

# Variational Methods for Computational Fluid Dynamics

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# Chapter 1

## Models for incompressible fluids

### 1.1 Some basics on viscous fluid models

#### Definition 1 (Stress tensor)

We consider a fluid, possibly in motion,  $\mathbf{x}$  a point in the domain occupied by the fluid,  $\mathbf{n}$  a unit vector, and  $D_\varepsilon(\mathbf{n})$  a disc (or a segment if the space dimension is 2) centered at  $\mathbf{x}$ , with area  $\varepsilon$  (length  $\varepsilon$  in dimension 2), and perpendicular to  $\mathbf{n}$ . We denote by  $\mathbf{F}_\varepsilon(\mathbf{n})$  the force exerted on  $D_\varepsilon(\mathbf{n})$  by the fluid located on the side of  $D_\varepsilon(\mathbf{n})$  to which  $\mathbf{n}$  points. If  $\mathbf{F}_\varepsilon(\mathbf{n})/\varepsilon$  goes to  $\mathbf{F}(\mathbf{n})$  when  $\varepsilon$  goes to 0, and if the mapping  $\mathbf{n} \mapsto \mathbf{F}(\mathbf{n})$  is linear, we call stress tensor at  $\mathbf{x}$  the tensor  $\boldsymbol{\sigma}$  that represents this linear mapping:

$$\mathbf{F}(\mathbf{n}) = \boldsymbol{\sigma} \cdot \mathbf{n}.$$

The motion of a fluid that admits a stress tensor follows an evolution equation that can be formalized in a very general way. We denote by  $\rho = \rho(\mathbf{x}, t)$  the local density, by  $\mathbf{u}$  the velocity field, and by  $\mathbf{f}$  a body force acting on the fluid (like gravity). Consider a fluid element  $\omega(t)$ , i.e. a set of particle which we follow in their motion. Newton's law write

$$\frac{d}{dt} \int_{\omega(t)} \rho \mathbf{u} = \text{sum of external forces.} \quad (1.1)$$

The right-hand side is the sum of the body force contribution  $\int_{\omega} \mathbf{f}$ , and the net force exerted on  $\omega$  by the fluid located outside  $\omega$ , which writes, according to Definition 1,

$$\int_{\partial\omega} \boldsymbol{\sigma} \cdot \mathbf{n} = \int_{\omega} \nabla \cdot \boldsymbol{\sigma}.$$

The left-hand side of Eq. 1.1 now writes

$$\frac{d}{dt} \int_{\omega(t)} \rho \mathbf{u} = \int_{\omega(t)} \frac{\partial(\rho \mathbf{u})}{\partial t} + \int_{\partial\omega(t)} \rho \mathbf{u} (\mathbf{u} \cdot \mathbf{n}),$$

and the last term can be transformed onto a volume integral:

$$\int_{\partial\omega(t)} \rho \mathbf{u} (\mathbf{u} \cdot \mathbf{n}) = \int_{\omega(t)} \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}),$$

where  $\mathbf{u} \otimes \mathbf{u}$  stands for the symmetric matrix  $(u_i u_j)_{i,j}$ . Since all integrals are now expressed as volume integrals, and since the fluid element is arbitrary, we deduce that the integrand vanishes identically.

**Model 1** (General evolution equation for an inertial fluid)

We consider a fluid with density  $\rho(\mathbf{x}, t)$  and velocity field  $\mathbf{u}(\mathbf{x}, t)$ , submitted to a body force  $\mathbf{f}$ . We assume that this fluid admits a stress tensor  $\boldsymbol{\sigma}(\mathbf{x}, t)$ . Newton's law writes

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla \cdot (\rho\mathbf{u} \otimes \mathbf{u}) - \nabla \cdot \boldsymbol{\sigma} = \mathbf{f}. \quad (1.2)$$

Mass conservation writes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0.$$

**Model 2** (Force balance equation for a non-inertial fluid)

Whenever inertia can be considered as negligible, Newton's law is replaced by instantaneous force balance for all fluid element, which expresses

$$-\nabla \cdot \boldsymbol{\sigma} = \mathbf{f}.$$

**Definition 2** (Perfect fluid)

A fluid is said to be perfect if it admits a stress tensor which is diagonal, i.e. there exists a scalar field  $p$ , called pressure, such that

$$\boldsymbol{\sigma}(\mathbf{x}) = -p \text{Id},$$

where  $\text{Id}$  is the identity tensor.

According to the previous models and definitions, an inertial perfect fluid is such that

$$-\nabla \cdot \boldsymbol{\sigma} = \nabla \cdot (p \text{Id}) = \nabla p,$$

and we obtain Euler's equation

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla \cdot (\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathbf{f}.$$

In the case where the fluid is homogeneous ( $\rho$  is uniform) and incompressible (the divergence of the velocity field is 0), one has

$$\nabla \cdot (\rho\mathbf{u} \otimes \mathbf{u}) = \rho(\mathbf{u} \cdot \nabla)\mathbf{u},$$

where  $(\mathbf{u} \cdot \nabla)\mathbf{u}$  is such that

$$((\mathbf{u} \cdot \nabla)\mathbf{u})_i = \sum_{j=1}^d u_j \frac{\partial u_i}{\partial x_j}.$$

**Remark 1** Consider the velocity field  $\mathbf{u}$  as stationary, and consider a scalar function  $f$  defined on the domain. Denote by  $X(t)$  the path of a particle (advected by the velocity field). Then

$$\frac{d}{dt}f(X(t)) = \sum_{j=1}^d u_j \frac{\partial f}{\partial x_j}.$$

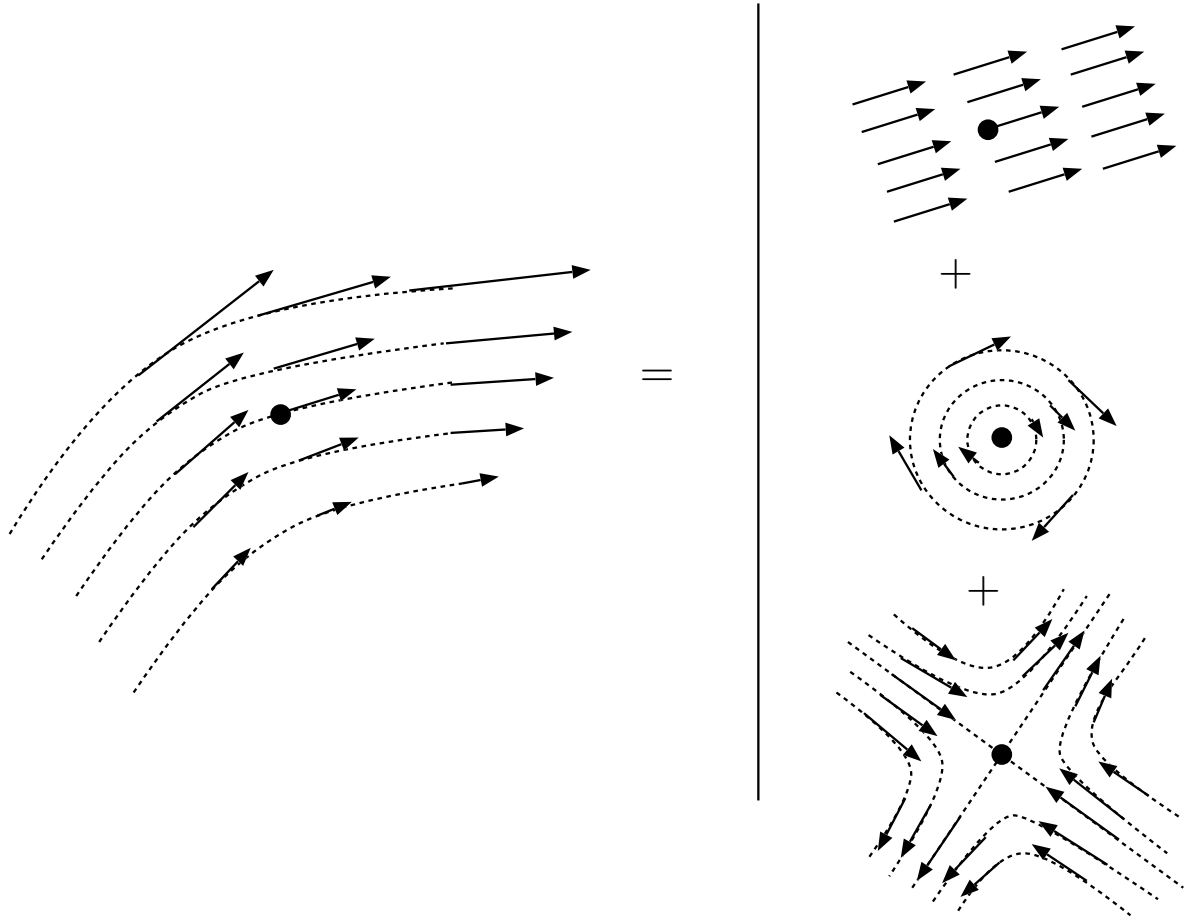


Figure 1.1: Local decomposition of a two-dimensional velocity field

**Newtonian fluids.** Real fluids are characterized by a resistance to deformation. We consider a fluid element moving in the neighborhood of  $t \mapsto \mathbf{x}(t)$ . The velocity field can be expanded in the neighborhood of  $\mathbf{x}$  as

$$\mathbf{u}(\mathbf{y}, t) \approx \mathbf{u}(\mathbf{x}, t) + \nabla \mathbf{u}(\mathbf{x}, t) \cdot (\mathbf{y} - \mathbf{x})$$

$$= \underbrace{\mathbf{u}(\mathbf{x}, t)}_{\text{Translation}} + \left( \underbrace{\frac{\nabla \mathbf{u} - {}^t \nabla \mathbf{u}}{2}}_{\text{Rotation}} + \underbrace{\frac{\nabla \mathbf{u} + {}^t \nabla \mathbf{u}}{2}}_{\text{Deformation}} \right) \cdot (\mathbf{y} - \mathbf{x}).$$

The motion of a material segment  $\overline{\mathbf{xy}}$  can therefore be decomposed into 3 contributions: a translational motion at the local velocity, a rotational motion (skew symmetric part of the velocity gradient), and a last contribution that accounts for local deformations (symmetric part of the velocity gradient). See Fig. 1.1 for an illustration of this decomposition for a two dimensional velocity field.

**Definition 3** (*Strain tensor*)

Considering a fluid moving according to the velocity field  $\mathbf{u}$ , the strain tensor is defined as

$$\mathbf{T} = \frac{\nabla \mathbf{u} + {}^t \nabla \mathbf{u}}{2} = \frac{1}{2} \begin{pmatrix} 2 \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_2} & 2 \frac{\partial u_2}{\partial x_2} \end{pmatrix} \quad (\text{in } 2d).$$

The simplest real fluid model is built by considering that the stress tensor is proportional to this strain tensor, up to a diagonal contribution:

**Definition 4** (*Newtonian fluid*)

A fluid is said *newtonian* if there exists a positive parameter  $\mu$ , called viscosity, and a scalar field  $p = p(\mathbf{x}, t)$  (pressure), such that

$$\boldsymbol{\sigma} = 2\mu \mathbf{T} - p \text{Id} = \mu (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) - p \text{Id}.$$

We consider now incompressible fluids, i.e. such that the volume of a fluid element remains constant:  $\nabla \cdot \mathbf{u} = 0$ . We furthermore assume that the fluid is homogeneous:  $\rho$  is supposed to be uniform and constant.

The incompressible Stokes and Navier-Stokes equations are straightforward expressions of the Newtonian character, in the non-inertial and inertial settings, respectively.

Since  $\rho$  is constant, it can be taken out of the time derivative. Besides, as

$$\nabla \cdot \mathbf{u} = \sum_{i=1}^d \frac{\partial u_i}{\partial x_i} = 0,$$

we have

$$\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = \nabla \cdot (u_i u_j)_{i,j} = \left( \sum_{i=1}^d u_i \frac{\partial u_j}{\partial x_i} \right)_{1 \leq j \leq d}.$$

The latter quantity expresses the derivative of the velocity in its own direction, and it is denoted by  $(\mathbf{u} \cdot \nabla) \mathbf{u}$ .

**Model 3** (*Incompressible Navier-Stokes equations*)

An inertial fluid that is newtonian, incompressible, and homogeneous (the density is constant), and which is subject to a body force  $\mathbf{f}$ , follows the incompressible Navier-Stokes equations

$$\begin{cases} \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) - \mu \Delta \mathbf{u} + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0. \end{cases}$$

**Adimensional form of the Navier-Stokes equations.** Let  $U$  be the order of magnitude of the velocity,  $L$  the length scale (a characteristic dimension of the domain), and  $T = L/U$  the associated time scale. We introduce the dimensionless quantities

$$\mathbf{u}^* = \frac{\mathbf{u}}{U}, \quad \mathbf{x}^* = \frac{\mathbf{x}}{L}, \quad t^* = \frac{t}{T}.$$

Denoting by  $\nabla^*$  (resp.  $\Delta^*$ ) the gradient (resp. Laplacian) operator with respect to the adimensional space variable, we obtain

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \mathbf{u}^* - \frac{\mu}{\rho U L} \Delta^* \mathbf{u}^* + \nabla^* p^* = \mathbf{f}^*,$$



where  $p^* = p/(\rho U^2)$  is the adimensional pressure, and  $\mathbf{f}^* = \mathbf{f}L/(\rho U^2)$  the adimensional forcing term.

The quantity  $Re = \rho UL/\mu$  is called the Reynolds number. It quantifies the relative importance of inertia compared to viscous effects. When this number is small compared to 1, it can be considered that inertial effects are negligible, so that Newton's law for fluid elements simplifies to local force balance

**Model 4** (*Incompressible Stokes equations*)

*A fluid that is newtonian and incompressible, subject to a body force  $\mathbf{f}$  in a regime where inertia can be neglected, follows the incompressible Stokes equations*

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \quad (1.3)$$

If we consider the situation where the fluid is enclosed in a domain delimited by physical, impermeable walls, it is usually considered<sup>1</sup> that the fluid sticks to the wall, which expresses as *homogeneous Dirichlet boundary conditions*  $\mathbf{u} = 0$  on the boundary  $\partial\Omega$ .

**Exercise 1** *One considers a circular domain  $\Omega$  centered at 0, and  $\omega$  a given angular velocity. Show that the rigid velocity field*

$$\mathbf{u} = \omega \begin{pmatrix} -y \\ x \end{pmatrix}$$

*is a steady solution to the incompressible Navier-Stokes equations, i.e. that there exists a pressure field  $p(x, y)$  such that  $(\mathbf{u}, p)$  is a solution to (3.20), with  $\partial\mathbf{u}/\partial t = 0$ .*

**Remark 2** *The description of phenomena associated to flows at high Reynolds number goes far beyond the scope of this course, and the understanding of turbulence (which refers to situations where complex motions occur over a large large a space scales) still raises many open questions. Let us simply say here that high Reynolds flows can be pictured as containing eddies over a large range of sizes, starting to the global size of the observed phenomenon, down to much smaller scales. It is usually considered that dissipation occurs at some space scale  $\eta$  (smaller size of the eddies). According to Kolmogorov's theory,  $L/\eta$  is of the order  $Re^{3/4}$ . This formula gives a precious indication in the context of numerical simulations. If one aims at discretizing the space in order to "capture" (i.e. represent on the mesh) the smallest eddies, the number of mesh vertices in each direction scales at  $Re^{3/4}$ , so that in 3d, the total number of vertices is  $Re^{9/4}$ . This remark concerns Direct Navier-Stokes simulations (called DNS). Other methods, like Large Eddy Simulation (LES) method, or methods based on the so-called  $k - \varepsilon$  model, have been introduced to limit the cost of numerical simulation by discretizing at a scale larger than  $\eta$ . Those approaches rely on assumptions regarding what happens at scales smaller than the mesh size, and involve extra unknown pertaining to those phenomena (like the kinetic energy  $k$  associated to smaller scales in the  $k - \varepsilon$  model).*

## 1.2 Mathematical framework

A variational formulation is obtained by considering a test function  $\mathbf{v}$  which vanishes on the boundary of the domain  $\Omega$ , taking the scalar product of the first equation of (1.3) by  $\mathbf{v}$  and

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<sup>1</sup>This strong assumption is sometimes ruled out. In some situations it is in particular more relevant to use the so-called *Navier* conditions, which still preserve the impervious character of the wall, but allow a non zero tangential velocity.

integrating by part, which leads to

$$\mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

Considering now a scalar test function  $q$ , the incompressibility constraint can be expressed similarly in a weak form. Note that, since only the *gradient* of the pressure appears in (1.3), only uniqueness up to an additive constant may be expected<sup>2</sup>. For this reason, we shall prescribe an extra constraint on this variable: the mean value over the domain must be zero. The problem may now be expressed in an appropriate mathematical sense as follows

$$\left\{ \begin{array}{l} V = H_0^1(\Omega)^d, \quad X = L^2(\Omega) \\ \mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \quad \forall \mathbf{v} \in V \\ \int_{\Omega} q \nabla \cdot \mathbf{u} = 0 \quad \forall q \in X. \end{array} \right. \quad (1.4)$$

where  $H_0^1(\Omega)^d$  is the Sobolev space of vector fields the components of which are  $L^2$  function with square integrable gradient, and which vanish on the boundary  $\partial\Omega$ , and

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega), \int_{\Omega} q = 0 \right\}.$$

**Proposition 1** *Let  $\mathbf{f}$  be given in  $L^2(\Omega)^d$ . Problem (1.4) admits a unique solution  $(\mathbf{u}, p) \in V \times X$ , where  $\mathbf{u}$  minimizes*

$$\mathbf{v} \mapsto J(\mathbf{v}) = \frac{\mu}{2} \int_{\Omega} |\nabla \mathbf{v}|^2 - \int_{\Omega} \mathbf{f} \cdot \mathbf{v},$$

*among all those fields in  $H^1(\Omega)$  which are divergence-free (i.e.  $\nabla \cdot \mathbf{u} = 0$ ).*

*Proof :* Problem (1.4) is the saddle-point formulation of a minimization problem of the type (A.2) (page 67), where  $K = \ker B$  is the set of divergence free fields. By Proposition 12, page 68, it is sufficient to prove that  $B$ , which is the (opposite of the) divergence operator is surjective. This enlightens the necessity to consider the set of pressure with zero mean. As  $V$  consists of velocity fields which vanish on the boundary, any field  $\mathbf{v} \in V$  is such that

$$\int_{\Omega} \nabla \cdot \mathbf{u} = \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n} = 0,$$

so that  $B$  maps  $V$  onto scalar fields with zero means. We refer to [10] for a proof of this property.

Because of the advective term  $(\mathbf{u} \cdot \nabla) \mathbf{u}$ , which comes from the fact that Newton's law is written for a fluid element in motion, the Navier-Stokes equations are nonlinear, and this non-linearity makes the problem much more difficult to solve. A huge literature is dedicated to this problem, which still presents unresolved issues. In particular, in the three-dimensional setting, i.e. in the most interesting case from the modeling standpoint, there is no general proof of existence of smooth solutions defined globally in time. Weak solutions (in the spirit of Definition 5 below) can be defined, and their existence under very general assumptions is known since the

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<sup>2</sup>In the context of lung modeling, this technical problem will actually disappear, as we shall deal with domain with inlet and outlets, so that this indeterminacy shall not be met.

celebrated paper by Leray [13], but uniqueness of those weak solutions is an open issue. The two standpoints are related, as uniqueness of smooth solutions holds. We shall focus in this chapter on the notion of weak solution, and in particular the role played by a priori estimates, which make it possible to build weak solutions by means of compactness arguments. We refer the reader to [19, 14, 9] for details on the analysis of the Navier-Stokes equations.

To emphasize the role played by a priori estimates, let us start with the so-called *energy balance*, which expresses how the different types of energy interact. In the case of a fluid enclosed in a domain with no-slip condition on the wall, this balance will contain three contribution:

1. time derivative of the kinetic energy (stored by the system);
2. power dissipated by viscous effects (lost by the system);
3. power of external forces (supplied to the system).

Note that the system is not closed in terms of energy as temperature is not accounted for. In practice, dissipation transforms kinetic energy into heat, which ends up in the system as internal energy (the temperature increases).

Energy balance is obtained by multiplying the momentum equation by the velocity itself, and by integrating by part. Assuming that the velocity is sufficiently regular to allow those operations, we obtain

$$\frac{d}{dt} \int_{\Omega} \frac{\rho}{2} |\mathbf{u}|^2 + \mu \int_{\Omega} |\nabla \mathbf{u}|^2 = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}.$$

Time integration over an interval  $(0, T)$  gives

$$\int_{\Omega} \frac{\rho}{2} |\mathbf{u}(\mathbf{x}, T)|^2 = \int_{\Omega} \frac{\rho}{2} |\mathbf{u}(\mathbf{x}, 0)|^2 - \int_0^T \mu \int_{\Omega} |\nabla \mathbf{u}|^2 + \int_0^T \int_{\Omega} \mathbf{f} \cdot \mathbf{u}, \quad (1.5)$$

which expresses that the kinetic energy is the sum of its initial value and the work of external forces, minus the dissipated energy during  $(0, T)$ . Assuming that a finite energy is supplied to the system, the kinetic energy will remain bounded, and so is the dissipated energy. Those physical considerations are integrated in the mathematical framework through the space in which the problem is set. We introduce

$$V = \left\{ \mathbf{v} \in H_0^1(\Omega)^d, \nabla \cdot \mathbf{v} = 0 \right\}, \quad H = \overline{V}^{L^2}, \quad (1.6)$$

where  $\overline{V}^{L^2}$  is the complete closure of  $V$  for the  $L^2$  norm. Considering now a divergence-free velocity field  $\mathbf{u}(\mathbf{x}, t)$  over  $\Omega \times (0, T)$ , the fact that the kinetic energy is bounded over  $(0, T)$  writes

$$\mathbf{u} \in L^\infty(0, T; H) = \left\{ \mathbf{v}, \operatorname{ess\,sup}_{(0, T)} \int_{\Omega} |\mathbf{v}(\cdot, t)|^2 dx < +\infty \right\}.$$

Similarly, boundedness of the dissipated energy over the interval  $(0, T)$  is equivalent to the requirement that  $\mathbf{u}$  belongs to

$$L^2(0, T; V) = \left\{ \mathbf{v}, \int_0^T \int_{\Omega} |\nabla \mathbf{v}|^2 dx dt < +\infty \right\}.$$

The theoretical framework is based on a variational formulation of the problem, which is obtained by multiplying the momentum equation by a test function  $\mathbf{v} \in V$ , and integrating by part the viscous term, which leads to the following formulation of the problem:

**Definition 5** (*Weak solution of the Navier-Stokes equations*)

Let  $V$  and  $H$  be defined by (1.6). We consider a forcing term  $\mathbf{f} \in L^2(0, T; V')$ , and an initial condition  $\mathbf{u}_0 \in H$ . Following [14], we say that  $(\mathbf{x}, t) \mapsto \mathbf{u}(\mathbf{x}, t)$  is a weak solution to the Navier-Stokes equations, with initial condition  $\mathbf{u}_0$ , if

$$\begin{aligned} \mathbf{u} &\in L^2(0, T; V) \cap L^\infty(0, T; H), \\ \rho \frac{d}{dt} \int_{\Omega} \mathbf{u} \cdot \mathbf{v} + \rho \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} + \mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \quad \forall \mathbf{v} \in V \\ \mathbf{u}(\cdot, 0) &= \mathbf{u}_0. \end{aligned}$$

**Theorem 1** *Let  $\Omega$  be a bounded, regular domain,  $T > 0$ . The incompressible Navier-Stokes equations admit at least a solution  $\mathbf{u}(\mathbf{x}, t)$  in  $(0, T)$ , in the sense of Definition 5.*

*Proof :* We give here a sketch of the proof, based on the Faedo-Galerkin approach, which consists in introducing a special basis of  $V$ . We refer to [14, 19] for a detailed proof. We aim out pointint out the difficulties that we can expect in extending this approach to the situation that we are interested in, namely the case of open domains (with fluid getting in or out of the domain). Thus, we shall restrict our attention here to the way a priori estimates allow to prove well-posedness of a finite dimensional version of the original problem. Let us first note that the problem can be written in an abstract form as

$$\frac{d}{dt}(u, v) + c(u, v, v) + a(u, v) = \langle \varphi, v \rangle \quad \forall v \in V, \quad (1.7)$$

where  $V$  and  $H$  are two Hilbert spaces,  $V \subset H$  with compact and dense injection, and  $(\cdot, \cdot)$  denotes the scalar product in  $H \times H$ . The bilinear form  $a(\cdot, \cdot)$  is symmetric, continuous, and elliptic over  $V \times V$ , so that we can use it as a scalar product in  $V$ . We denote by  $\|\cdot\|$  the norm in  $V$ , and by  $|\cdot|$  the norm in  $H$ .

