KOSZUL PROPERTY OF DIAGONAL SUBALGEBRAS

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ABSTRACT. Let $S = K[x_1,\dots,x_n]$ be a polynomial ring over a field K and I a homogeneous ideal in S generated by a regular sequence $f_1, f_2, \dots, f_k$ of homogeneous forms of degree d. We study a generalization of a result of Conca, Herzog, Trung, and Valla [9] concerning Koszul property of the diagonal subalgebras associated to I. Each such subalgebra has the form $K[(I^e)_{ed+c}]$, where $c, e \in \mathbb{N}$. For $k = 3$, we extend [9, Corollary 6.10] by proving that $K[(I^e)_{ed+c}]$ is Koszul as soon as $c \geq \frac{d}{2}$ and $e > 0$. We also extend [9, Corollary 6.10] in another direction by replacing the polynomial ring with a Koszul ring.

1. INTRODUCTION

Let $S = K[x_1,\dots,x_n]$ be a polynomial ring over a field K and I a homogeneous ideal in S. For large c, the algebra $K[(I^e)_c]$ is isomorphic to the coordinate ring of some embedding of the blow up of $\mathbb{P}_K^{n-1}$ along the ideal sheaf $\tilde{I}$ in a projective space [9].

Let $\mathrm{Rees}(I) = \bigoplus_{j\geq 0} I^j t^j$ be the *Rees algebra* of I. Since the polynomial ring $S[t]$ is a bigraded algebra with $S[t]_{(i,j)} = S_i t^j$, we may consider $\mathrm{Rees}(I)$ as a bigraded subalgebra of $S[t]$ with $\mathrm{Rees}(I)_{(i,j)} = (I^j)_i t^j$.

Let c and e be positive integers. Let $\triangle = \{(cs, es) : s \in \mathbb{Z}\}$. We call $\triangle$ the (c,e)-*diagonal* of $\mathbb{Z}^2$ [9]. Let $R = \oplus_{(i,j)\in\mathbb{Z}^2} R_{(i,j)}$ be a bigraded algebra, where $R_{(i,j)}$ denotes the (i,j)-th bigraded component of R. The (c,e)-*diagonal subalgebra* of R is defined as the $\mathbb{Z}$-graded algebra $R_\triangle = \bigoplus_{s\in\mathbb{Z}} R_{(cs,es)}$. Similarly, for every bigraded R-module M, one defines the (c,e)-*diagonal submodule* of M as $M_\triangle = \bigoplus_{s\in\mathbb{Z}} M_{(cs,es)}$. Notice that $M_\triangle$ is a module over $R_\triangle$.

When I is a homogeneous ideal in S generated by $f_1, f_2, \dots, f_k$, of homogeneous forms of degree d, then $\mathrm{Rees}(I)$ is a standard bigraded algebra by setting $\deg x_i = (1,0)$ and $\deg f_j t = (0,1)$. We observe that $K[(I^e)_{ed+c}]$ is the (c,e)-diagonal subalgebra of $\mathrm{Rees}(I)$. We may also view $K[(I^e)_{ed+c}]$ as a K-subalgebra of S generated by the forms of degree $ed + c$ in the ideal I^e.

Given a field K, a positively graded K-algebra $A = \bigoplus_{i\in\mathbb{N}} A_i$ with $A_0 = K$ is *Koszul* if the field K, viewed as a A-module via the identification $K = A/A_+$, has a linear free resolution. Koszul algebras were introduced by Priddy [15] in 1970. During the last four decades Koszul algebra have been studied in various contexts. Good survey on Koszul algebra is given by Fröberg in [12] during nineties and recently by Conca, De Negri and Rossi in [10].

Diagonal subalgebras have been studied intensively by several authors (e.g see [9], [16], [1]) because they naturally appear in Rees algebras and symmetric algebras. In [9] Conca, Herzog, Trung and Valla discuss some algebraic properties

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of diagonal subalgebras, such as Cohen-Macaulayness and Koszulness. In [14] Kurano et al. showed that Cohen-Macaulayness property holds in [9] even if the polynomial ring is replaced by a Cohen-Macaulay ring of dimension $d \geq 2$. In this article, we study the Koszul property of certain diagonal subalgebras of bigraded algebras, with focus and applications to diagonals of Rees algebras. We generalize some of the important results of [9] regarding the Koszulness of certain diagonal subalgebras of bigraded algebras.

