A posteriori error control for the heat equation

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Outline

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An optimal a posteriori estimate for evolutive problems

Guaranteed upper bound

- $\|u - u_{h\tau}\|^2_{?,\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}^n} \eta_K^n(u_{h\tau})^2$
- no undetermined constant: error control

Local efficiency

- $\eta_K^n(u_{h\tau}) \leq C_{\text{eff}} \|u - u_{h\tau}\|_{\mathfrak{T}_K\times(t^{n-1},t^n)}$
- optimal space–time mesh refinement
- will only be global in space

Asymptotic exactness

- $\sum_{n=1}^{N} \sum_{K\in\mathcal{T}^n} \eta_K^n(u_{h\tau})^2 / \|u - u_{h\tau}\|^2_{?,\Omega\times(0,T)} \searrow 1$
- overestimation factor goes to one with meshes size

Robustness

- C_{eff} indep. of data, domain, final time, meshes, or solution

Small evaluation cost

- estimators can be evaluated locally in space and in time

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$$\begin{aligned} \partial_t u - \Delta u &= f && \text{a.e. in } Q := \Omega \times (0, T), \\ u &= 0 && \text{a.e. on } \partial\Omega \times (0, T), \\ u(\cdot, 0) &= u_0 && \text{a.e. in } \Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$, polygonal domain
- $T > 0$, final simulation time
- $f \in L^2(Q)$, $u_0 \in L^2(\Omega)$

Spaces

- $X := L^2(0, T; H_0^1(\Omega))$, $X' = L^2(0, T, H^{-1}(\Omega))$
- $Y := \{y \in X; \partial_t y \in X'\}$

Weak solution

Find $u \in Y$ such that, for a.e. $t \in (0, T)$ and for all $v \in H_0^1(\Omega)$,

$$\langle \partial_t u, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}(t) + (\nabla u, \nabla v)(t) = (f, v)(t).$$

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Error and residual in the steady case

Laplace equation

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Residual of $u_h \in H_0^1(\Omega)$

- $\mathcal{R}(u_h) \in H^{-1}(\Omega)$, the misfit of u_h in the weak formulation:

$$\langle \mathcal{R}(u_h), \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} := (f, \varphi) - (\nabla u_h, \nabla \varphi) \qquad \varphi \in H_0^1(\Omega)$$

- dual norm of the residual

$$\|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)} := \sup_{\varphi \in H_0^1(\Omega);\, \|\nabla \varphi\|=1} \langle \mathcal{R}(u_h), \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$$

Energy error is the dual norm of the residual

$$\|\nabla(u - u_h)\| = \sup_{\varphi \in H_0^1(\Omega);\, \|\nabla \varphi\|=1} (\nabla(u - u_h), \nabla \varphi) = \|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)}$$

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Intrinsic norms for the heat equation

Norms

- energy norm: $\|v\|_X^2 := \int_0^T \|\nabla v\|^2(t)\,\mathrm{d}t,\ v \in X$
- augmented norm: (Verfürth (2003)) $\|v\|_Y := \|v\|_X + \|\partial_t v\|_{X'}$,

$$\|\partial_t v\|_{X'} = \left\{ \int_0^T \|\partial_t v\|^2_{H^{-1}(\Omega)}(t)\,\mathrm{d}t \right\}^{1/2},\ v \in Y$$

- link: $\|\partial_t v\|_{X'} = \sup_{\varphi \in X,\, \|\varphi\|_X = 1} \int_0^T \langle \partial_t v, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}(t)\mathrm{d}t$
- $\|v\|_{X^*} := \sup_{\varphi \in X,\, \|\varphi\|_X = 1} \int_0^T \{\langle \partial_t v, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + (\nabla v, \nabla \varphi)\}(t)\mathrm{d}t$

Theorem (Norm equivalence, cf. Verfürth (2003))

Let $v \in Y$. Then

$$\|v\|_Y \le 3\|v\|_{X^*} + 2^{1/2}\|v(\cdot,0)\|,$$

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Norm equivalence I

Proof.