Consider the eigenvalue problem

$$a(w, v) = \lambda(w, v) \quad \forall v \in V.$$

We define the operator  $A \in \mathcal{L}(V, V')$  by  $(Au, v) = a(u, v)$ , for all  $v \in V$ . For any  $f \in H$ , there exists a unique  $u$  such that  $a(u, v) = (f, v)$ , so that  $A^{-1}$  is in  $\mathcal{L}(H)$ . Since  $\|A^{-1}f\| \leq C|f|$ ,  $A^{-1}$  is compact. The eigenvalue problem therefore admits a sequence of solutions  $(w_n, \lambda_n)$ , where  $0 < \lambda_0 \leq \lambda_1 \leq \dots$ , the sequence  $(\lambda_n)$  goes to infinity, and  $(w_n)$  is an Hilbert basis of  $H$ . Denoting by  $V_m$  the linear space spanned by  $w_1, \dots, w_m$ , the variational problem (1.7) can be written in  $V_m$ , by looking for a solution of the form

$$u_m = \sum_{j=1}^m u_m^j(t) w_j,$$

such that (1.7) is verified for all test functions in  $V_m$ . We obtain a system of  $m$  ordinary differential equations with unknown functions  $t \mapsto u_m^j(t)$ , for which the Cauchy Lipschitz Theorem ensures existence and uniqueness of a maximal solution. Now the energy balance (1.5) can be written in the abstract setting, for any  $t \in (0, T)$ :

$$\frac{1}{2} |u_m(t)|^2 + \int_0^t a(u_m, u_m) = \frac{1}{2} |u_m(0)|^2 + \int_0^t \langle \varphi, u_m \rangle.$$

We have

$$|\langle \varphi, u_m \rangle| \leq \|\varphi\| \|u_m\| \leq \frac{1}{2} \|\varphi\|^2 + \frac{1}{2} \|u_m\|^2,$$

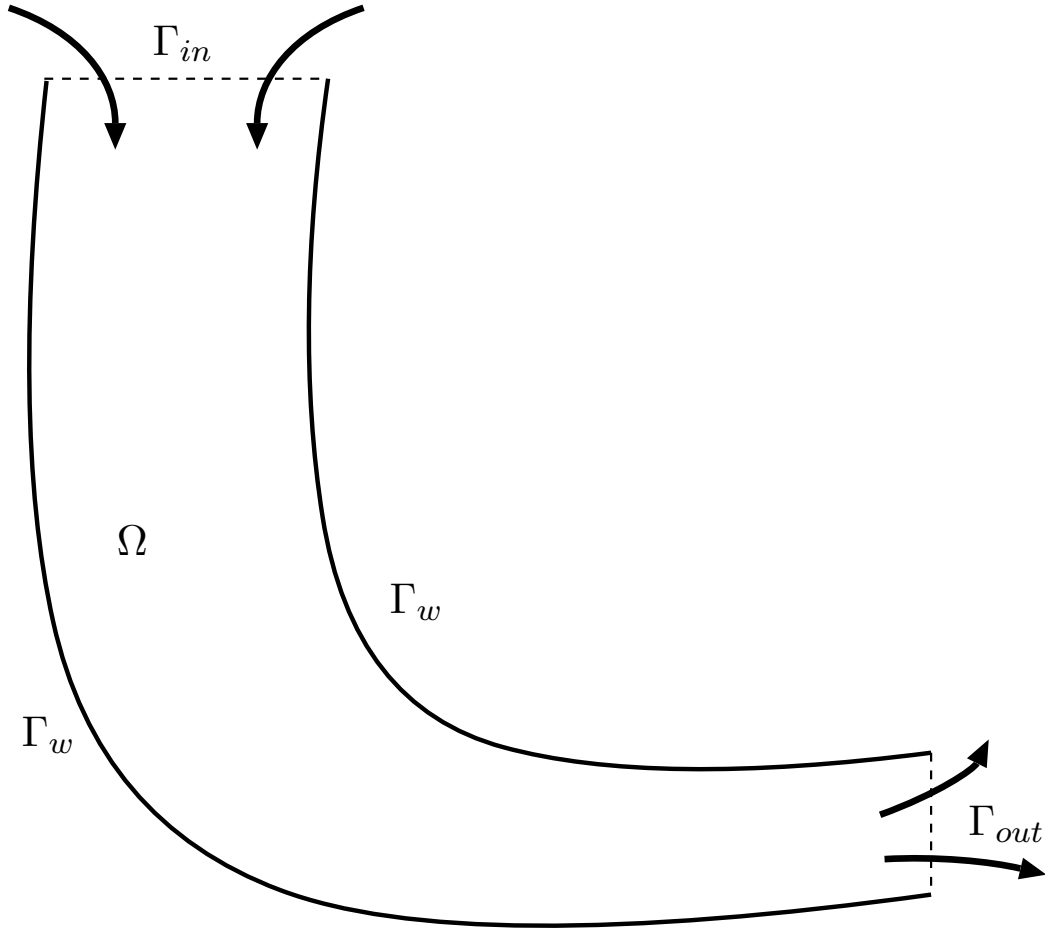


Figure 1.2: Pipe-like domain

with  $\|u_m\|^2 = a(u_m, u_m)$ , so that finally

$$|u_m(t)|^2 + \int_0^t a(u_m, u_m) = |u_m(0)|^2 + \int_0^t \|\varphi\|_{V'}^2.$$

As a consequence,  $|u_m(t)|$  can not blow up in finite time, and the solution given by the Cauchy Lipschitz theorem is global, i.e. defined up to the end  $T$  of the time interval. The sequence  $(u_m)$  is bounded in

$$L^2(0, T; V) \cap L^\infty(0, T; H).$$

The proof is ended by compactness arguments:  $u_m$  weakly converges (up to a subsequence) to a limit  $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$ , and the rest of the proof mainly consists in proving that  $u$  is a weak solution to the Navier-Stokes equations. We refer to [14, 19, 9] for details.

### 1.3 Free in-/outlet conditions

To fix the ideas, we consider a pipe-like domain  $\Omega$  whose boundary  $\partial\Omega = \Gamma$  is decomposed into three components: The lateral component  $\Gamma_w$  corresponds to a rigid wall, on which homogeneous

Dirichlet boundary conditions are prescribed,  $\Gamma_{in}$  is the inlet, and  $\Gamma_{out}$  the outlet. We aim at modeling the flow of some viscous fluid through the pipe, in a way which allows to keep the possibility to couple this pipe to other components of a multi-element model.

### 1.3.1 Prescribed normal stress

The simplest approach for setting boundary conditions consists in considering that the normal component of the stress tensor (see Definition 1, page 5) is known. The stress tensor is

$$\boldsymbol{\sigma} = \mu(\nabla \mathbf{u} + {}^t\nabla \mathbf{u}) - p \text{Id}$$

so that free outlet conditions (expressing force balance in the normal direction) reads

$$\mu(\nabla \mathbf{u} + {}^t\nabla \mathbf{u}) \cdot \mathbf{n} - p\mathbf{n} = -P_{ext}\mathbf{n}. \quad (1.8)$$

This option may make sense from a modeling point of view, yet it corresponds to a situation where the viscous fluid is separated by  $\Gamma_{out}$  (or  $\Gamma_{in}$ ) from a fluid in which the stress tensor is diagonal (like a perfect gas), and it would make clear sense if  $\Gamma_{out}$  were indeed a *free surface* separating the viscous fluid and such a medium.

But it might not be the case, if we are interested in a situation where the pipe continues beyond  $\Gamma_{out}$ . It turns out that a better way to account for this continuing pipe consists in prescribing a boundary condition of the type

$$\mu \nabla \mathbf{u} \cdot \mathbf{n} - p\mathbf{n} = -P_{ext}\mathbf{n} \quad (1.9)$$

which does not make clear sense from a physical point of view (the non symmetric tensor  $\nabla \mathbf{u}$  has no mechanical significance), but which makes it possible to recover the exact solution in case of Poiseuille's flow in a cylinder. As Conditions (1.9) correspond to the situation where the actual fluid domain continues beyond the boundary, we shall call those conditions *free outlet* (or *inlet*) conditions.

Figure 1.3 illustrates the difference between the 2 types of conditions. The velocity field on the left (parabolic profile) corresponds to free outlet conditions (1.9), and the plot on the right represents the same zone for free surface conditions (1.8). Computation have been performed with the software `FreeFem++`[8].

**Mathematical framework.** Neumann conditions (1.8) (free surface) or (1.9) (free outlet) raise theoretical issues. For the Navier-Stokes equations, as we shall see, the fact that some fluid is entering the domain will make it much more difficult to establish a priori estimates. As a first step, we start with the Stokes problem. We introduce

$$V = \left\{ \mathbf{v} \in H^1(\Omega)^d, \mathbf{v}|_{\Gamma_w} = 0, \right\}$$

and the set of pressure  $X = L^2(\Omega)$ . The variational formulation corresponding to the free outlet conditions is

$$\mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} - \int_{\Gamma_{in}} P_{in} \mathbf{n} \cdot \mathbf{v} - \int_{\Gamma_{out}} P_{out} \mathbf{n} \cdot \mathbf{v} \quad \forall \mathbf{v} \in V, \quad (1.10)$$

together with the weak expression of the divergence free constraint

$$\int_{\Omega} \mathbf{q} \nabla \cdot \mathbf{u} = 0 \quad \forall \mathbf{q} \in X.$$

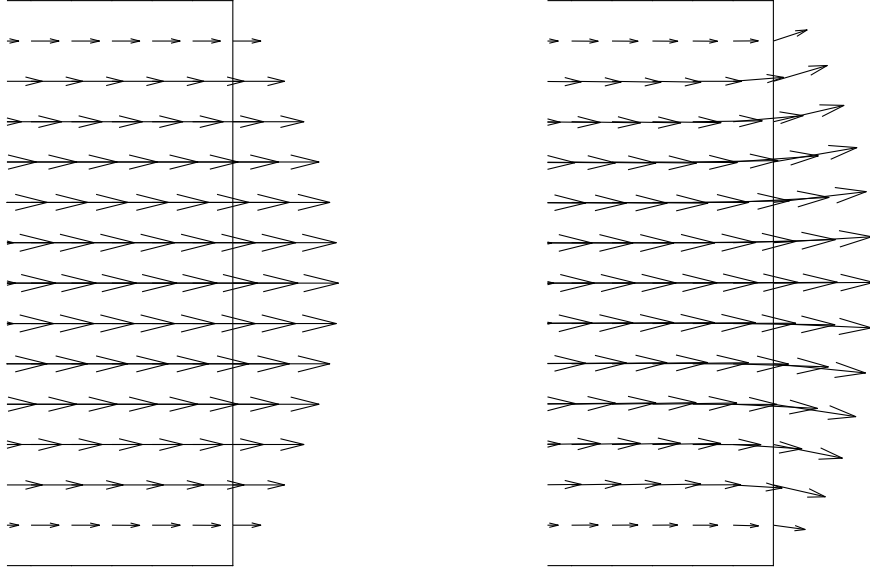


Figure 1.3: Free outlet conditions based on the velocity gradient (left) and on the strain tensor (right).

**Proposition 2** *We consider the Stokes problem on domain  $\Omega$  represented in Figure 1.2, with homogeneous Dirichlet on the lateral boundary  $\Gamma_w$  and free in-/out-let conditions (1.9) on  $\Gamma_{in} \cup \Gamma_{out}$ . The problem admits a unique solution.*

*Proof :* In the present situation, the velocity may have non zero values on some part of the boundary, and the pressure is no longer defined up to a constant. This makes the proof of the surjectivity of the divergence operator  $B$  slightly different. Considering a pressure field  $q \in L^2(\Omega)$ , the first step consists in building a velocity field  $\mathbf{v}$  such that  $\int \nabla \cdot \mathbf{v} = \int q$ , which is straightforward as  $\mathbf{v}$  may have non zero values on  $\Gamma_{in} \cup \Gamma_{out}$ . The rest of the proof is then identical to the case of homogenous boundary conditions, by considering the pressure minus its mean value over  $\Omega$ .

**Remark 3** *If one considers free surface conditions (1.8), the variational formulation is modified:*

$$\begin{aligned} & \mu \int_{\Omega} (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) : (\nabla \mathbf{v} + {}^t \nabla \mathbf{v}) - \int_{\Omega} p \nabla \cdot \mathbf{v} \\ &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} - \int_{\Gamma_{in}} P_{in} \mathbf{n} \cdot \mathbf{v} - \int_{\Gamma_{out}} P_{out} \mathbf{n} \cdot \mathbf{v} \quad \forall \mathbf{v} \in V, \end{aligned}$$

*The bilinear form involves the symmetrized velocity gradient. Ellipticity of this bilinear form is*

a consequence of Korn's inequality, which ensures existence of a constant such that

$$\int_{\Omega} |\nabla \mathbf{v} + {}^t\nabla \mathbf{v}|^2 \geq C \int_{\Omega} |\nabla \mathbf{v}|^2 \quad \forall \mathbf{v} \in V,$$

as soon as  $V$  is a subset of  $H^1(\Omega)^d$  which does not contain any rigid motion other than 0, which is the case here thanks to the no-slip condition on the lateral boundary. Those conditions complicate the numerical resolution. Whereas the matrix corresponding to free outlet conditions is block-diagonal (scalar Laplace operator for each component of the velocity), it is no longer block-diagonal for the form that is based on the symmetrized tensor.

## 1.4 Some typical problems

We describe here some situations for which the exact solution cannot be given analytically.

### 1.4.1 Flow around an obstacle

A great amount of computational strategies to approximate solutions to the Navier-Stokes equation have been developed during the last decades to address the following problem (represented in the two dimensional setting by Fig. 1.4): consider a flow of a viscous fluid around a fixed obstacle  $\mathcal{O}$ , one is interested in estimating the forces exerted by the fluid on the obstacle. The problem consists in solving incompressible Navier-Stokes equations in the fluid part  $\Omega$ , with no-slip conditions on the boundary  $\gamma$  of the obstacle. The question of boundary conditions on the outer boundary raises some interesting issues. A first model consists in assuming that the outer boundary is far enough from  $\omega$ , so that the flow there is not affected by the presence of the obstacle. In this case, it is reasonable to assume the velocity to be fixed on  $\Gamma = \partial\Omega$ , equal to a "velocity at infinity"  $\mathbf{U}_{\infty}$ . The approach may lead to a large exterior domain, and thereby high computational costs. A great amount of investigations has been undertaken to propose alternative strategies to overcome this problem, by elaborating appropriate boundary conditions on the outer boundary.

In the case where  $\omega$  represent an airfoil, quantities of interested are the lift force (*portance* in french)  $\mathbf{F}_y$ , and the drag (*traînée*)  $\mathbf{F}_x$

$$\mathbf{F}_y = -\mathbf{e}_y \cdot \int_{\gamma} \boldsymbol{\sigma} \cdot \mathbf{n}, \quad \mathbf{F}_x = -\mathbf{e}_x \cdot \int_{\gamma} \boldsymbol{\sigma} \cdot \mathbf{n}.$$

The lift force maintains the aircraft in the air (it balances the total weight of the flying plane), whereas drag is opposed to motion, thereby conditioning the price to pay to make the plane flying.

### 1.4.2 Some simple fluid-structure interaction problems

In real life application, one is commonly interested in modeling the way a fluid interacts with another medium, like an elastic structure. This problem raises issues which are still the object of active research, both on the theoretical and numerical aspects. We present here here two situations where the number of degrees of freedom for the structure is finite.

**Spring-mass-fluid system.** We consider the situation represented in Fig. 1.5 (left). The domain  $\Omega$  is bounded on the right side by a piston with mass  $m$ , attached to a spring with stiffness  $k > 0$ . Despite its formal simplicity, this example calls for a special care of boundary



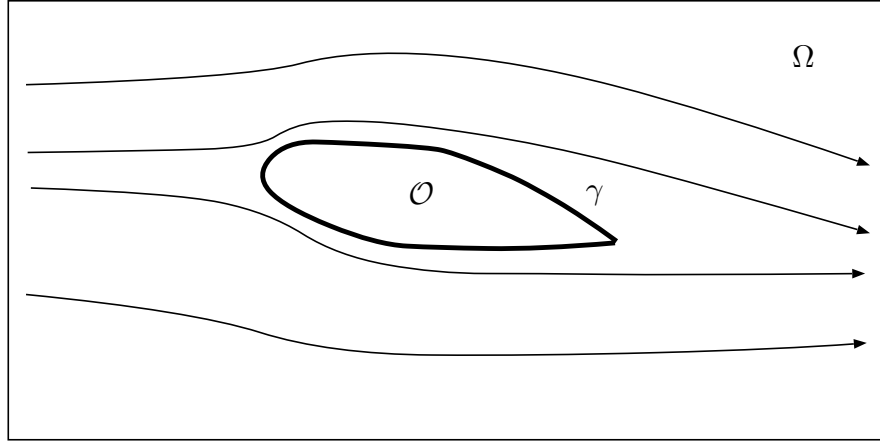


Figure 1.4: Flow around an obstacle

conditions : as the motion of the piston is not compatible with the no-slip condition on the lateral boundary, we shall assume that the fluid may slip freely along this part of the boundary. Assuming the fluid sticks to the piston, the global problem may be written

$$\begin{aligned} \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) - \mu \Delta \mathbf{u} + \nabla p &= 0 && \text{in } \Omega(t) \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega(t) \\ \mathbf{u} &= 0 && \text{on } \Gamma_0 \\ \boldsymbol{\sigma} \cdot \mathbf{n} &= 0 && \text{on } \Gamma_{in} \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{t} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} &= 0 && \text{on } \Gamma_\ell \\ \mathbf{u} &= \dot{x} \mathbf{e}_x && \text{on } \Gamma_m \end{aligned}$$

supplemented by Newton's law for the piston:

$$m \frac{d^2 x}{dt^2} = -k(x - x_0) - \int_{\Gamma_m} \boldsymbol{\sigma} \cdot \mathbf{n}.$$

**Fluid particle flow.** In this second example (see Fig. 1.5, right), we consider a rigid disc flowing freely in a viscous fluid. Denoting by  $\rho_s$  the density of the solid, and by  $m$  its mass, the fluid problem can be written in the form of incompressible Navier-Stokes equations in the moving domain  $\Omega(t)$  (with right-hand side  $\rho \mathbf{g}$ , supplemented by two types of coupling conditions:

1. Kinematic coupling. No-slip condition on the boundary of the body writes

$$\mathbf{u} = \mathbf{U} + \boldsymbol{\omega} \wedge \mathbf{r} \quad \text{on } \gamma.$$

2. Dynamic coupling. Newton's law for the particle translational and angular velocity write

$$m \frac{d\mathbf{U}}{dt} = - \int_\gamma \boldsymbol{\sigma} \cdot \mathbf{n} + m \mathbf{g}, \quad J \frac{d\boldsymbol{\omega}}{dt} = 0,$$

where  $J$  is the moment of inertia of the disc.

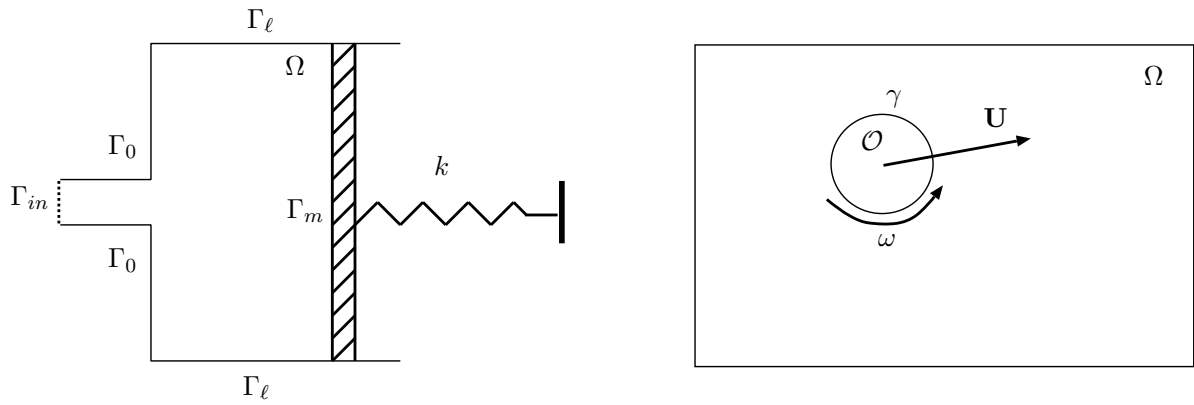


Figure 1.5: Interaction with a rigid structure

The problem is written here for a single particle, but it can be extended straightforwardly to the many-body situation (with 3 degrees of freedom per particle, together with a three dimensional coupling between each particle and the fluid).

## Chapter 2

# Navier-Stokes equations, numerics

Navier-Stokes equations possess several terms that need to be handled adequately in view of a full numerical scheme. Indeed, one faces three problems that seem quite uncoupled and follow different (though complementary) strategies:

- The incompressibility constraint  $\operatorname{div} \mathbf{u} = 0$  and the related pressure gradient ;
- The nonlinearity present in the convective term ;
- The time dependence. The equation is non stationary, and a suitable time marching algorithm needs to be written.

We hereafter discuss those three aspects of the problem and detail the natural links between them as far as the discretization by the finite element method is concerned.

### 2.1 The incompressibility constraint, Stokes equations

Let us start with the first problem, namely the discretization of the constraint  $\operatorname{div} \mathbf{u} = 0$ . To this aim, we consider only Stokes equations, where, in order to focus only on the mathematical aspects of the problem, the viscosity has been taken equal to unity :

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0. \end{cases} \quad (2.1)$$

We have already seen that, depending on the Neumann boundary conditions that one considers, there are typically two variational formulations associated to (2.1), namely

$$\begin{aligned} & \text{Find } (\mathbf{u}, p) \in V \times M, \text{ such that for all } (\mathbf{v}, q) \in V \times M, \\ & \begin{cases} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} d\mathbf{x} - \int_{\Omega} p \operatorname{div} \mathbf{v} d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}, \\ \int_{\Omega} q \operatorname{div} \mathbf{u} d\mathbf{x} = 0, \end{cases} \end{aligned} \quad (2.2)$$

or

$$\begin{aligned} & \text{Find } (\mathbf{u}, p) \in V \times M, \text{ such that for all } (\mathbf{v}, q) \in V \times M, \\ & \begin{cases} \int_{\Omega} (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) : \nabla \mathbf{v} d\mathbf{x} - \int_{\Omega} p \operatorname{div} \mathbf{v} d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}, \\ \int_{\Omega} q \operatorname{div} \mathbf{u} d\mathbf{x} = 0. \end{cases} \end{aligned} \quad (2.3)$$

Both variational formulation lead to the same Stokes equations inside the domain due to the fact that

$$\operatorname{div}({}^t\nabla\mathbf{u}) = \nabla(\operatorname{div}\mathbf{u}) = 0$$

since the vector field must be incompressible. They however lead to different Neumann boundary conditions when one integrates by part. It can be shown that both formulations leads to well-posed problems<sup>1</sup> provided  $V \subset H^1(\Omega, \mathbb{R}^3)$  and  $M = L^2(\Omega, \mathbb{R})/\mathbb{R}$ .

