Introduction to the Theory of the Navier–Stokes Equations for Incompressible Fluid

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References

1. Basic notions, equations and function spaces

A physical background

We assume that **the fluid is a continuum.** We denote

$\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$	$\ldots$ the **velocity**,	$p = p(\mathbf{x}, t)$	$\ldots$ the **pressure**,
$\rho = \rho(\mathbf{x}, t)$	$\ldots$ the **density**,	$\theta = \theta(\mathbf{x}, t)$	$\ldots$ the **temperature**

Stokes' postulates for the fluid (19th century) require that

a) the stress tensor $\mathbb{T}$ depends on the velocity only through the stretching tensor (= rate of strain tensor, rate of deformation tensor) $\mathbb{D} := (\nabla \mathbf{v})_{\mathrm{sym}}$,

b) the stress tensor $\mathbb{T}$ does not explicitly depend on position $\mathbf{x}$ and time t,

c) the continuum is isotropic, i.e. it contains no preferred directions,

d) if the fluid is at rest then $\mathbb{T}$ is a multiple of the identity tensor $\mathbb{I}$ by a scalar.

Nowadays, the postulates are usually formulated in another way:

a) the stress tensor $\mathbb{T}$ depends on the velocity only through the stretching tensor $\mathbb{D}$,

b) the stress tensor $\mathbb{T}$ does not explicitly depend on position $\mathbf{x}$ and time t,

c') the way tensor $\mathbb{T}$ depends on tensor $\mathbb{D}$ is material frame indifferent.

Postulate c' means that IF

- $Q = Q(t)$ is an arbitrary unitary matrix,
- $\mathbf{x}^* = \mathbf{c}(t) + Q(t) \cdot (\mathbf{x} - \overline{\mathbf{x}})$ is another observer's frame,
- $\mathbb{T}$ and $\mathbb{D}$ have the representations $\mathbb{T}^*$ and $\mathbb{D}^*$ in frame $\mathbf{x}^*$

THEN $\mathbb{T}^*$ depends on $\mathbb{D}^*$ in the same way as $\mathbb{T}$ depends on $\mathbb{D}$.

One can derive from these postulates that

$$\mathbb{T} \;=\; -p\,\mathbb{I} + \mathbb{T}_{\rm d},$$

where $\mathbb{T}_{\rm d}$ is the "dynamic stress tensor" (it equals the zero tensor if the fluid is at rest), and $\mathbb{T}_{\rm d}$ depends on tensor $\mathbb{D}$ through the formula

$$\mathbb{T}_{\rm d} \;=\; \alpha\,\mathbb{I} + \beta\,\mathbb{D} + \gamma\,\mathbb{D}^2.$$

The coefficients α, β and γ may generally depend on the so called state variables (pressure, density, temperature) and on the principal invariants of tensor $\mathbb{D}$.

Ideal fluid (= inviscid fluid): $\mathbb{T}_{\rm d} = \mathbb{O}$, i.e. $\mathbb{T} = -p\,\mathbb{I}$

Newtonian fluid: $\mathbb{T}$ depends linearly on $\mathbb{D}$. One can deduce from this assumption (also using Kirchhoff's formula $p = -\frac{1}{3}\mathbb{T} + \mu'\,\mathrm{div}\,\mathbf{v}$, where μ' is the so called *coefficient of bulk viscosity*) that

$$\mathbb{T} \;=\; \big[-p + (\mu' - \tfrac{2}{3}\mu)\,\mathrm{div}\,\mathbf{v}\big]\,\mathbb{I} + 2\mu\,\mathbb{D}, \qquad (1.1)$$

where μ is the *dynamic coefficient of viscosity*.

Condition of incompressibility: $$\operatorname{div} \mathbf{v} = 0 \tag{1.2}$$

Condition (1.2) expresses the fact that each part of the fluid preserves its volume.

Conservation of mass: $$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0 \quad \ldots \quad \textbf{equation of continuity} \tag{1.3}$$

Note that condition (1.2) follows from (1.3) in the special case when $\rho = const$. Due to this and also other historical reasons, equation (1.2) is often called **equation of continuity for incompressible fluid.**

However, note that there exist incompressible fluids with non–constant density (e.g. mixtures of several liquids). In these cases, (1.3) reduces to the transport equation for density ρ:

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho = 0.$$

Conservation of momentum: $$\rho\, \partial_t \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\operatorname{Div} \mathbb{T} + \rho \mathbf{f} \tag{1.4}$$

From now, we use the **IMPORTANT ASSUMPTIONS:**

- **the fluid is incompressible with the constant density** $\rho = const. = 1$,
- **the fluid is Newtonian,**
- $\mu = const. > 0$.

We denote $\nu := \mu/\rho = \mu$ ... the *kinematic coefficient of viscosity.*

Formula (1.1) now reduces to $\mathbb{T} \;=\; -p\,\mathbb{I} + 2\nu\,\mathbb{D}$.

Substituting this to (1.4), we obtain the so called **Navier–Stokes equation for viscous incompressible fluid** (H. Navier 1824, G. Stokes 1845)

$$\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} \;=\; -\nabla p + \nu\Delta\mathbf{v} + \mathbf{f}.$$

This vectorial equation is equivalent to the system of three scalar equations

$$\partial v_i + v_j\,\partial_j v_i \;=\; -\partial_i p + \nu\Delta v_i + f_i \qquad (i = 1, 2, 3).$$

Boundary conditions on a fixed material boundary:

the no–slip condition: $$\mathbf{v} \;=\; \mathbf{0}, \tag{1.5}$$

Navier's slip condition: $$[\mathbb{T}\cdot\mathbf{n}]_\tau + \gamma\mathbf{v}_\tau \;=\; 0. \tag{1.6}$$

Here, τ denotes the tangential component and $\mathbf{n}$ denotes the outer normal vector on the boundary. Condition (1.6) is used together with

the condition of impermeability: $$\mathbf{v}\cdot\mathbf{n} \;=\; 0. \tag{1.7}$$

γ	...	the coefficient of friction between the fluid and the wall
$\gamma\to\infty$	...	conditions (1.6) and (1.7) lead to (1.5)
$\gamma=0$	...	the so called *free slip*

Further, we mostly consider the no–slip condition (1.5).

The Navier–Stokes initial–boundary value problem

Ω ... a domain in $\mathbb{R}^3$ ... the domain where we consider the motion of the fluid

$(0,T)$ ... the time interval $(0 < T \leq \infty)$

$Q_T := \Omega \times (0,T)$

Thus, **the Navier–Stokes initial–boundary value problem** we deal with is:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} + \mathbf{f} \quad \text{in } Q_T, \tag{1.8}$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } Q_T, \tag{1.9}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T), \tag{1.10}$$

$$\mathbf{v} = \mathbf{v}_0 \quad \text{in } \Omega \times \{0\}. \tag{1.11}$$

Function space $\mathbf{L}^2_\sigma(\Omega)$

Let $\mathbf{C}^\infty_{0,\sigma}(\Omega)$ be the linear space of all infinitely differentiable divergence–free vector functions in Ω with a compact support in Ω. We denote by $\mathbf{L}^2_\sigma(\Omega)$ the closure of $\mathbf{C}^\infty_{0,\sigma}(\Omega)$ in $\mathbf{L}^2(\Omega)$.

Remark. Assume that Ω has a locally Lipschitzian boundary.

Then the space $\mathbf{L}^2_\sigma(\Omega)$ can be characterized as a space of functions from $\mathbf{L}^2(\Omega)$, whose divergence equals zero in Ω (in the sense of distributions), whose normal component on $\partial\Omega$) is zero (as an element of the space $W^{-1/2,2}(\partial\Omega)$) in the sense of traces. (See e.g. [1] or [11].)

Helmholtz decomposition (see e.g. [1] or [11] for more details):

Lemma. *Let Ω be any domain in $\mathbb{R}^3$. Then*

$$\mathbf{L}^2_\sigma(\Omega)^\perp \;=\; \mathbf{G}_2(\Omega) \;:=\; \big\{\mathbf{w} \in \mathbf{L}^2(\Omega);\; \mathbf{w} = \nabla\varphi \textit{ for some } \varphi \in W^{1,2}_{loc}(\Omega)\big\}.$$

Consequently, $\mathbf{L}^2(\Omega) = \mathbf{L}^2_\sigma(\Omega) \oplus \mathbf{G}_2(\Omega)$, *where* $\mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{G}_2(\Omega)$ *are closed orthogonal subspaces of* $\mathbf{L}^2(\Omega)$.

A part of the proof: Only the part $\boxed{\supset}$. (See e.g. [1] or [11] for the opposite inclusion.) Thus, let $\mathbf{w} = \nabla\varphi \in \mathbf{G}_2(\Omega)$. It suffices to show that $(\mathbf{u}, \mathbf{w})_2 = 0$ for all $\mathbf{u} \in \mathbf{C}^\infty_{0,\sigma}(\Omega)$. However,

$$(\mathbf{u}, \mathbf{w})_2 \;=\; \int_\Omega \mathbf{u} \cdot \nabla\varphi \, \mathrm{d}\mathbf{x} \;=\; -\int_\Omega \operatorname{div} \mathbf{u} \, \varphi \, \mathrm{d}\mathbf{x} \;=\; 0.$$

The orthogonal projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{L}^2_\sigma(\Omega)$ is called the **Helmholtz projection.** It is usually denoted by P_2, or P^2_σ or only by P_σ.

