Some applications of Nitsche's method

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Texte

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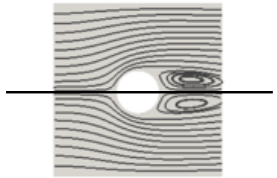
I. Nitsche's method

1.Motivation

- Complicated boundary conditions
- Optimal boundary control
- B.C. for singularly perturbed problems
- Interface problems

'Complicated' boundary conditions



Navier-Stokes flow in symmetrical domain

symmetry axis

$$\rho\left(\frac{\partial v}{\partial t} + v\cdot\nabla v\right) - \mu\Delta v + \nabla p = f$$

$$\operatorname{div} v = 0$$

Symmetry boundary condition:

$$v\cdot n = 0 \quad \text{on } \Gamma$$

$$\mu\frac{\partial v}{\partial n}\times n = 0 \quad \text{on } \Gamma$$

not very easy to implement in general

Nonlinear boundary conditions: (avalanche of water, blood flow in a vein):

$$v_n = 0, \quad -\sigma_t \in \alpha\partial(|v_t|)$$

$$(v_n := v\cdot n, \quad v_t := v - v_n n)$$

$$(\sigma = 2\mu\varepsilon(v) - pI, \quad \sigma_n := \sigma n\cdot n, \quad \sigma_t = \sigma n - \sigma_n n)$$

$$(\partial\psi(a) := \{b|\ \psi(h) - \psi(a) \geq (b-a)\cdot h\ \forall h\in\mathbb{R}^d\})$$

Optimal control

Elliptic (right-hand-side) control

$$-\Delta u = f + q \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$\inf \frac{1}{2}\|u-\overline{u}\|_X^2 + \alpha\frac{1}{2}\|q\|_Y^2$$

Functional spaces: $u, \bar{u}\in X = H_0^1(\Omega)$, $q\in Y = L^2(\Omega)$.

Variational framework for optimization:

$$\mathcal{L}(u,z,q) := J(q,u) + \langle f, z\rangle_{L^2(\Omega)} - b(q,z) - a(u,z)$$

$$J(q,u) := \frac{1}{2}\|u-\overline{u}\|_X^2 + \frac{\alpha}{2}\|q\|_Y^2$$

$$a(u,z) := \int_\Omega \nabla u\cdot\nabla z$$

$$b(q,z) := \langle q, z\rangle_{L^2(\Omega)}$$

Variational framework of optimal control

Variational framework for optimization:

$$\mathcal{L}(u,z,q) := J(q,u) + \langle f, z\rangle_{L^2(\Omega)} - b(q,z) - a(u,z)$$

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$$a(u,z) := \int_\Omega \nabla u \cdot \nabla z$$

$$b(q,z) := \langle q, z\rangle_{L^2(\Omega)}$$

Optimality system

$$\begin{aligned} a(u,v) &= \langle f, v\rangle - b(q,v) && \forall v \in X \\ a(v,z) &= J_u'(q,u)(v) && \forall v \in X \\ J_q'(q,u)(r) &= b(r,z) && \forall r \in Y \end{aligned}$$

Gradient of reduced functional

$$S : Y \to X \qquad q \to u(q) =: S(q)$$

$$j(q) := J(q, S(q))$$

$$j'(q)(r) = J_q'(q,u)(r) - b(r,z)$$

Advantages of variational framework

$$X_h \subset X, \qquad Y_h \subset Y$$

$$S_h : Y_h \to X_h \qquad q \to u_h(q) =: S_h(q)$$

same as on continuous level:

$$j_h(q) := J(q, S_h(q))$$

$$j_h'(q)(r) = J_q'(q,u_h)(r) - b(r, z_h)$$

- key for error estimates
- discretization and optimization commute (!)
  - i.e.
    - (1) discretizing the optimality system (first optimize)
    - (2) deriving the optimality system in the discrete setting (first discretize)
  - give the same result
  - important, since algorithms developed on the continuous level can be applied for the discrete system, mesh-independency principles...