When the discretization by the finite element method is concerned, it turns out that there is an additional problem due to the discretization of the Hilbert spaces  $V$  and  $M$ . Indeed, the discrete variational formulation associated with (2.2) for instance (the corresponding holds for (2.3)) reads as follows

$$\begin{aligned} &\text{Find } (\mathbf{u}_h, p_h) \in V_h \times M_h, \text{ such that for all } (\mathbf{v}_h, q_h) \in V_h \times M_h, \\ &\left[ \begin{aligned} &\int_{\Omega} \nabla\mathbf{u}_h : \nabla\mathbf{v}_h d\mathbf{x} - \int_{\Omega} p_h \operatorname{div}\mathbf{v}_h d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h d\mathbf{x}, \\ &\int_{\Omega} q_h \operatorname{div}\mathbf{u}_h d\mathbf{x} = 0, \end{aligned} \right. \end{aligned} \quad (2.4)$$

where  $V_h \subset V$  and  $M_h \subset M$  are two finite dimensional subspaces of  $V$  and  $M$  respectively. The preceding problem has a clear energetic formulation. Namely  $\mathbf{u}_h$  can be seen as minimizing

$$\mathcal{E}(\mathbf{u}_h) = \frac{1}{2} \int_{\Omega} |\nabla\mathbf{u}_h|^2 d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u}_h d\mathbf{x}$$

under the constraint(s)

$$\int_{\Omega} q_h \operatorname{div}\mathbf{u}_h d\mathbf{x} = 0, \forall q_h \in M_h. \quad (2.5)$$

One sees that if the constraints expressed by (2.5) are too numerous, there might be only one function satisfying all those which is  $\mathbf{u}_h = 0$ . This might for instance be the case if unfortunately  $\dim(M_h) > \dim(V_h)$ . If on the other hand, there are not *enough* constraints in (2.5), then there might be a convergence problem. It is unclear why the discrete solution found  $\mathbf{u}_h$  should be close to the continuous one  $\mathbf{u}$ . Therefore the finite element spaces  $V_h$  and  $M_h$  need not be chosen separately. Some choices are known to be *unstable* (they do not lead to a convergent discrete solution as the space step  $h$  goes to 0) like the natural couples

$$V_h = P^1, \quad M_h = P^1 \text{ or } V_h = P^1, \quad M_h = P^0.$$

Other choices lead to both convergent and well posed discrete formulations, the most known being

$$V_h = P^2 \text{ and } M_h = P^1.$$

In the former,  $P^k$  stands for the classical polynomial approximation of degree  $k$  on a triangulation, also known as the *Lagrange* finite element of degree  $k$ .

To finish for the time being on this question, let us say that *stable and convergent* choices for the finite element spaces  $V_h$  and  $M_h$  are those which satisfy the following *Inf-Sup condition*

$$\inf_{h>0} \inf_{q_h \in M_h} \sup_{\mathbf{v}_h \in V_h} \frac{\int_{\Omega} q_h \operatorname{div}\mathbf{v}_h d\mathbf{x}}{\|q_h\|_M \|\mathbf{v}_h\|_V} \geq \beta > 0. \quad (2.6)$$

---

<sup>1</sup>Of course, extra work is needed here. Lax-Milgram lemma does not apply directly because the bilinear form is not coercive, suitable boundary conditions need to be considered, Korn's inequality is needed to prove existence and uniqueness of the second formulation, etc. It is not the aim of the authors, at this level of the discussion to discuss these problems.

**Temporary conclusion:** Due to stability problems the velocity/pressure need to be computed using a stable finite element, typically  $(P^2, P^1)$ .

## 2.2 Time discretization

### 2.2.1 Generalities

Let us focus now on the time discretization. At first sight, Navier-Stokes equations are very similar to the scalar heat equation, in the sense that forgetting the nonlinearity for the time being, the remaining terms of the equations are parabolic.

For scalar and linear problems like the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f,$$

time discretization strategies can be described straightforwardly. The simplest time-discretization approach is the explicit Euler scheme

$$\frac{u^{n+1} - u^n}{\delta t} - \Delta u^n = f, \quad (2.7)$$

where  $\delta t$  is the time step. The implicit version writes

$$\frac{u^{n+1} - u^n}{\delta t} - \Delta u^{n+1} = f.$$

Extensions of those two approaches can be built to obtain a better accuracy in the time resolution. As an example, the Crank-Nicholson scheme (which can be described as half explicit - half implicit)

$$\frac{u^{n+1} - u^n}{\delta t} - \frac{1}{2} (\Delta u^n + \Delta u^{n+1}) = f,$$

is more accurate (second order in time), whereas it preserves the stability properties of the implicit scheme.

Of course, all these schemes must be thought in weak form. Namely, calling  $V_h$  the finite element space in which the solution is sought, we have for the explicit Euler scheme (2.7)

$$\int_{\Omega} u^{n+1} v \, d\mathbf{x} = \int_{\Omega} u^n v \, d\mathbf{x} - \delta t \int_{\Omega} \nabla u^n \nabla v \, d\mathbf{x} + \delta t \int_{\Omega} f v \, d\mathbf{x}.$$

Similar expressions can be derived for the implicit or Crank-Nicholson schemes.

All these schemes enter in the class of the so-called  $\theta$ -scheme which for a parameter  $\theta \in [0, 1]$  writes

$$\frac{u^{n+1} - u^n}{\delta t} - ((1 - \theta)\Delta u^n + \theta\Delta u^{n+1}) = f, \quad (2.8)$$

where  $\theta = 0$  corresponds to the explicit scheme,  $\theta = 1$  the implicit scheme and  $\theta = \frac{1}{2}$  is the Crank-Nicholson scheme.

There is again a stability issue given by the following proposition.

**Proposition 3** *After a finite element space discretization, the  $\theta$ -scheme is uniformly and unconditionally stable (and convergent) provided  $\theta \geq \frac{1}{2}$ .*

*Proof* : (Hint.) A recipe to prove the stability is to look for a discrete energy bound. The solution to the heat equation, verifies, when the right-hand side  $f$  vanishes the two following estimates obtained respectively by multiplying the equation by  $u$  and  $\frac{du}{dt}$ :

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 d\mathbf{x} + \int_{\Omega} |\nabla u|^2 d\mathbf{x} = 0,$$

and

$$\int_{\Omega} \left| \frac{du}{dt} \right|^2 d\mathbf{x} + \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 d\mathbf{x} = 0.$$

Both equations give after integration in time a bound of a positive quantity, namely  $L^2$  and the  $l_0$  norms respectively

$$\|u(T)\|_{L^2}^2 + \int_0^T \int_{\Omega} |\nabla u|^2 d\mathbf{x} = \|u(0)\|_{L^2}^2$$

and

$$\|\nabla u(T)\|_{L^2}^2 + \int_0^T \int_{\Omega} \left( \frac{du}{dt} \right)^2 d\mathbf{x} = \|\nabla u(0)\|_{L^2}^2.$$

The idea is to mimic those results in the discrete case. For the  $L^2$  stability, multiplying the scheme (2.8) by  $u_{\theta} := (1 - \theta)u^n + \theta u^{n+1}$  (with  $f = 0$ ) and integrating by parts leads to

$$\int_{\Omega} \frac{u^{n+1} - u^n}{\delta t} u_{\theta} + \int_{\Omega} |\nabla u_{\theta}|^2 d\mathbf{x} = 0,$$

which, noticing that  $u_{\theta} = \frac{u^n + u^{n+1}}{2} + (2\theta - 1) \frac{u^{n+1} - u^n}{2}$  leads to the estimate

$$\int_{\Omega} \frac{(u^{n+1})^2}{2\delta t} d\mathbf{x} + (2\theta - 1) \int_{\Omega} \frac{(u^{n+1} - u^n)^2}{2\delta t} d\mathbf{x} + \int_{\Omega} |\nabla u_{\theta}|^2 d\mathbf{x} = \int_{\Omega} \frac{(u^n)^2}{2\delta t} d\mathbf{x}.$$

When  $\theta \geq \frac{1}{2}$  this gives the uniform bound (for all  $n, \delta t, h$ ) for the  $L^2$  norm

$$\int_{\Omega} (u^n)^2 d\mathbf{x} \leq \int_{\Omega} (u^0)^2 d\mathbf{x}.$$

Similarly, one can use the second estimate. Namely, multiplying now (2.7) by  $\frac{u^{n+1} - u^n}{\delta t}$  gives

$$\int_{\Omega} \left( \frac{u^{n+1} - u^n}{\delta t} \right)^2 d\mathbf{x} + \int_{\Omega} \nabla u_{\theta} \cdot \nabla \left( \frac{u^{n+1} - u^n}{\delta t} \right) d\mathbf{x} = 0,$$

which, writing again  $u_{\theta} = \frac{u^n + u^{n+1}}{2} + (2\theta - 1) \frac{u^{n+1} - u^n}{2}$  leads now to the estimate

$$\int_{\Omega} \frac{|\nabla u^{n+1}|^2}{2\delta t} d\mathbf{x} + \int_{\Omega} \left( \frac{u^{n+1} - u^n}{\delta t} \right)^2 d\mathbf{x} + \left( \theta - \frac{1}{2} \right) \delta t \int_{\Omega} \left| \nabla \left( \frac{u^{n+1} - u^n}{\delta t} \right) \right|^2 d\mathbf{x} = \int_{\Omega} \frac{|\nabla u^n|^2}{2\delta t} d\mathbf{x},$$

Again, when  $\theta \geq \frac{1}{2}$ , we obtain the uniform estimate (for all  $n, \delta t, h$ )

$$\int_{\Omega} |\nabla u^n|^2 d\mathbf{x} \leq \int_{\Omega} |\nabla u^0|^2 d\mathbf{x}.$$

**Exercise 2** *Extend those results for the case of a non vanishing external force  $f$ .*

**Temporary conclusion:** Implicitizing the Laplace term in the heat equation increases the stability of the underlying finite element scheme. It is typically unconditional between the Crank-Nicolson scheme ( $\theta = \frac{1}{2}$ ) and the fully implicit scheme ( $\theta = 1$ ).

### 2.2.2 Going back to Navier-Stokes equations

From the preceding section, we deduce that the Laplace term is more stable when discretized implicitly. However, there is still the problem of the non linear term that can be treated explicitly or implicitly.

**Fully explicit scheme.** The fully explicit scheme is based on an explicit evaluation of the non linear term in the Navier Stokes equations

$$\rho \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} + \rho (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n - \mu \Delta \mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f},$$

with  $\nabla \cdot \mathbf{u}^{n+1} = 0$ . As for Burger's equation, or the even simpler transport equations, this approach is expected to raise stability issues. In particular, it obviously requires a so called CFL condition, i.e. condition of the type

$$u \delta t \leq Ch,$$

where  $C$  is a dimensionless constant smaller than 1.

Note that it leads to a so called generalized Stokes problem

$$\begin{cases} \alpha \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = \tilde{\mathbf{f}}, \\ \nabla \cdot \mathbf{u} = 0, \end{cases}$$

where  $\alpha = \rho/\delta t$ , and the new right-hand side  $\tilde{\mathbf{f}} = \mathbf{f} - \rho (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n + \frac{\rho}{\delta t} \mathbf{u}^n$  accounts for inertial terms. This problem has exactly the same mathematical structure as the standard Stokes problem, and can be handled numerically as such. Note that complete space discretization may necessitate special quadrature formulae to compute (or at least estimate) the term

$$\int (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n \cdot \mathbf{v},$$

in the variational formulation.

#### Semi-explicit scheme.

The semi explicit scheme is also very popular, it consists in writing the nonlinear term as an advection of the unknown velocity by the known one:

$$\rho \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} + \rho (\mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1} - \mu \Delta \mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f},$$

with  $\nabla \cdot \mathbf{u}^{n+1} = 0$  which leads to a problem of the type

$$\begin{cases} \alpha \mathbf{u} - \mu \Delta \mathbf{u} + (\mathbf{U} \cdot \nabla) \mathbf{u} + \nabla p = \tilde{\mathbf{f}}, \\ \nabla \cdot \mathbf{u} = 0, \end{cases}$$

where  $\mathbf{U} = \rho \mathbf{u}^n$  is known. The problem is still linear, but the associated bilinear form is no longer symmetric, it is defined by

$$a(\mathbf{u}, \mathbf{v}) = \alpha \int_{\Omega} \mathbf{u} \cdot \mathbf{v} + \mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} + \int_{\Omega} \mathbf{v} \cdot (\mathbf{U} \cdot \nabla) \mathbf{u}.$$

**Fully implicit scheme.** The fully implicit scheme is based on an implicit evaluation of the non linear term in the Navier Stokes equations

$$\rho \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} + \rho (\mathbf{u}^{n+1} \cdot \nabla) \mathbf{u}^{n+1} - \mu \Delta \mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f},$$

with  $\nabla \cdot \mathbf{u}^{n+1} = 0$ .

Note that this approach requires to solve a nonlinear problem at each time step.

**Exercise 3** Notice that if  $\mathbf{u}$  and  $\mathbf{v}$  are two vector fields with  $\operatorname{div} \mathbf{u} = 0$  (and suitable, e.g. homogeneous Dirichlet boundary conditions), one has

$$\int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{v} \, d\mathbf{x} = 0.$$

Then apply the energy method to prove unconditional stability of the semi explicit and implicit schemes.

### 2.2.3 The method of characteristics

The method of characteristics can be seen as a direct discretization of the total derivative. Compared to the previous methods, it is also a way of handling the non linear term explicitly but which enjoys better stability properties than the previous explicit scheme.

Consider

$$t \longmapsto \chi(t)$$

the trajectory of a fluid particle located at  $\mathbf{x}$  at time  $\tau$  For any function  $\Phi$  (scalar or vector), the total derivative of  $\Phi$  at  $(\mathbf{x}, \tau)$  is

$$\begin{aligned} \left. \frac{D\Phi}{Dt} \right|_{(\mathbf{x}, \tau)} &= \left. \frac{d}{dt} \Phi(\chi(t), t) \right|_{t=\tau} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\Phi(\chi(\tau + \varepsilon), \tau + \varepsilon) - \Phi(\chi(\tau), \tau)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\Phi(\chi(\tau), \tau) - \Phi(\chi(\tau - \varepsilon), \tau - \varepsilon)}{\varepsilon} \\ &\sim \frac{\Phi(\chi(\tau), \tau) - \Phi(\chi(\tau - \delta t), \tau - \delta t)}{\delta t}. \end{aligned}$$

Now consider that the solution  $\mathbf{u}^n$  at time  $n\delta t$  is known, for any point of the domain. Using the preceding formula, we see that

$$\left. \frac{D\mathbf{u}}{Dt} \right|_{(\mathbf{x}, \tau)} \sim \frac{\mathbf{u}^{n+1}(\mathbf{x}) - \mathbf{u}^n(\mathbf{y})}{\delta t}$$

where  $\mathbf{y}$  is the position at time  $n\delta t$  of the characteristic which ends at  $\mathbf{x}$  at time  $(n+1)\delta t$  ( $\mathbf{y}$  is the so-called “foot” of the characteristic). Notice that  $\mathbf{y}$  is obtained by integrating *backward* along the vector field  $\mathbf{u}^n$  which is known.

Calling

$$\begin{aligned} \mathbf{X} : \Omega &\longrightarrow \Omega \\ \mathbf{x} &\longmapsto \mathbf{y} \end{aligned}$$



the scheme is usually written as

$$\alpha \mathbf{u}^{n+1} - \alpha \mathbf{u}^n \circ \mathbf{X} - \mu \Delta \mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f}.$$

As for the explicit scheme, the term containing  $\mathbf{X}$  goes to the right hand side, which yields a generalized Stokes problem:

$$\begin{cases} \alpha \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p & = \tilde{\mathbf{f}}, \\ \nabla \cdot \mathbf{u} & = 0 \end{cases}$$

where  $\tilde{\mathbf{f}}$  carries the initial forcing term supplemented by the inertial term  $\alpha \mathbf{u}^n \circ \mathbf{X}$ .

This method can be implemented in Freefem++ using the command `convect` as follows.

```
+int2d(Th)(-convect([ux,uy],-dt,ux)*tux-convect([ux,uy],-dt,uy)*tuy)
```

In the preceding command, the first argument (`[ux,uy]`) of the `convect` command is the vector field along which the quantity needs to be advected, the second argument (`-dt`) is the time of the advection and the third argument (`ux` or `uy`) what has to be advected. The variables (`tux` and `tuy`) stand for the test functions in the weak formulation. Notice that the integration of the characteristic flow is done backward.

## 2.3 Projection methods

Projection methods are very popular, as they split the saddle point problem onto two Poisson like problems. The approach described here does not rely to the way the inertial term is handled. We shall present the approach coupled with the method of characteristics, but the other approaches where it is treated explicitly or implicitly could be used as well:

1. (Prediction) The velocity is predicted by solving

$$\alpha \tilde{\mathbf{u}}^{n+1} - \alpha \mathbf{u}^n \circ \mathbf{X} - \mu \Delta \tilde{\mathbf{u}}^{n+1} + \nabla p^n = \mathbf{f},$$

without the divergence constraint (notice that the pressure has be *frozen* to its previous value). This step is nothing that an implicit heat flow time step. It is clear that the incompressibility constraint has been lost during this step.

2. The velocity is now corrected by adding a pressure gradient correction term  $\nabla r^{n+1}$  to  $\tilde{\mathbf{u}}^{n+1}$  in order that  $\mathbf{u}^{n+1}$  be divergence free. More precisely, one writes

$$\begin{cases} \alpha \mathbf{u}^{n+1} - \alpha \tilde{\mathbf{u}}^{n+1} + \nabla r^{n+1} = 0, \\ \nabla \cdot \mathbf{u}^{n+1} = 0, \end{cases} \quad (2.9)$$

which yields

$$-\nabla \cdot \nabla r^{n+1} = -\Delta r^{n+1} = -\alpha \nabla \cdot \tilde{\mathbf{u}}^{n+1}.$$

and

$$\begin{cases} \mathbf{u}^{n+1} = \tilde{\mathbf{u}}^{n+1} - \frac{1}{\alpha} \nabla r^{n+1} \\ p^{n+1} = p^n + r^{n+1}. \end{cases}$$

This discretization approach therefore to Poisson like problems only : a vector problem for the momentum equation, and a scalar one for the incompressibility constraint.

Notice that (2.9) is nothing than the projection of  $\tilde{\mathbf{u}}^{n+1}$  on divergence free vector fields. Indeed, one can rewrite (2.9) as

$$\begin{cases} \mathbf{u}^{n+1} + \nabla \left( \frac{1}{\alpha} r^{n+1} \right) = \tilde{\mathbf{u}}^{n+1}, \\ \nabla \cdot \mathbf{u}^{n+1} = 0, \end{cases} \quad (2.10)$$

which is the Helmholtz's decomposition of  $\tilde{\mathbf{u}}^{n+1}$  as

$$\tilde{\mathbf{u}}^{n+1} = \nabla \phi + \text{curl } \psi.$$

Notice also that, although there are two seemingly uncoupled steps, this splitting algorithm is consistent with the original Navier-Stokes equations since one has

$$\alpha \mathbf{u}^{n+1} - \alpha \mathbf{u}^n \circ \mathbf{X} - \mu \Delta \tilde{\mathbf{u}}^{n+1} + \nabla(p^n + r^{n+1}) = \mathbf{f}.$$

# Chapter 3

## Moving domains

### 3.1 Lagrangian standpoint

For the sake of clarity, the kinematical approach is applied to a particular, and somewhat idealized, situation. We shall consider here a 2D free surface flow, with no inlet nor outlet, nor any contact with any wall. All surfaces and functions are supposed to be sufficiently regular. The domain  $\Omega(t)$  is delimited by

$$\partial\Omega(t) = \Gamma(t), \quad (3.1)$$

where  $\Gamma(t)$  is the free surface. We shall disregard here the equations to which the velocity  $\mathbf{u}(\mathbf{x}, t)$  is a solution, and focus on the kinematical aspects. The fact that the domain  $\Omega(t)$  is “moved” by the physical velocity can be expressed the following way (Lagrangian approach): one can build mappings

$$\begin{aligned} L(\cdot, t) &: \Omega(0) \longrightarrow \Omega(t) \\ \mathbf{x}_0 &\longmapsto \mathbf{x}_t = L(\mathbf{x}_0, t) \end{aligned} \quad (3.2)$$

where  $s \longmapsto L(\mathbf{x}_0, s)$  is the characteristic curve from  $\mathbf{x}_0$  to  $\mathbf{x}_t$  in  $\mathbb{R}^2$ :

$$\begin{cases} \frac{\partial}{\partial s} L(\mathbf{x}_0, s) = \mathbf{u}(L(\mathbf{x}_0, s), s) \\ L(\mathbf{x}_0, 0) = \mathbf{x}_0. \end{cases} \quad (3.3)$$

It can be written in integral form

$$L(\mathbf{x}_0, t) = \mathbf{x}_0 + \int_0^t \mathbf{u}(L(\mathbf{x}_0, s), s) ds. \quad (3.4)$$

The domain at time  $t$  can be written as

$$\Omega(t) = \{L(\mathbf{x}_0, t), \mathbf{x}_0 \in \Omega(0)\}. \quad (3.5)$$

**Navier-Stokes equations.** We consider that the domain moves according to Navier-Stokes equations. Let us denote by  $\mathbf{u}_0$  the Lagrangian velocity with respect to the initial configuration, i.e.

$$\mathbf{u}_0(\mathbf{x}_0, t) = \mathbf{u}(L(\mathbf{x}_0, t), t).$$

Since the space variable is Lagrangian, the partial time derivative of  $\mathbf{u}_0$  is the total derivative. Indeed, for  $\mathbf{x}_0$  fixed,

$$\begin{aligned} \frac{d}{dt}\mathbf{u}_0(\mathbf{x}_0, t) &= \frac{d}{dt}(\mathbf{u}(L(\mathbf{x}_0, t), t)) = \frac{\partial}{\partial t}\mathbf{u} + \left(\frac{d}{dt}L(\mathbf{x}_0, t) \cdot \nabla\right)\mathbf{u} \\ &= \frac{\partial}{\partial t}\mathbf{u}(L(\mathbf{x}_0, t), t) + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{D\mathbf{u}}{Dt}. \end{aligned}$$

As a consequence, the Navier-Stokes equation expressed in the Lagrangian variable at  $t = 0$  write

$$\rho \frac{\partial \mathbf{u}_0}{\partial t} - \mu \Delta \mathbf{u}_0 + \nabla p_0 = \mathbf{f}, \quad (3.6)$$

with  $\nabla \cdot \mathbf{u}_0 = 0$ .

This linearization of Navier-Stokes equations is only valid instantaneously. If one aims at expressing it at a time  $t$  different from that of the reference domain, the partial time derivative will still express the total derivative (i.e., the term  $(\mathbf{u} \cdot \nabla)\mathbf{u}$  is dropped), but the terms that correspond to space derivatives will be deeply affected, in a nonlinear way. Indeed, the eulerian coordinate at time  $t$  (according to which the Laplacian et gradient operators are defined) and the Lagrangian one  $\mathbf{x}_0$  are related by

$$\mathbf{x}_0 \mapsto \mathbf{x} = L(\mathbf{x}_0, t).$$

Nevertheless, assuming that (3.6) is valid at the first order in a neighborhood of  $t = 0$ , it is possible to design a numerical method of the Lagrangian type, by using the Lagrangian coordinates that corresponds to the current configuration. The time stepping approach can be described in the following way.

Denote by  $\Omega^n$  and  $\mathbf{u}^n$  the domain and velocity at time  $t^n$ . We denote by  $\mathbf{u}_n^{n+1}$  the next velocity, defined in the lagrangian coordinates associated with  $\Omega^n$ . It can be computed as the solution to

$$\rho \frac{\mathbf{u}_n^{n+1} - \mathbf{u}^n}{\delta t} - \mu \Delta \mathbf{u}_n^{n+1} + \nabla p_n^{n+1} = f. \quad (3.7)$$

The domain  $\Omega^{n+1}$  is now defined the as the push forward of  $\Omega^n$  by  $\delta t \mathbf{u}^n$ :

$$\Omega^{n+1} = \{\mathbf{x} + \delta t \mathbf{u}^n(\mathbf{x}), \mathbf{x} \in \Omega^n\}.$$

The velocity  $\mathbf{u}^{n+1}$  is then defined by

$$\mathbf{u}^{n+1}(\mathbf{x} + \delta t \mathbf{u}^n(\mathbf{x})) = \mathbf{u}_n^{n+1}(\mathbf{x}).$$

Note that the latter operation, as tricky as it may look, is actually trivial in terms of computation. In the discrete setting, e.g. for piecewise affine functions,  $\mathbf{u}_n^{n+1}$  is an array of values associated to the vertices of the triangulation  $T^n$  of  $\Omega^n$ . The next triangulation is obtained by performing

$$T^{n+1} = T^n + \delta t \mathbf{u}^n.$$

The equation defining  $\mathbf{u}^{n+1}$  from  $\mathbf{u}_n^{n+1}$  simply consists in *doing nothing* to the array of velocities<sup>1</sup>.

The method that has been presented is quite simple to implement, and is based on solutions of linear systems (the nonlinear term has been dropped). Yet, it is restricted to very few situations, because in practice the underlining mesh is likely to degenerate, up to creation of reversed triangles, that will stop the computation.