Remark. Let $\mathbf{v} \in \mathbf{L}^2(\Omega)$. The Helmholtz decomposition of $\mathbf{v}$ is: $\boxed{\mathbf{v} \;=\; P_\sigma \mathbf{v} + \nabla\varphi}$ where φ is a weak solution of the Neumann problem

$$\Delta\varphi = \operatorname{div} \mathbf{v} \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\mathbf{n}} = \mathbf{v} \cdot \mathbf{n} \quad \text{on } \partial\Omega.$$

Remark. An analogous decomposition in $\mathbf{L}^q(\Omega)$ (for $1 < q < \infty$) is possible $\Longleftrightarrow$ this Neumann problem has a weak solution φ in $\mathbf{D}^{1,q'}(\Omega) := \{w \in L^1_{loc}(\Omega);\ \nabla w \in \mathbf{L}^{q'}(\Omega)\}$, where $1/q + 1/q' = 1$.

2. Weak solution to the Navier–Stokes equations I

(First observations and definition)

Further important function spaces:

$\mathbf{W}^{1,2}_{0,\sigma}(\Omega) := \mathbf{W}^{1,2}_0(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$ (a closed subspace of $\mathbf{W}^{1,2}_0(\Omega)$), dense in $\mathbf{L}^2_\sigma(\Omega)$)

$\mathbf{W}^{-1,2}_{0,\sigma}(\Omega) \ldots$ the dual to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$

Formal considerations (steps 1 and 2)

Step 1: the integral equation. Let $\boldsymbol{\phi}$ be an arbitrary test function from the space $C_0^\infty\big([0,T);\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$. If we multiply (1.8) by $\boldsymbol{\phi}$, integrate in Ω, and apply the integration by parts, we obtain the integral equation

$$\boxed{\begin{aligned}\int_0^T \int_\Omega \big[\mathbf{v}\cdot\partial_t\boldsymbol{\phi} - \nu\nabla\mathbf{v} : \nabla\boldsymbol{\phi} - \mathbf{v}\cdot\nabla\mathbf{v}\cdot\boldsymbol{\phi}\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ = -\int_0^T \langle\mathbf{f},\boldsymbol{\phi}\rangle\,\mathrm{d}t - \int_\Omega \mathbf{v}_0\cdot\boldsymbol{\phi}(0)\,\mathrm{d}\mathbf{x}\end{aligned}} \tag{2.1}$$

Step 2: a priori estimates. Let us multiply formally the Navier–Stokes equation (1.8) by function $\mathbf{v}$, integrate in Ω, and apply the integration by parts. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\mathbf{v}|^2\,\mathrm{d}\mathbf{x} + \nu\int_\Omega \Delta\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} \;=\; \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\mathbf{v}\|_2^2 + \nu\,\|\nabla\mathbf{v}\|_2^2 \;=\; (\mathbf{f},\mathbf{v})_2 \;\le\; \|\mathbf{f}\|_{-1,2}\,\|\mathbf{v}\|_{1,2}$$

$$\le\; C\,\|\mathbf{f}\|_{-1,2}\,\|\nabla\mathbf{v}\|_2 \;\le\; \frac{\nu}{2}\|\nabla\mathbf{v}\|_2^2 + C(\nu)\,\|\mathbf{f}\|_{-1,2}^2,$$

$$\|\mathbf{v}(t)\|_2^2 + \nu\int_0^t \|\nabla\mathbf{v}\|_2^2\,\mathrm{d}\tau \;\le\; \|\mathbf{v}_0\|_2^2 + \int_0^t \|\mathbf{f}\|_{-1,2}^2\,\mathrm{d}\tau$$

$$\le\; \|\mathbf{v}_0\|_2^2 + C(\nu)\int_0^T \|\mathbf{f}\|_{-1,2}^2\,\mathrm{d}\tau$$

Omitting at first the second term on the left hand side, we get

$$\|\mathbf{v}(t)\|_2^2 \;\le\; \|\mathbf{v}_0\|_2^2 + C(\nu)\int_0^T \|\mathbf{f}\|_{-1,2}^2\,\mathrm{d}\tau \qquad \text{for all } t\in(0,T).$$

This a priori estimate indicates that solution $\mathbf{v}$ should be in $L^\infty(0,T;\,\mathbf{L}^2_\sigma(\Omega))$.

Then, omitting the first term on the left hand side and considering $t = T$, we obtain

$$\nu \int_0^T \|\nabla \mathbf{v}\|_2^2 \, \mathrm{d}\tau \ \le \ \|\mathbf{v}_0\|_2^2 + C(\nu) \int_0^T \|\mathbf{f}\|_{-1,2}^2 \, \mathrm{d}\tau.$$

This a priori estimate indicates that solution $\mathbf{v}$ should be in $L^2(0,T;\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$.

Both the estimates require $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{f} \in L^2(0,T;\ \mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$.

Thus, we arrive at the definition:

A weak formulation of the Navier–Stokes problem (1.8)–(1.11). Let $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{f} \in L^2(0,T;\mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$. A vector function $\mathbf{v} \in L^2(0,T;\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega)) \cap L^\infty(0,T;$ $\mathbf{L}^2_\sigma(\Omega))$ is said to be a **weak solution** of the problem (1.8)–(1.11) if it satisfies integral equation (2.1) for all $\boldsymbol{\phi} \in C_0^\infty\big([0,T);\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$.

Remark. In order to show that no important information on the solution was lost when passing from the "classical formulation" (1.8)–(1.11) of the considered Navier–Stokes initial–boundary value problem to the weak formulation, we assume that $\mathbf{v}$ is a "sufficiently smooth" weak solution. Applying the backward integration by parts to (2.1), we get

$$\int_\Omega [\mathbf{v}(0) - \mathbf{v}_0] \cdot \boldsymbol{\phi}(0) \, \mathrm{d}\mathbf{x} + \int_0^T \int_\Omega \big[\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} - \mathbf{f} \big] \cdot \boldsymbol{\phi} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \;=\; 0 \qquad (2.2)$$

for all $\boldsymbol{\phi} \in C_0^\infty\big([0,T);\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$. Considering at first $\boldsymbol{\phi} \in C_0^\infty\big((0,T);\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$ (which guarantees that $\boldsymbol{\phi}(0) = \mathbf{0}$), we obtain

$$\int_0^T \int_\Omega \big[\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} - \mathbf{f} \big] \cdot \boldsymbol{\phi} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \;=\; 0.$$

Hence $\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} - \mathbf{f} \in \mathbf{G}_2(\Omega)$ for a.a. $t \in (0,T)$. Consequently, there exists p (also depending on t) such that

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} - \mathbf{f} \;=\; -\nabla p,$$

which is equation (1.8).

Now we return to $\boldsymbol{\phi} \in C_0^\infty\big([0,T);\, \mathbf{W}_{0,\sigma}^{1,2}(\Omega)\big)$ and to equation (2.2). It gives

$$\begin{aligned} 0 &= \int_\Omega [\mathbf{v}(0) - \mathbf{v}_0] \cdot \boldsymbol{\phi}(0)\, \mathrm{d}\mathbf{x} + \int_0^T \int_\Omega \big[\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} - \mathbf{f}\big] \cdot \boldsymbol{\phi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \\ &= \int_\Omega [\mathbf{v}(0) - \mathbf{v}_0] \cdot \boldsymbol{\phi}(0)\, \mathrm{d}\mathbf{x} - \int_0^T \int_\Omega \nabla p \cdot \boldsymbol{\phi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \\ &= \int_\Omega [\mathbf{v}(0) - \mathbf{v}_0] \cdot \boldsymbol{\phi}(0)\, \mathrm{d}\mathbf{x}. \end{aligned}$$

As this holds for all $\boldsymbol{\phi} \in C_0^\infty\big([0,T);\, \mathbf{W}_{0,\sigma}^{1,2}(\Omega)\big)$, i.e. for all $\boldsymbol{\phi}(0) \in \mathbf{W}_{0,\sigma}^{1,2}(\Omega)$, we obtain $\mathbf{v}(0) = \mathbf{v}_0$.

Equation (1.9) (i.e. $\operatorname{div} \mathbf{v} = 0$) is satisfied due to the condition $\mathbf{v} \in L^\infty(0,T;\, \mathbf{L}_\sigma^2(\Omega))$.

Boundary condition (1.10) (i.e. $\mathbf{v} = \mathbf{0}$ on $\partial\Omega \times (0,T)$) holds due to the condition $\mathbf{v} \in L^2(0,T;\, \mathbf{W}_{0,\sigma}^{1,2}(\Omega))$.

Conclusion: A "sufficiently smooth" weak solution $\mathbf{v}$ satisfies the system (1.8)–(1.11) in a classical sense.