- straightforwardly:

$$\begin{aligned}\|v\|_{X^*} &= \sup_{\varphi\in X,\,\|\varphi\|_X=1} \int_0^T \{\langle \partial_t v, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + (\nabla v, \nabla\varphi)\}(t)\mathrm{d}t \\ &\leq \sup_{\varphi\in X,\,\|\varphi\|_X=1} \int_0^T \langle \partial_t v, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}(t)\mathrm{d}t \\ &\quad + \sup_{\varphi\in X,\,\|\varphi\|_X=1} \int_0^T (\nabla v, \nabla\varphi)(t)\mathrm{d}t \\ &= \|\partial_t v\|_{X'} + \|v\|_X = \|v\|_Y\end{aligned}$$

- classically:

$$\frac{1}{2}\|v(\cdot,T)\|^2 = \frac{1}{2}\|v(\cdot,0)\|^2 + \int_0^T \langle \partial_t v, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}(t)\mathrm{d}t$$

- thus $\|v\|_X^2 \leq \frac{1}{2}\|v(\cdot,T)\|^2 + \|v\|_X^2 = \frac{1}{2}\|v(\cdot,0)\|^2 + \int_0^T \{\langle \partial_t v, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + (\nabla v, \nabla v)\}(t)\mathrm{d}t$

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Norm equivalence II

Proof.

- passing to supremum:

$$\|v\|_X^2 \leq \sup_{\varphi\in X,\, \|\varphi\|_X=1} \int_0^T \{\langle \partial_t v, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + (\nabla v, \nabla\varphi)\}(t)\mathrm{d}t \|v\|_X + \frac{1}{2}\|v(\cdot,0)\|^2$$

- $x^2 \leq ax + b^2 \Rightarrow x \leq a + b$, $a, b \geq 0$:

$$\|v\|_X \leq \|v\|_{X^*} + 2^{-1/2}\|v(\cdot,0)\|$$

- for any $\varphi \in X$:

$$\int_0^T \langle \partial_t v, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}(t)\mathrm{d}t = \int_0^T \{\langle \partial_t v, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + (\nabla v, \nabla\varphi) - (\nabla v, \nabla\varphi)\}(t)\mathrm{d}t$$

- thus $\|\partial_t v\|_{X'} \leq \|v\|_{X^*} + \|v\|_X$
- finally $\|v\|_Y = \|\partial_t v\|_{X'} + \|v\|_X \leq 3\|v\|_{X^*} + 2^{1/2}\|v(\cdot,0)\|$

Error and residual for the heat equation

Residual and its dual norm for the heat equation

For $u_{h\tau} \in Y$, the residual $\mathcal{R}(u_{h\tau}) \in X'$:

$$\langle \mathcal{R}(u_{h\tau}), \varphi \rangle_{X',X} := \int_0^T \{(f,\varphi) - \langle \partial_t u_{h\tau}, \varphi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - (\nabla u_{h\tau}, \nabla \varphi)\}(t)\mathrm{d}t \qquad \varphi \in X,$$

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{\varphi \in X,\, \|\varphi\|_X = 1} \langle \mathcal{R}(u_{h\tau}), \varphi \rangle_{X',X}.$$

Corollary (Equivalence of the $\|\cdot\|_Y$ error and of the residual)

Let u be the weak solution. Let $u_{h\tau} \in Y$ be arbitrary. Then

$$\|u - u_{h\tau}\|_Y \le 3\|\mathcal{R}(u_{h\tau})\|_{X'} + 2^{1/2}\|(u - u_{h\tau})(\cdot, 0)\|,$$

$$\|\mathcal{R}(u_{h\tau})\|_{X'} \le \|u - u_{h\tau}\|_Y.$$

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Residual and its dual norm for the heat equation