---

<sup>1</sup>With some softwares like Freefem++, a special attention to this step has to be paid. Indeed, when a field is defined on a certain mesh, if the mesh is moved (e.g. with the command `movemesh`), when the initial field is called, an interpolation between the two meshes is automatically performed. In a way, Freefem++ favors the *function* represented by a discrete field before the *array* of values that represented. It is necessary to use auxiliary arrays to bypass this automatic interpolation (see Section 3.4).

### 3.2 Arbitrary Lagrangian Eulerian (ALE) point of view

In what follows, we investigate the possibility to build mappings between the domains (similar to (3.5)), with a field which is not the fluid velocity. The approach is based on a definition of the domain velocity that is partially decoupled from the fluid velocity: it has to satisfy some consistency relations so that the computational domain will follow the actual fluid domain (globally Lagrangian behavior), but it will be defined within the domain in order to minimize the domain motion (Eulerian tendency: move the vertices as little as possible). The fact that the second step can be performed arbitrarily explains the name of the approach: Arbitrary Lagrangian Eulerian.

The definition of the domain velocity is based on a mapping

$$(\mathbf{u}, \Omega) \longmapsto \mathbf{c} = \mathcal{C}(\mathbf{u}, \Omega), \quad (3.8)$$

where  $\Omega$  is a smooth domain,  $\mathbf{u}$  and  $\mathbf{c}$  are regular fields defined in  $\Omega$  (domains and fields are supposed at least  $C^1$ ).

**Definition 6 (Consistent ALE mapping, continuous case)** *The ALE mapping*

$$\mathbf{u} \mapsto \mathbf{c} = \mathcal{C}(\mathbf{u}, \Omega)$$

is said to be consistent if, for any  $\Omega$ ,  $\mathbf{u}$ ,

$$\mathbf{u}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{c}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \quad , \quad \forall \mathbf{x} \in \Gamma, \quad (3.9)$$

where  $\mathbf{n}(\mathbf{x})$  is the normal vector to  $\Gamma = \partial\Omega$  at  $\mathbf{x}$ .

We consider a consistent ALE mapping  $\mathcal{C}$ , and the associated domain velocity field  $\mathbf{c}(\mathbf{x}, t)$ .

**Proposition 4** *The characteristics associated to the space-time field  $(\mathbf{c}(\mathbf{x}, t), 1)$  remain in the space-time domain*

$$S = \bigcup_{t \in [0, T]} \Omega(t) \times \{t\}.$$

In other words, the 3D field  $(\mathbf{c}(\mathbf{x}, t), 1)$  can be integrated in the physical space-time domain  $S$  corresponding to the time interval  $[0, T]$ .

*Proof*: Let us show how equation (3.9) ensures that this field is tangent to the ‘‘lateral’’ boundary of  $S$  (boundary of  $S$  excepting  $\Gamma(0) \times \{0\}$  and  $\Gamma(T) \times \{T\}$ ). Let  $\mathbf{t} = (t_x, t_y)$  be the vector tangent to  $\Gamma(t)$  at  $\mathbf{x}$ . The normal vector to the space-time domain  $S$  at  $(\mathbf{x}, t)$  can be expressed

$$\mathcal{N} = \begin{pmatrix} t_x \\ t_y \\ 0 \end{pmatrix} \times \begin{pmatrix} u_x \\ u_y \\ 1 \end{pmatrix} = \begin{pmatrix} t_y \\ -t_x \\ t_x u_y - t_y u_x \end{pmatrix}, \quad (3.10)$$

so that, on the lateral boundary,

$$\begin{pmatrix} \mathbf{c} \\ 1 \end{pmatrix} \cdot \mathcal{N} = \begin{pmatrix} c_x \\ c_y \\ 1 \end{pmatrix} \cdot \begin{pmatrix} t_y \\ -t_x \\ t_x u_y - t_y u_x \end{pmatrix} = c_x t_y - c_y t_x + t_x u_y - t_y u_x \quad (3.11)$$

$$= \mathbf{u} \cdot \mathbf{n} - \mathbf{c} \cdot \mathbf{n} \quad (3.12)$$

$$= 0 \quad (3.13)$$

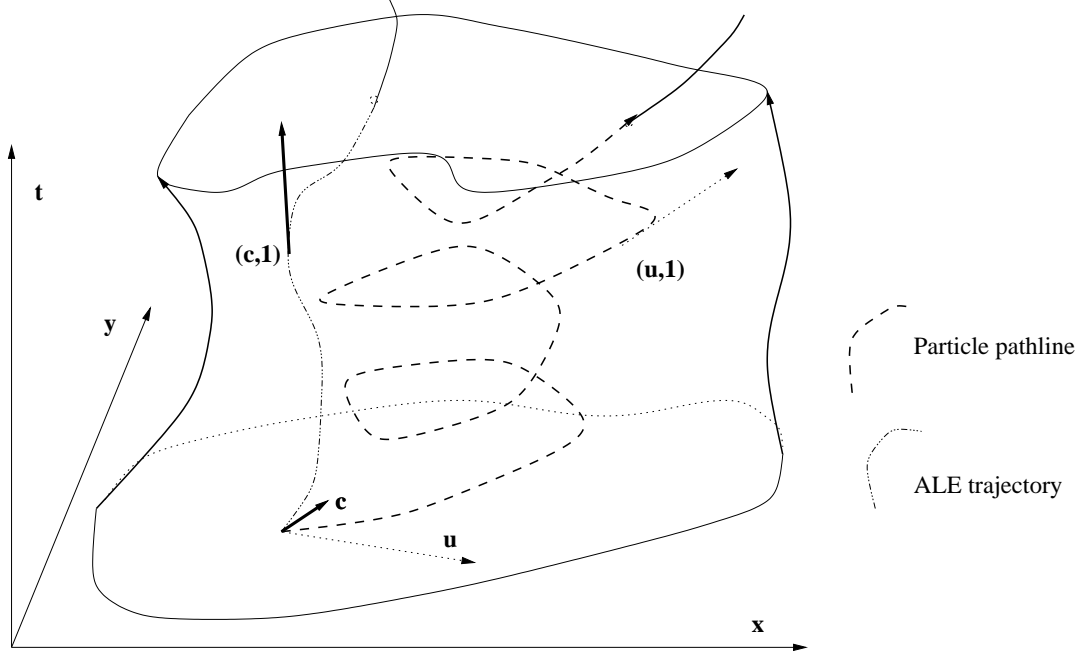


Figure 3.1: Space time domain

as soon as  $\mathbf{u}$  and  $\mathbf{c}$  verify the consistency relation (3.9). □

Proposition 4 makes it possible to define applications between the  $\Omega(t)$ :

$$\begin{aligned} \gamma(\cdot, t_1; t_2) &: \Omega(t_1) \longrightarrow \Omega(t_2) \\ \mathbf{x}_1 \in \Omega(t_1) &\longmapsto \mathbf{x}_2 = \gamma(\mathbf{x}_1, t_1; t_2) \end{aligned}$$

where  $(\gamma(\mathbf{x}_1, t_1; t), t)$  is the characteristic curve from  $(\mathbf{x}_1, t_1)$  to  $(\mathbf{x}_2, t_2)$  in  $S$ :

$$\begin{cases} \frac{d}{dt}(\gamma(\mathbf{x}_1, t_1; t), t) = [\mathbf{c}(\gamma(\mathbf{x}_1, t_1; t), t), 1] \\ \gamma(\mathbf{x}_1, t_1; t_1) = (\mathbf{x}_1, t_1) \end{cases}$$

For each  $\tau$ , the ALE velocity is then defined by

$$\mathbf{u}_\tau(\mathbf{x}, t) = \mathbf{u}(\gamma(\mathbf{x}, \tau; t), t) \quad (\mathbf{x} \in \Omega(\tau), \gamma(\mathbf{x}, \tau; t) \in \Omega(t)) \quad (3.14)$$

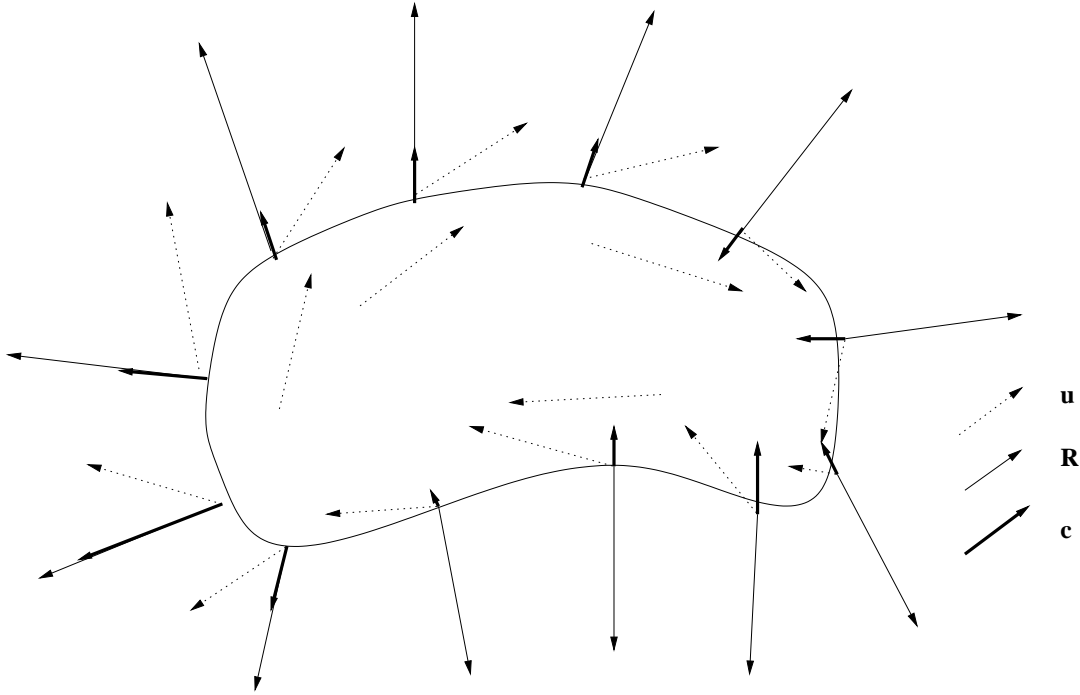
which is equivalent to

$$\mathbf{u}_\tau(\gamma(\mathbf{x}, t; \tau), t) = \mathbf{u}(\mathbf{x}, t) \quad (\mathbf{x} \in \Omega(t)). \quad (3.15)$$

Partial derivation with respect to  $t$  of (3.15) at  $t = \tau$  gives

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= \frac{\partial \mathbf{u}_\tau}{\partial t} + \nabla \mathbf{u}_\tau \cdot \frac{\partial \gamma}{\partial t} \Big|_{t=\tau} \\ &= \frac{\partial \mathbf{u}_\tau}{\partial t} - (\mathbf{c}_\tau \cdot \nabla) \mathbf{u}_\tau \end{aligned} \quad (3.16)$$

Note that identity (3.15) can be used to define the ALE version of any scalar or vector field  $\Phi(\mathbf{x}, t)$  defined in the space time domain (for  $\mathbf{x} \in \Omega(t)$ ).

Figure 3.2: Fields  $\mathbf{u}$ ,  $\mathbf{R}$ , and  $\mathbf{c}$ .

**Remark 4** *Evolution problems in a fixed domain lead to a cylindrical space–time domain (set product of the space domain by the time interval). When the domain moves, this space–time  $S$  is non longer cylindrical. The approach which has been carried out can be seen as a way to straighten  $S$  locally by introducing the ALE velocity: the partial time derivative of the new field follows the lateral boundary of  $S$ .*

### 3.2.1 Actual computation of the domain velocity

The proper way to define the velocity  $\mathbf{c}$  on the boundary is highly dependent on the type of flow which is considered. We shall restrict ourself here to a simple way which consists in prescribing the direction  $\mathbf{R}$  of the boundary velocity. For a given class of domains, let  $\mathbf{R}(\cdot, \Omega)$  be a given family of vector fields : for any  $\Omega$ ,  $\mathbf{R}$  is defined on  $\Gamma = \partial\Omega$ , and it is such that  $\mathbf{R} \cdot \mathbf{n} > 0$  on  $\Gamma$ . If we prescribe  $\mathbf{R}$  to be the direction of the boundary velocity, then the consistency relation (3.9) imposes the consistent ALE mapping

$$(\mathbf{u}, \Omega) \longmapsto \mathbf{c} = \mathcal{C}(\mathbf{u}, \Omega) \quad (3.17)$$

to be defined uniquely (on the boundary) as

$$\mathbf{c} = \left( \frac{\mathbf{u} \cdot \mathbf{n}}{\mathbf{R} \cdot \mathbf{n}} \right) \mathbf{R}. \quad (3.18)$$

See Fig. 2 for an example of field  $\mathbf{c}$ .

Definition of the domain velocity within the domain plays little role in the present approach. We chose to solve a Laplace equation for each component of the domain velocity  $\mathbf{c}$ , which can

be expressed in vector form

$$\Delta \mathbf{c} = 0 \quad (3.19)$$

with boundary conditions given by (3.18).

### 3.2.2 Navier–Stokes equations in the ALE context

We consider now a fluid motion governed by the incompressible Navier–Stokes equations

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{g} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \quad (3.20)$$

with conditions on the free boundary

$$\sigma \cdot \mathbf{n} = -p_e \mathbf{n} - \kappa \frac{\mathbf{n}}{R} \quad (3.21)$$

where  $R$  is the radius of curvature of  $\Gamma(t)$ ,  $\kappa$  is the surface tension coefficient (between the fluid and the external gas),  $p_e$  the external pressure (which we shall take equal to 0 in the following), and  $\sigma$  is the total stress tensor

$$\sigma = \nu(\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) - p \mathbf{1}. \quad (3.22)$$

ALE velocity  $\mathbf{u}_\tau(\mathbf{x}, t)$  and ALE pressure, which are defined on  $\Omega(\tau)$  and related to the Eulerian velocity by (3.14), verify the ALE Navier–Stokes equations at  $t = \tau$ , for every  $\tau \in ]0, T]$ :

$$\begin{cases} \rho \frac{\partial \mathbf{u}_\tau}{\partial t} + \rho((\mathbf{u}_\tau - \mathbf{c}_\tau) \cdot \nabla) \mathbf{u}_\tau - \mu \Delta \mathbf{u}_\tau + \nabla p_\tau = \mathbf{g} \\ \nabla \cdot \mathbf{u}_\tau = 0 \end{cases} \quad (3.23)$$

The ALE Navier–Stokes equations can be discretized in time. Whereas in the Lagrangian approach, the nonlinear term disappears, it does not in the present context, since the motion of the domain is partially decoupled from the actual motion of fluid particles. As a consequence, a modified nonlinear term remains in the equations, and it has to be handled in some way. The method of characteristics can be used to handle it. The difference with the full eulerian case is that the velocity must be advected by the velocity that is *relative* to the domain velocity, i.e.  $\mathbf{u} - \mathbf{c}$ . In the time discretized setting, the scheme consists in computing the domain velocity  $\mathbf{c}^n$ , and then solving

$$\rho \frac{\mathbf{u}_n^{n+1} - \mathbf{u}^n \circ X}{\delta t} - \mu \Delta \mathbf{u}_n^{n+1} + \nabla p_n^{n+1} = \mathbf{g},$$

in the current domain  $\Omega^n$ , with  $\nabla \cdot \mathbf{u}_n^{n+1} = 0$ , and  $X$  is the foot of the characteristic, which is obtained by upstream integration of  $\mathbf{u}^n - \mathbf{c}^n$  during  $\delta t$ . The next domain is then obtained as

$$\Omega^{n+1} = \Omega^n + \delta t \mathbf{c}^n.$$

The last step consists in transferring ALE quantities to the new domain but, as explained at the end of the previous section, this operation can be performed at zero cost.



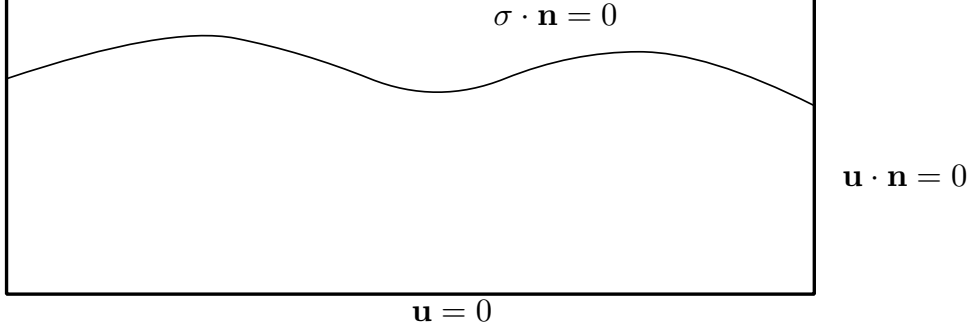


Figure 3.3: Fluid in a container

### 3.3 Some applications of the ALE approach

#### 3.3.1 Water waves

We consider the situation illustrated by Fig. 3.3. A viscous fluid is located in a container, with an open top (free surface), in contact with a gas considered as perfect. The fluid is subject to gravity forces. It is assumed to stick to the bottom boundary, and to slip with no friction on the lateral walls:

$$\left\{ \begin{array}{l} \rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = \mathbf{g} \quad \text{in } \Omega(t) \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega(t) \\ \boldsymbol{\sigma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_3(t) \\ \mathbf{u} \cdot \mathbf{n} = 0, \mathbf{t} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_2(t) \cup \Gamma_4(t) \\ \mathbf{u} = 0 \quad \text{on } \Gamma_1. \end{array} \right.$$

In this context, it is natural to define a vertical velocity of the domain. Since its normal component must equal that of the fluid velocity, the vertical component is determined:

$$\mathbf{c} = c \mathbf{e}_y, \quad \mathbf{c} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} \implies c = \frac{\mathbf{u} \cdot \mathbf{n}}{\mathbf{e}_y \cdot \mathbf{n}}. \quad (3.24)$$

A natural way to define the interior domain velocity is to interpolate between the upper boundary and 0. It may not be so straightforward in practice: for any mesh vertex inside the domain, one has to determine the velocity of the corresponding boundary point. This task is not trivial for a general mesh. An alternative way consists in solving a Laplace problem over the whole domain, for the vertical component of the domain velocity. In order to avoid diffusion in the horizontal direction, it is possible to solve use an anisotropic diffusion tensor, i.e. to solve

$$-\nabla \cdot D \nabla c = 0,$$

with  $c = 0$  on  $\Gamma_1$ ,  $\partial c / \partial n = 0$  on  $\Gamma_2 \cup \Gamma_4$ , and relation (3.24) on  $\Gamma_3$ .

#### 3.3.2 Falling water

The next situation corresponds to water falling out of a tap (see Fig. 3.4, left), and falling down a corner (see Fig. 3.4, right).

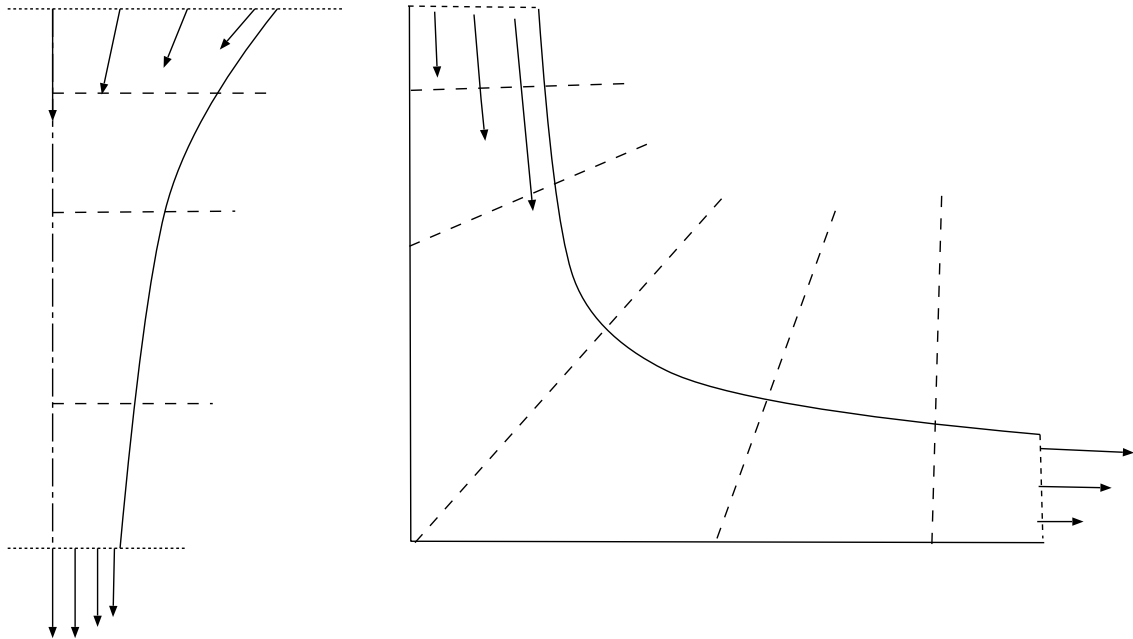


Figure 3.4: Falling water

In the first situation, the domain velocity can be taken horizontal. In the second case, the direction cannot be taken constant: it can be prescribed in the way indicated by the dotted lines in Fig. 3.4, right.

### 3.4 Implementation issues

As previously pointed out, Freefem considers a discrete field as a function more than an array of values. Consider a field like  $u_x$  (first component of the velocity) that has been computed as a piecewise affine function on a mesh  $\text{Th}$ , and another field  $p_{ux}$  that is defined in the space  $P_1$  associated to the same mesh. If the mesh is moved according to

```
Th = movemesh(Th,[x+dt*cx,y+dt*cy]);
```

and if one updates  $p_{ux}$  with the instruction  $p_{ux} = u_x$ , Freefem++ will interpolate the values between the two meshes. More precisely, to compute the nodal value of  $p_{ux}$  at a vertex  $i$  of the new mesh, the position of the vertex will be located in the previous configuration, and the corresponding value of the field will be computed by interpolation of  $u_x$  in the previous configuration. To keep the same nodal values for  $u_x$  and  $p_{ux}$ , the following trick can be used:

```
tmp=ux[]; pux=0; pux[]=tmp ;
```

where  $\text{tmp}$  has been defined previously as an auxiliary array :

```
real[int] tmp(ux[].n);
```

# Chapter 4

## Fluid-structure interactions

Many problems involve the interaction between a fluid and a solid. The solid can be either a deformable boundary for the fluid like for the blood flow in the arteries and veins, or pieces immersed in the fluid or at its surface like for instance a boat on a lake or the sand in concrete, particles in the air, etc.

The coupling between both elements, the fluid and the solid is not obvious since, at first sight the physics involved is different. Indeed, for a non deformable solid immersed in a fluid, one has to write Newton's laws for the solid where the external forces contains the ones given by the fluid. This is a discrete system, in which only a few degrees of freedom need to be computed, while the fluid is modeled through e.g. Navier-Stokes equations which describe a continuum.

### 4.1 A non deformable solid in a fluid

We focus for the time being on the case of a non deformable solid  $\Omega_S$  in the fluid. The fluid domain will be denoted by  $\Omega_F$  while the total domain (fluid+solid) is  $\Omega = \Omega_S \cup \Omega_F$ . Classical mechanics, given by Newton's law describes the motion of the solid by the system of differential equations

$$\begin{cases} m \frac{d\mathbf{V}}{dt} = \mathbf{F}, \\ \frac{d\mathbf{L}}{dt} = \mathbf{T}. \end{cases} \quad (4.1)$$

In the former equations,  $m$  stands for the mass of the solid,  $\mathbf{V}$  the velocity of its center of mass and  $\mathbf{L}$  its angular momentum. On the right hand sides,  $\mathbf{F}$  and  $\mathbf{T}$  are respectively the total force and torque induced by external forces, here due to the fluid and external forces density<sup>1</sup>  $\mathbf{f}_{ext}$ , so that assuming that the fluid obeys Navier-Stokes equations

$$\begin{cases} \mathbf{F} = - \int_{\partial\Omega_S} \boldsymbol{\sigma} \cdot \mathbf{n} d\sigma(\mathbf{x}) + \int_{\Omega_S} \mathbf{f}_{ext} d\mathbf{x}, \\ \mathbf{T} = - \int_{\partial\Omega_S} (\mathbf{x} - \mathbf{G}(t)) \times \boldsymbol{\sigma} \cdot \mathbf{n} d\sigma(\mathbf{x}) + \int_{\Omega_S} (\mathbf{x} - \mathbf{G}(t)) \times \mathbf{f}_{ext} d\mathbf{x}. \end{cases} \quad (4.2)$$

where  $\mathbf{G}(t)$  stands for the position of the center of mass of the solid  $\Omega_S$ , which depends on time, and the normal  $\mathbf{n}$  points from the fluid to the solid.