Optimal boundary control

Elliptic boundary control

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= q \quad \text{on } \partial\Omega \end{aligned}$$

$$\inf \frac{1}{2}\|u-\overline{u}\|_X^2 + \frac{\alpha}{2}\|q\|_Y^2$$

Functional spaces X, Y ?
Variational formulation ?

May not be straightforward for $X \subset H^1(\Omega)$ and classical P_h^k approximations since $u_h = I_h q_h$ on $\partial\Omega$ is not variational

Singularly perturbed problems

Convection-Diffusion (for example in stationary heat flow)

$$\begin{aligned} -\varepsilon\Delta u + v \cdot \nabla u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

ε : real positive number, possibly very small
v : given velocity

If $\varepsilon > 0$ we can look for a solution in $H_0^1(\Omega)$, but general estimates depend on ε !

For robustness, we have to investigate the limit problem $\varepsilon = 0$:

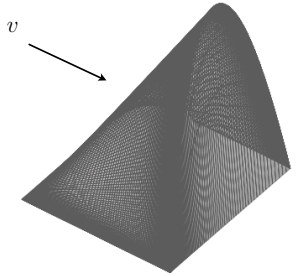
$$v \cdot \nabla u = f.$$

This problem has a completely different theory, and at least the boundary conditions can only be imposed on the inflow boundary

$$\partial\Omega^{\text{in}} := \{x \in \partial\Omega |\ v(x) \cdot n(x) < 0\}$$

Singularly perturbed problems

What happens when diffusion gets small ?



The solution develops a boundary layer at the outflow

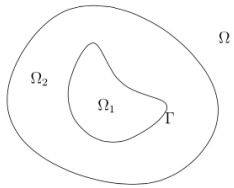
The boundary layer can only be resolved on a *sufficiently* fine mesh, what to do if the mesh is not fine enough ?

Interface problems

$$-\operatorname{div}(k\nabla u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$k > 0$ and piecewise constant

Describes heterogenous media



$$\bar\Omega = \bar\Omega_1 \cup \bar\Omega_2$$
$$\Gamma = \bar\Omega_1 \cap \bar\Omega_2$$
$$k|_{\Omega_i} = k_i$$

$$[u] = 0, \quad [k\frac{\partial u}{\partial n}] = k_1\frac{\partial u_1}{\partial n_1} + k_2\frac{\partial u_2}{\partial n_2} = 0$$

Solution cannot be smooth if k discontinuous !

Interface problems

Solutions are not smooth at the interface: 1D

$$-\operatorname{div}(k\nabla u) = 0, \quad u(-1) = -1, \; u(1) = 1$$

Example. *Let* $\Omega =]-1,1[$, $k = k_1$ *if* $x \le 0$ *and* $K = k_2$ *else. If* $f = 0$, $u(-1) = -1$, *and* $u(1) = 1$, *the solution is piecewise linear, i.e.*

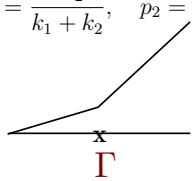
$$u(x) = -1 + p_1(x+1) \quad \textit{for } x \le 0 \textit{ and else} \quad u(x) = 1 + p_2(x-1)$$

with $([k\frac{\partial u}{\partial n}] = 0)$ $([u] = 0)$

$$k_1 p_1 = k_2 p_2 \quad \textit{and} \quad -1 + p_1 = 1 - p_2, \quad \textit{i.e.} \quad p_1 + p_2 = 2,$$

such that

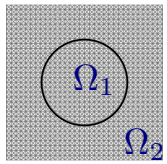
$$(1 + \frac{k_1}{k_2})p_1 = 2 \quad \textit{i.e.} \quad p_1 = \frac{2k_2}{k_1 + k_2}, \quad p_2 = \frac{2k_1}{k_1 + k_2}, \quad \sigma = \frac{k_1 k_2}{(k_1 + k_2)/2}.$$



