Linear Algebra- Tutorial Sheet 1

1. Let A and B be symmetric matrices of the same size. Show that AB is a symmetric matrix if and only if $AB = BA$.

2. The trace of a square matrix $A = (a_{ij})$ is defined as the sum of its diagonal elements, that is, $\operatorname{tr} A := \sum_i a_{ii}$. Prove that if A, B are square matrices of the same order and α, β are scalars then
   (i) $\operatorname{tr}(\alpha A + \beta B) = \alpha \operatorname{tr}(A) + \beta \operatorname{tr}(B)$; (ii) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
   (iii) If A is invertible, then $\operatorname{tr}(ABA^{-1}) = \operatorname{tr}(B)$.

3. A square matrix A is called **nilpotent** if $A^m = O$ for some positive integer m. Show that an $n \times n$ matrix $A = (a_{ij})$ in which $a_{ij} = 0$ for $i \geq j$, is nilpotent. In fact, $A^n = O$.

4. The **conjugate transpose** (also called **Hermitian adjoint**) A^* of a complex $m \times n$ matrix A is defined as the transpose of its conjugate (or equivalently, the conjugate of its transpose). Prove that the properties of the conjugate transpose are analogous to that of the transpose: e.g., $(A+B)^* = A^*+B^*$ and $(AB)^* = B^*A^*$. Note, however, that $(\alpha A)^* = \overline{\alpha} A^*$.

5. A square matrix A is called **Hermitian** if $A^* = A$ and **skew-Hermitian** if $A^* = -A$. Show that every square matrix can be uniquely expressed as the sum of a Hermitian and a skew-Hermitian matrix.

6. If A and B are $n \times n$ Hermitian matrices and α and β are any real numbers, show that $C = \alpha A + \beta B$ is a Hermitian matrix.

7. Let A and B be skew-Hermitian matrices. For arbitrary complex scalars a and b, under what conditions is the matrix $C = aA + bB$ skew-Hermitian?

8. Let A, B be nilpotent matrices of the same order.
   (i) Show by an example that $A + B$ need not be nilpotent.
   (ii) However, prove that this is the case if A and B commute with each other (i.e., if $AB = BA$). [Hint: In this case, show that the binomial theorem holds for the expansion of $(A + B)^n$.]

9. Given a polynomial $p(x)$ and a square matrix A, let $p(A)$ denote the matrix obtained by 'substituting' A for the variable x. Thus, if $p(x) = a_0 + a_1x + \cdots + a_kx^k$, then $p(A) = a_0I + a_1A + \cdots + a_kA^k$. Given polynomials p and q, show that

(i) $(p+q)(A) = p(A) + q(A)$.

(ii) $(pq)(A) = p(A)q(A)$.

10. Show that any diagonal matrix D whose only possible entries are 0 and 1 is idempotent, that is, $D^2 = D$.

11. If $A^2 = A$, prove that $(A+I)^n = I + (2^n - 1)A$.

12. A square matrix $A = (a_{ij})$ is called **upper triangular** if $a_{ij} = 0$ for all $j < i$. A **lower triangular** matrix is defined similarly (or equivalently, as the transpose of an upper triangular matrix). Prove that the sum as well as the product of two upper triangular matrices (of equal orders) is upper triangular.

13. Show that $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$ for all $\lambda \in \mathbb{R}, n \geq 1$.

14. An $m \times n$ matrix, all whose entries are 1 is often denoted by $J_{m\times n}$ or simply by J if m, n are understood. Let $A = J_{n\times n}$ and $B = J_{n\times 1}$. Prove that $AB = nB$, $A^2B = n^2B, \ldots$ and, in general, for any polynomial $p(x) = a_0 + a_1x + \cdots + a_rx^r$, we have $p(A)B = p(n)B$.

15. Let $N_{n\times n}$ be an upper triangular matrix with diagonal entries zero. Show that $I + N$ is invertible and $(I+N)^{-1} = I - N + N^2 - + \cdots + (-1)^{n-1}N^{n-1}$.

16. Solve the following system of linear equations in the unknowns $x_1, \ldots, x_5$ by the Gauss Elimination Method.

(i)
$$\begin{array}{rrrrl} & 2x_3 & -2x_4 & +x_5 & = 2 \\ 2x_2 & -8x_3 & +14x_4 & -5x_5 & = 2 \\ x_2 & +3x_3 & & +x_5 & = 8 \end{array}$$

(ii)
$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & -x_4 & = 2 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & = -2 \end{array}$$

(iii)
$$\begin{array}{rrrrl} & & -2x_4 & +x_5 & = 2 \\ 2x_2 & -2x_3 & +14x_4 & -x_5 & = 2 \\ 2x_2 & +3x_3 & +13x_4 & +x_5 & = 3 \end{array}$$

(iv)
$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & -x_4 & = 2 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & = -2 \end{array}$$