3. The Stokes problem (see e.g. Galdi [1], Sohr [11] or Temam [12])

The steady Stokes problem

$$-\nu\Delta\mathbf{v} = -\nabla p + \mathbf{f} \qquad \text{in } \Omega, \tag{3.1}$$

$$\operatorname{div}\mathbf{v} = 0 \qquad \text{in } \Omega, \tag{3.2}$$

$$\mathbf{v} = \mathbf{0} \qquad \text{on } \partial\Omega. \tag{3.3}$$

Recall the function spaces

$\mathbf{W}^{1,2}_{0,\sigma}(\Omega) := \mathbf{W}^{1,2}_0(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$

$\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ ... the dual to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$

Remark. As $\mathbf{W}^{1,2}_{0,\sigma}(\Omega) \hookrightarrow \mathbf{L}^2_\sigma(\Omega)$, we have $\mathbf{L}^2_\sigma(\Omega) \subset \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. If $\mathbf{g} \in \mathbf{L}^2_\sigma(\Omega)$ then $\langle \mathbf{g}, \mathbf{w}\rangle := (\mathbf{g}, \mathbf{w})_2$ for all $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$.

A weak solution formulation of the problem (3.1)–(3.3). Let $\mathbf{f} \in \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Function $\mathbf{v} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ is said to be a **weak solution** of the problem (3.1)–(3.3) if

$$\boxed{\nu\,(\nabla\mathbf{v}, \nabla\mathbf{w})_2 \;\equiv\; \nu\int_\Omega \nabla\mathbf{v} : \nabla\mathbf{w}\; \mathrm{d}\mathbf{x} \;=\; \langle\mathbf{f}, \mathbf{w}\rangle \qquad \forall\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).} \tag{3.4}$$

Remark. (3.4) follows formally from (3.1)–(3.3) if we multiply (3.1) by $\mathbf{w}$ and integrate in Ω. On the other hand, if $\mathbf{v}$ is a "smooth" weak solution, then we can we can apply the backward integration by parts to (3.4) and get

$$\nu\int_\Omega [\nu\Delta\mathbf{v} + \mathbf{f}] \cdot \mathbf{w}\; \mathrm{d}\mathbf{x} \;=\; 0 \quad \forall\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

Hence $\nu\Delta\mathbf{v} + \mathbf{f} \in \mathbf{G}_2(\Omega)$. Thus, there exists p such that $\nu\Delta\mathbf{v} + \mathbf{f} = \nabla p$, which is equation (3.1).

Operator $\mathcal{A}$: Define a linear operator $\mathcal{A} : \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \to \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ by the equation

$$\langle\mathcal{A}\mathbf{v}, \mathbf{w}\rangle \;=\; (\nabla\mathbf{v}, \nabla\mathbf{w})_2 \qquad \forall\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

Now, (3.4) is equivalent to the equation $\boxed{\nu\mathcal{A}\mathbf{v} = \mathbf{f}}$ (in space $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$).

Basic properties of operator $\mathcal{A}$:

- $D(\mathcal{A}) = \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ *(follows from the definition of $\mathcal{A}$)*

- *Operator $\mathcal{A}$ is 1–1.*

$$\Big(\mathbf{v} \in N(\mathcal{A}) \implies \forall \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega) : (\nabla\mathbf{v}, \nabla\mathbf{w})_2 = 0 \implies \|\nabla\mathbf{v}\|_2 = 0 \implies \mathbf{v} = \mathbf{0}\Big)$$

- *Operator $\mathcal{A}$ is bounded (as an operator from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$).*

$$\Big(\|\mathcal{A}\mathbf{v}\|_{-1,2} = \sup_{\mathbf{w}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega),\ \mathbf{w}\neq\mathbf{0}} \frac{|\langle\mathcal{A}\mathbf{v}, \mathbf{w}\rangle|}{\|\mathbf{w}\|_{1,2}} = \sup_{\mathbf{w}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega),\ \mathbf{w}\neq\mathbf{0}} \frac{|(\nabla\mathbf{v},\ \nabla\mathbf{w})_2|}{\|\mathbf{w}\|_{1,2}} \leq \|\nabla\mathbf{v}\|_2\Big)$$

- *The range of $\mathcal{A}$ need not be generally the whole space $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$.*

Proof. Operator $\mathcal{A}$ is closed because its domain is the whole $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ and it is bounded. Hence $\mathcal{A}^{-1}$ is also closed. **By contradiction:** Assume that $R(\mathcal{A}) = D(\mathcal{A}^{-1}) = \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Then operator $\mathcal{A}^{-1}$ is bounded (by the closed graph theorem).

Choose $\mathbf{z}_n \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ so that $\|\nabla \mathbf{z}_n\|_2 \longrightarrow 0$ and $\|\mathbf{z}_n\|_2 \longrightarrow 1$. (This choice is possible e.g. if Ω is an exterior domain or $\Omega = \mathbb{R}^3$.) Let $\mathbf{f}_n \in \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ be defined by the equation

$$\langle \mathbf{f}_n, \mathbf{w} \rangle \;:=\; (\nabla \mathbf{z}_n, \nabla \mathbf{w})_2 + (\mathbf{z}_n, \mathbf{w})_2 \qquad \forall \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega). \tag{3.5}$$

Then $\{\mathbf{f}_n\}$ is a bounded sequence in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Put $\mathbf{u}_n := \mathcal{A}^{-1}\mathbf{f}_n$. It means that $\mathbf{f}_n = \mathcal{A}\mathbf{u}_n$. Hence

$$\langle \mathbf{f}_n, \mathbf{w} \rangle \;:=\; (\nabla \mathbf{u}_n, \nabla \mathbf{w})_2 \qquad \forall \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega). \tag{3.6}$$

Equations (3.5) and (3.6) (with $\mathbf{w} = \mathbf{z}_n$ yield

$$(\nabla \mathbf{z}_n, \nabla \mathbf{z}_n)_2 + (\mathbf{z}_n, \mathbf{z}_n)_2 \;=\; (\nabla \mathbf{u}_n, \nabla \mathbf{z}_n)_2 \;\leq\; \|\nabla \mathbf{u}_n\|_2 \, \|\nabla \mathbf{z}_n\|_2$$

The left hand side tends to one, while the right hand side tends to zero (for $n \longrightarrow \infty$). This is the contradiction. ■

- *The range of $\mathcal{A}$ need not generally contain $\mathbf{L}^2_\sigma(\Omega)$.*

Proof. By contradiction, assume that $\mathbf{L}^2_\sigma(\Omega) \subset R(\mathcal{A})$. Denote by A the restriction of $\mathcal{A}$ to $\mathcal{A}^{-1}(\mathbf{L}^2_\sigma(\Omega))$. Operator A is closed as an operator from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{L}^2_\sigma(\Omega)$. (This can be proven similarly as the fact that $\mathcal{A}$ is closed.) Hence A^{-1} is a bounded operator from $\mathbf{L}^2_\sigma(\Omega)$ to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ (by the closed graph theorem), because $D(A^{-1}) = \mathbf{L}^2_\sigma(\Omega)$.

$\mathbf{f} \in \mathbf{L}^2_\sigma(\Omega) \implies \mathbf{f} \in \mathbf{W}^{-1,2}_{0,\sigma}(\Omega),\ \langle \mathbf{f}, \mathbf{w} \rangle = (\mathbf{f}, \mathbf{w})_2\ \forall \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$

$\mathbf{L}^2_\sigma(\Omega) \subset R(\mathcal{A}) \implies \exists \mathbf{u} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega) : \langle \mathbf{f}, \mathbf{w} \rangle = (\nabla \mathbf{u}, \nabla \mathbf{w})_2\ \forall \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$

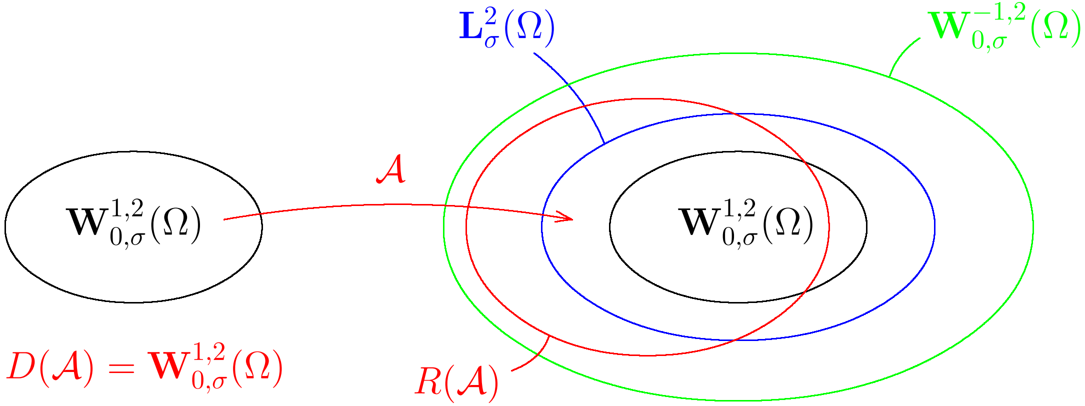
Hence $(\mathbf{f}, \mathbf{w})_2 = (\nabla \mathbf{u}, \nabla \mathbf{w})_2\ \ \forall \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$, where $\mathbf{u} = A^{-1}\mathbf{f}$.

