Some applications of Nitsche’s method

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I. Nitsche’s method

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1. Motivation

- Complicated boundary conditions
- Optimal boundary control
- B.C. for singularly perturbed problems
- Interface problems

‘Complicated’ boundary conditions

Navier-Stokes flow in symmetrical domain

\[ \mu \left( \frac{\partial v}{\partial n} + v \cdot \nabla v \right) - \mu \Delta v + \nabla p = f \]
\[ \text{div} \, v = 0 \]
\[ v \cdot n = 0 \quad \text{on} \quad \Gamma \]
\[ \mu \frac{\partial v}{\partial n} \times n = 0 \quad \text{on} \quad \Gamma \]

Symmetry boundary condition: not very easy to implement in general

Nonlinear boundary conditions: (avalanche of water, blood flow in a vein):
\[ v_n = 0, \quad -\sigma_t \in \alpha \partial(v_i) \]
\[ v_n := \frac{v \cdot n}{n}, \quad v_i := v - v_n n \]
\[ \sigma := 2 \mu \varepsilon(v) - \mu I, \quad \sigma_n := \sigma n \cdot n, \quad \sigma_t = \sigma n - \sigma_n n \]
\[ (\partial \psi(h) := \{ h | \psi(h) - \psi(a) \geq (b - a) \cdot h \quad \forall h \in \mathbb{R}^2 \}) \]

Optimal control

Elliptic (right-hand-side) control
\[ -\Delta u = f + q \quad \text{in} \quad \Omega \]
\[ u = 0 \quad \text{on} \quad \partial \Omega \]
\[ \inf \frac{1}{2} \| u - \pi \|_X^2 + \frac{1}{2} \| q \|_Y^2 \]

Functional spaces: \( u, \bar{u} \in X = H_0^1(\Omega), \quad q \in Y = L^2(\Omega) \).

Variational framework for optimization:
\[ \mathcal{L}(u, z, q) := J(q, u) + (f, z)_{L^2(\Omega)} - b(q, z) - a(u, z) \]
\[ J(q, u) := \frac{1}{2} \| u - \pi \|_X^2 + \frac{1}{2} \| q \|_Y^2 \]
\[ a(u, z) := \int_{\Gamma} \nabla u \cdot \nabla z \]
\[ b(q, z) := (q, z)_{L^2(\Omega)} \]
Variational framework of optimal control

Variational framework for optimization:

\[ \mathcal{L}(u, z) := J(q, u) + (f, z)_H^0\Omega - b(q, z) - a(u, z) \]

\[ J(q, u) := \frac{1}{2} \| u - \pi_h^u \|^2_X + \frac{\alpha}{2} \| q \|^2_Y \]

\[ a(u, z) := \int_{\Omega} \nabla u \cdot \nabla z \]

\[ b(q, z) := (q, z)_{L^2(\Omega)} \]

Optimality system

\[ \alpha(u, v) = (f, v) - b(q, v) \quad \forall v \in X \]

\[ a(v, z) = J'_u(q, u)(v) \quad \forall v \in X \]

\[ J'_u(q, u)(r) = b(r, z) \quad \forall r \in Y \]

Gradient of reduced functional

\[ S : Y \to X \quad q \to u(q) := S(q) \]

\[ j(q) := J(q, S(q)) \]

\[ j'(q)(r) = J'_u(q, u)(r) - b(r, z) \]

Advantages of variational framework

\[ X_h \subset X, \quad Y_h \subset Y \]

\[ S_h : Y_h \to X_h \quad q \to u_h(q) =: S_h(q) \]

same as on continuous level:

\[ j_h(q)(r) = J'_u(q, u_h)(r) - b(r, z) \]

\[ \star \text{key for error estimates} \]
\[ \star \text{discretization and optimization commute (!)} \]
\[ \star \text{i.e.} \]
\[ \star (1) \text{discretizing the optimality system (first optimize)} \]
\[ \star (2) \text{deriving the optimality system in the discrete setting (first discretize)} \]
\[ \star \text{give the same result} \]
\[ \star \text{important, since algorithms developed on the continuous level can be applied for the discrete system, mesh-independency principles...} \]

Optimal boundary control

Elliptic boundary control

\[ -\Delta u = f \quad \text{in } \Omega \]

\[ u = g \quad \text{on } \partial \Omega \]

\[ \inf \left\{ \frac{1}{2} \| u \|^2_X + \frac{\alpha}{2} \| q \|^2_Y \right\} \]

Functional spaces \( X, Y \)?