For $u_{h\tau} \in Y$, the residual $\mathcal{R}(u_{h\tau}) \in X'$:

$$\langle \mathcal{R}(u_{h\tau}), \varphi \rangle_{X',X} := \int_0^T \{(f,\varphi) - \langle \partial_t u_{h\tau}, \varphi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - (\nabla u_{h\tau}, \nabla\varphi)\}(t)\mathrm{d}t \qquad \varphi \in X,$$

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{\varphi\in X,\, \|\varphi\|_X=1} \langle \mathcal{R}(u_{h\tau}), \varphi \rangle_{X',X}.$$

Corollary (Equivalence of the $\|\cdot\|_Y$ error and of the residual)

Let u be the weak solution. Let $u_{h\tau} \in Y$ be arbitrary. Then

$$\|u - u_{h\tau}\|_Y \le 3\|\mathcal{R}(u_{h\tau})\|_{X'} + 2^{1/2}\|(u - u_{h\tau})(\cdot,0)\|,$$

$$\|\mathcal{R}(u_{h\tau})\|_{X'} \le \|u - u_{h\tau}\|_Y.$$

Outline

Time-dependent meshes and discrete solutions

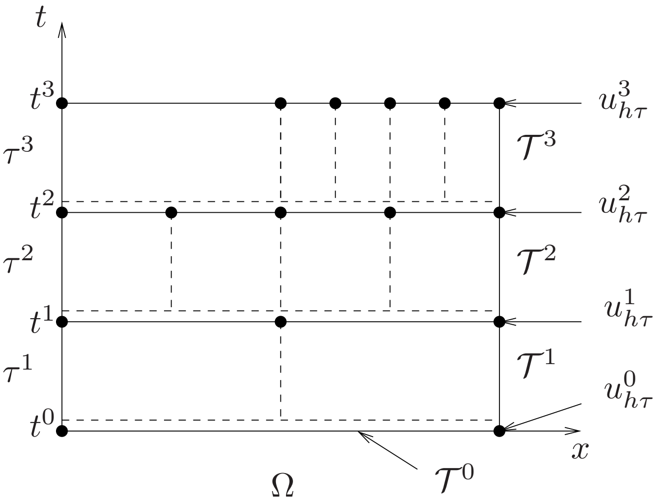
Approximate solutions

- discrete times $\{t^n\}_{0 \le n \le N}$, $t^0 = 0$ and $t^N = T$
- $I_n := (t^{n-1}, t^n]$, $\tau^n := t^n - t^{n-1}$, $1 \le n \le N$
- a different simplicial mesh $\mathcal{T}^n$ on all $0 \le n \le N$
- $u_h^n \in V_h^n$, $0 \le n \le N$, piecewise polynomial for simplicity
- u_h^n possibly nonconforming, $V_h^n \not\subset H_0^1(\Omega)$
- $u_{h\tau} : Q \to \mathbb{R}$ continuous and piecewise affine in time

$$u_{h\tau}(\cdot, t) := (1 - \varrho(t))u_h^{n-1} + \varrho(t)u_h^n, \qquad \varrho(t) = \frac{1}{\tau^n}(t - t^{n-1})$$

- tailored to the backward Euler discretization

Time-dependent meshes and discrete solutions



Time-dependent meshes and discrete solutions

Outline

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Potential and flux reconstructions

General form

- potential reconstruction $s_{h\tau}$ is continuous and piecewise affine in time with $s_h^n \in H_0^1(\Omega)$ for all $0 \le n \le N$
- flux reconstruction $\boldsymbol{\sigma}_{h\tau}$ is piecewise constant in time with $\boldsymbol{\sigma}_h^n := \boldsymbol{\sigma}_{h\tau}|_{I_n} \in \mathbf{H}(\mathrm{div}, \Omega)$ for all $1 \le n \le N$
- $\mathcal{T}^{n,n+1}$ is a common refinement of $\mathcal{T}^n$ and $\mathcal{T}^{n+1}$
- $\tilde{f}^n := \int_{I_n} f(\cdot, t)\mathrm{d}t/\tau^n$ is piecewise constant in time