A second difficulty stands for the fact that at the boundary of the solid, the fluid must also fulfill the no-slip boundary condition which imposes that the velocity of the fluid equates a rigid

---

<sup>1</sup>Those external forces are commonly the gravity, but could be as well magnetic or electric forces if the body is magnetized or charged for instance.

motion, namely those of the solid. However this rigid motion is unknown and precisely obtained through the coupling.

Putting everything together, we therefore consider the system

$$\rho_F \frac{\partial \mathbf{u}}{\partial t} + \rho_F (\mathbf{u} \cdot \nabla) \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega_F, \quad (4.3)$$

$$\operatorname{div} \mathbf{u} = 0 \text{ in } \Omega_F, \quad (4.4)$$

$$\mathbf{u} = \dot{\mathbf{G}} + \omega \times (\mathbf{x} - \mathbf{G}) \text{ on } \partial\Omega_S, \quad (4.5)$$

$$m \ddot{\mathbf{G}} = - \int_{\partial\Omega_S} \boldsymbol{\sigma} \cdot \mathbf{n} \, d\sigma + \mathbf{F}_{ext}, \quad (4.6)$$

$$\frac{d}{dt}(J\omega) = - \int_{\partial\Omega_S} (\mathbf{x} - \mathbf{G}) \times \boldsymbol{\sigma} \cdot \mathbf{n} \, d\sigma + \mathbf{T}_{ext}, \quad (4.7)$$

where we have denoted by  $\rho_F$  the density of the fluid, and by  $\omega$  the rotational vector of the solid, and

$$\mathbf{F}_{ext} = \int_{\Omega_S} \mathbf{f}_{ext} \, d\mathbf{x}, \quad \text{and} \quad \mathbf{T}_{ext} = \int_{\Omega_S} (\mathbf{x} - \mathbf{G}) \times \mathbf{f}_{ext} \, d\mathbf{x}$$

respectively stand for the total force and torque on the solid due to external forces.

## 4.2 Rigid body motion

The coupling of the PDE and the ODE in the preceding problem needs very specific numerical development. Indeed, the mathematical features of the equations are different, and the numerical methods for the PDE and the ODE as well. The idea we aim at explaining hereafter is to consider the solid as a *very viscous* fluid. Therefore, we extend the velocity field *inside* the body by setting

$$\mathbf{u}(\mathbf{x}) = \dot{\mathbf{G}} + \omega \times (\mathbf{x} - \mathbf{G}) \text{ for all } \mathbf{x} \text{ in } \Omega_S. \quad (4.8)$$

Notice that the continuity of the velocity at the boundary  $\partial\Omega_S$  is naturally enforced. Moreover, since  $\mathbf{u}$  is a rigid body motion in the solid, one has

$$\nabla \mathbf{u} + {}^t \nabla \mathbf{u} = 0 \text{ in } \Omega_S.$$

The total momentum of the solid is then given by

$$m \mathbf{V} = \int_{\Omega_S} \rho_S \mathbf{u} \, d\mathbf{x},$$

where  $\rho_S$  is the density of the solid<sup>2</sup>, and  $\mathbf{V} = \dot{\mathbf{G}}$  is the velocity of the center of mass  $\mathbf{G}$  of the solid. Similarly, the angular momentum of the solid is given by

$$\mathbf{L} = \int_{\Omega_S} \rho_S (\mathbf{x} - \mathbf{G}) \times \mathbf{u} \, d\mathbf{x}.$$

Doing so, we can rewrite the balances of momentum and angular momentum respectively as

$$\frac{d}{dt} \int_{\Omega_S} \rho_S \mathbf{u} \, d\mathbf{x} = - \int_{\partial\Omega_S} \boldsymbol{\sigma} \cdot \mathbf{n} \, d\sigma + \mathbf{F}_{ext} \quad (4.9)$$

$$\frac{d}{dt} \mathbf{L} = \frac{d}{dt} \int_{\Omega_S} \rho_S (\mathbf{x} - \mathbf{G}) \times \mathbf{u} \, d\mathbf{x} = - \int_{\partial\Omega_S} (\mathbf{x} - \mathbf{G}) \times \boldsymbol{\sigma} \cdot \mathbf{n} \, d\sigma + \mathbf{T}_{ext}. \quad (4.10)$$

---

<sup>2</sup>One has typically  $\rho_S = \frac{m}{|\Omega_S|}$  if the solid is homogeneous, but this is not mandatory.

On the left hand side, both expression can be written as

$$\frac{d}{dt} \int_{\Omega_S} \rho_S \mathbf{u} \, d\mathbf{x} = \int_{\Omega_S} \rho_S \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \, d\mathbf{x} \quad (4.11)$$

$$\frac{d}{dt} \int_{\Omega_S} \rho_S (\mathbf{x} - \mathbf{G}) \times \mathbf{u} \, d\mathbf{x} = \int_{\Omega_S} \rho_S (\mathbf{x} - \mathbf{G}) \times \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \, d\mathbf{x}, \quad (4.12)$$

and we recognize terms that are present in Navier-Stokes equations, though with a different density ( $\rho_S$  instead of  $\rho_F$ ). In view of this, we define the density in all  $\Omega$  by setting

$$\rho(\mathbf{x}) = \begin{cases} \rho_F & \text{if } \mathbf{x} \in \Omega_F, \\ \rho_S & \text{if } \mathbf{x} \in \Omega_S, \end{cases}$$

we also extend the pressure inside the body by setting  $p = 0$  in  $\Omega_S$  and the force  $\mathbf{f}$  that we take to be equal to  $\mathbf{f}_{ext}$  inside  $\Omega_S$ . We now consider a test function  $\phi$  (smooth or at least in  $H^1(\Omega, \mathbb{R}^3)$ ) defined on all  $\Omega$  such that  $\phi$  is a rigid body motion in  $\Omega_S$

$$\phi(\mathbf{x}) = \bar{\phi} + \phi_\omega \times (\mathbf{x} - \mathbf{G}) \text{ in } \Omega_S,$$

where both  $\bar{\phi}$  and  $\phi_\omega$  are constant vectors in  $\mathbb{R}^3$ .

We now compute

$$\int_{\Omega} \left( \rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \cdot \phi \, d\mathbf{x} = \int_{\Omega_F} \left( \rho_F \frac{\partial \mathbf{u}}{\partial t} + \rho_F (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \cdot \phi \, d\mathbf{x} + \bar{\phi} \cdot \frac{d}{dt} (m\mathbf{V}) + \phi_\omega \cdot \frac{d\mathbf{L}}{dt}$$

because of (4.11) and (4.12). On the other hand

$$\mu \int_{\Omega} (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) : \nabla \phi \, d\mathbf{x} = \mu \int_{\Omega_F} (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) : \nabla \phi \, d\mathbf{x}.$$

Indeed,  $\phi$  and  $\mathbf{u}$  being rigid body motions in  $\Omega_S$  we have

$$\nabla \mathbf{u} + {}^t \nabla \mathbf{u} = \nabla \phi + {}^t \nabla \phi = 0 \text{ in } \Omega_S.$$

Similarly

$$- \int_{\Omega} p \operatorname{div} \phi \, d\mathbf{x} = - \int_{\Omega_F} p \operatorname{div} \phi \, d\mathbf{x}$$

since  $\operatorname{div} \phi = 0$  in  $\Omega_S$ , and eventually

$$\begin{aligned} - \int_{\Omega} \mathbf{f} \cdot \phi \, d\mathbf{x} &= - \int_{\Omega_F} \mathbf{f} \cdot \phi \, d\mathbf{x} - \int_{\Omega_S} \mathbf{f}_{ext} \cdot \phi \, d\mathbf{x} \\ &= - \int_{\Omega_F} \mathbf{f} \cdot \phi \, d\mathbf{x} - \mathbf{F}_{ext} \cdot \bar{\phi} - \mathbf{T}_{ext} \cdot \phi_\omega. \end{aligned}$$

Summing up all contributions, and noticing that

$$\begin{aligned} \mu \int_{\Omega_F} (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) : \nabla \phi \, d\mathbf{x} - \int_{\Omega_F} p \operatorname{div} \phi \, d\mathbf{x} &= \int_{\Omega_F} (-\mu \Delta \mathbf{u} + \nabla p) \cdot \phi \, d\mathbf{x} \\ &\quad + \bar{\phi} \cdot \int_{\partial \Omega_F} \boldsymbol{\sigma} \cdot \mathbf{n} \, d\sigma + \phi_\omega \cdot \int_{\partial \Omega_F} (\mathbf{x} - \mathbf{G}) \times \boldsymbol{\sigma} \cdot \mathbf{n} \, d\sigma, \end{aligned}$$

we obtain

$$\begin{aligned}
& \int_{\Omega} \left( \rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \cdot \phi \, d\mathbf{x} + \mu \int_{\Omega} (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) : \nabla \phi \, d\mathbf{x} - \int_{\Omega} p \operatorname{div} \phi \, d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \phi \, d\mathbf{x} \\
&= \int_{\Omega_F} \left( \rho_F \frac{\partial \mathbf{u}}{\partial t} + \rho_F (\mathbf{u} \cdot \nabla) \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p - \mathbf{f} \right) \cdot \phi \, d\mathbf{x} + \bar{\phi} \cdot \left( \frac{d}{dt} (m \mathbf{V}) - \int_{\partial \Omega_F} \boldsymbol{\sigma} \cdot \mathbf{n} \, d\sigma - \mathbf{F}_{ext} \right) \\
&\quad + \phi_{\omega} \cdot \left( \frac{d \mathbf{L}}{dt} - \int_{\partial \Omega_F} (\mathbf{x} - \mathbf{G}) \times \boldsymbol{\sigma} \cdot \mathbf{n} \, d\sigma - \mathbf{T}_{ext} \right) \\
&= 0.
\end{aligned}$$

Therefore the variational formulation, which encodes the system (4.3–4.7) in a whole is given by testing with functions that are rigid motions inside the solid. The situation is very symmetric since at a time  $t$  the solution  $\mathbf{u}$  also belongs to the space of velocity fields that are rigid body motions inside the solid. The time discretization can then be obtained using one of the strategies that were described in chapter 2.

### 4.3 Enforcing the constraint

Nevertheless, the question of implementation of the constraint of being a rigid motion inside the solid remains entire. What is the easiest way to take the constraint into account? Should we use special finite element spaces? How to handle such a constraint inside a `FREEFEM++` code?

A possible way that we explain below consists in *penalizing* the rigid body motion constraint. Namely, the idea consists in changing the variational formulation in order that the constraint will be naturally approximately satisfied. In order to do so we modify the viscous term in the variational formulation as follows:

$$\int_{\Omega} \mu (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) : \nabla \phi \, d\mathbf{x} \longrightarrow \int_{\Omega} \mu_{\varepsilon} (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) : \nabla \phi \, d\mathbf{x}$$

where the viscosity has been extended to all  $\Omega$  by

$$\mu_{\varepsilon}(\mathbf{x}) = \begin{cases} \mu & \text{if } \mathbf{x} \in \Omega_F, \\ \frac{1}{\varepsilon} & \text{otherwise.} \end{cases} \quad (4.13)$$

In the preceding definition,  $\varepsilon > 0$  is meant to be a small parameter.

hence, a typical discrete variational formulation at time  $n\delta t$  for the whole problem will be given by

$$\begin{cases} \text{Find } (\mathbf{u}^{n+1}, p^{n+1}) \text{ such that for all } (\mathbf{v}, q) \\ \int_{\Omega} \rho \left( \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} + (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n \right) \cdot \phi \, d\mathbf{x} + \int_{\Omega} \mu_{\varepsilon} (\nabla \mathbf{u}^{n+1} + {}^t \nabla \mathbf{u}^{n+1}) : \nabla \phi \, d\mathbf{x} \\ \quad - \int_{\Omega} p^{n+1} \operatorname{div} \phi \, d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \phi \, d\mathbf{x} = 0, \\ \int_{\Omega} q \operatorname{div} \mathbf{u}^{n+1} = 0, \end{cases}$$

Formally, this variational formulation discretizes the continuous variational formulation

$$\begin{cases} \int_{\Omega} \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \cdot \phi \, d\mathbf{x} + \int_{\Omega} \mu_{\varepsilon} (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) : \nabla \phi \, d\mathbf{x} - \int_{\Omega} p \operatorname{div} \phi \, d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \phi \, d\mathbf{x} = 0, \\ \int_{\Omega} q \operatorname{div} \mathbf{u} = 0, \end{cases}$$

for which one has the energy inequality (test formally with  $\phi = \mathbf{u}$ )

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |\mathbf{u}|^2 d\mathbf{x} + \int_{\Omega} \mu_{\varepsilon} |\nabla \mathbf{u} + {}^t\nabla \mathbf{u}|^2 d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}, \quad (4.14)$$

from which we deduce a uniform  $L^2$  bound for  $\mathbf{u}$  and the estimation

$$\exists C > 0, \text{ such that } \forall t > 0, \int_0^t \int_{\Omega} \mu_{\varepsilon} |\nabla \mathbf{u} + {}^t\nabla \mathbf{u}|^2 d\mathbf{x} \leq C. \quad (4.15)$$

This leads to

$$\int_0^t \int_{\Omega_S} |\nabla \mathbf{u} + {}^t\nabla \mathbf{u}|^2 d\mathbf{x} \leq C\varepsilon. \quad (4.16)$$

which means that  $\nabla \mathbf{u} + {}^t\nabla \mathbf{u}$  tends to 0 as  $\varepsilon$  tends to 0 *inside the solid*. Therefore, in the solid,  $\mathbf{u}$  is approximately a rigid motion.

Physically speaking, the method above amounts to saying that the solid is a fluid with very high viscosity (actually  $1/\varepsilon$ ). As we know, due to the incompressibility constraint and the conservation of mass, both fluids never mix, and it is expected that the fluid with very high viscosity only deforms very little. At the limit  $\varepsilon \rightarrow 0$  it seems reasonable to think that the solution converges towards the solution of the previous variational formulation<sup>3</sup>

## 4.4 Yet other non deformable constraints

The method described above can be easily generalized to other type of constraints. For instance, if the solid can not rotate but only translate, we can use the previous approach, in the continuous setting by restricting both the unknown and the test functions to be constant in the solid  $\Omega_S$ , in the penalized setting by replacing the term

$$\frac{1}{\varepsilon} \int_{\Omega_S} (\nabla \mathbf{u} + {}^t\nabla \mathbf{u}) : \nabla \phi d\mathbf{x},$$

by

$$\frac{1}{\varepsilon} \int_{\Omega_S} \nabla \mathbf{u} : \nabla \phi d\mathbf{x}.$$

Similarly to the former case, the solution  $\mathbf{u}$  satisfies an estimate

$$\int_0^t \int_{\Omega_S} |\nabla \mathbf{u}|^2 d\mathbf{x} \leq C\varepsilon \quad (4.17)$$

which ensures that  $\mathbf{u}$  is close to a constant inside the solid (though this constant may depend on time).

Eventually, a fixed rigid body can be modeled by  $\mathbf{u} = 0$  inside the solid. This can be also enforced by penalty by replacing again the same term by

$$\frac{1}{\varepsilon} \int_{\Omega_S} \mathbf{u} \cdot \phi d\mathbf{x}.$$

Proofs of convergence as  $\varepsilon$  tends to 0 of the corresponding formulation to the constrained Navier-Stokes problem are possible although not obvious. This field is still active at the research level.

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<sup>3</sup>This is true although not obvious to prove. However, proving the very same statement but for Stokes equations is much easier and left as an interesting exercise to the curious reader.

## 4.5 Conclusion

The aim of this chapter was to design slight changes that can be made in the classical variational formulation of Navier-Stokes equations in order to handle different material. We have focused our attention to the case of a non deformable solid in a viscous (incompressible) fluid, and we have seen that it can be seen as a mixture of two incompressible fluids: the original one, and a very viscous one which becomes solid when its viscosity tends to infinity.



## Chapter 5

# Duality methods

This chapter describes an general approach to deal with constraints in fluid mechanics. Some typical situations have already been presented, e.g.

1. Incompressibility: this condition can be seen as a constraint on the fluid velocity<sup>1</sup>.
2. Obstacle: a fixed body in a fluid domain can be considered as a subdomain where the velocity is zero.
3. Rigid body: when a rigid object moves in a fluid, it can be considered at each instant as a subdomain where the velocity field is that of a rigid motion (translation + rotation), by prescribing that the strain tensor identically vanishes.
4. Constrained rigid body: consider the typical case of an helix in a fluid flow. It can be modeled as a rigid body attached at one of its point. The attachment can be modeled by prescribing that the mean velocity in a ball contained in the rigid part, and centered at the fixed point.
5. No slip conditions: this type of boundary condition has been considered in an *essential* manner, by integrating them to the space where the unknown velocity is defined. But it can be also considered as a constraint<sup>2</sup>.
6. Global constraints. In some situations, it can be of great interest to prescribe the value of some average quantity. For example, on the outlet of a pipe flow, it may be more relevant to prescribe the global flow rather than the whole velocity profile.

Some of these problems can be addressed by the penalty approach presented in the previous chapter. Yet, this approach, presents some drawbacks. Firstly, it modifies the underlying operator (like the Laplace operator) by addition of a term in  $1/\varepsilon$ . Discretization of the corresponding problem will lead to matrices that are ill-conditioned<sup>3</sup>, therefore the systems will be harder to solve. Secondly, in some cases, the penalty approach significantly affects the structure of the matrix, and necessitates a deep change in the assembling procedure. We present here a general approach based on a dual expression of the constraints, and we shall see that it circumvents those problems.

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<sup>1</sup>The pressure has been introduced in the first chapter according to physical considerations. We shall see in this chapter that it can be formalized from a constrained minimization point of view. In particular the pressure will be interpreted as a Lagrange multiplier of the incompressibility constraint.

<sup>2</sup>This is actually the way it is handled in Freefem++.

<sup>3</sup>It means that the condition number, that is the ratio between extremal eigenvalues, is large.

## 5.1 General presentation of the approach

Consider a smooth functional  $J$  in a Hilbert space  $V$ . We want to find a minimizer of this functional on a subspace  $K$ , defined as the kernel of a continuous linear operator  $B \in \mathcal{L}(V, \Lambda)$ , where  $\Lambda$  is also a Hilbert space.

Suppose that such a minimizer  $u$  exists. It holds that

$$J(u + tv) \geq J(u) \quad \forall v \in K, \forall t \in \mathbb{R},$$

from which we deduce that

$$(\nabla J(u), v) = 0 \quad \forall v \in K,$$

i.e.  $\nabla J(u) \in (\ker B)^\perp$ . If the dimension of  $\Lambda$  is finite, we have  $(\ker B)^\perp = \text{im } B^*$ , so that there exists  $\lambda \in \Lambda$  such that

$$\begin{cases} \nabla J(u) + B^* \lambda = 0 \\ Bu = 0, \end{cases}$$

In the case where  $J$  is a quadratic functional,

$$J(v) = \frac{1}{2}(v, v) - \langle \varphi, v \rangle,$$

it takes the form

$$\begin{cases} Au + B^* \lambda = \varphi \\ Bu = 0, \end{cases}$$

It can also be written in a variational way:

$$\begin{cases} a(u, v) + (Bv, \lambda) = \langle \varphi, v \rangle & \forall v \in V \\ (Bu, \mu) = 0 & \forall \mu \in \Lambda. \end{cases} \quad (5.1)$$

The latter system is called a *saddle point formulation* of the minimization problem (see Proposition 10, page 68, for an explanation of this term).

In general ( $\Lambda$  is infinite dimensional),  $(\ker B)^\perp = \text{im } B^*$  no longer holds, but we have

$$(\ker B)^\perp = \overline{\text{im } B^*}.$$

In the case where  $\text{im } B^*$  is closed (which is equivalent to saying that  $\text{im } B$  is closed), the saddle-point formulation is still valid, i.e. existence of a saddle point  $(u, \lambda)$  is guaranteed. On the other hand, existence of a saddle-point  $(u, \lambda)$  implies that  $u$  minimizes the quadratic functional  $J$  over  $K$  (see Proposition 10, page 68). Note also that if  $B$  is onto,  $B^*$  is one-to-one, thus the Lagrange multiplier is unique.

When the image of  $B$  is not closed, the existence of a solution to (5.1) might be ruled out. Yet, in the context of PDE's, the corresponding formulation can be discretized in space, which leads to a finite dimensional problem for which existence of a saddle point is guaranteed. The question of the behavior of the discrete Lagrange multiplier when the mesh size goes to 0 is delicate, since the limit does not exist in  $\Lambda$ , but it can be rigorously proven that the approach is efficient in terms of approximation of the primal part  $u$ , i.e. the minimizer of  $J$  over  $K$ .

### Interpretation of the Lagrange multipliers in a mechanical context.

Consider a horizontal system of  $n + 1$  masses  $0, 1, 2, \dots, n$ , linked to each other (0 to 1, 1 to 2, etc...) by springs with stiffness  $k$  and length at rest 0. The positions are represented by the vector  $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ . The potential energy writes

$$J(\mathbf{x}) = \frac{1}{2}k \sum_{i=1}^n |x_i - x_{i-1}|^2 = \frac{1}{2}k(A\mathbf{x}, \mathbf{x}),$$

where  $A$  is, up to a multiplicative constant, the discrete Laplacian with Neuman boundary conditions. Any configuration  $(x, x, \dots, x)$  de  $\mathbb{R}^{n+1}$  obviously minimizes this energy. Let us now consider the situation where 0 is fixed at  $x_0 = 0$ , and the  $n$ -th mass to the point  $x_n = L > 0$ . We now want to minimize  $J$  over the affine space

$$E = \{\mathbf{x}, x_0 = 0, x_n = L\} = X + \ker B, \quad \text{avec } B : \mathbf{x} \in \mathbb{R}^{n+1} \mapsto (x_0, x_n) \in \mathbb{R}^2.$$

The matrix  $B$  writes

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

There exists  $\lambda = (\lambda_0, \lambda_1) \in \mathbb{R}^2$  such that

$$Ax + B\lambda = 0.$$

The two end lines of this system are

$$\begin{aligned} k(x_0 - x_1) + \lambda_0 &= 0 \\ k(-x_{n-1} + x_n) + \lambda_1 &= 0. \end{aligned}$$

These relations express the balance of end masses, and allows to interpret  $-\lambda_0$  (resp.  $-\lambda_1$ ) as the force exert by the wall at 0 upon the mass 0 (resp. by the wall at 1 upon the mass  $n$ ).

**Uzawa algorithm.** We shall end this introductory section by describing a practical way to solve the saddle-point problem. The Uzawa algorithm is an iterative procedure to solve Problem (5.1). Given a parameter  $\rho > 0$ , and an initial guess  $\lambda^0$ , it consists in computing  $\lambda^1, \lambda^2, \dots$  according to the following procedure

1. Assuming that  $\lambda^n$  is known, compute  $u^{n+1}$  as the solution to

$$a(u^{n+1}, v) + (Bv, \lambda^n) = \langle \varphi, v \rangle \forall v \in V.$$

2. Update the Lagrange multiplier

$$\lambda^{n+1} = \lambda^n + \rho B u^{n+1}.$$

This algorithm can be shown to converge in the following sense: if the saddle-point problem admits a solution  $(u, \lambda)$  (not necessarily unique), and if  $\rho > 0$  is sufficiently small ( $\rho < 2\alpha / \|B\|$ , where  $\alpha$  is the ellipticity constraint of  $A$ ), then the sequence  $u^n$  converges to the solution  $u$  to the constrained minimization problem.