Variational formulation?

May not be straightforward for \( X \subset H^1(\Omega) \) and classical \( P_h^0 \) approximations since \( u_h = I_h(q_h) \) on \( \partial \Omega \) is not variational

Singularly perturbed problems

Convection-Diffusion (for example in stationary heat flow)

\[ -\varepsilon \Delta u + v \cdot \nabla u = f \quad \text{in } \Omega \]

\[ u = 0 \quad \text{on } \partial \Omega \]

\[ \varepsilon : \text{real positive number, possibly very small} \]

\[ v : \text{given velocity} \]

If \( \varepsilon > 0 \) we can look for a solution in \( H^1_0(\Omega) \), but general estimates depend on \( \varepsilon \)!

For robustness, we have to investigate the limit problem \( \varepsilon = 0 \):

\[ v \cdot \nabla u = f. \]

This problem has a completely different theory, and at least the boundary conditions can only be imposed on the inflow boundary

\[ \partial \Omega^\text{in} := \{ x \in \partial \Omega \mid v(x) \cdot n(x) < 0 \} \]
Solutions are not smooth at the interface: 1D such that

\[
(1 + v_1) = u \quad \text{for} \quad x \leq 0 \quad \text{and} \quad K = k_2 \quad \text{else.}
\]

The boundary layer can only be resolved on a sufficiently fine mesh, what to do if the mesh is not fine enough?

The solution develops a boundary layer at the outflow.

### Interface problems

Solutions are not smooth at the interface: ID

\[- \text{div}(k \nabla u) = 0 \quad \text{in} \quad \Omega \]

\[u = 0 \quad \text{on} \quad \partial \Omega \]

\[k > 0 \quad \text{and} \quad \text{piecewise constant} \]

Describes heterogeneous media

\[
\Omega = \Omega_1 \cup \Omega_2
\]

\[\Gamma = \Omega_1 \cap \Omega_2
\]

\[k_{|\Omega_1} = k_i
\]

\[|u| = 0, \quad [k \frac{\partial u}{\partial n}] = k_1 \frac{\partial u_1}{\partial n} + k_2 \frac{\partial u_2}{\partial n} = 0
\]

Solution cannot be smooth if \(k\) discontinuous!

### Interface problems

**Example.** Suppose that \(\Omega_1 \subset \Omega_2\).

The interface system reads

\[- \Delta u_1 = f/k_1 \quad \text{in} \quad \Omega_1, \quad - \Delta u_2 = f/k_2 \quad \text{in} \quad \Omega_2
\]

\[u_1 = u_2, \quad \frac{k_1}{k_2} \partial_n u_1 = \partial_n u_2 \quad \text{on} \quad \Gamma
\]

Let \(u_2|_{\partial \Omega_2 \setminus \Gamma} = 0, k_1 = 1 \quad \text{and} \quad k_2 \to \infty. \) Then \(u_2\) solves

\[- \Delta u_2 = 0 \quad \text{in} \quad \Omega_2, \quad \partial_n u_2 = 0 \quad \text{on} \quad \Gamma, \quad u_2 = 0 \quad \text{on} \partial \Omega_2 \setminus \Gamma.
\]

thus \(u_2 = 0\) and we end up with a **Dirichlet problem for** \(u_1\).