Assumption A (Potential and flux reconstructions)

Reconstructions s_h^n **preserve mean values** *of u_h^n on $\mathcal{T}^{n,n+1}$*

$$(s_h^n, 1)_K = (u_h^n, 1)_K \qquad \forall K \in \mathcal{T}^{n,n+1}, \forall 1 \le n \le N.$$

Reconstructions $\boldsymbol{\sigma}_h^n$ satisfy a **local conservation property**

$$(\tilde{f}^n - \partial_t u_{h\tau}|_{I_n} - \nabla\cdot\boldsymbol{\sigma}_h^n, 1)_K = 0 \qquad \forall K \in \mathcal{T}^n, \forall 1 \le n \le N.$$

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Practical construction of $s_{h\tau}$ and $\sigma_{h\tau}$

Possible construction of s_h^n

$$s_h^n := \mathcal{I}_{\mathrm{av}}^n(u_h^n) + \sum_{K \in \mathcal{T}^{n,n+1}} \alpha_K^n b_K,$$

$$\alpha_K^n := \frac{1}{(b_K, 1)_K}(u_h^n - \mathcal{I}_{\mathrm{av}}^n(u_h^n), 1)_K$$

- $\mathcal{I}_{\mathrm{av}}^n$: the averaging interpolate on the mesh $\mathcal{T}^n$
- b_K standard (time-independent) bubble function supported on the element K
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Construction of σ_h^n

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Outline

A posteriori error estimate

Theorem (A posteriori error estimate)

Let

- *u be the weak solution*
- *$u_{h\tau}$ given by $u_h^n \in V_h^n$, $0 \le n \le N$, be arbitrary*
- *$s_{h\tau}$ be the mean values-preserving potential reconstruction and $\boldsymbol{\sigma}_{h\tau}$ locally conservative flux reconstruction.*

Then

$$\|u - u_{h\tau}\|_Y \le 3 \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{K \in \mathcal{T}^n} (\eta_{\mathrm{E},K}^n + \eta_{\mathrm{CR},K}^n(t))^2 \, \mathrm{d}t \right\}^{1/2} + \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{K \in \mathcal{T}^n} (\eta_{\mathrm{NC1},K}^n)^2(t) \, \mathrm{d}t \right\}^{1/2} + \left\{ \sum_{n=1}^{N} \tau^n \sum_{K \in \mathcal{T}^n} (\eta_{\mathrm{NC2},K}^n)^2 \right\}^{1/2} + \eta_{\mathrm{IC}} + 3\|f - \tilde{f}\|_{X'}.$$

Estimators

Estimators

- *constitutive relation estimator*
  - $\eta^n_{\mathrm{CR},K}(t) := \|\nabla s_{h\tau}(t) + \boldsymbol{\sigma}^n_h\|_K, \quad t \in I_n$
  - evaluates that $-\nabla s^n_h \notin \mathbf{H}(\mathrm{div},\Omega)$ & temporal error
- *equilibrium estimator*
  - $\eta^n_{\mathrm{E},K} := \frac{h_K}{\pi}\|\widetilde{f}^n - \partial_t s_{h\tau}|_{I_n} - \nabla\cdot\boldsymbol{\sigma}^n_h\|_K$
  - equilibrium evaluated for $\partial_t s_{h\tau}|_{I_n}$
- *nonconformity (constraint) estimators*
  - $\eta^n_{\mathrm{NC1},K}(t) := \|\nabla(s_{h\tau} - u_{h\tau})(t)\|_K, \quad t \in I_n$
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  - evaluate the fact that $u^n_h \notin H^1_0(\Omega)$
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Outline

Separating the space and time error components

Separating the space and time error components

- $\int_{I_n}(\eta^n_{\mathrm{CR},K}(t))^2 \mathrm{d}t \leq (\eta^n_{\mathrm{CR},K,1})^2 + (\eta^n_{\mathrm{CR},K,2})^2,$