**Non-homogeneous constraint.** In the car where the constraint reads  $Bu = z \neq 0$ , a similar approach can be carried out, and the corresponding saddle point problem reads simply

$$\begin{cases} Au + B^* \lambda = \varphi \\ Bu = z, \end{cases}$$

The first step of the Uzawa algorithm is unchanged, and the second one writes

$$\lambda^{n+1} = \lambda^n + \rho(Bu^{n+1} - z).$$

**Small number of constraints.** When the number of scalar constraints is small, the Lagrange multiplier can be directly computed. For instance, in the case of a unique linear constraint, denoting by  $u_0$  (resp.  $u_1$ ) the solution to the primal problem associated to  $\lambda = 0$  (resp.  $\lambda = 1$ ), the exact solution can be written

$$u = (1 - \theta)u_0 + \theta u_1.$$

The value of  $\theta$  is obtained by prescribing that the constrained is met:

$$\theta = \frac{Bu_0}{Bu_0 - Bu_1}.$$

## 5.2 The divergence-free constraint for velocity fields

This constraint plays a special role in the context of contained minimization. Consider the Stokes problem:

$$-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f},$$

with  $\nabla \cdot \mathbf{u} = 0$ , and homogeneous Dirichlet boundary conditions. The simplest way to handle it consists in taking divergence free test-functions, to obtain

$$\mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

Thus, the problem which consists in finding a function in  $H_0^1$ , that is divergence free, directly fits in Lax-Milgram framework. Yet, this approach presents two main drawbacks:

1. The pressure, that has a meaning in terms of modeling, has disappeared from the formulation.
2. Numerical implementation of this approach necessitates to account for the divergence free constraint in the discretization space for the velocity. It can be done in practice, but it is very delicate in terms of implementation, and it is rarely done in practice.

From the mathematical standpoint, the pressure can be recovered by considering that the problem consists in minimizing the functional

$$J(\mathbf{v}) = \frac{\mu}{2} \int_{\Omega} |\mathbf{v}|^2 - \int_{\Omega} \mathbf{f} \cdot \mathbf{v},$$

over

$$K = \ker B, \quad B : \mathbf{v} \in H_0^1(\Omega)^d \mapsto -\nabla \cdot \mathbf{v} \in L^2(\Omega).$$

The fact that the constrained is distributed over the whole fluid domain requires to pay a special attention to the well-posedness of the saddle-point formulation. At the continuous level, existence of a pressure as a Lagrange multiplier of incompressibility constraint is a consequence of the surjective character of the divergence operator from  $H^1(\Omega)^d$  to  $L^2(\Omega)$ . As the discrete level, good approximation properties required special conditions between the two discretization spaces (in velocity and pressure). This so called *inf-sup* (or *Babuska-Brezzi*) condition expresses in some way that the discrete divergence operator asymptotically behaves like a surjection between two infinite dimensional Hilbert spaces.

## 5.3 Other types of constraints

### 5.3.1 Obstacle problem

We consider here the flow around a fixed cylinder. We denote by  $\omega$  the subdomain occupied by the cylinder. Consider the Stokes problem in a domain  $\Omega \setminus \bar{\omega}$  around the cylinder, and homogeneous Dirichlet boundary condition on the external boundary  $\partial\Omega$ .

The (double) saddle-point formulation can be written

$$\begin{aligned} \mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v} + \int_{\omega} \lambda \cdot \mathbf{v} &= \int \mathbf{f} \cdot \mathbf{v} \quad \forall \mathbf{v} \quad \forall \mathbf{v} \in H_0^1(\Omega), \\ \int_{\Omega} q \nabla \cdot \mathbf{u} &= 0 \quad \forall q \in L^2(\Omega), \\ \int_{\omega} \mu \cdot \mathbf{u} &= 0 \quad \forall \mu \in L^2(\omega). \end{aligned}$$

If one considers that the Stokes problem can be directly solved (i.e. without the use of an iterative algorithm), the Uzawa algorithm on  $\lambda$  writes:

1. Choose  $\lambda^0 \in L^2(\omega)$ .
2. Compute  $\mathbf{u}^{n+1}$  (together with  $p^{n+1}$ ) as the solution to

$$\begin{cases} \mu \int_{\Omega} \nabla \mathbf{u}^{n+1} : \nabla \mathbf{v} - \int_{\Omega} p^{n+1} \nabla \cdot \mathbf{v} + \int_{\omega} \lambda \cdot \mathbf{v} = \langle \varphi, v \rangle & \forall v \in H_0^1(\Omega) \\ - \int_{\Omega} q \nabla \cdot \mathbf{u}^{n+1} = 0 & \forall q \in L^2(\Omega). \end{cases}$$

3. Compute  $\lambda^{n+1}$  as

$$\lambda^{n+1} = \lambda^n + \rho \mathbf{u}^{n+1}|_{\omega}.$$

Note that the continuous saddle-point formulation is not well-posed, since  $B$  maps a field in  $H^1$  to its restriction in  $\omega$  as a function of  $L^2$ . Yet, the approach can be shown to work in practice, and it can even be proved that the discretized velocity field  $\mathbf{u}_h$  converges to the exact solution.

Another approach would consist here in defining  $B$  as the restriction operator to the space  $H^1(\omega)^d$ . In that case, the saddle-point problem writes

$$\begin{aligned} \mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v} + \int_{\omega} \lambda \cdot \mathbf{v} + \int_{\omega} \nabla \lambda \cdot \nabla \mathbf{v} &= \int \mathbf{f} \cdot \mathbf{v} \quad \forall \mathbf{v} \quad \forall \mathbf{v} \in H_0^1(\Omega), \\ \int_{\Omega} q \nabla \cdot \mathbf{u} &= 0 \quad \forall q \in L^2(\Omega), \\ \int_{\omega} \mu \cdot \mathbf{u} + \int_{\omega} \nabla \mu \cdot \nabla \mathbf{u} &= 0 \quad \forall \mu \in H^1(\omega), \end{aligned}$$

and the Uzawa algorithm can be written accordingly.

### 5.3.2 Prescribed flux

Consider the flow in a pipe-like domain, with two end boundaries  $\Gamma_{in}$  and  $\Gamma_{out}$ . We aim at prescribing the flux through  $\Gamma_{out}$ , without prescribing the whole velocity profile. For the Stokes problem, it consists in minimizing the functional

$$J(\mathbf{v}) = \frac{\mu}{2} \int_{\Omega} |\nabla \mathbf{v}|^2$$

among all those fields  $\mathbf{v}$  that are divergence free, and that verify

$$\int_{\Gamma_{out}} \mathbf{v} \cdot \mathbf{n} = Q,$$

where  $Q$  is prescribed.

The (double) saddle-point formulation can be written

$$\begin{aligned} \mu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v} + \int_{\Gamma_{out}} \lambda \mathbf{v} \cdot \mathbf{n} &= \int \mathbf{f} \cdot \mathbf{v} \quad \forall \mathbf{v} \in H_0^1(\Omega), \\ \int_{\Omega} q \nabla \cdot \mathbf{u} &= 0 \quad \forall q \in L^2(\Omega), \\ \int_{\Gamma_{out}} \mu \mathbf{u} \cdot \mathbf{n} &= Q \mu \quad \forall \mu \in L^2(\Gamma_{out}). \end{aligned}$$

Note that, in this situation, it is not necessary to run the Uzawa algorithm. Indeed, denoting by  $\mathbf{u}_0$  (resp.  $\mathbf{u}_1$ ) the velocity field corresponding to  $\lambda = 0$  (resp.  $\lambda = 1$ ), the exact solution  $\mathbf{u}$  can be written

$$\mathbf{u} = (1 - \lambda)\mathbf{u}_0 + \lambda\mathbf{u}_1.$$

The value of  $\lambda$  is then obtained by requiring that the constraint is verified, i.e.

$$\int_{\Gamma_{out}} \mu \mathbf{u} \cdot \mathbf{n} = Q \implies \lambda = \frac{Q - \int_{\Gamma_{out}} \mathbf{u}_0 \cdot \mathbf{n}}{\int_{\Gamma_{out}} \mathbf{u}_1 \cdot \mathbf{n} - \int_{\Gamma_{out}} \mathbf{u}_0 \cdot \mathbf{n}}.$$

### 5.3.3 Prescribed mean velocity

Consider a Stokes (or Navier-Stokes) flow in a domain  $\Omega$ , coupled with the motion of a rigid object, say, a disc (subdomain  $\omega \subset \Omega$ ). Consider the case where the disc is attached at its center, and is free to rotate around it. Prescribing a zero velocity at the center is not sound in the present context, since pointwise values are not<sup>4</sup> defined in  $H^1$ . It is more appropriate to prescribe a zero mean velocity over the disc, i.e.

$$\int_{\omega} \mathbf{u} = 0.$$

This could be done by penalizing this quantity. It would lead to an extra term in the Stokes (or Navier-Stokes) variational formulation of the type

$$\frac{1}{\varepsilon} \left( \int_{\omega} \mathbf{u} \right) \left( \int_{\omega} \mathbf{v} \right).$$

---

<sup>4</sup>In the present case, where the point lies within a zone that is considered rigid, it would actually make sense. But it would involve linear functionals that are not defined over the unconstrained space, which makes it tricky to formalize properly.

Note that this term will couple all the degrees of freedom associated to vertices within the particle. Besides, it cannot be straightforwardly implemented in a software like `freefem++`. This motivates the use of a duality method to account for this constraint. In order to illustrate the possibility to couple different approaches, we will handle the rigid body constraint by following a penalty approach. In case of homogeneous Dirichlet boundary conditions on the outer boundary, we obtain the variational formulation (for the Stokes problem)

$$\begin{aligned} & \frac{1}{2}\mu \int_{\Omega} (\nabla \mathbf{u} + {}^t\nabla \mathbf{u}) : (\nabla \mathbf{v} + {}^t\nabla \mathbf{v}) - \int_{\Omega} p \nabla \cdot \mathbf{v} + \\ & \frac{1}{\varepsilon} \int_{\omega} (\nabla \mathbf{u} + {}^t\nabla \mathbf{u}) : (\nabla \mathbf{v} + {}^t\nabla \mathbf{v}) + \lambda \cdot \int_{\omega} \mathbf{v} = \int \mathbf{f} \cdot \mathbf{v} \quad \forall \mathbf{v} \in H_0^1(\Omega)^2, \\ & \int_{\Omega} q \nabla \cdot \mathbf{u} = 0 \quad \forall q \in L^2(\Omega), \\ & \mu \cdot \int_{\omega} \mathbf{u} = 0 \quad \forall \mu \in \mathbb{R}^2. \end{aligned}$$

Note that, as in the previous case, the space of Lagrange multiplier has low dimension (2 in the present case), so that a direct strategy can be followed: Denote by

$$\lambda_{00} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \lambda_{10} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_{01} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and by  $\mathbf{u}_{00}$ ,  $\mathbf{u}_{10}$ ,  $\mathbf{u}_{01}$  the corresponding velocity fields (solutions of the Stokes problem with the correspond choice of  $\lambda$ ). By linearity of the Stokes problem, the exact velocity writes

$$\mathbf{u} = \mathbf{u}_{00} + \lambda_1 (\mathbf{u}_{10} - \mathbf{u}_{00}) + \lambda_2 (\mathbf{u}_{01} - \mathbf{u}_{00}).$$

Finally,  $\lambda = (\lambda_1, \lambda_2)^T$  can be determined as a solution to a  $2 \times 2$  linear system, obtained by writing that  $\mathbf{u}$  (expressed as above), verifies the (two) constraints.

**Interpretation as forces.** The corresponding  $\lambda$  accounts for the force exerted by the fluid on the particle. More precisely, the variational term in  $\lambda$  corresponds to an extra term in the right-hand side of the momentum equation:

$$\dots = -\lambda \mathbf{1}_{\omega},$$

so that  $|\omega| \lambda$  is the resultant force exerted by the obstacle on the fluid.

Following the principle of virtual power, the penalty term can also be interpreted as a force. Indeed, consider the linear form

$$\Psi_{\varepsilon} : \mathbf{v} \mapsto \langle \Psi_{\varepsilon}, \mathbf{v} \rangle = \frac{1}{\varepsilon} \int_{\omega} (\nabla \mathbf{u} + {}^t\nabla \mathbf{u}) : (\nabla \mathbf{v} + {}^t\nabla \mathbf{v}).$$

For any test function considered as a virtual velocity field,  $-\langle \Psi_{\varepsilon}, \mathbf{v} \rangle$  is the power that the forces exerted on the fluid to maintain a quasi rigid motion would develop. It follows immediately that the corresponding force field has a zero resultant and a zero torque.

**Implementation issues.** The penalty method does not require a conforming mesh. Yet, using a conforming mesh (i.e. without any element crossed by the boundary) can improve space accuracy. In this spirit, the approach presented above can be discretized over a conforming mesh, i.e. a mesh that covers the whole computational domain (including  $\omega$ ), but that contains a piecewise approximation of the disc boundary. Such a mesh can be build in the following way

```

int Np = 20, NB = 2*Np ;
real L = 2, xB = 0.5 , yB = 0.5 , rB = 0.1 ;
border Gs(t=0,L){x=t;y=0;label=1;}
border Ge(t=0,1){x=L;y=t;label=2;}
border Gn(t=0,L){x=L-t;y=1;label=3;}
border Gw(t=0,1){x=0;y=1-t;label=4;}
border Gin (t=0,2*pi){x=xB+rB*cos(t);y=yB+rB*sin(t);label=5;}
mesh Th=buildmesh(Gs(L*Np)+Ge(Np)+Gn(L*Np)+Gw(Np)+Gin(NB));

```

Note that  $Gin(NB)$  is written without a minus sign, so that a mesh will be created within the particle. To compute integrals over the discretized disc, a robust way consists in defining the characteristic function of the disc as a piecewise constant function, and to multiply the integrand by this function when the integration is to be limited to the disc.



## Chapter 6

# Elements of numerical analysis for saddle-point problems

### 6.1 Mixed finite element formulations

We consider here the general saddle-point formulation of the problem which consists in minimizing a quadratic functional

$$J(v) = \frac{1}{2}a(v, v) - \langle \varphi, v \rangle,$$

defined in a Hilbert space  $V$ , when the minimizers is subject to belong to a linear space that is written as the  $V$ -kernel of a bilinear form:

$$(v, \mu) \in V \times \Lambda \longmapsto b(v, \mu), \quad K = \{v \in V, b(v, \mu) = 0 \quad \forall \mu \in \Lambda\}.$$

The formulation reads

$$\begin{cases} a(u, v) + b(v, p) = \langle \varphi, v \rangle & \forall v \in V \\ b(u, \mu) = 0 & \forall \mu \in \Lambda. \end{cases} \quad (6.1)$$

One denotes by  $B \in \mathcal{L}(V, \Lambda)$  the operator defined by  $(Bv, \mu) = b(v, \mu)$ . Note that  $b(v, p)$  can also be written  $\langle B^*p, v \rangle$ .

It is natural to introduce  $V_h$  and  $\Lambda_h$ , approximation spaces for  $V$  and  $\Lambda$ , respectively, and the associated discretized problem

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) = \langle \varphi, v_h \rangle & \forall v_h \in V_h \\ b(u_h, \mu_h) = 0 & \forall \mu_h \in \Lambda_h. \end{cases} \quad (6.2)$$

We define the operator  $B_h \in \mathcal{L}(V, \Lambda_h)$  by

$$(B_h v, \mu_h) = (Bv, \mu_h) \quad \forall \mu_h \in \Lambda_h,$$

and the discrete constrained space is denoted by

$$K_h = V_h \cap \ker B_h = \{v_h, b(v_h, \mu_h) = 0 \quad \forall \mu_h \in \Lambda_h\}$$

The problem 6.2 consists in minimizing

$$J(v) = \frac{1}{2}a(v, v) - \langle \phi, v \rangle$$

over  $K_h$ . Considering finite dimensional approximation spaces  $V_h$  and  $\Lambda_h$ ,  $K_h$  has finite dimension, therefore it is closed, so that Problem 6.2 is obviously well-posed, at least in terms of existence. Uniqueness holds for the primal variable  $u_h$ , but not necessarily for the Lagrange multiplier  $p_h$ .

The question is now: assuming that  $(V_h)$  and  $(\Lambda_h)$  are suitable sequences of approximation spaces, i.e.  $\text{dist}(v, V_h)$  and  $\text{dist}(q, \Lambda_h)$  both go to zero as the discretization parameter  $h$  goes to 0, can  $u_h$  be expected to converge to the solution  $u$  of the continuous constrained minimization problem? As for the dual part  $p_h$ , in case it is uniquely defined, one may also wonder whether it converges to  $p$ .

Extension of the theory developed for standard (unconstrained) variational problems is in fact not straightforward. Informally, if  $\Lambda_h$  is much “richer” than  $V_h$  in terms of approximation, the discrete problem might be overconstrained, so that  $K_h$  does not approximate  $V$  properly (it might even be reduced to  $\{0\}$ ). On the other way, if  $\Lambda_h$  is too poor, the problem might be underconstrained, so that the solution  $u_h$  to the discrete constrained minimization problem might converge (if ever) outside  $K$ . Both approximations have obviously to be *balanced* in some way. This balance necessitates a fine tuning between the two discretization spaces, which will be expressed by the discrete inf-sup (or BBL) condition.

**Remark 5** *In the case of Stokes problem, as the velocity is  $H^1$  and the pressure  $L^2$ , it would be very natural to approximate velocities with continuous, piecewise  $P^1$  functions (as for the poisson problem), while pressures would be approximated by  $P^0$  functions. As we shall see, this straightforward approach does not work, for reasons that we shall detail below.*

### 6.1.1 A general estimate without inf-sup condition

To enlight how this inf-sup condition allows a proper error estimation of primal and dual quantities (i.e.  $|u - u_h|$  and  $|p - p_h|$ ), we shall start the numerical analysis by a very general estimation of the sole primal quantity, under very loose conditions (in particular without the inf-sup condition). Although this estimate will not be used as such in the context of divergence free constraint, it proves powerful in other contexts, e.g. for the fictitious domain method.

We assume here that  $a(\cdot, \cdot)$  is a continuous, symmetric and coercive bilinear form over  $V \times V$ , and  $B \in \mathcal{L}(V, \Lambda)$ , so that  $K = \ker B$  is closed. The problem which consists in minimizing

$$J(v) = \frac{1}{2}a(v, v) - \langle \phi, v \rangle$$

over  $K$  is well-posed by Lax Milgram theorem, and we denote by  $u$  its solution. We do not make extra assumptions on  $B$ , so that the saddle-point problem may be ill-posed in terms of existence of a Lagrange multiplier. Concerning approximation spaces, we simply assume that  $V_h \subset V$ , but  $\Lambda_h$  is any finite dimensional space, and  $B_h \in \mathcal{L}(V, \Lambda_h)$ . Of course, it will be necessary to make extra assumptions on  $\Lambda_h$  and  $B_h$  to obtain useful estimates, but it is not necessary for the time being.

We consider the (abstract) discretized saddle-point problem:

$$\begin{cases} a(u_h, v_h) + (B_h v_h, p_h) = \langle \varphi, v_h \rangle & \forall v_h \in V_h \\ (B_h u_h, q_h) = 0 & \forall q_h \in \Lambda_h. \end{cases} \quad (6.3)$$

This problems consists in minimizing

$$J(v) = \frac{1}{2}a(v, v) - \langle \phi, v \rangle$$

over  $K_h = \ker B_h$ . As  $\Lambda_h$  is finite dimensional, the saddle-point problem is well-posed in the following sense: it admits a solution  $(u_h, p_h)$ , and  $u_h$  is uniquely defined as the minimizer of  $J$  over  $K_h$ .

Let us start by a very simple remark on the continuous problem: there exists a unique  $\xi \in V'$  (more precisely in  $K^\perp$ ) such that

$$Au + \xi = \phi.$$

As  $u$  is uniquely defined,  $\xi$  is uniquely defined as  $\phi - Au$ . The basis of the following estimate will be to prove that, even if  $\xi$  cannot be written  $B^*p$ , some convergence of  $u_h$  toward  $u$  can be expected as soon as  $\xi$  can be approximated by  $B_h^*q_h$ .

**Proposition 5** *Let  $a(\cdot, \cdot)$  be a continuous, symmetric, coercive, bilinear form over  $V \times V$ ,  $\phi \in V'$ ,  $B \in \mathcal{L}(V, \Lambda)$ . We denote by  $u$  the unique minimizer of  $J$  over  $K = \ker B$ . Let  $V_h \subset V$  and  $\Lambda_h$  be finite-dimensional spaces,  $B_h \in \mathcal{L}(V, \Lambda_h)$ , and  $K_h = \ker B_h \cap V_h$ . Let  $(u_h, p_h)$  be a solution of the saddle point-problem (6.2). Then*

$$|u - u_h| \leq C \left( \inf_{w_h \in K_h} |w_h - u| + \inf_{q_h \in \Lambda_h} \|\xi - B_h^*q_h\|_{V_h'} \right) \quad (6.4)$$

*Proof :* For any  $v_h \in K_h$ ,  $(B_h v_h, p_h) = 0$ , so that

$$a(u_h, v_h) = \langle \phi, v_h \rangle.$$

The core of the proof consists in taking  $v_h$  in the form  $u_h - w_h$ , where  $w_h$  is in  $K_h$ . It comes

$$a(v_h, v_h) = \langle \phi, v_h \rangle - a(w_h, v_h).$$

The exact solution  $u$  verifies

$$a(u, v_h) + \langle \xi, v_h \rangle = \langle \phi, v_h \rangle,$$

so that

$$a(v_h, v_h) = a(u - w_h, v_h) + \langle \xi, v_h \rangle.$$

As  $v_h$  is in  $K_h$ ,  $\langle B_h^*q_h, v_h \rangle = (B_h v_h, \mu_h) = 0$  for any  $\mu_h \in \Lambda_h$ , so that this quantity can be subtracted to the right-hand side.

$$a(v_h, v_h) = a(u - w_h, v_h) + \langle \xi - B_h^*\mu_h, v_h \rangle.$$

One obtains

$$\alpha |v_h|^2 \leq \|a\| |u - w_h| |v_h| + \|B_h^*q_h - \xi\| |v_h| \Rightarrow \alpha |u_h - w_h| \leq \|a\| |u - w_h| + \|B_h^*q_h - \xi\|$$

and therefore

$$|u_h - u| \leq C \left( |u - w_h| + \|B_h^*q_h - \xi\|_{V_h'} \right).$$

for any  $w_h \in K_h$ , any  $q_h \in \Lambda_h$ , which ends the proof.

Estimate (6.4) expresses the required balance between both approximation spaces. Firstly,  $\Lambda_h$  has to be rich enough (and  $B_h$  has to approximate  $B$  in some sense) for the second term to be small. But if  $\Lambda_h$  is too rich, it may constrain excessively the problem, so that  $K_h$  may not be a good approximation space for  $K$ , preventing the first term to go to 0.

### 6.1.2 Estimates with the inf-sup condition

We may now introduce the framework which we will favor for Stokes equation. We consider here a saddle point formulation which is well-posed at the continuous level, i.e.  $B$  is surjective.

**Proposition 6** *Notations and assumptions are those of Proposition 5. We furthermore assume here that  $B \in \mathcal{L}(V, \Lambda)$  is surjective, so that the continuous saddle point problem admits a unique solution  $(u, p)$ . We also assume that both approximations are conforming, i.e.  $V_h \subset V$  and  $\Lambda_h \subset \Lambda$ , that  $B_h$  is defined by*

$$(B_h v, q_h) = (B v, q_h) = b(v, q_h),$$

and that it verifies the discrete inf-sup condition:

$$\inf_{q_h \in \Lambda_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{|v_h| |q_h|} \geq \beta > 0,$$

where  $\beta$  is independent<sup>1</sup> of  $h$ . Then we have the following error estimate

$$|u - u_h| + |p - p_h| \leq C \left( \inf_{w_h \in V_h} |w_h - u| + \inf_{q_h \in \Lambda_h} |q_h - p| \right) \quad (6.5)$$

*Proof :* The proof is based on estimate (6.4). As preliminary step, let us first note that the infimum over  $\Lambda_h$  of  $\|B_h^* q_h - \xi\|_{V_h'}$  takes the form (as  $\xi = B^* p$ )

$$\inf_{q_h \in \Lambda_h} |B^*(q_h - p)| \leq \inf_{q_h \in \Lambda_h} \|B^*\| |q_h - p|.$$

The first step of the proof, which is essential to use approximation properties of  $V_h$ , consists in showing that the first infimum in (6.4) can be replaced by an infimum over the unconstrained approximation space  $V_h$ .

In a second step, we will show that the estimate on  $|u - u_h|$  induces an estimate on  $|p - p_h|$ .

**Step 1.** As for the first term in estimate (6.4), we proceed as follows: consider an approximation  $v_h \in V_h$  of  $u$ , we build an approximation  $w_h = v_h + z_h \in K_h$  with the same approximation properties (up to a multiplicative constant which does not depend on  $h$ ).