Choose $\{\mathbf{f}_n\}$ in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ so that $\|\mathbf{f}_n\|_2 \to 1$ and $\|\nabla \mathbf{f}_n\|_2 \to 0$. (A sequence with these properties exists e.g. if Ω is an exterior domain or $\Omega = \mathbb{R}^3$.) Put $\mathbf{w}_n = \mathbf{f}_n$. Then

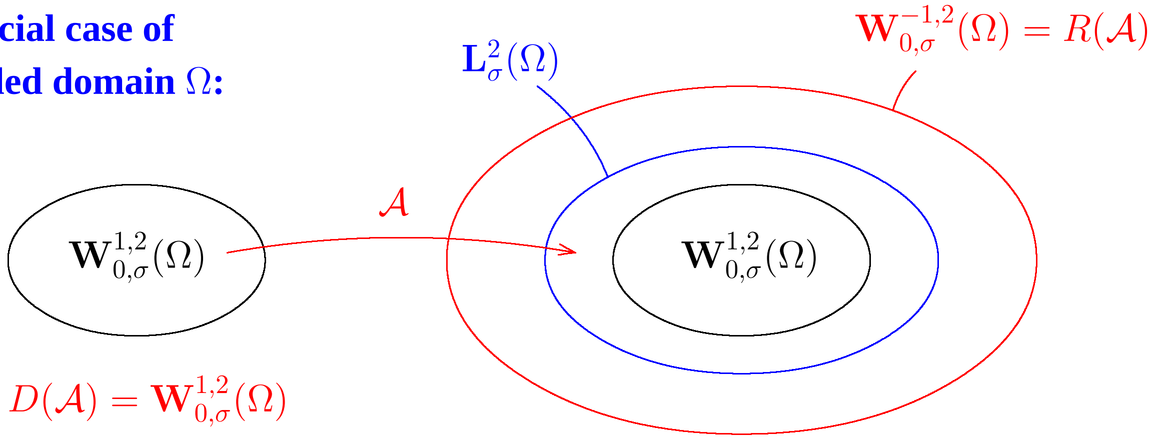
$$\|\mathbf{f}_n\|_2 \ = \ (\nabla \mathbf{u}_n, \nabla \mathbf{f}_n)_2 \ \leq \ \|\nabla \mathbf{u}_n\|_2\, \|\mathbf{f}_n\|_2,$$

where $\mathbf{u}_n = A^{-1}\mathbf{f}_n$. The right hand side tends to zero, while the left hand side tends to one (for $n \to \infty$). This is a contradiction. ■

Domain and range of operator $\mathcal{A}$:



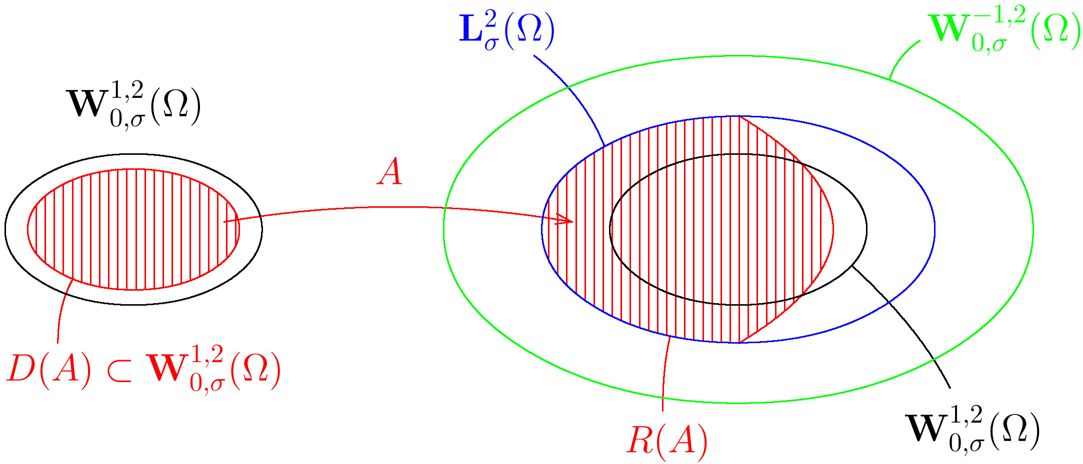
The special case of
a bounded domain Ω:



- *If Ω is bounded then* $R(\mathcal{A}) = \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$.

Proof. The scalar product $(\nabla\mathbf{v}, \nabla\mathbf{w})_2$ is equivalent to the scalar product $(\mathbf{v}, \mathbf{w})_{1,2}$ in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. Hence, given $\mathbf{f} \in \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, there exists $\mathbf{u} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ such that $\langle \mathbf{f}, \mathbf{w} \rangle = (\nabla\mathbf{u}, \nabla\mathbf{w})_2$ for all $\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ (by the Riesz lemma). It means that $\mathbf{f} = \mathcal{A}\mathbf{u}$ (the identity in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$). ■

Corollary: *If Ω is bounded then $\mathcal{A}^{-1}$ is bounded from $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$.*



Denote by A the part of operator $\mathcal{A}$ with the range $R(\mathcal{A}) \cap \mathbf{L}^2_\sigma(\Omega)$. Thus, A is the restriction of $\mathcal{A}$ to

$$D(A) \;:=\; \big\{\mathbf{u} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega);\; \mathcal{A}\mathbf{u} \in \mathbf{L}^2_\sigma(\Omega)\big\} \;=\; \mathcal{A}^{-1}[R(\mathcal{A}) \cap \mathbf{L}^2_\sigma(\Omega)].$$

Operator A is an operator in $\mathbf{L}^2_\sigma(\Omega)$. It is often called the **Stokes operator.**

Some properties of operator A: (see, e.g., [11])

- *A is a 1–1 positive and self–adjoint operator in $\mathbf{L}^2_\sigma(\Omega)$. Its domain satisfies the inclusions $\mathbf{C}^\infty_{0,\sigma}(\Omega) \subset D(A) \subset \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$.*

- *If $\mathbf{f} \in \mathbf{L}^2_\sigma(\Omega)$ then the steady Stokes problem is equivalent to $\boxed{\nu A\mathbf{v} = \mathbf{f}}$. It means that there exists $p \in L^2_{loc}(\Omega)$ (unique up to an additive constant), such that*

$$-\nu\Delta\mathbf{v} + \nabla p \;=\; \mathbf{f} \quad \textit{(in the sense of distributions in } \Omega\textit{)}. \tag{3.7}$$

Principle of the proof. The equation $\nu A\mathbf{v} = \mathbf{f}$ means that

$$\nu\,(\nabla\mathbf{v}, \nabla\mathbf{w})_2 \;=\; (\mathbf{f}, \mathbf{w})_2 \qquad \forall\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

Hence

$$\nu\,\langle -\Delta\mathbf{v}, \mathbf{w}\rangle \;=\; (\mathbf{f}, \mathbf{w})_2 \qquad \forall\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega),$$

$$\langle -\nu\Delta\mathbf{v} - \mathbf{f}, \mathbf{w}\rangle \;=\; 0 \qquad \forall\mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega).$$

Then there exists $p \in L^2_{loc}(\Omega)$ such that $\nu\Delta\mathbf{v} - \mathbf{f} = \nabla p$ (in the sense of distributions in Ω); see [11]. ∎

- *If Ω is bounded then $R(A) \equiv D(A^{-1}) = \mathbf{L}^2_\sigma(\Omega)$ and operator A^{-1} is bounded from $\mathbf{L}^2_\sigma(\Omega)$ to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$.*

  Proof. **1st possibility:** If $\mathbf{u} \in D(A)$ then $(A\mathbf{u}, \mathbf{w})_2 = (\nabla\mathbf{u}, \nabla\mathbf{w})_2 \;\; \forall \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. Hence $\|\nabla\mathbf{u}\|_2^2 \le \|A\mathbf{u}\|_2 \, \|\mathbf{u}\|_2 \le c \, \|A\mathbf{u}\|_2 \, \|\nabla\mathbf{u}\|_2$. (Constant c is the constant from Poincaré's inequality $\|\mathbf{u}\|_2 \le c \, \|\nabla\mathbf{u}\|_2$.) This yields

  $$\|\nabla A^{-1}\mathbf{f}\|_2 \le c \, \|\mathbf{f}\|_2 \quad \text{for } \mathbf{f} = A\mathbf{u}.$$

  2nd possibility: by the closed graph theorem ■

- If Ω is a bounded C^2–domain then $D(A) = \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{W}^{2,2}(\Omega)$, $A = -P_\sigma\Delta$, and

  $$\|\mathbf{u}\|_{2,2} + \|\nabla p\|_2 \;\le\; c \, \|\mathbf{f}\|_2 \;=\; c \, \|A\mathbf{u}\|_2$$

  for all $\mathbf{u}$, p and $\mathbf{f}$ satisfying $-\Delta\mathbf{u} + \nabla p = \mathbf{f}$ (i.e. $A\mathbf{u} = \mathbf{f}$).