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- temporal estimator η^n_{tm} uses $\eta^n_{\mathrm{CR},K,2}$
- spatial estimator η^n_{sp} uses $\eta^n_{\mathrm{CR},K,1}$, $\eta^n_{\mathrm{E},K}$, $\eta^n_{\mathrm{NC1},K}$, and $\eta^n_{\mathrm{NC2},K}$

Corollary (Estimate separating the space and time errors)

$$\|u - u_{h\tau}\|_Y \leq \left\{\sum_{n=1}^{N} (\eta^n_{\mathrm{sp}})^2\right\}^{1/2} + \left\{\sum_{n=1}^{N} (\eta^n_{\mathrm{tm}})^2\right\}^{1/2} + \eta_{\mathrm{IC}} + 3\|f - \tilde{f}\|_{X'}$$

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Outline

Assumptions for efficiency

Assumption B (Piecewise polynomials, data, and meshes)

There holds:

- *the meshes $\{\mathcal{T}^n\}_{0\le n\le N}$ are shape-regular uniformly in n;*
- *the meshes cannot be refined or coarsened too quickly;*
- *for nonconforming methods on time-varying meshes, $(h^n)^2 \lesssim \tau^n$ (mild inverse parabolic CFL on time step).*

Local lower bound for the classical residual estimators:

Assumption C (Approximation property)

For all $1 \le n \le N$ and all $K \in \mathcal{T}^n$, there holds

$$\|\nabla u_h^n + \sigma_h^n\|_K^2 \lesssim \sum_{K'\in\mathfrak{T}_K} h_{K'}^2 \|\tilde{f}^n - \partial_t u_{h\tau}|_{I_n} + \Delta u_h^n\|_{K'}^2 + \sum_{e\in\mathfrak{E}_K^{\mathrm{int}}} h_e \|[\![\nabla u_h^n]\!]\cdot\mathbf{n}_e\|_e^2 + \sum_{e\in\mathfrak{E}_K} h_e^{-1} \|[\![u_h^n]\!]\|_e^2$$

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Efficiency

Theorem (Efficiency)

Let Assumptions B and C hold. Then, for all $1 \le n \le N$,

$$\eta_{\mathrm{sp}}^n + \eta_{\mathrm{tm}}^n \lesssim \|u - u_{h\tau}\|_{Y(I_n)} + \mathcal{J}^n(u_{h\tau}) + \mathcal{E}_f^n.$$

Notation

- jump seminorm:

$$\mathcal{J}^n(u_{h\tau})^2 := \tau^n \sum_{e\in\mathcal{E}^{n-1}} h_e^{-1} \|[\![u_h^{n-1}]\!]\|_e^2 + \tau^n \sum_{e\in\mathcal{E}^n} h_e^{-1} \|[\![u_h^n]\!]\|_e^2$$

- $(\mathcal{E}_f^n)$ space-time data oscillation term

Comments

- $\mathcal{J}^n(u_{h\tau}) \lesssim \|u - u_{h\tau}\|_{X(I_n)}$ if the jumps in $u_{h\tau}$ have zero mean values
- $\mathcal{J}^n$ can be added to error: $(\mathcal{J}^n(u_{h\tau}) = \mathcal{J}^n(u - u_{h\tau}))$
- efficiency is local in time but only global in space

Efficiency

Theorem (Efficiency)

Let Assumptions B and C hold. Then, for all $1 \le n \le N$,

$$\eta_{\mathrm{sp}}^n + \eta_{\mathrm{tm}}^n \lesssim \|u - u_{h\tau}\|_{Y(I_n)} + \mathcal{J}^n(u_{h\tau}) + \mathcal{E}_f^n.$$

Notation

- jump seminorm:

$$\mathcal{J}^n(u_{h\tau})^2 := \tau^n \sum_{e\in\mathcal{E}^{n-1}} h_e^{-1}\|[\![u_h^{n-1}]\!]\|_e^2 + \tau^n \sum_{e\in\mathcal{E}^n} h_e^{-1}\|[\![u_h^n]\!]\|_e^2$$

