof is that it easily extends to any elliptic eigenvalue problem of the form: find $(\lambda, u) \in \mathbb{R} \times (H_0^1(\Omega)\backslash\{0\})$ such that for all $v \in H_0^1(\Omega)$,

$$\int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \; dx + \int_\Omega a_0 uv \; dx \;=\; \lambda \int_\Omega uv \; dx$$

with the matrix (a_{ij}) being symmetric and satisfying the ellipticity condition (2.6) and $a_0 \geq 0$ so that the corresponding Rayleigh quotient characterization and the strong maximum principle work. ■

Remark 3.7 To apply the strong maximum principle we also need that any eigenfunction u satisfies $u \in \mathcal{C}(\overline{\Omega})$. This is true even for Lipschitz domains when $N \leq 3$ since $u \in H^2(\Omega)$ by the regularity theorem which is imbedded in $\mathcal{C}(\overline{\Omega})$ when $N \leq 3$ by the Sobolev imbedding theorem. Otherwise, we need to assume more smoothness of the boundary so that we can successively apply the regularity theorem to ultimately get $u \in \mathcal{C}(\overline{\Omega})$ by Sobolev inclusion. ■

Remark 3.8 Just as the stationary problem $-\Delta u = f$ in Ω and $u = 0$ on Γ models the displacement of a membrane fixed along Γ and acted upon by a force f, the eigenvalue problem describes the vibration of a membrane fixed along Γ, when $\Omega \subset \mathbb{R}^2$. The harmonics of a membrane fixed along Γ are given by $w_m(x)$ and $\sqrt{\lambda_m} t$. ■

Theorem 3.4 *(Monotonicity of the spectrum with respect to the domain) Let Ω_1 and Ω_2 be two bounded domains in $\mathbb{R}^N$ such that $\Omega_1 \subset \Omega_2$. Let*

$\{\lambda_n(\Omega_i)\}, i = 1, 2,$ *denote the sequence of eigenvalues of the Laplace operator with homogeneous Dirichlet boundary conditions, numbered in increasing order of magnitude, for the domains* $\Omega_i, i = 1, 2$. *Then, for every positive integer* n, *we have*

$$\lambda_n(\Omega_1) \;\geq\; \lambda_n(\Omega_2).$$

Proof: Observe that if $u \in H_0^1(\Omega_1)$, then its extension by zero outside Ω_1, denoted $\widetilde{u}$, is in $H^1(\mathbb{R}^N)$ and its restriction to Ω_2 is in $H_0^1(\Omega_2)$. Further

$$\int_{\Omega_1} |\nabla u|^2 \; dx \;=\; \int_{\Omega_2} |\nabla \widetilde{u}|^2 \; dx \text{ and } \int_{\Omega_1} u^2 \; dx \;=\; \int_{\Omega_2} \widetilde{u}^2 \; dx.$$

The result is now an immediate consequence of the min-max characterization of eigenvalues given by (3.4). ■

We saw that any eigenfunction corresponding to the first eigenvalue λ_1 has to be of constant sign. Consequently, since eigenfunctions of different eigenvalues are orthogonal to each other in $L^2(\Omega)$, it follows that any eigenfunction of an eigenvalue λ_n, where $n \geq 2$, must change sign in Ω. A **nodal line** of an eigenfunction is a curve in $\overline{\Omega}$, other than Γ, along which an eigenfunction vanishes. A **nodal domain** is a connected open subset of Ω where the eigenfunction has a constant sign. Thus, an eigenfunction corrsponding to λ_1 has exactly one nodal domain, *viz.* Ω itself and has no nodal lines.

Proposition 3.1 *Let* $\lambda_k, k \geq 2$ *be an eigenvalue such that* $\lambda_k < \lambda_{k+1}$. *Then an eigenfunction* u *of* λ_k *has at most* k *nodal domains.*

Proof: Let u be an eigenfunction for λ_k with ℓ nodal domains denoted $\Omega_i, 1 \leq i \leq \ell$. In each Ω_i, the function u satisfies $-\Delta u = \lambda_k u$ and it vanishes on the boundary of this domain. Define, for $1 \leq i \leq \ell$,

$$u^i \;=\; \begin{cases} u|_{\Omega_i} & \text{in } \Omega_i, \\ 0, & \text{outside } \Omega_i. \end{cases}$$

Then $u^i \in H_0^1(\Omega)$ and clearly $\int_\Omega u^i u^j \; dx = 0$ whenever $i \neq j$. Thus the dimension of the space $V = \text{span}\{u^1, \cdots, u^\ell\}$ is ℓ. Now, since $-\Delta u^i = \lambda_k u^i$ in Ω_i and $u^i = 0$ on $\partial\Omega^i$, we have

$$\int_{\Omega_i} |\nabla u^i|^2 \; dx \;=\; \lambda_k \int_{\Omega_i} (u^i)^2 \; dx$$

and so

$$\int_{\Omega} |\nabla u^i|^2 \; dx \;=\; \lambda_k \int_{\Omega} (u^i)^2 \; dx$$

for each $1 \leq i \leq \ell$. If $v = \sum_{k=1}^{\ell} \alpha_k u^k$ is an arbitrary element of V, then it is immediate to verify that $R(v) = \lambda_k$ as well. Thus, since $\dim(V) = \ell$, it follows from (3.4) that $\lambda_\ell \leq \lambda_k$. Since $\lambda_{k+1} > \lambda_k$, it follows that $\ell \leq k$, which completes the proof. ■

Remark 3.9 A slight refinement of the above proof, involving subtler properties of solutions to elliptic equations, implies in fact that for *every* positive integer k, an eigenfunction corresponding to the eigenvalue λ_k has at most k nodal domains. Thus the additional hypothesis $\lambda_k < \lambda_{k+1}$ is not necessary. This is the famous *Courant's nodal line theorem* and a proof can be found in Courant and Hilbert [2], Volume I. In particular, an eigenvalue corresponding to λ_2 must have at least 2 nodal domains since it is orthogonal to any eigenfunction of λ_1 (which will be of constant sign) and, on the other hand, it will have at most 2 nodal domains, by Courant's theorem. Thus it will have exactly two nodal domains. This is not true in general for an eigenfunction corresponding to λ_k when $k \geq 3$. For example, in the unit square in $\mathbb{R}^2$, we have $\lambda_2 = \lambda_3$ and so eigenfunctions of λ_3 will have only 2 nodal domains and never 3. Thus Courant's theorem is optimal. ■

We thus see that the min-max characterization of eigenvalues (3.4) is very powerful and leads to several interesting properties of eigenvalues and eigenfunctions.

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