  This is a deep statement. It shows that operator A has the so called **maximum regularity property.**

The non–steady Stokes problem

$$\begin{aligned} \partial_t \mathbf{v} - \nu \Delta \mathbf{v} &= -\nabla p + \mathbf{f} && \text{in } Q_T, && (3.8) \\ \operatorname{div} \mathbf{v} &= 0 && \text{in } Q_T, && (3.9) \\ \mathbf{v} &= \mathbf{0} && \text{on } \partial\Omega \times (0,T), && (3.10) \\ \mathbf{v} &= \mathbf{v}_0 && \text{in } \Omega \times \{0\}. && (3.11) \end{aligned}$$

A weak formulation of the non–steady Stokes problem (3.8)–(3.11). Let $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{f} \in L^2(0,T;\, \mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$. A vector function $\mathbf{v} \in L^2(0,T;\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$ is said to be a **weak solution** of the problem (3.8)–(3.10) if

$$\boxed{\int_0^T \int_\Omega \big[-\mathbf{v} \cdot \partial_t \boldsymbol{\phi} + \nu \nabla \mathbf{v} : \nabla \boldsymbol{\phi}\big]\, \mathrm{d}\mathbf{x}\, \mathrm{d}t = \int_0^T \langle \mathbf{f}, \boldsymbol{\phi} \rangle\, \mathrm{d}t + \int_\Omega \mathbf{v}_0 \cdot \boldsymbol{\phi}(0)\, \mathrm{d}\mathbf{x}} \qquad (3.12)$$

for all $\boldsymbol{\phi} \in C_0^\infty\big([0,T);\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$.

Remark. As each function $\phi = \phi(\mathbf{x},t) \in C_0^\infty([0,T);\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$ can be approximated, with an arbitrary accuracy in the norm of $C_0^1([0,T);\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$, by a sum of finitely many functions of the type $\boldsymbol{\varphi}(\mathbf{x})\vartheta(t)$, where $\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ and $\vartheta \in C_0^\infty([0,T))$, (3.12) is equivalent to

$$\begin{aligned} \forall \boldsymbol{\varphi}\ \forall \vartheta\ :\quad & \int_0^T \int_\Omega \big[-\mathbf{v}\cdot\boldsymbol{\varphi}\,\dot{\vartheta}(t) + \nu\nabla\mathbf{v} : \nabla\boldsymbol{\varphi}\,\vartheta(t)\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &= \int_0^T \langle \mathbf{f}, \boldsymbol{\varphi}\rangle\,\vartheta(t)\,\mathrm{d}t + \vartheta(0)\int_\Omega \mathbf{v}_0\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}. \end{aligned}$$

This can also be written in the form

$$\forall \boldsymbol{\varphi}\ \forall \vartheta\ :\quad \int_0^T (\mathbf{v},\boldsymbol{\varphi})_2\,\dot{\vartheta}(t)\,\mathrm{d}t - \int_0^T \big\langle \nu\mathcal{A}\mathbf{v} - \mathbf{f},\, \boldsymbol{\varphi}\big\rangle\,\vartheta(t)\,\mathrm{d}t \;=\; -\big(\mathbf{v}_0,\boldsymbol{\varphi}\big)_2\,\vartheta(0),$$

which is equivalent to

$$\boxed{\forall \boldsymbol{\varphi}\ :\quad \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v},\boldsymbol{\varphi})_2 + \big\langle \nu\mathcal{A}\mathbf{v} - \mathbf{f},\boldsymbol{\varphi}\big\rangle \;=\; 0 \quad \text{a.e. in } (0,T).} \tag{3.13}$$

The derivative with respect to t is understood in the sense of distributions.

Lemma. (follows e.g. from [12, Lemma III.1.1]) *Let H be a Hilbert space. Let $u,\ g \in L^1(0,T;\ H)$. Then the next conditions are equivalent:*

- $\dot{u} = g$ *a.e. in $(0,T)$, where $\dot{u}$ is the distributional derivative of u in $(0,T)$,*
- $\frac{\mathrm{d}}{\mathrm{d}t}(u,\varphi)_H = (g,\varphi)_H$ *a.e. in $(0,T)$, for all $\varphi \in H$, where the derivative with respect to t is the distributional derivative in $(0,T)$.*

Due to this lemma, (3.13) is equivalent to

$$\boxed{\dot{\mathbf{v}} + \nu\mathcal{A}\mathbf{v} \; = \; \mathbf{f} \qquad \text{a.e. in } (0,T),} \tag{3.14}$$

which is an equation in space $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$.

Equivalent definition of the weak solution to the problem (3.8)–(3.11). Let $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{f} \in L^2(0,T;\mathbf{W}_{0,\sigma}^{-1,2}(\Omega))$. A vector function $\mathbf{v} \in L^2(0,T;\ \mathbf{W}_{0,\sigma}^{1,2}(\Omega))$ is said to be a **weak solution** of the problem (3.8)–(3.11) if it satisfies differential equation (3.13) (or, alternatively, differential equation (3.14)), with the initial condition $\mathbf{v}(0) = \mathbf{v}_0$.

Remark (in which sense the weak solution satisfies the initial condition). It follows from (3.14) that $\dot{\mathbf{v}} \in L^2(0,T;\ \mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$. Hence $\mathbf{v}$ is continuous as a mapping from $(0,T)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. The initial condition $\mathbf{v}(0) := \mathbf{v}_0$ now means that $\mathbf{v}_0$ should be equal to the limit (for $t \to 0+$) of $\mathbf{v}(t)$ in the norm of $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. (The next theorem shows that $\lim_{t\to 0+} \mathbf{v}(t) = \mathbf{v}_0$ even in the norm of $\mathbf{L}^2_\sigma(\Omega)$.)

Theorem 1. *Given $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{f} \in L^2(0,T;\ \mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$, the problem (3.8)–(3.11) has a unique weak solution $\mathbf{v}$. The solution satisfies the energy equality*

$$\begin{aligned} &\|\mathbf{v}(t)\|_2^2 + 2\nu \int_0^t \|\nabla\mathbf{v}(\tau)\|_2^2 \,\mathrm{d}\tau \\ &\quad = \|\mathbf{v}_0\|_2^2 + 2\int_0^t \big\langle \mathbf{f}(\tau), \mathbf{v}(\tau)\big\rangle \,\mathrm{d}\tau \quad \textit{for all } t \in [0,T). \end{aligned} \tag{3.15}$$

Moreover, $\mathbf{v} \in C([0,T);\ \mathbf{L}^2_\sigma(\Omega))$.

Proof. **Existence:** the proof by means of Galerkin's method will be shown later in a more complicated non–linear case of the Navier–Stokes equation.

Lemma (see e.g. Lions, Magenes [8]). *Let* $V \hookrightarrow H \hookrightarrow V'$ *be three Hilbert spaces such that* V' *is a dual to* V*. Let* $u \in L^2(0,T;\,V)$ *and* $\dot{u} \in L^2(0,T;\,V')$*. Then* u *is (after a possible redefinition on a set of measure zero) continuous as a function from* $[0,T]$ *to* H *and*

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|u(t)\|_H^2 \;=\; \langle \dot{u}(t), u(t)\rangle \quad \textit{(in the sense of distributions in } (0,T)\textit{).} \tag{3.16}$$

Energy equality: equation (3.14) implies

$$\begin{aligned} \langle \dot{\mathbf{v}}, \mathbf{v}\rangle + \nu\,\langle \mathcal{A}\mathbf{v}, \mathbf{v}\rangle \;&=\; \langle \mathbf{f}, \mathbf{v}\rangle, \\ \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathbf{v}\|_2^2 + \nu\,(\nabla\mathbf{v}, \nabla\mathbf{v})_2 \;&=\; \langle \mathbf{f}, \mathbf{v}\rangle. \end{aligned}$$

Integrating this identity from 0 to t, we obtain (3.15).