For any homogeneous ideal I, there exists integers c_0, e_0 such that the K-algebra $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq c_0$ and $e \geq e_0$, see [9, Corollary 6.9]. If I is a complete intersection ideal generated by $f_1, f_2, \ldots, f_k$, of homogeneous forms of degree d, then the K-algebra $K[(I^e)_{ed+c}]$ is quadratic if $c \geq \frac{d}{2}$ and $e > 0$; furthermore $K[(I^e)_{ed+c}]$ is Koszul if $c \geq \frac{d(k-1)}{k}$ and $e > 0$, see [9, Corollary 6.10].

The main results of this paper are the following:

(i) Let I be an ideal of the polynomial ring $K[x_1, \ldots, x_n]$ generated by a regular sequence f_1, f_2, f_3, of homogeneous forms of degree d. Then $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq \frac{d}{2}$ and $e > 0$.

(ii) Let A be a standard graded Koszul ring. Let I be an ideal of A generated by a regular sequence $f_1, f_2, \ldots, f_k$, of homogeneous forms of degree d. Then $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq \frac{d(k-1)}{k}$ and $e > 0$.

Let R be a standard bigraded K-algebra. In Section 2, we study homological properties of shifted modules $R(-a,-b)_\triangle$, which play an important role in the transfer of homological information from R to $R_\triangle$. It is important to bound the homological invariants of the shifted diagonal module $R(-a,-b)_\triangle$ as an $R_\triangle$-module. For a bigraded polynomial ring R, it is proved in [9] that $R(-a,-b)_\triangle$ has a linear $R_\triangle$ resolution. Proposition 2.10 is an extension of [9, Theorem 6.2] for certain bigraded complete intersection ideal and crucial in proving Theorem 3.1.

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring. In [9, p.900] the authors mentioned that for a complete intersection ideal I in S generated by $f_1, f_2, \ldots, f_k$, of homogeneous forms of degree d, the algebra $K[(I^e)_{ed+c}]$ is expected to be Koszul as soon as $c \geq \frac{d}{2}$. For $k = 1,2$, it is obvious. The first nontrivial case is $k = 3$. In Section 3, we answer their expectation affirmatively for $k = 3$, see Theorem 3.1. The motivation for such generalization came from the work of Caviglia [7], and Caviglia and Conca [8]. Note that for $k = 3$, the result of [8] is just the case: $d = 2$ and $c = 1$; furthermore the main result of [7] correspond to the case: $d = 2$, $c = 1$, $f_1 = x_1^2, f_2 = x_2^2, f_3 = x_3^2$ and $n = 3$.

In Section 4, we generalize [9, Theorem 6.2] and some of its relevant corollaries. The main result of this section is Theorem 4.5, which is a generalization of [9, Corollary 6.10] regarding the koszulness of certain diagonals of the Rees algebra of an ideal in the polynomial ring. We show that the Koszulness property holds even if the polynomial ring is replaced by a Koszul ring. The reason for such generalization comes from the fact that from certain point of view, Koszul algebras behave homologically as polynomial rings.

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2. Generalities and preliminary results

Let A be a standard graded K-algebra i.e. $A = \oplus A_i = S/I$, where S is a polynomial ring and I a homogeneous ideal of S. For a finitely generated graded A-module $M = \oplus M_i$, set
$$t_i^A(M) = \sup\{j : \operatorname{Tor}_i^A(M,K)_j \neq 0\},$$
with $t_i^A(M) = -\infty$ if $\operatorname{Tor}_i^A(M,K) = 0$.

Definition 2.1. The *Castelnuovo-Mumford regularity* $\operatorname{reg}_A(M)$ of an A-module M is defined to be
$$\operatorname{reg}_A M = \sup\{t_i^A(M) - i : i \geq 0\}.$$
When $A = S$ is a polynomial ring, one has
$$\operatorname{reg}_S M = \max\{t_i^S(M) - i : i \geq 0\}.$$
For a polynomial ring $S = K[x_1,\dots,x_n]$, one can also compute $\operatorname{reg}_S M$ via the local cohomology modules $H^i_{\mathfrak{m}}(M)$ for $i = 0,1,\dots,n$. One has
$$\operatorname{reg}_S M = \max\{j+i : H^i_{\mathfrak{m}}(M)_j \neq 0\}.$$

Definition 2.2. Koszul algebra: A standard graded K-algebra A is said to be a *Koszul algebra* if the residue field K has a linear A-resolution. Equivalently, A is Koszul when $\operatorname{reg}_A(K) = 0$.