**Example.** Suppose that \(\Omega_1 \subset \Omega_2\).

The interface system reads

\[- k_1 \Delta u_1 = f_1 \quad \text{in} \quad \Omega_1, \quad - k_2 \Delta u_2 = f_2 \quad \text{in} \quad \Omega_2
\]

\[u_1 = u_2, \quad \partial_n u_1 = \frac{k_1}{k_2} \partial_n u_2 \quad \text{on} \quad \Gamma
\]

Let \(f_2 = 0, u_2|_{\partial \Omega_2 \setminus \Gamma} = 0, k_1 = 1 \quad \text{and} \quad k_2 \to \infty. \) Then \(u_1\) solves

\[- \Delta u_1 = f_1 \quad \text{in} \quad \Omega_1, \quad \partial_n u_1 = 0 \quad \text{on} \quad \Gamma, \quad u_1 = u_2 \quad \text{on} \quad \Gamma.
\]

thus we end up with a **Neumann problem for** \(u_1, u_2|_{\Gamma}\) is the Steklov-Poincaré multiplier.
Interface problems

- If the mesh coincides with the interface: All methods are OK ?!
- If not?
  - bad approximation
  - pollution effect
- We want methods robust w.r.t.
  - jump in coefficient
  - cut geometry
  - least modification of FE spaces
- Interesting for domain decomposition/ fictitious domains

References

There is a saying “The boundary is the invention of the devil” attributed to Werner Karl Heisenberg in connection with his research on turbulence modelling. We paraphrase Heisenberg’s words and formulate the question: "Is the no-slip boundary in incompressible fluid dynamics an invention of the devil?"


Alan Turing is reported as saying that PDE’s are made by God, the boundary conditions by the Devil! The situation has changed, Devil has changed places...We can say that the main challenges are in the interfaces, with Devil not far away from them...“

Jacques-Louis Lions

References


2. Definition of Nitsche’s method

- Notation
- General approach
- Abstract error analysis
- The penalty method
- Nitsche’s method
- 1D
**Notation**

\( \mathcal{H} = \) family of admissible simplicial finite element meshes  
\( h \in \mathcal{H} : \) covering \( \Omega \) exactly (no curved boundaries)  
\( \mathcal{K}_h : \) set of cells, \( d_K(p_K) = \) diameter (inscribed circle radius) of \( K \) (\( K \in \mathcal{K}_h \))  
\( S_h = S^v_h \cup S^{sl}_h : \) set of sides (edges/faces)  
shape-regularity: \( \max_{h \in \mathcal{H}} \max_{K \in \mathcal{K}_h} \frac{d_K}{p_K} < \infty \)

\( P^k_h = \) space of continuous FE (\( k \geq 1 \))  
\( D^k_h = \) space of (completely) discontinuous FE (\( k \geq 0 \))  
\( h \) also denotes the piecewise constant function \( h|_K = d_K \) (and \( h|_S = d_K \) if \( S \subset K \))  
\( a \leq b \iff \) constant(?) \( C \) such that \( a \leq C b \)  
\( a \geq b \iff \) constant(?) \( C \) such that \( a \geq C b \)  
\( a \approx b \iff a \leq b \) and \( a \geq b \)

**General approach**

Consider the Poisson (model) problem: 
\[ -\Delta u = f \quad \text{in} \quad \Omega, \quad u = u_D \quad \text{on} \quad \partial \Omega, \]
\[ f \in L^2(\Omega), u_D \in H^{1/2}(\partial \Omega) \quad \text{given data}. \]

The standard weak formulation is: 
Find \( u \in \tilde{u}_D + H^1_0(\Omega) \) such that for all \( v \in H^1_0(\Omega) \)
\[ \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v, \]
\( u_D \in H^1(\Omega) : \quad \gamma(u_D) = u_D, \quad \|\nabla u_D\| = \|u_D\|_{H^{1/2}(\partial \Omega)} \)

Equivalent to energy minimization: 
\[ \inf_{u \in \tilde{u}_D + H^1_0(\Omega)} J(u) \]
\[ J(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u \]

If we want to relax the constraint of boundary condition, we need to modify the energy (on the discrete level)!
For example (Penalty method)

\[ L_h(u_h) = \psi_h(u_h - u_0), \quad \psi_h \approx h^{-m} \]

Discrete minimization (still convex-quadratic):

\[ \inf_{u_h \in V_h} J_h(u_h) \]