Uniqueness: let $\mathbf{v}_1$, $\mathbf{v}_2$ be two solutions, corresponding to the same data $\mathbf{v}_0$ and $\mathbf{f}$. Then $\mathbf{v} := \mathbf{v}_1 - \mathbf{v}_2$ is a solution of the same problem, with the body force $\mathbf{f} - \mathbf{f} = \mathbf{0}$ and with the initial condition $\mathbf{v}(0) = \mathbf{v}_0 - \mathbf{v}_0 = \mathbf{0}$. Due to (3.15), $\mathbf{v} \equiv \mathbf{0}$. ■

More regular solutions:

Theorem 2. *Let* Ω *be a bounded* C^2*–domain,* $\mathbf{v}_0 \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ *and* $\mathbf{f} \in L^2(0,T;\, \mathbf{L}^2_\sigma(\Omega))$*. Then solution* $\mathbf{v}$*, given by Theorem 1, is in* $L^2(0,T;\, \mathbf{W}^{2,2}(\Omega))$*. Its derivative with respect to* t *is in* $L^2(0,T;\, \mathbf{L}^2_\sigma(\Omega))$ *and an associated pressure* p *is in* $L^2(0,T;\, W^{1,2}(\Omega))$*.*

Proof – based on the a priori estimate: we multiply formally equation (3.8) by $P_\sigma\Delta\mathbf{v}$ and integrate in Ω. Applying the integration by parts, we get

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\nabla\mathbf{v}\|_2^2 + \nu\|P_\sigma\Delta\mathbf{v}\|_2^2 \;=\; (\mathbf{f}, P_\sigma\Delta\mathbf{v})_2$$

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\nabla\mathbf{v}\|_2^2 + \nu\|A\mathbf{v}\|_2^2 \;=\; (\mathbf{f}, A\mathbf{v})_2 \;\leq\; \frac{\nu}{2}\,\|A\mathbf{v}\|_2^2 + \frac{1}{2\nu}\,\|\mathbf{f}\|_2^2$$

Integrating this estimate with respect to t, we get

$$|\!|\!|\nabla\mathbf{v}|\!|\!|_{\infty;\,0,2} + |\!|\!|A\mathbf{v}|\!|\!|_{2;\,0,2} \;\leq\; c\,|\!|\!|\mathbf{f}|\!|\!|_{2;\,0,2} + c\,\|\mathbf{v}_0\|_{1,2}$$

where $|\!|\!|\mathbf{g}|\!|\!|_{r;\,k,s} \;:=\; \|\mathbf{g}\|_{L^r(0,T;\,\mathbf{W}^{k,s}(\Omega))}$. ∎

4. Weak solution to the Navier–Stokes equations II

(other equivalent definitions, subtler properties)

Define

$$\mathcal{B} \, : \, \mathbf{W}^{1,2}_{0,\sigma}(\Omega)^2 \longrightarrow \mathbf{W}^{-1,2}_{0,\sigma}(\Omega) \qquad \ldots \qquad \langle \mathcal{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle := \int_\Omega \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x}$$

for $\mathbf{u},\ \mathbf{v},\ \mathbf{w} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. By analogy with (2.33), we get

$$\boxed{\forall\,\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \ : \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v},\boldsymbol{\varphi})_2 + \big\langle \nu\mathcal{A}\mathbf{v} + \mathcal{B}(\mathbf{v},\mathbf{v}) - \mathbf{f},\boldsymbol{\varphi}\big\rangle \ = \ 0.} \qquad (4.1)$$

This is a differential equation in $(0,T)$. The derivative with respect to t is understood in the sense of distributions.

Operator $\mathcal{B}$ satisfies

$$\|\mathcal{B}(\mathbf{u},\mathbf{v})\|_{-1,2} = \sup_{\mathbf{w}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega),\ \mathbf{w}\neq\mathbf{0}} \frac{|\langle\mathcal{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle|}{\|\mathbf{w}\|_{1,2}} = \sup_{\mathbf{w}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega),\ \mathbf{w}\neq\mathbf{0}} \frac{|(\mathbf{u}\cdot\nabla\mathbf{v},\ \mathbf{w})_2|}{\|\mathbf{w}\|_{1,2}}$$

$$\leq \sup_{\mathbf{w}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega),\ \mathbf{w}\neq\mathbf{0}} \frac{\|\nabla\mathbf{v}\|_2\,\|\mathbf{u}\|_2^{\frac{1}{2}}\,\|\mathbf{u}\|_6^{\frac{1}{2}}\,\|\mathbf{w}\|_6^{\frac{1}{2}}}{\|\mathbf{w}\|_{1,2}} \leq c\,\|\nabla\mathbf{v}\|_2\,\|\mathbf{u}\|_2^{\frac{1}{2}}\,\|\mathbf{u}\|_6^{\frac{1}{2}}$$

$$\leq c\,\|\nabla\mathbf{v}\|_2\,\|\mathbf{u}\|_2^{\frac{1}{2}}\,\|\nabla\mathbf{u}\|_2^{\frac{1}{2}}.$$

Thus, if $\mathbf{v}\in L^\infty(0,T;\,\mathbf{L}^2_\sigma(\Omega))\cap L^2(0,T;\,\mathbf{W}^{1,2}_0(\Omega))$ (which is the class for weak solutions) then $\mathcal{A}\mathbf{v}\in L^2(0,T;\,\mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$ and $\mathcal{B}(\mathbf{v},\mathbf{v})\in L^{4/3}(0,T;\,\mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$.

By analogy with (3.14), we deduce that (4.1) is equivalent to

$$\boxed{\dot{\mathbf{v}}+\nu\mathcal{A}\mathbf{v}+\mathcal{B}(\mathbf{v},\mathbf{v}) = \mathbf{f}} \qquad \text{(an equation in } \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)). \tag{4.2}$$

Remark (on the initial condition). As $\dot{\mathbf{v}}\in L^{4/3}(0,T;\,\mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$, $\mathbf{v}$ (after a possible redefinition on a set of measure zero) is continuous as a mapping from $(0,T)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. The initial condition $\mathbf{v}(0):=\mathbf{v}_0$ now means that $\mathbf{v}_0$ equals the limit (for $t\to 0+$) of $\mathbf{v}(t)$ in the norm of $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$.

Equivalent definition of the weak solution to the Navier–Stokes initial–boundary value problem (1.8)–(1.11).

Let $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{f} \in L^2(0,T;\, \mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$.

A vector function $\mathbf{v} \in L^2(0,T;\, \mathbf{W}^{1,2}_0(\Omega)) \cap L^\infty(0,T;\, \mathbf{L}^2_\sigma(\Omega))$ is said to be a **weak solution** of the problem (1.8)–(1.11) if it satisfies

$$\boxed{\forall\, \boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \ : \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}, \boldsymbol{\varphi})_2 + \big\langle \nu\mathcal{A}\mathbf{v} + \mathcal{B}(\mathbf{v},\mathbf{v}) - \mathbf{f}, \boldsymbol{\varphi} \big\rangle \ = \ 0} \tag{4.1}$$

or, alternatively,

$$\boxed{\dot{\mathbf{v}} + \nu\mathcal{A}\mathbf{v} + \mathcal{B}(\mathbf{v},\mathbf{v}) \ = \ \mathbf{f}} \qquad \text{(an equation in } \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)) \tag{4.2}$$

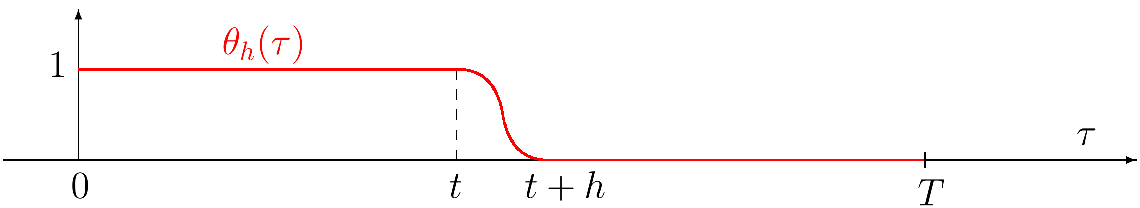
a.e. in $(0,T)$, with the initial condition $\mathbf{v}(0) = \mathbf{v}_0$.

Lemma (Hopf 1951, Prodi 1959, Serrin 1963). *The weak solution* $\mathbf{v}$ *to problem (1.3)–(1.4) can be redefined on a set of zero Lebesgue measure so that* $\mathbf{v}(\,.\,,t) \in \mathbf{L}^2(\Omega)$ *for all* $t \in [0,T)$ *and for all* $\boldsymbol{\phi} \in C_0^\infty([0,T);\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$*:*

$$\int_0^t \int_\Omega \big[\mathbf{v}\cdot\partial_\tau\boldsymbol{\phi} - \nu\nabla\mathbf{v} : \nabla\boldsymbol{\phi} - \mathbf{v}\cdot\nabla\mathbf{v}\cdot\boldsymbol{\phi}\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau$$
$$= -\int_0^t \langle\mathbf{f},\boldsymbol{\phi}\rangle\,\mathrm{d}\tau + \int_\Omega \mathbf{v}(t)\cdot\boldsymbol{\phi}(t)\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{v}_0\cdot\boldsymbol{\phi}(0)\,\mathrm{d}\mathbf{x}. \tag{4.3}$$

Corollary. *The weak solution* $\mathbf{v}$ *is weakly continuous as a mapping from* $[0,T)$ *to* $\mathbf{L}^2_\sigma(\Omega)$.

Principle of the proof of the lemma: We use a C^1 function θ_h as on the next figure. We use (2.1) with $\boldsymbol{\phi}(\mathbf{x},\tau)\,\theta_h(\tau)$ instead of $\boldsymbol{\phi}(\mathbf{x},\tau)$, and we consider the limit for $h \to 0$, see Galdi [2].