Interface problems

Example. *Suppose that* $\overline{\Omega_1} \subset \Omega_2$.
The interface system reads

$$-\Delta u_1 = f/k_1 \quad \textit{in } \Omega_1, \quad -\Delta u_2 = f/k_2 \quad \textit{in } \Omega_2$$
$$u_1 = u_2, \quad \frac{k_1}{k_2}\partial_n u_1 = \partial_n u_2 \quad \textit{on } \Gamma.$$



Let $u_2|_{\partial\Omega_2\setminus\Gamma} = 0$, $k_1 = 1$ *and let* $k_2 \to \infty$. *Then* u_2 *solves*

$$-\Delta u_2 = 0 \quad \textit{in } \Omega_2, \quad \partial_n u_2 = 0 \quad \textit{on } \Gamma, \quad u_2 = 0 \quad \textit{on } \partial\Omega_2 \setminus \Gamma.$$

thus $u_2 = 0$ *and we end up with a Dirichlet problem for* u_1.

Example. *Suppose that* $\overline{\Omega_1} \subset \Omega_2$.
The interface system reads

$$-k_1\Delta u_1 = f_1 \quad \textit{in } \Omega_1, \quad -k_2\Delta u_2 = f_2 \quad \textit{in } \Omega_2$$
$$u_1 = u_2, \quad \partial_n u_1 = \frac{k_2}{k_1}\partial_n u_2 \quad \textit{on } \Gamma.$$

Let $f_2 = 0$, $u_2|_{\partial\Omega_2\setminus\Gamma} = 0$, $k_1 = 1$ *and let* $k_2 \to 0$. *Then* u_1 *solves*

$$-\Delta u_1 = f_1 \quad \textit{in } \Omega_1, \quad \partial_n u_1 = 0 \quad \textit{on } \Gamma, \quad u_1 = u_2 \quad \textit{on } \Gamma.$$

thus we end up with a Neumann problem for u_1. $u_2|_\Gamma$ *is the Steklov-Poincaré multiplier.*

Interface problems

- If the mesh coincides with the interface: All methods are OK ?!
- If not ?
  - bad approximation
  - pollution effect
- We want methods robust w.r.t
  - jump in coefficient
  - cut geometry
  - least modification of FE spaces
- Interesting for domain decomposition/ fictitious domains

References

There is a saying "The boundary is the invention of the devil" attributed to *Werner Karl Heisenberg* in connection with his research on turbulence modelling. We paraphrase Heisenberg's words and formulate the question: "Is the no-slip boundary in incompressible fluid dynamics an invention of the devil?"

Jens Frehse and Joseph Malek. *Problems due to the no-slip boundary in incompressible fluid dynamics.* In Geometric analysis and nonlinear partial differential equations, pages 559–571. Springer, Berlin, 2003.

Alan Turing is reported as saying that PDE's are made by God, the boundary conditions by the Devil ! The situation has changed, Devil has changed places...We can say that the main challenges are in the interfaces, with Devil not far away from them..."
Jacques-Louis Lions

References

J. Nitsche. *Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind.* Abh. Math. Univ. Hamburg, 36:9–15, 1971.
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A. Hansbo and P. Hansbo. *An unfitted finite element method, based on Nitsche's method, for elliptic interface problems.* Comput. Methods Appl. Mech. Eng., 191(47-48):5537–5552, 2002.

2.Definition of Nitsche's method

- Notation
- General approach
- Abstract error analysis
- The penalty method
- Nitsche's method
- 1D

Notation

$\mathcal{H}$ = family of admissible simplicial finite element meshes

$h \in \mathcal{H}$: covering Ω exactly (no curved boundaries)

$\mathcal{K}_h$: set of cells, $d_K(\rho_K)$ = diameter (inscribed circle radius) of K $(K \in \mathcal{K}_h)$

$\mathcal{S}_h = \mathcal{S}_h^{\partial} \cup \mathcal{S}_h^{int}$ set of sides (edges/faces)

shape-regularity: $\max_{h\in\mathcal{H}} \max_{K\in\mathcal{K}_h} \dfrac{d_K}{\rho_K} < \infty$