The minimizer satisfies:

\[ \int \nabla u_h \cdot \nabla v_h + \int (u_h L'_h(v_h) + L_h(u_h)v_h) = \int f v_h + \int (u_0 L'_h(v_h) - L_h(0)v_h) \]

Proposition. Let us suppose the coercivity condition. Then we have (for any \( I_h : V \to V_h \)) the error estimate

\[ ||u - u_h||_h \leq ||u - I_h u||_h + \frac{1}{\alpha} C_h(u, I_h). \]  \hspace{1cm} (1)

Therefore, the consistency error should be bounded by the interpolation error

Proof. Let \( v_h = I_h u - u_h \). Suppose \( ||v_h||_h > 0 \) (otherwise we have by the triangle inequality \( ||u - u_h||_h \leq ||u - I_h u||_h \)). Then

\[ \alpha ||I_h u - u_h||_h^2 \leq a_h(I_h u - u_h, v_h) = a_h(I_h u, v_h) - l_h(v_h) \]

\[ \leq \sup_{u_h \in V_h, (0)} \frac{a_h(I_h u, u_h) - l_h(u_h)}{||u_h||_h} ||v_h||_h = C_h(u)||v_h||_h \]

Division by \( ||v_h||_h \) gives the result. \( \square \)
Poincaré

Remember: we have a nonconforming method.

$$||u_h||^2 = \int_{\Omega} |\nabla u_h|^2 + \int_{\partial \Omega} \psi_h u_h$$

Equivalent to full norm for any strictly positive weight:

$$c_h ||u_h||^2 \leq ||u_0||^2 + (1) \leq C_h ||u_h||^2$$

...gives discrete well-posedness. But when are the constants $h$-independent?

$$\psi_h = h^{-m}$$

$$\Omega \rightarrow \Omega, \quad x = \lambda \hat{x}$$

$$dx = \lambda^2 d\hat{x}, \quad \nabla u_h = \lambda^{-2} \nabla \hat{u}_h, \quad h = \lambda h$$

Scaling:

$$\int_{\Omega} |\nabla u_h|^2 dx = \int_{\Omega} h^{-m} u_h^2 \Rightarrow m = 1$$

The penalty method

Suppose that $u$ satisfies:

$$\int_{\Omega} f v_h = -\int_{\Omega} \Delta u v_h = \int_{\Omega} \nabla u \cdot \nabla v_h - \int_{\partial \Omega} \frac{\partial u}{\partial n} v_h$$

Then we have:

$$a_h(I_h u, v_h) - l_h(v_h) = \left( \int_{\partial \Omega} \nabla I_h u \cdot \nabla v_h + \int_{\Omega} (I_h u L^0_{\nu}(v_h) + v_h L^0_{\nu}(I_h u)) \right)$$

$$- \left( \int_{\partial \Omega} f v_h + \int_{\Omega} (u_0 L^0_{\nu}(v_h) - L^0_{\nu}(0) v_h) \right)$$

$$= \int_{\partial \Omega} (I_h u - u) \cdot \nabla v_h + \int_{\Omega} v_h \left( l_h(I_h u) - \frac{\partial u_0}{\partial n} \right) + \int_{\Omega} (I_h u - u) \psi_h$$

$$= \int_{\Omega} (I_h u - u) \cdot \nabla v_h + \int_{\Omega} v_h \left( \psi_h (I_h u - u) + \frac{\partial u_0}{\partial n} \right)$$

Nitsche’s method


$$L_h(u_h) = -\frac{\partial u_h}{\partial n} + \psi_h \frac{u_h - u_0}{2}, \quad L_h^0(v_h) = -\frac{\partial u_0}{\partial n} + \psi_h \frac{v_h - u_0}{2}.$$

• The method is globally conservative

Take $\psi_h = 1$ !

$$a_h(u_h, 1) = l_h(1) \quad \Leftrightarrow \quad -\int_{\partial \Omega} \left( \frac{\partial u_h}{\partial n} - \psi_h (u_h - u_0) \right) = \int f$$

$$\left( a_h(u_h, 1) = \int_{\Omega} \nabla u_h \cdot \nabla 1 + \int_{\partial \Omega} (u_h L^0_{\nu}(1) + L^0_{\nu}(u_h)) \right) = \int_{\Omega} \psi_h u_h - \frac{\partial u_0}{\partial n}$$