5. Global in time existence of the so called Leray–Hopf weak solution

Theorem 3 (existence of a weak solution – Leray 1934, Hopf 1951, et al). *Let* Ω *be a domain in* $\mathbb{R}^3$, $T > 0$, $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{f} \in \mathbf{L}^2(Q_T)$. *Then there exists at least one weak solution* $\mathbf{v}$ *to problem (1.3)–(1.4). The solution satisfies*

- *the energy inequality (EI)*

$$\|\mathbf{v}(\,.\,,t)\|_2^2 + 2\nu \int_0^t \|\nabla\mathbf{v}(\,.\,,\tau)\|_2^2\, \mathrm{d}\tau \\ \leq \|\mathbf{v}_0\|_2^2 + 2\int_0^t \big\langle \mathbf{f}(\,.\,,\tau), \mathbf{v}(\,.\,,\tau)\big\rangle\, \mathrm{d}\tau, \qquad (5.1)$$

  holds for all $t \in [0,T)$,

- $\lim_{t\to 0+} \|\mathbf{v}(\,.\,,t) - \mathbf{v}_0\|_2 = 0.$

Principles of the proof of Theorem 3

Assume, for simplicity, that Ω is bounded and Lipschitzian. Then $\mathbf{W}^{1,2}_{0,\sigma}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2_\sigma(\Omega)$. Consequently, A^{-1} is a compact operator in $\mathbf{L}^2_\sigma(\Omega)$.

Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ be the eigenvalues of operator A and $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\dots$ be the corresponding orthonormal eigenfunctions. Put $\mathbf{V}_n := \mathcal{L}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$.

1) Galerkin's approximations

Let $\mathbf{v}_n$ have the form $\mathbf{v}_n(t) = \sum_{i=1}^{n} \alpha_i(t)\, \mathbf{u}_i$ and let it satisfy

$$\partial_t(\mathbf{v}_n, \boldsymbol{\varphi})_2 + \nu\langle A\mathbf{v}_n, \boldsymbol{\varphi}\rangle + \langle \mathcal{B}(\mathbf{v}_n, \mathbf{v}_n), \boldsymbol{\varphi}\rangle \;=\; \langle \mathbf{f}, \boldsymbol{\varphi}\rangle \qquad \text{for all } \boldsymbol{\varphi} \in \mathbf{V}_n. \tag{5.2}$$

This is equivalent to

$$\partial_t(\mathbf{v}_n, \mathbf{u}_i) + \nu\langle A\mathbf{v}_n, \mathbf{u}_i\rangle + \langle \mathcal{B}(\mathbf{v}_n, \mathbf{v}_n), \mathbf{u}_i\rangle \;=\; \langle \mathbf{f}, \mathbf{u}_i\rangle \qquad \text{for } i = 1, 2, \dots, n.$$

It means that

$$\dot{\alpha}_i + \nu\lambda_i\alpha_i + \sum_{k,l=1}^{n} \alpha_k\alpha_l \langle \mathcal{B}(\mathbf{u}_k, \mathbf{u}_l), \mathbf{u}_i\rangle \;=\; \langle \mathbf{f}, \mathbf{u}_i\rangle \qquad \text{for } i = 1, 2, \dots, n. \tag{5.3}$$

This is a system of n ODE's for the unknown coefficients $\alpha_1(t)$, $\dots$, $\alpha_n(t)$. The system is solved with the initial conditions

$$\alpha_i(0) \;=\; (\mathbf{v}_0, \mathbf{u}_i)_2 \qquad i = 1, \dots, n. \tag{5.4}$$

We denote $\mathbf{v}_{0n} := \sum_{i=1}^{n} \alpha_i(0)\, \mathbf{u}_i$ (= the orthogonal projection of $\mathbf{v}_0$ into $\mathbf{V}_n$).

2) A priori estimates and existence of Galerkin's approximation $\mathbf{v}_n$

Multiply i–th equation by α_i and sum for $i = 1, \dots, n$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\sum_{i=1}^{n}\alpha_i^2 + \nu\sum_{i=1}^{n}\lambda_i\alpha_i^2 \;&=\; \sum_{i=1}^{n}\alpha_i\langle\mathbf{f},\mathbf{u}_i\rangle \;=\; \left\langle\mathbf{f},\sum_{i=1}^{n}\alpha_i\mathbf{u}_i\right\rangle \\
&\le\; \|\mathbf{f}\|_{-1,2}\left\|\sum_{i=1}^{n}\alpha_i\mathbf{u}_i\right\|_{1,2} \;\le\; C\,\|\mathbf{f}\|_{-1,2}\left\|\nabla\sum_{i=1}^{n}\alpha_i\mathbf{u}_i\right\|_2 \\
&=\; C\,\|\mathbf{f}\|_{-1,2}\left[\left(\sum_{i=1}^{n}\alpha_i\nabla\mathbf{u}_i,\sum_{j=1}^{n}\alpha_j\nabla\mathbf{u}_j\right)_2\right]^{\frac{1}{2}} \\
&=\; C\,\|\mathbf{f}\|_{-1,2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\langle A\mathbf{u}_i,\mathbf{u}_j\rangle\right]^{\frac{1}{2}} \;=\; C\,\|\mathbf{f}\|_{-1,2}\left[\sum_{i=1}^{n}\alpha_i^2\lambda_i\right]^{\frac{1}{2}} \\
&\le\; \frac{\nu}{2}\sum_{i=1}^{n}\lambda_i\alpha_i^2 + C\,\|\mathbf{f}\|_{-1,2}^2,
\end{aligned}
$$

where $C = C(\Omega,\nu)$. Integrating from 0 to t, we get

$$\sum_{i=1}^{n} \alpha_i^2(t) + \nu \int_0^t \sum_{i=1}^{n} \lambda_i \alpha_i^2(\tau)\, \mathrm{d}\tau \;\leq\; C \int_0^t \|\mathbf{f}\|_{-1,2}^2\, \mathrm{d}\tau + \sum_{i=1}^{n} \alpha_i^2(0),$$

$$\sum_{i=1}^{n} \alpha_i^2(t) + \nu \int_0^t \sum_{i=1}^{n} \lambda_i \alpha_i^2(\tau)\, \mathrm{d}\tau \;\leq\; C \int_0^t \|\mathbf{f}\|_{-1,2}^2\, \mathrm{d}\tau + \|\mathbf{v}_0\|_2^2, \tag{5.5}$$

$$\|\mathbf{v}_n(t)\|_2^2 + \nu \int_0^t \|\nabla \mathbf{v}_n(\tau)\|_2^2\, \mathrm{d}\tau \;\leq\; C \int_0^t \|\mathbf{f}\|_{-1,2}^2\, \mathrm{d}\tau + \|\mathbf{v}_0\|_2^2. \tag{5.6}$$

One can deduce from these estimates that the initial–value problem (5.3), (5.4) has a solution $\alpha_1, \dots, \alpha_n$ on $(0,T)$. The solution satisfies inequality (5.5) for all $t \in (0,T)$. Hence the approximate solution $\mathbf{v}_n$ satisfies inequality (5.6) for all $t \in (0,T)$.

Note that returning to the first line on the previous page, we also obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \frac{1}{2}\, \|\mathbf{v}_n\|_2^2 + \nu\, \|\nabla \mathbf{v}_n\|_2^2 \;=\; \langle \mathbf{f}, \mathbf{v}_n \rangle,$$

$$\|\mathbf{v}_n(t)\|_2^2 + 2\nu \int_0^t \|\nabla \mathbf{v}_n(\tau)\|_2^2\, \mathrm{d}\tau \;\leq\; 2 \int_0^t \langle \mathbf{f}, \mathbf{v}_n \rangle\, \mathrm{d}\tau + \|\mathbf{v}_0\|_2^2. \tag{5.7}$$

3) Convergent subsequences of $\{\mathbf{v}_n\}$

Inequality (5.6) provides uniform estimates of $\mathbf{v}_n$ in $L^\infty(0,T;\ \mathbf{L}^2_\sigma(\Omega))$ and in $L^2(0,T;$ $\mathbf{W}^{1,2}_{0,\sigma}(\Omega))$. Hence there exists a sub–sequence of $\{\mathbf{v}_n\}$ (we denote it in the same way) and $\mathbf{v} \in L^\infty(0,T;\ \mathbf{L}^2_\sigma(\Omega)) \cap L^2(0,T;\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$ such that

$$\mathbf{v}_n \longrightarrow \mathbf{v} \qquad \text{weakly–* in } L^\infty(0,T;\ \mathbf{L}^2_\sigma(\Omega)), \tag{5.8}$$

$$\mathbf{v}_n \longrightarrow \mathbf{v} \qquad \text{weakly in } L^2(0,T;\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega)). \tag{5.9}$$