P_h^k = space of continuous FE $(k \geq 1)$

D_h^k = space of (completely) discontinuous FE $(k \geq 0)$

h also denotes the piecewise constant function $h|_K = d_K$

(and $h|_S = d_K$ if $S \subset K$)

$$a \lesssim b \quad \Leftrightarrow \quad \exists \text{constant}(?)\ C \text{ such that } \quad a \leq C\, b$$
$$a \gtrsim b \quad \Leftrightarrow \quad \exists \text{constant}(?)\ C \text{ such that } \quad a \geq C\, b$$
$$a \approx b \quad \Leftrightarrow \quad a \lesssim b \quad \text{and} \quad a \gtrsim b$$

General approach

Consider the Poisson (model) problem:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = u_{\mathrm{D}} \quad \text{on } \partial\Omega,$$
$$f \in L^2(\Omega), u_{\mathrm{D}} \in H^{1/2}(\partial\Omega) \quad \text{given data.}$$

The standard weak formulation is:

Find $u \in \tilde{u_{\mathrm{D}}} + H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v,$$
$$\tilde{u_{\mathrm{D}}} \in H^1(\Omega): \quad \gamma(\tilde{u_{\mathrm{D}}}) = u_{\mathrm{D}},\ \|\nabla \tilde{u_{\mathrm{D}}}\| = \|u_{\mathrm{D}}\|_{H^{1/2}(\partial\Omega)}$$

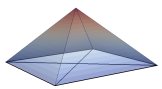
Equivalent to energy minimization:

$$\inf_{u\in\tilde{u_{\mathrm{D}}}+H_0^1(\Omega)} J(u)$$
$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f u$$

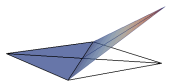
General approach

Standard discrete energy minimization:

$$\inf_{u_h\in\tilde{u_{\mathrm{D}h}}+V_h\cap H_0^1(\Omega)} J(u_h)$$
$$V_h = P_h^k, \quad \tilde{u_{\mathrm{D}h}} \in V_h \quad \tilde{u_{\mathrm{D}h}} = I_h \tilde{u_{\mathrm{D}}}?$$



$v_h \in V_h \cap H_0^1(\Omega)$



$v_h \notin V_h \cap H_0^1(\Omega)$

If we want to relax the constraint of boundary condition, we need to modify the energy (on the *discrete* level) !

General approach

Remind that for the standard method $\inf_{u_h\in\tilde{u_{\mathrm{D}h}}+V_h\cap H_0^1(\Omega)} J(u_h) \quad \to \quad u_h^{\mathrm{st}}$

suppose $\tilde{u_{\mathrm{D}h}} = \tilde{u_{\mathrm{D}}}$

$$J(u_h^{\mathrm{st}}) - J(u) = \underbrace{J'(u)(\underbrace{u - u_h^{\mathrm{st}}}_{\in H_0^1(\Omega)})}_{=0!} + \frac{1}{2}J''(u - u_h^{\mathrm{st}}, u - u_h^{\mathrm{st}}) = \frac{1}{2}\|\nabla(u - u_h^{\mathrm{st}})\|^2$$

General approach

Modified energy (discrete):

$$J_h(u_h) := \frac{1}{2}\int_\Omega |\nabla u_h|^2 - \int_\Omega f u_h + \int_{\partial\Omega} (u_h - u_{\mathrm{D}}) L_h(u_h)$$

L_h affine operator: $\quad L_h(u) = L_h(0) + L_h'(u)$

For example (Penalty method): $\quad L_h(u_h) = \psi_h(u_h - u_{\mathrm{D}}), \quad \psi_h \approx h^{-m}$

Discrete minimization (still convex-quadratic):

$$\inf_{u_h \in V_h} J_h(u_h)$$

The minimizer satisfies:

$$\underbrace{\int_\Omega \nabla u_h \cdot \nabla v_h + \int_{\partial\Omega} (u_h L_h'(v_h) + L_h'(u_h) v_h)}_{=:a_h(u_h, v_h)} = \underbrace{\int_\Omega f v_h + \int_{\partial\Omega} (u_{\mathrm{D}} L_h'(v_h) - L_h(0) v_h)}_{=:l_h(v_h)}$$