Improved flux:

$$\sigma_h := \frac{\partial u_h}{\partial n} - \psi_h (u_h - u_0) \int_{\partial \Omega} \sigma_h \cdot n = \int f$$

(for penalty: $\sigma_h = -\psi_h (u_h - u_0)$)
Nitsche’s method

To make it simple:
\[ \int_{\Omega} \nabla u_k \cdot \nabla v_k + \int_{\partial \Omega} (u_k L_0'(y_k) + (y_k) v_k) = \int_{\Omega} f v_k + \int_{\partial \Omega} (u_k L_0'(y_k) - L_0(y_k) v_k) \]
\[ \int_{\Omega} \nabla u_k \cdot \nabla v_k - \int_{\partial \Omega} \frac{\partial u_k}{\partial n} v_k - \int_{\partial \Omega} v_k \frac{\partial u_k}{\partial n} + \int_{\Omega} \psi_k u_k v_k = \int_{\Omega} f v_k - \int_{\partial \Omega} u_0 \left( \frac{\partial v_k}{\partial n} - \gamma \frac{v_k}{h} \right) \]

consistency symmetry stability

\[ \psi_k = \frac{\gamma}{h}, \quad \gamma > 0, \quad \text{we will see later} \]

3. A priori error analysis

- Stability
- Consistency
- L2 estimate

Nitsche’s method in 1D

\[ \int_{\Omega} \nabla u_k \cdot \nabla v_k - \int_{\partial \Omega} \frac{\partial u_k}{\partial n} v_k - \int_{\partial \Omega} u_0 \frac{\partial v_k}{\partial n} + \int_{\Omega} \gamma \frac{u_k}{h} v_k = \int_{\Omega} f v_k - \int_{\partial \Omega} u_0 \left( \frac{\partial v_k}{\partial n} - \gamma \frac{v_k}{h} \right) \]

(and trapezoidal rule)

\[ \frac{u_0 - u_1}{h} - \frac{u_0 - u_2}{h} + \gamma \frac{u_0}{h} - \frac{h}{2} (f(x_0) + f(x_1)) - \frac{u_0}{h} + \gamma \frac{u_0}{h} + \frac{h}{2} (f(x_1) + f(x_1)) \]

\[ \frac{u_1 - u_2}{h} + \frac{u_1 - u_0}{h} - \frac{h}{2} (f(x_0) + 2f(x_1) + f(x_1)) + \frac{u_0}{h} \]

FD-scheme:

\[ \frac{2u_0 - u_2 - u_0}{h} - \frac{h}{2} (f(x_0) + f(x_1) + f(x_1)) + \frac{u_0}{h} \]

\[ -u_1 + 2u_2 - u_2 - \frac{h}{2} (f(x_1) + 2f(x_1) + f(x_1)) \]

Stability

By definition:

\[ a_h(u_h, u_h) = \int_{\Omega} \nabla u_h \cdot \nabla u_h + 2 \int_{\partial \Omega} u_h L_0'(y_h) = \int_{\Omega} \nabla u_h \cdot \nabla u_h + \int_{\partial \Omega} \psi_k u_h^2 - 2 \int_{\partial \Omega} u_h \frac{\partial u_h}{\partial n} \]

\[ \int_{\partial \Omega} u_h \frac{\partial u_h}{\partial n} \leq \left( \int_{\partial \Omega} u_h^2 \right)^{1/2} \left( \int_{\partial \Omega} \psi_k \frac{1}{2} \frac{\partial u_h^2}{\partial n} \right)^{1/2} \]

Inverse estimate: \[ \| h^{1/2} \frac{\partial u_h}{\partial n} \|_{\partial \Omega} \leq C_{inv} \| \nabla u_h \| \]

Clearly we should have:

\[ \psi_h = \frac{\gamma}{h}, \quad \gamma > 0, \quad \left( \gamma_h = \frac{\psi_h}{h} \right) \]