In order to obtain an information on a strong convergence of the sequence $\{\mathbf{v}_n\}$, we still need an information on $\partial_t \mathbf{v}_n$. Since $\partial_t \mathbf{v}_n \in \mathbf{V}_n$, we have

$$\begin{aligned} \|\partial_t \mathbf{v}_n\|_{-1,2} &= \sup_{\mathbf{w}\in \mathbf{W}^{1,2}_{0,\sigma}(\Omega),\ \mathbf{w}\neq \mathbf{0}} \frac{|\langle \partial_t \mathbf{v}_n, \mathbf{w}\rangle|}{\|\mathbf{w}\|_{1,2}} = \sup_{\mathbf{w}\in \mathbf{V}_n,\ \mathbf{w}\neq \mathbf{0}} \frac{|(\partial_t \mathbf{v}_n, \mathbf{w})_2|}{\|\mathbf{w}\|_{1,2}} \\ &= \sup_{\mathbf{w}\in \mathbf{V}_n,\ \mathbf{w}\neq \mathbf{0}} \frac{|\langle -\mathcal{A}\mathbf{v}_n - \mathcal{B}(\mathbf{v}_n,\mathbf{v}_n) + \mathbf{f}, \mathbf{w}\rangle|}{\|\mathbf{w}\|_{1,2}} \\ &\leq \|\mathcal{A}\mathbf{v}_n\|_{-1,2} + \|\mathcal{B}(\mathbf{v}_n,\mathbf{v}_n)\|_{-1,2} + \|\mathbf{f}\|_{-1,2} \\ &\leq \|\nabla \mathbf{v}_n\|_2 + C\,\|\nabla \mathbf{v}_n\|_2^{\frac{3}{2}}\,\|\mathbf{v}_n\|_2^{\frac{1}{2}} + \|\mathbf{f}\|_{-1,2}. \end{aligned}$$

From this, we observe that $\{\partial_t \mathbf{v}_n\}$ is uniformly bounded in $L^{4/3}(0,T;\, \mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$.

Lemma (Lions, Aubin). (see e.g. Lions [7] or Temam [12]) *Let X_0, X, X_1 be three Banach spaces such that X_0 and X_1 are reflexive and $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$. Let $0 < T < \infty$, $1 < \alpha_1 < \infty$, $1 < \alpha_2 < \infty$. Denote*

$$\mathcal{Y} \;:=\; \big\{ z \in L^{\alpha_0}(0,T;\, X_0),\ \dot{z} \in L^{\alpha_1}(0,T;\, X_1) \big\}$$

the Banach space with the norm $\|z\|_{\mathcal{Y}} \;:=\; \|z\|_{L^{\alpha_0}(0,T;\, X_0)} + \|\dot{z}\|_{L^{\alpha_1}(0,T;\, X_1)}$.

Then $\mathcal{Y} \hookrightarrow\hookrightarrow L^{\alpha_0}(0,T;\, X)$ (i.e. the injection of $\mathcal{Y}$ into $L^{\alpha_0}(0,T;\, X)$ is compact.

We use the lemma with $X_0 = \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$, $X = \mathbf{L}^2_\sigma(\Omega)$, $X_1 = \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, $\alpha_0 = 2$, $\alpha_1 = \frac{4}{3}$.

As $\{\mathbf{v}_n\}$ is a bounded sequence in $\mathcal{Y}$, it is compact in $L^2(0,T;\, \mathbf{L}^2_\sigma(\Omega))$. Hence there exists a sub–sequence (denoted again $\{\mathbf{v}_n\}$) that, in addition to (5.8) and (5.9), satisfies

$$\mathbf{v}_n \;\longrightarrow\; \mathbf{v} \qquad \text{strongly in } L^2(0,T;\, \mathbf{L}^2_\sigma(\Omega)). \tag{5.10}$$

4) Verification that v satisfies equation (4.2)

Equation (5.2) means that

$$\int_0^T \int_\Omega \big[-\mathbf{v}_n \cdot \boldsymbol{\varphi}\, \dot{\vartheta} + \nu \nabla \mathbf{v}_n : \nabla \boldsymbol{\varphi}\, \vartheta + \mathbf{v}_n \cdot \nabla \mathbf{v}_n \cdot \boldsymbol{\varphi}\, \vartheta\big]\, \mathrm{d}\mathbf{x}\, \mathrm{d}t$$

$$= \int_0^T \langle \mathbf{f}, \boldsymbol{\varphi} \rangle\, \vartheta\, \mathrm{d}t + \vartheta(0) \int_\Omega \mathbf{v}_{0n} \cdot \boldsymbol{\varphi}\, \mathrm{d}\mathbf{x} \qquad (5.11)$$

for all $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{x}) \in \mathbf{V}_n$ and all $\vartheta = \vartheta(t) \in C_0^\infty([0,T))$. Particularly, (5.11) also holds for all $\boldsymbol{\varphi} \in \mathbf{V}_m$, where $m \leq n$. Assume, for a while, that $\boldsymbol{\varphi} \in \mathbf{V}_m$ is fixed. Using all the types (5.8), (5.9), (5.10) of convergence of $\mathbf{v}_n$ to $\mathbf{v}$, one can pass to the limit (for $n \to \infty$) in (5.11) and show that

$$\int_0^T \int_\Omega \big[-\mathbf{v} \cdot \boldsymbol{\varphi}\, \dot{\vartheta} + \nu \nabla \mathbf{v} : \nabla \boldsymbol{\varphi}\, \vartheta + \mathbf{v} \cdot \nabla \mathbf{v} \cdot \boldsymbol{\varphi}\, \vartheta\big]\, \mathrm{d}\mathbf{x}\, \mathrm{d}t$$

$$= \int_0^T \langle \mathbf{f}, \boldsymbol{\varphi} \rangle\, \vartheta\, \mathrm{d}t + \vartheta(0) \int_\Omega \mathbf{v}_0 \cdot \boldsymbol{\varphi}\, \mathrm{d}\mathbf{x} \qquad (5.12)$$

for all $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{x}) \in \mathbf{V}_m$ and all $\vartheta = \vartheta(t) \in C_0^\infty([0,T))$. Passing now to the limit for $m \to \infty$, we deduce that (5.12) holds for all $\boldsymbol{\varphi} = \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ and all functions ϑ.

5) The energy inequality

Recall inequality (5.7):

$$\|\mathbf{v}_n(t)\|_2^2 + 2\nu \int_0^t \|\nabla \mathbf{v}_n(\tau)\|_2^2 \,\mathrm{d}\tau \;\leq\; 2\int_0^t \langle \mathbf{f}, \mathbf{v}_n \rangle \,\mathrm{d}\tau + \|\mathbf{v}_0\|_2^2.$$

The limit of the right hand side (for $n \to \infty$) is

$$=\; 2\int_0^t \langle \mathbf{f}(\tau), \mathbf{v} \rangle \,\mathrm{d}\tau + \|\mathbf{v}_0\|_2^2.$$

The limit inferior of the left hand side (for $n \to \infty$) is

$$\geq\; \|\mathbf{v}(t)\|_2^2 + 2\nu \int_0^t \|\nabla \mathbf{v}_n(\tau)\|_2^2 \,\mathrm{d}\tau.$$

This yields the energy inequality

$$\|\mathbf{v}(t)\|_2^2 + 2\nu \int_0^t \|\nabla \mathbf{v}(\tau)\|_2^2 \,\mathrm{d}\tau \;\leq\; \|\mathbf{v}_0\|_2^2 + 2\int_0^t \big\langle \mathbf{f}(\tau), \mathbf{v}(\tau) \big\rangle \,\mathrm{d}\tau. \tag{5.1}$$

6) The strong right L^2–continuity of $\mathbf{v}$ at time $t = 0$

The energy inequality implies that

$$\limsup_{t\to 0+} \|\mathbf{v}(t)\|_2^2 \;\leq\; \|\mathbf{v}_0\|_2^2.$$

On the other hand, as $\mathbf{v}$ is weakly continuous from $[0,T)$ to $\mathbf{L}^2_\sigma(\Omega)$, we have

$$\liminf_{t\to 0+} \|\mathbf{v}(t)\|_2^2 \;\geq\; \|\mathbf{v}_0\|_2^2.$$

These inequalities yield

$$\lim_{t\to 0+} \|\mathbf{v}(t)\|_2^2 \;=\; \|\mathbf{v}_0\|_2^2.$$

This identity, together with the weak L^2–continuity, enables us to conclude that

$$\lim_{t\to 0+} \|\mathbf{v}(t) - \mathbf{v}_0\|_2^2 \;=\; 0.$$

It means that $\mathbf{v}(t) \to \mathbf{v}_0$ in $\mathbf{L}^2_\sigma(\Omega)$ for $t \to 0+$. ■

Natural questions:

- **Does every weak solution satisfy (EI), or even the energy equality (EE)?**

  Note that we cannot apply formula (3.16) from the Lions–Magenes lemma as in the linear case of the Stokes problem because we do not have $\dot{\mathbf{v}} \in L^2(0,T;\mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$ – now, we only have $\dot{\mathbf{v}} \in L^{4/3}(0,T;\mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$. Consequently, the energy equality (or at least the energy inequality) does not automatically follow from the definition of the weak solution.

- **Is the weak solution unique?**

- **Is the weak solution regular provided that v_0 and f are regular?**

and many others

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