Abstract error analysis

Symmetric discrete variational problem:

$$u_h \in V_h : \qquad a_h(u_h, v_h) = l_h(v_h) \quad \forall v_h \in V_h.$$

$|||u_h|||_h$ norm on V_h

coercivity: $\quad \exists \alpha > 0 : \quad a_h(u_h, u_h) \geq \alpha |||u_h|||_h^2 \quad \forall u_h \in V_h.$

Consistency error (depending on interpolation)

$$C_h(u, I_h) := \sup_{v_h \in V_h \setminus \{0\}} \frac{a_h(I_h u, v_h) - l_h(v_h)}{|||v_h|||_h}$$

Abstract error analysis

Proposition. *Let us suppose the coercivity condition. Then we have (for any $I_h : V \to V_h$) the error estimate*

$$|||u - u_h|||_h \leq |||u - I_h u|||_h + \frac{1}{\alpha} C_h(u, I_h). \tag{1}$$

Therefore, the consistency error should be bounded by the interpolation error

Proof. Let $v_h = I_h u - u_h$. Suppose $|||v_h|||_h > 0$ (otherwise we have by the triangle inequality $|||u - u_h|||_h \leq |||u - I_h u|||_h$). Then

$$\begin{aligned} \alpha |||I_h u - u_h|||_h^2 &\leq a_h(I_h u - u_h, v_h) = a_h(I_h u, v_h) - l_h(v_h) \\ &\leq \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(I_h u, w_h) - l_h(w_h)}{|||w_h|||_h} |||v_h|||_h = C_h(u) |||v_h|||_h \end{aligned}$$

Division by $|||v_h|||_h$ gives the result. $\square$

The penalty method

J.-P. Aubin. *Approximation of elliptic boundary-value problems.* Wiley-Interscience , New York-London-Sydney, 1972. Pure and Applied Mathematics, Vol. XXVI.
I. Babuska. *The finite element method with penalty.* Math. Comp., 27:221–228, 1973.

$$L_h(u_h) = \frac{\psi_h}{2}(u_h - u_{\mathrm{D}}), \quad L_h'(v_h) = \frac{\psi_h}{2} v_h.$$

Coercivity:

$$a_h(u_h, u_h) = \int_\Omega \nabla u_h \cdot \nabla u_h + 2\int_{\partial\Omega} u_h L_h'(u_h) = \underbrace{\int_\Omega |\nabla u_h|^2 + \int_{\partial\Omega} \psi_h u_h^2}_{=:|||u_h|||_h^2}$$

Poincaré

Remember: we have a nonconforming method..

$$|||u_h|||_h^2 = \int_\Omega |\nabla u_h|^2 + \int_{\partial\Omega} \psi_h u_h^2$$

Equivalent to full norm for any strictly positive weight:

$$c_h \, |||u_h|||_h \le \|u_h\|_{H^1(\Omega)} \le C_h \, |||u_h|||_h$$

…gives discrete well-posedness. But when are the constants h-independent ?

$\psi_h = h^{-m}$

$$\hat\Omega \to \Omega, \quad x = \lambda \hat x$$

$$dx = \lambda^d \, d\hat x, \quad \nabla u_h = \lambda^{-2} \hat\nabla \hat u_h, \quad h = \lambda \hat h$$

Scaling:

$$\underbrace{\int_\Omega |\nabla u_h|^2 \, dx}_{\approx \lambda^d \times \lambda^{-2}} \overbrace{\approx}^{!} \underbrace{\int_{\partial\Omega} h^{-m} u_h^2}_{\approx \lambda^{d-1} \times \lambda^{-m}} \quad \Rightarrow \quad m = 1$$

The penalty method

Suppose that u satisfies:

$$\int_\Omega f v_h = -\int_\Omega \Delta u v_h = \int_\Omega \nabla u \cdot \nabla v_h - \int_{\partial\Omega} \frac{\partial u}{\partial n} v_h$$