Let  $v_h \in V_h$  be given. We denote by  $z_h$  the element of  $V_h$  which verifies

$$B_h z_h = -B_h v_h$$

and which minimizes the norm (i.e.  $w_h = v_h + z_h$  is the projection of  $v_h$  onto  $K_h$ ). This  $z_h$  is the primal part of the solution to the saddle point problem

$$\begin{aligned} (z_h, y_h) + (\eta_h, B_h y_h) &= 0 \quad \forall y_h \in V_h \\ (\mu_h, B_h z_h) &= -(\mu_h, B_h v_h) \quad \forall \mu_h \in \Lambda_h. \end{aligned}$$

Thanks to the inf-sup condition, we have

$$|\eta_h| \leq \frac{1}{\beta} \sup_{y_h} \frac{\langle B_h^* \eta_h, y_h \rangle}{|y_h|} = \frac{1}{\beta} |z_h|.$$

---

<sup>1</sup>As always in such contexts, we implicitly consider sequences of approximation spaces  $(V_h)$  and  $(\Lambda_h)$ , indexed by a parameter  $h$  (which will represent the mesh diameter in actual Finite Element discretization, which goes to 0 as the dimension of the spaces goes to infinity).

On the other hand, taking  $y_h = z_h$ , it comes

$$|z_h|^2 \leq |(\eta_h, B_h z_h)| \leq |\eta_h| |B_h v_h| = |\eta_h| |B_h(u - v_h)| \leq C |z_h| |B_h(u - v_h)|.$$

The new approximant  $w_h = v_h + z_h$  is then such that

$$|u - w_h| \leq |u - v_h| + |z_h| \leq C |u - v_h|.$$

**Step 2.** To estimate  $|p - p_h|$ , we subtract the discrete variational formulation from the continuous one. We get

$$(p_h, Bv_h) = a(u - u_h, v_h) + (p, Bv_h),$$

so that, for any  $q_h \in \Lambda_h$

$$(p_h - q_h, Bv_h) = a(u - u_h, v_h) + (p - q_h, Bv_h) \quad \forall q_h \in \Lambda_h.$$

By the inf-sup condition, we have

$$\begin{aligned} |p_h - q_h| &\leq \frac{1}{\beta} \sup_{v_h \in V_h} \frac{|a(u - u_h, v_h) + (p - q_h, Bv_h)|}{|v_h|} \\ &\leq \frac{1}{\beta} (\|a\| |u - u_h| + \|B\| |p - q_h|), \end{aligned}$$

so that, finally,

$$|p - p_h| \leq \frac{\|a\|}{\beta} |u - u_h| + \left(1 + \frac{\|B\|}{\beta}\right) \inf_{q_h \in \Lambda_h} |p - q_h|.$$

The discrete inf-sup condition, which is not easily verified in its native form, is equivalent to the existence of an operator from  $V$  to  $V_h$  which preserves the discrete constraint, with a bounded norm<sup>2</sup>. More precisely, we have

**Proposition 7** (*Fortin's criterium*)

We assume that  $B \in \mathcal{L}(V, \Lambda)$  verifies the inf-sup condition. Then the sequence  $(V_h, \Lambda_h)$  verifies the discrete inf-sup condition iff there exists a constant  $C > 0$  and a family  $(\Pi_h)_h$ , with  $\Pi_h \in \mathcal{L}(V, V_h)$ , such that

$$b(\Pi_h v - v, q_h) = 0 \quad \forall (v, q_h) \in V \times \Lambda_h,$$

with

$$|\Pi_h v| \leq C |v|.$$

*Proof :* Assume that  $b(\cdot, \cdot)$  verifies the inf-sup condition. For any  $v \in V$ , we build  $v_h$  which minimizes the norm among all those  $w_h$  that verify  $Bw_h = Bv$  (as in the beginning of Proposition 6). The saddle point formulation of the problem writes

$$\begin{aligned} (v_h, w_h) + (p_h, Bw_h) &= 0 \quad \forall w_h \in V_h \\ (q_h, Bv_h) &= (q_h, Bv) \quad \forall q_h \in \Lambda_h. \end{aligned}$$

By the inf-sup condition, one has  $|p_h| \leq |v_h|/\beta$ , and by taking  $w_h = v_h$  in the first line, we obtain  $|v_h| \leq C |v|$ . We define  $\Pi_h v$  as  $v_h$  (orthogonal projection onto  $B^*(\Lambda_h)$ ).

<sup>2</sup>As always, this assertion refers to a family  $(\Pi_h)$  of operators which is uniformly bounded with respect to  $h$ .

On the other way, consider  $q_h \in \Lambda_h \subset \Lambda$ , denote by  $v$  its reciprocal image by  $B$  which minimizes the norm. This norm is controlled by that of  $q_h$  because the range of  $B$  is closed. We have

$$\sup_{v_h} \frac{b(v_h, q_h)}{|v_h|} \geq \frac{b(\Pi_h v, q_h)}{|\Pi_h v|} \geq \frac{1}{C} |q_h|,$$

which ends the proof.

### 6.1.3 Some stable finite elements for Stokes equations

A finite element  $(V_h, \Lambda_h)$  for the mixed formulation of Stokes problem is said to be *stable* whenever it verifies the discrete inf-sup condition uniformly with respect to the discretization parameter  $h$ . Among natural candidates, neither  $(P^1, P^1)$  (piecewise affine functions for the velocity and the pressure), nor  $(P^1, P^0)$  (piecewise constant functions for the pressure) meet this requirement.

**Bubble element.** The simplest stable element is the so called *mini* element, or *bubble* element. It is obtained from the unstable family  $(P^1, P^1)$  by adding a degree of freedom in the center of each element. It is usually performed by considering the so-called bubble function associated to each element, which is the product of the barycentric coordinates. Let us denote by  $w_K$  the bubble function associated to element  $K$ . For a triangulation  $\Omega_h$ ,  $V_h$  is defined as the set of all those vector fields which can be written

$$\mathbf{u}_h = \sum_{\alpha=1}^d \left( \sum_i u_i^\alpha w_i \mathbf{e}_\alpha + \sum_K \beta_K^\alpha w_K \mathbf{e}_\alpha \right)$$

where the first sum goes over “free” vertices (vertices of  $\Omega_h$  which do not belong to a boundary on which Dirichlet boundary condition is prescribed), the second sum goes over all simplices of  $\Omega_h$ , and  $w_i$  is the hat function associated to vertex  $i$ . The approximation space for pressure is simply

$$\Lambda_h = \left\{ q_h = \sum_i q_i w_i \right\}$$

**Proposition 8** *We consider a polygonal domain, and a regular family  $(\Omega_h)$  of triangulations. The family  $(V_h, \Lambda_h)$  defined previously is stable.*

*Proof:* The proof is based on the Fortin’s criterium (see Prop. 7): we give an explicit construction of  $\Pi_h$ . For a given  $\mathbf{v}$  in  $H_0^1(\Omega)^d$ , the idea consists in building  $\Pi_h \mathbf{v}$  with a norm controlled by the norm of  $\mathbf{v}$  (uniformly with respect to  $h$ ), such that the mean value of each of its component equals the mean value of the corresponding component of  $\mathbf{v}$ . If we are able to build such an operator, we shall have

$$b(\Pi_h \mathbf{v} - \mathbf{v}, q_h) = \int_{\Omega} q_h \nabla \cdot (\mathbf{v} - \Pi_h \mathbf{v}) = - \int_{\Omega} (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \nabla q_h.$$

As  $q_h$  is piecewise affine, its gradient is constant in each element  $K$ , so that the latter quantity is 0 by construction.

Let us now detail how  $\Pi_h$  can be built. It is actually built component by component, and we shall keep the same notation  $\Pi_h$  to denote the operator which acts on scalar functions. Consider

$v \in H_0^1(\Omega)$ . The first step consists in associating a piecewise affine function  $v_h$  which approximates  $v$ . The difficulty is that standard pointwise interpolation is impossible, as pointwise values of  $v$  are not defined (as soon as  $d \geq 2$ ). To overcome this difficulty, we consider instead  $R_h v$ , where  $R_h$  is the so-called Clément operator (the existence of which we admit here), which is such that

$$|v - R_h v|_{0,K} \leq C h_K |v|_{1,\Delta_K},$$

where  $\Delta_K$  is the set of elements in contact with  $K$ . The approach consists then in correcting  $R_h v$  within each element (using the bubble degree of freedom) to respect the mean value. More precisely, for any element  $K$ , we write

$$\Pi_h v = R_h v + w_h,$$

where  $w_h$  is a bubble function, chosen in such a way that

$$\int_K \Pi_h v = \int_K v.$$

Let us denote by  $\bar{w}_h$  the mean value of  $w_h$  over  $K$ . One has

$$\bar{w}_h = \frac{\int_K w_h}{|K|} = |K|^{-1} \left| \int_K v - \int_K R_h v \right| \leq |K|^{-1/2} |v - R_h v|_{0,K} \leq C |K|^{-1/2} h_K |v|_{1,\Delta_K}$$

Besides, for elements with a controlled aspect ratio (i.e.  $h_K/\rho_K$  is uniformly bounded), the  $H^1$  semi-norm of  $w_h$  verifies

$$|w_h|_{1,K} \leq C h_K^{-1} \bar{w}_h |K|^{1/2}.$$

We finally have

$$|w_h|_{1,K} \leq C |v|_{1,\Delta_K}.$$

As the semi-norm in the right-hand side involves a neighborhood of  $K$ , we have

$$|w_h|_{1,\Omega} \leq |v|_{1,\Omega}.$$

We therefore have  $|\Pi_h v|_{1,\Omega} \leq C |v|_{1,\Omega}$ , which ends the proof

**Remark 6** *We assumed that  $B$  verifies the continuous inf-sup condition, which means that  $B$  is surjective. It requires to restrict pressure fields to scalar fields with zero mean value. As a consequence, the set of discrete pressures is that of piecewise affine fields with zero mean value over  $\Omega$ . This constraint is a priori difficult to handle, as hat functions do not verify this constraint. In practice, one commonly considers a modified saddle-point problem of the form*

$$\begin{aligned} Au + B^* p &= \phi, \\ Bu + \varepsilon p &= 0, \end{aligned}$$

where  $\varepsilon > 0$  is a small parameter.

Note that, in the case where Neuman conditions are prescribed on some part of the boundary, then the pressure is no longer defined up to a constant, so that the couple  $(V_h, \Lambda_h)$  verifies directly the inf-sup condition (without any additional requirement on discrete pressure fields).

**Taylor-Hood element.** An higher order element is the so-called Taylor-Hood, or  $P2 - P^1$ , element. As suggested, it is based on a quadratic description of the velocity in each element, whereas the pressure is piecewise affine.





# Chapter 7

## Complements

### 7.1 Estimation of interaction forces

We consider the situation illustrated by figure 7.1. the domain is a  $2 \times 1$  rectangle. A newtonian fluid flows around a fixed cylinder centered at  $(0.5, 0.5)$  (radius 0.1), according to Navier-Stokes equations

$$\begin{aligned} \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) - \mu \Delta \mathbf{u} + \nabla p &= 0 \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega \end{aligned}$$

Boundary conditions will be taken as follows: Dirichlet boundary conditions on the inlet boundary  $\Gamma_4$  (uniform horizontal velocity), and free outlet boundary conditions on the rest of the boundary

$$\mu \nabla \mathbf{u} \cdot \mathbf{n} - p \mathbf{n} = 0 \quad \text{on } \Gamma_1, \Gamma_2, \Gamma_3.$$

No slip conditions ( $\mathbf{u} = 0$ ) will be prescribed on the boundary of the disc.

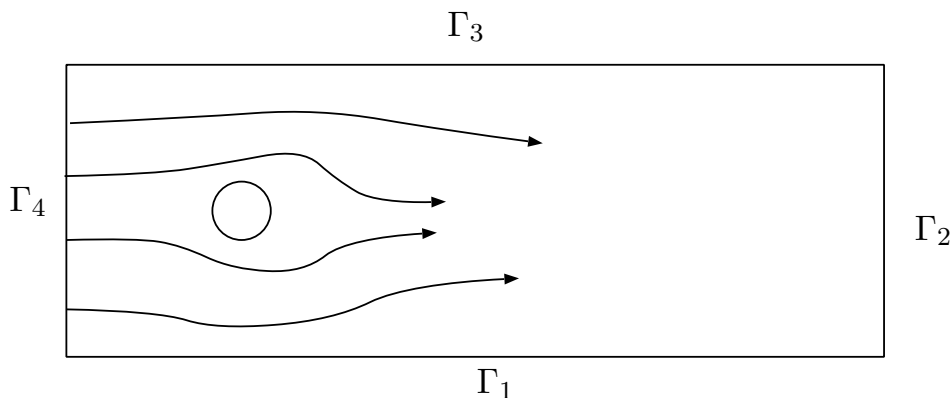


Figure 7.1: Flow around a cylinder

We propose here to use the method of characteristics to handle the advection term. The time discretized problem leads to the generalized Stokes problem

$$\begin{aligned} \frac{\rho}{\delta t} \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p &= \frac{\rho}{\delta t} \tilde{\mathbf{u}} \circ X \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, \end{aligned} \tag{7.1}$$

where  $\tilde{\mathbf{u}}$  is the velocity at the previous time step.

We aim at estimating the forces exerted by the cylinder on the fluid. Freefem++ makes it possible to directly estimate the boundary integral

$$\int_{\gamma} (\mu \nabla \mathbf{u} \cdot \mathbf{n} - p \mathbf{n}),$$

but it is more stable to use a volume integral.

To that purpose, we consider a smooth vector field  $\mathbf{v}$  that is  $\mathbf{e}_x$  on  $\gamma$  and 0 on other parts of the fluid domain boundary. In the expression of the horizontal (drag) force

$$D = \int_{\gamma} (\mu \nabla \mathbf{u} \cdot \mathbf{n} - p \mathbf{n}) \cdot \mathbf{e}_x,$$

$\mathbf{e}_x$  can be replaced by  $\mathbf{v}$ . It can be integrated by parts, which yields

$$\int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{v} + \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} \nabla p \cdot \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v}.$$

Thanks to Eq. (7.1), we have that

$$\Delta \mathbf{u} - \nabla p = \frac{\rho}{\delta t} \mathbf{u} - \frac{\rho}{\delta t} \tilde{\mathbf{u}} \circ X,$$

so that  $D$  can be expressed

$$D = \frac{\rho}{\delta t} \mathbf{u} - \frac{\rho}{\delta t} \tilde{\mathbf{u}} \circ X + \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} p \nabla \cdot \mathbf{v}.$$

The form we obtain is exactly the variational formulation of the force balance equation, so that effective computation of the force is straightforward under `freefem++`, by re-using the implemented variational formulation.

## 7.2 Energy balance

The models proposed in these lecture notes express standard physical laws (mainly mass conservation and Newton's law), therefore a relevant energy balance can be expected. This section describes the way such an energy balance can be established in some typical situation. This standpoint can be fruitful in the context of numerical simulation. Since an analytical solution is rarely known, checking that a "numerical" energy balance can be obtained is an efficient way to check the physical relevancy of the computed solution.

**Homogeneous Dirichlet boundary conditions.** The simplest situation is that of a Navier-Stokes fluid in a closed container  $\Omega$ , with no-slip conditions on the wall. Multiplying the momentum equations by the velocity itself and integrating over the domain yields

$$\rho \int_{\Omega} \partial_t \mathbf{u} \cdot \mathbf{u} + \rho \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \mathbf{u} - \mu \int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{u} + \int_{\Omega} \Omega \mathbf{u} \cdot \nabla p = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}.$$

The right-hand side is the power of external forces. The first term can be written

$$\frac{d}{dt} \int_{\Omega} \frac{\rho}{2} |\mathbf{u}|^2,$$

it is the time derivative of the kinetic energy. For the second one, we can use the formula

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} \nabla |\mathbf{u}|^2 - \mathbf{u} \times (\nabla \times) \mathbf{u},$$

which leads to

$$\rho \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \mathbf{u} = \rho \int_{\Omega} \frac{1}{2} \nabla |\mathbf{u}|^2 \cdot \mathbf{u} = \int_{\Gamma} \frac{\rho}{2} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{n},$$

that vanishes since  $\mathbf{u}$  is zero on the boundary. The viscous term is

$$-\mu \int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{u} = \mu \int_{\Omega} |\mathbf{u}|^2,$$

it corresponds to viscous dissipation.

We finally obtain that the time derivative of kinetic energy equals the power of external forces (rate of energy injected into the system) minus the rate of energy lost as heat by viscous effects.

**Free boundary problem.** Let us consider a richer situation: the fluid domain moves, and its boundary is assumed to separate the viscous fluid in consideration from an outside fluid that is assumed to be perfect, at constant pressure  $P$ . We consider the free fall a drop of viscous fluid:

$$\begin{cases} \rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = \mathbf{g} & \text{in } \Omega(t) \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega(t) \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mu (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) \cdot \mathbf{n} - p \mathbf{n} = -P \mathbf{n} & \text{on } \Gamma(t) \end{cases}$$

We multiply again by the velocity  $\mathbf{u}$ , integrate over  $\Omega$ , with two main differences: since the domain is moving according to the fluid velocity  $\mathbf{u}$ , the time derivative of an integral quantity writes

$$\frac{d}{dt} \int_{\Omega(t)} \Psi(x, t) = \int_{\Omega(t)} \partial_t \Psi(x, t) + \int_{\Gamma(t)} \Psi(x, t) \mathbf{u} \cdot \mathbf{n}. \quad (7.2)$$

Besides, the term

$$\rho \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \mathbf{u} = \rho \int_{\Omega} \frac{1}{2} \nabla |\mathbf{u}|^2 \cdot \mathbf{u} = \int_{\Gamma} \frac{\rho}{2} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{n}$$

is no longer 0. Putting things together, we obtain

$$\int_{\Omega} \left( \rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \cdot \mathbf{u} = \int_{\Omega} \partial_t \left( \frac{\rho}{2} |\mathbf{u}|^2 \right) + \int_{\Gamma} \frac{\rho}{2} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{n}$$

that is exactly, thanks to (7.2), the quantity

$$\frac{d}{dt} \int_{\Omega} \frac{\rho}{2} |\mathbf{u}|^2,$$

that is the time derivative of the kinetic energy.

The power of external forces writes

$$\int_{\Omega} \mathbf{g} \cdot \mathbf{u} = - \int_{\Omega} g \mathbf{e}_y \cdot \mathbf{u} = - \int_{\Omega} g \nabla y \cdot \mathbf{u} = \int_{\Omega} g y \underbrace{\nabla \cdot \mathbf{u}}_{=0} - \int_{\Gamma(t)} y \mathbf{u} \cdot \mathbf{n}.$$

The latter quantity can be written (using again (7.2))

$$-\int_{\Gamma(t)} y \mathbf{u} \cdot \mathbf{n} = -\frac{d}{dt} \int_{\Omega(t)} gy.$$

We finally obtain that the derivative of the total energy (kinetic + potential) balances the viscous losses:

$$\frac{d}{dt} \left( \int_{\Omega} \frac{\rho}{2} |\mathbf{u}|^2 + \int_{\Omega(t)} gy \right) = -\frac{\mu}{2} \int |\nabla \mathbf{u} + {}^t \nabla \mathbf{u}|^2.$$

### 7.3 Visualization

The graphic representation of the velocity field is often not sufficient to give a clear idea of the fluid flow structures. An alternative way consists in representing the vorticity, that is scalar in 2d:

$$\omega = \partial u_y / \partial x - \partial u_x / \partial y.$$

This can be done under Freefem++ by directly computing the partial derivatives:

```
om = dy(ux)-dx(uy)
```

Since the discrete velocities are not  $C^1$  over the domain, this function is a priori discontinuous. It can be defined in a natural way as a  $P^0$  function, which rules out the possibility to draw isolines. Defining it as a  $P^1$  function will force `freefem++` to perform some interpolation to define nodal values. Another approach consists in solving a variational problem discretized in the space of piecewise affine functions.

### 7.4 Surface tension

When two immiscible fluids are separated by an interface, short range forces between the fluid induce a non zero resulting force whenever the interface is not flat. The effect can be modeled by a pressure drop through the interface. This pressure drop is the product of the mean curvature of the interface and a coefficient  $\kappa$ . This coefficient depends of the properties of both fluids, and on the chemical state of the interface itself. Consider the case of a (two-dimensional) viscous fluid flowing freely in a perfect fluid, which we will consider at pressure zero. The corresponding boundary condition can be written

$$\sigma \cdot \mathbf{n} = -\kappa C \mathbf{n},$$

where  $C$  is the curvature, i.e. the inverse of the radius of curvature. This curvature will be considered positive in case of a convex body, so that the force points toward the center of the radius of curvature.

This effect is delicate to be accounted for numerically, since it involves a quantity that involves the second derivative of the boundary. Let us show here that there is a natural way to integrate it in the finite element method. We shall consider the case of a boundary with no ends, i.e. the interface is the whole boundary of the domain.

The approach is based on elementary calculus in differential geometry, namely the formula

$$C \mathbf{n} = \frac{\mathbf{n}}{R} = \frac{d}{ds},$$

where  $\mathbf{t}$  is the tangent vector, and  $s$  the curvilinear abscissa. Now consider the term that corresponds to the surface tension effect in the variational formulation:

$$\int_{\Gamma} C \mathbf{n} \cdot \mathbf{v},$$

where  $\mathbf{v}$  is a vector test function.



# Chapter 8

## Projects

In the following, we present subjects that are open to students for personal projects. It is expected that the students choose one of them and spend 10 to 15 hours of **personal** time working on them. At the end, they are asked to present their own work in a short 15 mn defense.

All the subjects are described very shortly. This is intentional since we expect the students to bring the chosen subject into personal directions. Both professors are available for discussion upon request during the total duration of the projects.

### 8.1 General

#### 8.1.1 Which mesh for which Reynolds number ?

Given a problem to solve, e.g. the flow around a cylinder at a certain Reynolds number (say  $Re=1000$ ), propose an approach (possibly a heuristic/experimental one) to determine the “right” mesh to use in order to provide relevant simulations. By “right” we mean the cheapest one that allows to properly describe the features of the flow, and to accurately compute relevant quantities.

### 8.2 Drag and lift of an obstacle

Describe a method to compute the two components of the instantaneous force exerted by a Navier-Stokes fluid on an obstacle: in the direction of the flow (drag), and perpendicular to this direction (lift).

#### 8.2.1 Benchmark

A benchmark is proposed in

[http://www.math.u-psud.fr/~maury/paps/NUM\\_NS.BenchmarkTurek.pdf](http://www.math.u-psud.fr/~maury/paps/NUM_NS.BenchmarkTurek.pdf)

for the 2d flow around a cylinder, in a specific geometry. Check your Freefem++ implementation by comparing your results in terms of drag and lift forces. Investigate the roles of the time step, the mesh diameter, and possibly the chosen approach (direct approach, penalty, saddle-point formulation).

### 8.2.2 Rotating cylinder

We consider that the obstacle is a rotating cylinder (disc in 2d). Investigate numerically the lift induced by the rotation of the cylinder around its center.

### 8.2.3 Airfoil

For a given shape of airfoil, and a given regime (choice of a Reynolds number between 20 and 500), investigate the dependence of the mean lift force upon the angle of attack.

## 8.3 Moving objects

### 8.3.1 Free fall

Propose and implement a numerical strategy to compute (in 2d) the motion of a rigid object (different from a disc) falling freely in a Navier-Stokes fluid. The case of an elongated body (ellipse or rectangle) is of particular interest.

### 8.3.2 Fluid structure interaction

Propose a way to compute the motion of a rigid disc in a Navier-Stokes flow, when the disc is assumed to be connected to a fixed point by a spring of a certain stiffness and a zero length. Investigate the interaction between the fluid flow and the moving object, in the case where the Reynolds number is sufficiently high to induce a oscillations downstream the cylinder.

### 8.3.3 oscillating cylinder

Propose a numerical approach to compute the problem presented at page 25 in

[http://pastel.archives-ouvertes.fr/docs/00/54/59/37/PDF/Duclercq-Marion\\_these.pdf](http://pastel.archives-ouvertes.fr/docs/00/54/59/37/PDF/Duclercq-Marion_these.pdf)

## 8.4 Free surface

### 8.4.1 Glass forming process

The production of flat glass is based on the so called *float* process: glass at high temperature (i.e. liquid glass) is poured on a glass of tin (liquid metal, with a very small viscosity), and pulled mechanically at the other end of the bath. We aim here at modeling the motion of the glass sheet over the tin bath, by considering the tin as a perfect fluid, with a pressure that is considered hydrostatic.