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Outline

Applications

Space discretization

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- various finite volumes
- mixed finite elements

Time discretization

- backward Euler

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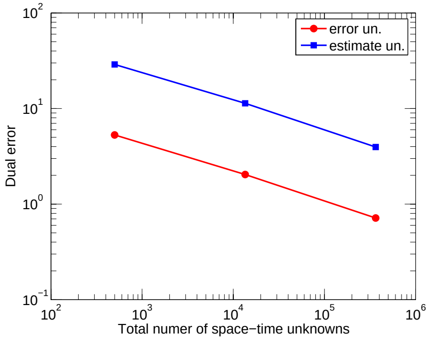
Outline

Numerical experiment

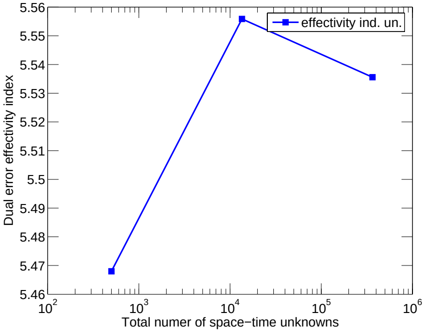
Numerical experiment

- exact solution $u = e^{x+y+t-3}$ on square domain $\Omega =]0,3[^2$, $T = 1.5$ or $T = 3$
- square meshes: 10×10, 30×30, 90×90
- time steps: 0.3, 0.1, 0.3333
- vertex-centered finite volumes

Error, estimate, and effectivity index, $T = 1.5$

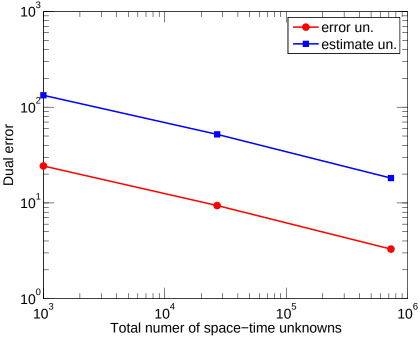


Dual error and estimator

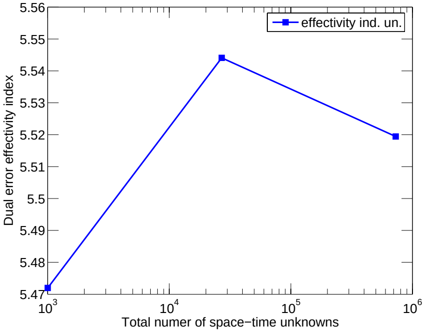


Effectivity index

Error, estimate, and effectivity index, $T = 3$



Dual error and estimator



Effectivity index

Outline

Previous results

Continuous finite elements

- Bieterman and Babuška (1982), introduction
- Eriksson and Johnson (1991), duality techniques
- Picasso (1998), no derefinement allowed
- Babuška, Feistauer, and Šolín (2001), continuous-in-time discretization
- Strouboulis, Babuška, and Datta (2003), guaranteed estimates
- Verfürth (2003), efficiency, robustness with respect to the final time
- Makridakis and Nochetto (2003), elliptic reconstruction
- Bergam, Bernardi, and Mghazli (2005), efficiency
- Lakkis and Makridakis (2006), elliptic reconstruction

Previous results

Finite volumes

- Ohlberger (2001), non-energy-norm estimates
- Amara, Nadau, and Trujillo (2004), energy-norm estimates

Discontinuous Galerkin finite elements

- Sun and Wheeler (2005, 2006), non-energy-norm estimates
- Georgoulis and Lakkis (2009)

Nonconforming finite elements

- Nicaise and Soualem (2005)

Mixed finite elements

- Cascón, Ferragut, and Asensio (2006)

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Bibliography

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Thank you for your attention!