Then we have:

$$\begin{aligned}
a_h(I_h u, v_h) - l_h(v_h) &= \left(\int_\Omega \nabla I_h u \cdot \nabla v_h + \int_{\partial\Omega} (I_h u L_h'(v_h) + v_h L_h'(I_h u)) \right) \\
&\quad - \left(\int_\Omega f v_h + \int_{\partial\Omega} (u_{\mathrm{D}} L_h'(v_h) - L_h(0) v_h) \right) \\
&= \int_\Omega \nabla (I_h u - u) \cdot \nabla v_h + \int_{\partial\Omega} v_h \left(L_h(I_h u) + \frac{\partial u}{\partial n} \right) + \int_{\partial\Omega} (I_h u - u) L_h'(v_h) \\
&= \int_\Omega \nabla (I_h u - u) \cdot \nabla v_h + \int_{\partial\Omega} v_h \left(\frac{\psi_h}{2} (I_h u - u) + \frac{\partial u}{\partial n} \right) + \int_{\partial\Omega} (I_h u - u) \frac{\psi_h}{2} v_h \\
&= \int_\Omega \nabla (I_h u - u) \cdot \nabla v_h + \int_{\partial\Omega} v_h \left(\psi_h (I_h u - u) + \frac{\partial u}{\partial n} \right)
\end{aligned}$$

The penalty method

$$\begin{aligned}
\int_{\partial\Omega} v_h \left(\psi_h (I_h u - u) + \frac{\partial u}{\partial n} \right) &= \int_{\partial\Omega} (I_h u - u) \psi_h v_h + \int_{\partial\Omega} v_h \frac{\partial u}{\partial n} \\
&\le \left(\underbrace{\left(\int_{\partial\Omega} \psi_h (I_h u - u)^2 \right)^{\frac12}}_{\lesssim h^{-\frac{m}{2} + k + \frac12}} + \underbrace{\left(\int_{\partial\Omega} \psi_h^{-1} (\frac{\partial u}{\partial n})^2 \right)^{\frac12}}_{\lesssim h^{\frac{m}{2}}} \right) \left(\int_{\partial\Omega} \psi_h v_h^2 \right)^{\frac12}
\end{aligned}$$

This can however be improved to: $C_h(u, I_h) \lesssim h^{\min\{k, m, k + (m-1)/2\}}$

OK for *m=k=1 (with higher regularity)*,
but ill conditioned for higher order, suboptimal in *L2* (not shown here)

To do better, L_h should be (a better) approximation of $-\frac{\partial u}{\partial n}$

Nitsche's method

J. Nitsche. *Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind.* Abh. Math. Univ. Hamburg, 36:9–15, 1971.

$$L_h(u_h) = -\frac{\partial u_h}{\partial n} + \frac{\psi_h}{2}(u_h - u_{\mathrm{D}}), \quad L_h'(v_h) = -\frac{\partial u_h}{\partial n} + \frac{\psi_h}{2} u_h.$$

- The method is globally conservative

  Take $v_h = 1$!

$$a_h(u_h, 1) = l_h(1) \quad \Leftrightarrow \quad -\int_{\partial\Omega} \left(\frac{\partial u_h}{\partial n} - \psi_h (u_h - u_{\mathrm{D}}) \right) = \int_\Omega f$$

$$\left(a_h(u_h, 1) = \int_\Omega \nabla u_h \cdot \nabla 1 + \int_{\partial\Omega} (u_h L_h'(1) + L_h'(u_h)) = \int_{\partial\Omega} \psi_h u_h - \frac{\partial u_h}{\partial n} \right)$$

Improved flux: $\sigma_h := \frac{\partial u_h}{\partial n} - \psi_h (u_h - u_{\mathrm{D}})$ $\qquad \int_{\partial\Omega} \sigma_h \cdot n = \int_\Omega f$

(for penalty: $\sigma_h = -\psi_h (u_h - u_{\mathrm{D}})$)