### 8.4.2 Density waves

Propose a way to simulate the motion of two layers of immiscible fluids, a light one and a heavier one, one on the top of the other, and investigate the propagation of waves on the interface between the two fluids.

### 8.4.3 Inclined plane

Situation: a viscous and inertial fluid flows down an inclined plane. Investigate the effect of bumps in the supporting surface upon the shape of the free boundary.



#### 8.4.4 Waves on the beach

Investigate the effect of a depth change upon a traveling water wave. Consider for example the case of a bottom profile that is flat on the left hand of the domain, and raises up to another height in the right hand part of the domain.

#### 8.4.5 Rayleigh-Taylor instability

A container is filled with two fluids of different densities, the heavier being on top of the lighter. If the free surface between them were flat, the situation would be stationary (though highly unstable). At time  $t = 0$ , we thus consider that the velocity vanishes, but the surface between both fluids is sinusoidal. Give a method which can compute this evolving situation.

### 8.5 Free convection

We consider the following situation: The bottom of a container filled with a viscous fluid is maintained at temperature 1, whereas its top is maintained at 0 (no-flux conditions on the lateral boundaries). Investigate numerically the so-called *Rayleigh Bénard* instabilities, due to the tendency of warm fluid to go up.



# Appendix A

## Appendix

### A.1 Hilbert spaces, minimization under constraint

We collect here some basic properties of Hilbert spaces, and we refer e.g. to [?] for a more detailed presentation.

**Theorem 2** (*Riesz- Fréchet*)

Let  $V$  be a Hilbert space and  $\varphi \in V'$  a continuous linear functional over  $V$ . There exists a unique  $u$  such that

$$(u, v) = \langle \varphi, v \rangle \quad \forall v \in V.$$

**Theorem 3** (*Lax Milgram*)

Let  $V$  be a Hilbert space, and  $a(\cdot, \cdot)$  a bilinear form which is continuous and elliptic, i.e. there exists a positive constants  $M < +\infty$  and  $\alpha > 0$

$$|a(u, v)| \leq M |u| |v|, \quad a(u, u) \geq \alpha |u|^2.$$

For any  $\varphi \in V'$ , there exists a unique  $u$  such that

$$a(u, v) = \langle \varphi, v \rangle \quad \forall v \in V. \tag{A.1}$$

If  $a(\cdot, \cdot)$  is symmetric, the solution  $u$  can be characterized as the unique minimizer of the functional

$$v \longmapsto J(v) = \frac{1}{2}a(v, v) - \langle \varphi, v \rangle.$$

Note that, in case the bilinear form is symmetric, continuity and ellipticity ensure that the corresponding scalar product defines a norm which is equivalent to the native one. In this case, the Lax-Milgram theorem is a straightforward consequence of Riesz-Fréchet theorem.

We consider the following set of assumptions

$$\left. \begin{array}{l} V \text{ and } \Lambda \text{ Hilbert spaces,} \\ a(\cdot, \cdot) \text{ bilinear, symmetric, elliptic functional in } V \times V, \\ \varphi \in V', \quad B \in \mathcal{L}(V, \Lambda), \\ K = \ker B = \{u \in V, Bu = 0\} \\ J(v) = \frac{1}{2}a(v, v) - \langle \varphi, v \rangle, \end{array} \right\} \tag{A.2}$$

and the constrained optimization problem:

$$\text{Find } u \in K \text{ such that } J(u) = \inf_{v \in K} J(v). \quad (\text{A.3})$$

**Definition 7** (*Saddle point*)

A couple  $(u, \lambda)$  is said to be a saddle point to  $L(\cdot, \cdot) : V \times \Lambda \mapsto \mathbb{R}$ , if

$$L(u, \mu) \leq L(u, \lambda) \leq L(v, \lambda) \quad \forall v \in V, \forall \mu \in \Lambda.$$

We define the Lagrangian of the problem as

$$L : (v, \mu) \in V \times \Lambda \mapsto L(v, \mu) = J(v) + (\mu, Bv). \quad (\text{A.4})$$

**Proposition 9** Under the set of assumptions (A.2),  $(u, \lambda)$  is a saddle-point for  $L$  (defined by (A.4)) if and only if

$$\begin{cases} Au + B^*\lambda = \varphi \\ Bu = 0, \end{cases} \quad (\text{A.5})$$

where  $A \in \mathcal{L}(V, V')$  is defined by

$$\langle Au, v \rangle = a(u, v) \quad \forall v \in V.$$

The saddle-point problem (A.5) is tightly related to the original problem, which consists in minimizing  $J$  over  $K$ .

**Proposition 10** Let  $(u, \lambda)$  be a saddle point for  $L$  (or, equivalently,  $(u, \lambda)$  is a solution to (A.5)). Then  $u$  minimizes  $J$  over  $K$ .

On the other hand, expressing that  $u$  minimizes  $J$  over  $K$ , i.e.  $J(u + th) \geq J(u)$  for all  $t \in \mathbb{R}$  and  $h \in K$ , we obtain  $(\nabla J, w) = 0$  for all  $w \in \ker B$ , i.e.  $\nabla J \in (\ker B)^\perp$ . In the finite dimension setting, we have  $(\ker B)^\perp = \text{im } B^*$ , so that existence of  $\lambda$  such that  $\nabla J + B^*\lambda$  is straightforward. But in general (infinite dimensional setting), we simply have  $\text{im } B^* \subset (\ker B)^\perp$ , so that full equivalence between the saddle point formulation (A.5) and the initial minimization problem does not hold in general.

**Proposition 11** Under the set of assumptions (A.2), if  $B$  has a closed range, there exists  $\lambda \in \Lambda$  such that  $(u, \lambda)$  is a saddle point for the Lagrangian  $L$ , i.e.  $(u, \lambda)$  is a solution to (A.5).

**Proposition 12** In the previous proposition, if  $B$  is furthermore assumed to be surjective, then the Lagrange multiplier  $\lambda$  is unique.

## A.2 Elliptic problems and Finite Element Method

We give here some basic facts on the discretization of an elliptic problem by the Finite Element approach. We shall restrict this section to the simplest case, namely the first order Lagrange Finite Element. We refer e.g. to [17, 7] for a general presentation of the approach, and to [16] for its application to fluid flows.

Let  $\Omega$  be a domain of  $\mathbb{R}^d$ , we consider the Poisson problem in  $\Omega$ :

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

The variational formulation of the problem can be formally obtained by multiplying the equation by a test function that vanishes on  $\Gamma$ , and integrating by part:

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v.$$

**Remark 7** *The elaboration of the variational formulation is not covered by strict mathematical rules, so it does not make sense to make precise assumptions on the regularity of the test functions, and all operation (like integration by parts) are performed formally. The process allows to obtain a new formulation of the problem, which fits into a precise mathematical framework as soon as functional spaces are defined. The initial phase which consists in obtaining the variational formulation does not guarantee any rigorous link between the initial problem and its new formulation. If one aims at establishing properly that the solution to the variational problem is a solution to the initial problem in some sense, a rigorous proof has to be provided after the solution to the variational problem has been identified and characterized.*

Considering the Sobolev space  $V = H_0^1(\Omega)$ , space of all those functions in  $L^2(\Omega)$  with a gradient in  $L^2(\Omega)^d$ , the problem fits into the framework of the Lax-Milgram theorem. Assume that  $f$  is in  $L^2(\Omega)$ , the problem can be written

$$\left\{ \begin{array}{l} \text{Find } u \in V \text{ such that} \\ \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v \quad \forall v \in V \end{array} \right. \quad (\text{A.6})$$

which can also be written in an abstract way as (A.1).

The Finite Element Method is based on the introduction of a finite dimensional approximation space  $V_h \in V$ , in which the variational formulation above makes sense. Such a space can be built as follows: We first consider a *triangulation* of  $\Omega$  in the sense of the following definition.

**Definition 8** (*Conforming triangulation*)

*A conforming triangulation  $T_h$  is a set of non degenerated simplices (triangles in the two-dimensional setting, tetrahedra for  $d = 3$ ), with*

$$\bar{\Omega} = \bigcup_{K \in T_h} \bar{K},$$

*such that, for any two simplices  $K$  and  $K'$  in  $T_h$ , the intersection  $\bar{K} \cap \bar{K}'$  is either empty, a common vertex, a common edge, or (in the case  $d = 3$ ), a common face.*

We now define  $V_h$  as the set of all those functions of  $V$  such that their restriction to any element of  $T_h$  is affine. Note that, as they are required to belong to  $V$ , they are continuous over  $\bar{\Omega}$ . The discrete problem is now

$$\left\{ \begin{array}{l} \text{Find } u_h \in V_h \text{ such that} \\ \int_{\Omega} \nabla u_h \cdot \nabla v_h = \int_{\Omega} f_h \quad \forall v_h \in V_h \end{array} \right. \quad (\text{A.7})$$

If one considers a sequence of such spaces  $V_h$  which converges in some sense to the infinite dimensional space  $V$ , it is natural to expect a convergence of  $u_h$  to  $u$ . This behavior is ensured by the following lemma, which can be seen as the abstract core of the Finite Element Method:

**Lemma 1** (*Céa's*)

Let  $V$  be a Hilbert space, and  $V_h \subset V$  a subspace. Let  $a(\cdot, \cdot)$  a bilinear form over  $V \times V$  which is continuous and elliptic. We define  $u$  and  $u_h$  as solutions to the problems

$$a(u, v) = \langle \varphi, v \rangle \quad \forall v \in V, \quad a(u_h, v_h) = \langle \varphi, v_h \rangle \quad \forall v_h \in V_h.$$

It holds

$$|u - u_h| \leq C \inf_{v_h \in V_h} |u - v_h|,$$

where  $C > 0$  does not depend on  $h$ .

It remains to establish that the latter infimum converges to 0 as  $h$  goes to 0. This property can be established in the present situation thanks to the  $H^2$  regularity of the solution.

**Definition 9** (*Regular sequence of triangulations*)

For a simplex  $K$ , we denote by  $h_K$  its diameter, and by  $\rho_K$  the diameter of the largest ball included in  $K$ . The maximum of  $h_K$  over all elements of a given triangulation  $T_h$  is called the diameter of the triangulation, and it is simply<sup>1</sup> denoted by  $h$ . A sequence  $(T_h)_h$  is said to be regular if the ratio  $h_K/\rho_K$  is bounded uniformly with respect to all the triangulations of the sequence.

Considering regular families of triangulation consists in ruling out the possibility that simplices may degenerated (i.e. become flat).

**Proposition 13** Let  $u \in H^2(\Omega) \cap H^1(\Omega)$  be given. We denote by  $I_h$  the interpolation operator, which maps any function in  $H^2$  to the unique element in  $V_h$  which takes the same values<sup>2</sup> at the vertices of the triangulation  $T_h$ . It holds

$$|u - I_h u| \leq Ch |u|_2.$$

Convergence of  $u_h$  toward  $u$  (for the  $H^1$  norm) is now a direct consequence of Proposition 13 and Lemma 1.

**Finite element method for constrained problems.** Handling constraints in the finite element framework is a delicate matter. Consider Problem (A.3, under assumptions (A.2)). It is natural to introduce discretization spaces  $V_h \subset V$  and  $\Lambda_h \subset \Lambda$  of the primal and dual spaces, respectively, and to compute the minimum of the same functional

$$v \mapsto J(v) = \frac{1}{2}a(v, v) - \langle \varphi, v \rangle,$$

over the the subspace of  $V_h$  of all elements that satisfy the discrete counterpart of the constraint, i.e. which belong to

$$K_h = \{v_h \in V_h, (Bv_h, \mu_h) = 0 \quad \forall \mu_h \in \Lambda_h\}.$$

The discrete counterpart of Problem (A.5) writes, in a variational form,

$$\begin{cases} a(u_h, v_h) + (Bv_h, \lambda_h) = \langle \varphi, v_h \rangle & \forall v_h \in V_h, \\ (Bu_h, \mu_h) = 0 & \forall \mu_h \in \Lambda_h. \end{cases} \quad (\text{A.8})$$

<sup>1</sup>Committing here a common abuse of notations, we denote by  $h$  both the *index* (i.e. the label) and the diameter (which is a positive number).

<sup>2</sup>For the physical dimensions  $d = 1, 2$ , or  $3$ , functions of the Sobolev space  $H^2(\Omega)$  can be identified with continuous functions, so that pointwise values are well-defined.

The choice of  $\Lambda_h$  is obviously related to  $V_h$ . If  $\Lambda_h$  it is too poor (i.e. if its dimension is too small compared to that of  $V_h$ ), it might harm the convergence of  $\lambda_h$  toward the exact Lagrange multiplier  $\lambda$ . Even if one is interested in the primal part  $u_h$  only, the problem might be underconstrained, and the solution  $u_h$  might converge to something which does not verify the constraint. On the other hand, if  $\Lambda_h$  is too rich, a first consequence can be the non uniqueness of the discrete Lagrange multiplier  $\lambda_h$ . Besides, the discrete problem can become overconstrained, reducing the discrete constrained space  $K_h$  to  $\{0\}$  in some situations. The perfect balance is met whenever the so called inf-sup condition is verified.

**Definition 10** (*Inf-sup condition*)

The sequence of discretization spaces  $(V_h, \Lambda_h)_h$  is said to verify the inf-sup condition (also called Ladyzenskaya-Babuska-Brezzi, or LBB condition) there exists  $C > 0$  such that

$$\inf_{\mu_h \in \Lambda_h} \sup_{v_h \in V_h} \frac{(Bv_h, \mu_h)}{|v_h| |\mu_h|} \geq C \quad \forall h. \quad (\text{A.9})$$

Note that the surjective character of an operator  $B \in \mathcal{L}(V, \Lambda)$  is equivalent to

$$\inf_{\mu \in \Lambda} \sup_{v \in V} \frac{(Bv, \mu)}{|v| |\mu|} > 0.$$

Inf-sup condition (A.9) can therefore be seen as a discrete counterpart of this characterization.

When such a condition is met, convergence of primal and dual components of the solution can be proven

**Proposition 14** Under assumptions (A.2), we assume that  $B$  is surjective, and we denote by  $(u, \lambda)$  the solution to Problem (A.5). Considering now a sequence  $(V_h, \Lambda_h)$  which verifies inf-sup condition (A.9), and  $(u_h, \lambda_h)$  the solution to Problem A.8, there exists  $C > 0$  such that

$$|u - u_h| + |\lambda - \lambda_h| \leq C \left( \inf_{v_h \in V_h} |u - v_h| + \inf_{\mu_h \in \Lambda_h} |\lambda - \mu_h| \right).$$

### A.3 Differential calculus. Integration by parts

**Notations.**

Let  $q$  be a scalar field. Its gradient is the vector

$$\nabla q = \begin{pmatrix} \frac{\partial q}{\partial x_1} \\ \frac{\partial q}{\partial x_2} \end{pmatrix}$$

Let  $\mathbf{u} = (u_1, u_2)^T$  be a vector field. Its divergence is

$$\nabla \cdot \mathbf{u} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}.$$

Let  $\mathbf{u} = (u_1, u_2)^T$  be a vector field. Its gradient is the matrix

$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{pmatrix}$$

For any vector  $\mathbf{n}$ , we have

$$\nabla \mathbf{u} \cdot \mathbf{n} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} n_1 + \frac{\partial u_1}{\partial x_2} n_2 \\ \frac{\partial u_2}{\partial x_1} n_1 + \frac{\partial u_2}{\partial x_2} n_2 \end{pmatrix},$$

which is the derivative of  $\mathbf{u}$  in the  $\mathbf{n}$  direction, so that

$$\mathbf{u}(\mathbf{x} + \varepsilon \mathbf{n}) = \mathbf{u}(\mathbf{x}) + \varepsilon \nabla \mathbf{u} \cdot \mathbf{n} + o(\varepsilon).$$

If  $\mathbf{n}$  a unit vector, we write

$$\nabla \mathbf{u} \cdot \mathbf{n} = \frac{\partial \mathbf{u}}{\partial n}.$$

Let  $\mathbf{u}$  be a vector field. Its Laplacian  $\Delta \mathbf{u}$  is the vector

$$\Delta \mathbf{u} = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix}.$$

For  $A = (a_{ij})$  and  $B = (b_{ij})$  two square matrices,  $A : B$  denotes the scalar

$$A : B = \sum_{i,j} a_{ij} b_{ij}.$$

Note that  $|A| = (A : B)^{1/2}$  is a Euclidean norm on the space of matrices (called *Frobenius* norm). For  $\mathbf{u}$  and  $\mathbf{v}$  two vector fields

$$\nabla \mathbf{u} : \nabla \mathbf{v} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{pmatrix} : \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} \end{pmatrix} = \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}.$$

The notation  $|\nabla \mathbf{u}|^2$  is used to denote  $\nabla \mathbf{u} : \nabla \mathbf{u}$ .

Let  $\boldsymbol{\sigma}$  be a matrix field. Its divergence is a vector, each component of which is the divergence (in the usual sense) of the corresponding matrix line :

$$\nabla \cdot \boldsymbol{\sigma} = \nabla \cdot \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} \end{pmatrix}$$

Let  $\mathbf{u} = (u_1, u_2)^T$  be a vector field. The matrix  $\mathbf{u} \otimes \mathbf{u}$  is the matrix  $(u_i u_j)_{i,j}$ .



If  $\nabla \cdot \mathbf{u} = 0$ , then

$$\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = (\mathbf{u} \cdot \nabla) \mathbf{u} = \begin{pmatrix} u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} \\ u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} \end{pmatrix}$$

Still under the condition that  $\nabla \cdot \mathbf{u} = 0$ ,

$$(\nabla \cdot (\mathbf{u} \otimes \mathbf{u})) \cdot \mathbf{u} = ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \mathbf{u} = \nabla \cdot \left( \frac{|\mathbf{u}|^2}{2} \mathbf{u} \right).$$

If  $\nabla \cdot \mathbf{u} = 0$ , then

$$\nabla \cdot {}^t \nabla \mathbf{u} = 0.$$

As a consequence, if  $\nabla \cdot \mathbf{u} = 0$ , then

$$\nabla \cdot (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) = \nabla \cdot \nabla \mathbf{u} = \Delta \mathbf{u}.$$

### Integration by parts.

Let  $\mathbf{v}$  be a vector field. It holds (Divergence Theorem)

$$\int_{\Omega} \nabla \cdot \mathbf{v} = \int_{\Gamma} \mathbf{v} \cdot \mathbf{n} \quad (\text{A.10})$$

Let  $\boldsymbol{\sigma}$  be a matrix field. It holds (Divergence Theorem for a matrix)

$$\int_{\Omega} \nabla \cdot \boldsymbol{\sigma} = \int_{\Gamma} \boldsymbol{\sigma} \cdot \mathbf{n} \quad (\text{A.11})$$

Let  $q$  be a scalar field. It holds

$$\int_{\Omega} \nabla q = \int_{\Gamma} q \mathbf{n}. \quad (\text{A.12})$$

Let  $\mathbf{v}$  be a vector field, and  $q$  a scalar field. It holds

$$\int_{\Omega} q \nabla \cdot \mathbf{u} + \int_{\Omega} \mathbf{u} \cdot \nabla q = \int_{\Gamma} q \mathbf{u} \cdot \mathbf{n}. \quad (\text{A.13})$$

Let  $u$  and  $v$  be scalar fields. It holds (Green's identity for scalar functions)

$$\int_{\Omega} v \Delta u = \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Gamma} v \frac{\partial u}{\partial n}, \quad (\text{A.14})$$

where  $n$  is the *outward* normal vector.

Let  $\mathbf{u}$  and  $\mathbf{v}$  be vector fields. It holds (Green's identity)

$$\int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{v} + \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} = \int_{\Gamma} \mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial n}. \quad (\text{A.15})$$

And, if  $\nabla \cdot \mathbf{u} = 0$ ,

$$0 + \int_{\Omega} {}^t \nabla \mathbf{u} : \nabla \mathbf{v} = \int_{\Gamma} \mathbf{v} \cdot ({}^t \nabla \mathbf{u} \cdot \mathbf{n}) \quad (\text{A.16})$$

As a consequence, if  $\nabla \cdot \mathbf{u} = 0$ , then

$$\int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{v} + \int_{\Omega} \nabla \mathbf{u} : (\nabla \mathbf{v} + {}^t \nabla \mathbf{v}) = \int_{\Gamma} \mathbf{v} \cdot (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) \cdot \mathbf{n}. \quad (\text{A.17})$$

(note that extra parentheses are not necessary in the boundary integral above, since  $(\tilde{c} \cdot \boldsymbol{\sigma}) \cdot \mathbf{n} = \tilde{c} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n})$  as soon as  $\boldsymbol{\sigma}$  is symmetric, which is the case here).

For any two vector fields  $\mathbf{u}$  and  $\mathbf{v}$ , we have

$$\int_{\Omega} \nabla \mathbf{u} : {}^t \nabla \mathbf{v} = \int_{\Omega} {}^t \nabla \mathbf{u} : \nabla \mathbf{v}, \quad (\text{A.18})$$

so that

$$\int_{\Omega} \nabla \mathbf{u} : (\nabla \mathbf{v} + {}^t \nabla \mathbf{v}) = \frac{1}{2} \int_{\Omega} (\nabla \mathbf{u} + {}^t \nabla \mathbf{u}) : (\nabla \mathbf{v} + {}^t \nabla \mathbf{v}) \quad (\text{A.19})$$

Let  $\omega$  be a material system advected by the velocity field  $\mathbf{u}(x, t)$ , and  $F(x, t)$  a scalar function. It holds

$$\frac{d}{dt} \int_{\omega(t)} F(x, t) = \int_{\omega(t)} \frac{\partial F}{\partial t} - \int_{\partial \omega(t)} F(x, t) \mathbf{u} \cdot \mathbf{n}. \quad (\text{A.20})$$

**Proposition 15** *Soient  $\mathbf{u}$  et  $\mathbf{v}$  deux champs de vecteurs réguliers définis sur  $\Omega$ . On suppose que  $\mathbf{u}$  est à divergence nulle. On a alors*

$$0 = - \int_{\omega} {}^t \nabla \mathbf{u} : \nabla \mathbf{v} + \int_{\partial \omega} \mathbf{v} \cdot ({}^t \nabla \mathbf{u} \cdot \mathbf{n})$$

*Proof* : On écrit

$$\begin{aligned} \int_{\partial \omega} \mathbf{v} \cdot ({}^t \nabla \mathbf{u} \cdot \mathbf{n}) &= \int_{\partial \omega} \mathbf{n} \cdot (\nabla \mathbf{u} \cdot \mathbf{v}) = \int_{\omega} \nabla \cdot (\nabla \mathbf{u} \cdot \mathbf{v}) \\ &= \sum_i \partial_i \sum_j v_j \partial_j u_i = \sum_i \sum_j \partial_i v_j \partial_j u_i + \sum_j v_j \partial_j \sum_i \partial_i u_i. \end{aligned}$$

Le second terme ci-dessus est nul car  $\mathbf{u}$  est à divergence nulle, d'où l'on déduit l'identité annoncée.

**Proposition 16** *Soient  $\mathbf{u}$  et  $\mathbf{v}$  deux champs réguliers sur  $\Omega$ . On a*

$$\int_{\Omega} \nabla \mathbf{u} : {}^t \nabla \mathbf{v} = \int_{\Omega} (\nabla \cdot \mathbf{u}) (\nabla \cdot \mathbf{v}) + \int_{\Gamma} (\nabla \cdot \mathbf{u}) \mathbf{v} \cdot \mathbf{n} - \int_{\Gamma} (\nabla \mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{n}$$

*Proof* : On a

$$\begin{aligned} \int_{\Gamma} (\nabla \mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{n} &= \int_{\Omega} \nabla \cdot (\nabla \mathbf{u} \cdot \mathbf{v}) \\ &= \int_{\Omega} \sum_i \partial_i \sum_j v_j \partial_j u_i \\ &= \int_{\Omega} \sum_i \sum_j ((\partial_i \partial_j u_i) v_j + \partial_j u_i \partial_i v_j) \\ &= \int_{\Omega} \mathbf{v} (\nabla \nabla \cdot \mathbf{u}) + \int_{\Omega} \nabla \mathbf{u} : {}^t \nabla \mathbf{v} \\ &= \int_{\Omega} (\nabla \cdot \mathbf{u}) \mathbf{v} \cdot \mathbf{n} - \int_{\Omega} (\nabla \cdot \mathbf{u}) (\nabla \cdot \mathbf{v}) + \int_{\Omega} \nabla \mathbf{u} : {}^t \nabla \mathbf{v} \end{aligned}$$

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