Nitsche's method

To make it simple:

$$\int_\Omega \nabla u_h \cdot \nabla v_h + \int_{\partial\Omega} (u_h L_h'(v_h) + L_h'(u_h) v_h) = \int_\Omega f v_h + \int_{\partial\Omega} (u_{\mathrm{D}} L_h'(v_h) - L_h(0) v_h)$$

$$\Longleftrightarrow$$

$$\int_\Omega \nabla u_h \cdot \nabla v_h - \int_{\partial\Omega} \frac{\partial u_h}{\partial n} v_h - \int_{\partial\Omega} u_h \frac{\partial v_h}{\partial n} + \int_{\partial\Omega} \psi_h u_h v_h = \int_\Omega f v_h - \int_{\partial\Omega} u_{\mathrm{D}} \left(\frac{\partial v_h}{\partial n} - \psi_h v_h \right)$$

consistency symmetry stability

$\psi_h = \frac{\gamma}{h} \quad \gamma > 0,$ we will see later

Nitsche's method in 1D

$$\int_\Omega \nabla u_h \cdot \nabla v_h - \int_{\partial\Omega} \frac{\partial u_h}{\partial n} v_h - \int_{\partial\Omega} u_h \frac{\partial v_h}{\partial n} + \int_{\partial\Omega} \frac{\gamma}{h} u_h v_h = \int_\Omega f v_h - \int_{\partial\Omega} u_{\mathrm{D}} \left(\frac{\partial v_h}{\partial n} - \frac{\gamma}{h} v_h \right)$$

(and trapezoidal rule)

$$\begin{aligned}
\frac{u_0 - u_1}{h} - \frac{u_0 - u_1}{h} - \frac{u_0}{h} + \gamma \frac{u_0}{h} &= \frac{h}{2} \left(f(x_0) + f(x_1) \right) - \frac{u_D}{h} + \gamma \frac{u_D}{h} \\
\frac{u_1 - u_2}{h} + \frac{u_1 - u_0}{h} + \frac{u_0}{h} &= \frac{h}{2} \left(f(x_0) + 2 f(x_1) + f(x_1) \right) + \frac{u_D}{h} \\
\frac{u_2 - u_3}{h} + \frac{u_2 - u_1}{h} &= \frac{h}{2} \left(f(x_1) + 2 f(x_2) + f(x_3) \right)
\end{aligned}$$

FD-scheme:

$$\begin{aligned}
(\gamma - 1) \frac{u_0}{h} &= \frac{h}{2} \left(f(x_0) + f(x_1) \right) + (\gamma - 1) \frac{u_D}{h} \\
\frac{2u_1 - u_2}{h} &= \frac{h}{2} \left(f(x_0) + 2 f(x_1) + f(x_1) \right) + \frac{u_D}{h} \\
\frac{-u_1 + 2u_2 - u_3}{h} &= \frac{h}{2} \left(f(x_1) + 2 f(x_2) + f(x_3) \right)
\end{aligned}$$

3.A priori error analysis

- Stability
- Consistency
- L2 estimate

Stability

By definition:

$$a_h(u_h, u_h) = \int_\Omega \nabla u_h \cdot \nabla u_h + 2 \int_{\partial\Omega} u_h L_h'(u_h) = \underbrace{\int_\Omega \nabla u_h \cdot \nabla u_h + \int_{\partial\Omega} \psi_h u_h^2}_{|||u_h|||_h^2} - 2 \int_{\partial\Omega} u_h \frac{\partial u_h}{\partial n}$$

$$\int_{\partial\Omega} u_h \frac{\partial u_h}{\partial n} \leq \left(\int_{\partial\Omega} \psi_h u_h^2 \right)^{1/2} \left(\int_{\partial\Omega} \psi_h^{-1} (\frac{\partial u_h}{\partial n})^2 \right)^{1/2}$$

Inverse estimate : $\|h^{1/2} \frac{\partial u_h}{\partial n}\|_{\partial\Omega} \leq C_{\mathrm{inv}} \|\nabla u_h\|$

Clearly we should have: $\psi_h = \frac{\gamma}{h} \quad \gamma > 0,$ $\quad (\gamma_h = \frac{\gamma}{h})$