Example 2.3. Let $A = K[x]/(x^2)$, then K has a linear A-resolution
$$\cdots \to A(-2) \xrightarrow{\bar{x}} A(-1) \xrightarrow{\bar{x}} A \to K \to 0.$$

The property of being a Koszul algebra is preserved under various constructions, in particular under taking tensor products, Segre products and Veronese subrings, see Backelin and Fröberg [4].

Let A be a Koszul algebra, and S be the polynomial ring mapping onto A. Then the regularity of any finitely generated graded module M over A is always finite; in fact, $\operatorname{reg}_A M \leq \operatorname{reg}_S M$, see Avramov and Eisenbud [3, Theorem 1]. If $M = \oplus_{i=a}^b M_i$ with $M_b \neq 0$, then
$$\operatorname{reg}_A M \leq \operatorname{reg}_S M = b. \tag{1}$$
If $A = \oplus_{i\geq 0} A_i$ is a graded algebra, then the c-th *Veronese subalgebra* is $A^{(c)} = \oplus_{i\geq 0} A_{ic}$. An element in A_{ic} is considered to have degree i.

Definition 2.4. Consider a standard graded K-algebra A. Given $k,m \in \mathbb{N}$, and $0 \leq k < m$, we set
$$V_A(m,k) = \bigoplus_{i\in\mathbb{N}} A_{im+k}.$$
We observe that $A^{(m)} = V_A(m,0)$ is the usual m-th Veronese subring of A, and that the $V_A(m,k)$ are $A^{(m)}$-modules known as the *Veronese modules* of A. For a finitely generated graded A-module M, similarly we define
$$M^{(m)} = \bigoplus_{i\in\mathbb{Z}} M_{im}.$$
We consider $A^{(m)}$ as a standard graded K-algebra with homogeneous component of degree i equal to A_{im}, and $M^{(m)}$ as a graded $A^{(m)}$-module with homogeneous components M_{im} of degree i.

Definition 2.5. Let A and B be positively graded K-algebra. Denote by $A\underline{\otimes}B$ the *Segre product*

$$A\underline{\otimes}B = \bigoplus_{i\in\mathbb{N}} A_i \otimes_K B_i,$$

of A and B. Given graded modules M and N over A and B, one may form the Segre product

$$M\underline{\otimes}N = \bigoplus_{i\in\mathbb{Z}} M_i \otimes_K N_i,$$

of M and N. Clearly $M\underline{\otimes}N$ is a graded $A\underline{\otimes}B$- module.

A beautiful introduction to construction of multigraded objects including Segre products is given by Goto and Watanabe in [13, Chapter 4]. Segre products have been studied in the sense of Koszulness by several authors e.g. Backelin et al. [4], Eisenbud et al. [11], Conca et al. [9], Fröberg [12] and Blum [5]. We will use their results at several occasions in this paper.

Let A and B be Koszul K-algebras. Let M be a finitely generated graded A-module and N be a finitely generated graded B-module. Assume M and N have linear resolutions over A and B respectively. Also assume $M\underline{\otimes}N \neq 0$. Then by [9, Lemma 6.5], $M\underline{\otimes}N$ has a linear $A\underline{\otimes}B$-resolution and

$$\operatorname{reg}_{A\underline{\otimes}B} M\underline{\otimes}N = \max\{\operatorname{reg}_A M,\ \operatorname{reg}_B N\}. \tag{2}$$

We will use this relation on regularity for Segre products in the proof of Theorem 4.1 and Theorem 4.5.

Remark 2.6. It is to be noticed that [9, Lemma 6.5] also needs the hypothesis $M\underline{\otimes}N \neq 0$. For instance, Assume $M = K$ and $N = K(-1)$, then $M\underline{\otimes}N = 0$ and this leads to inconsistency in (2).

The following lemma is very useful. We will use this lemma in the proof of Proposition 2.10, Theorem 3.1 and Theorem 4.5.

Lemma 2.7. (Technical Lemma)
Let

$$\mathbf{M} : \cdots \to M_{k+1} \to M_k \to M_{k-1} \to \cdots \to M_1 \to M_0 \to 0,$$

be a complex of graded A-modules with maps of degree 0. *Set $H_i = H_i(\mathbf{M})$. Then for every $i \geq 0$ one has*

$$t_i^A(H_0) \leq \max\{\alpha, \beta\}, \tag{3}$$

where $\alpha = \sup\{t_{i-j}^A(M_j) : j = 0, \ldots, i\}$ and $\beta = \sup\{t_{i-j-1}^A(H_j) : j = 1, \ldots, i-1\}$. Moreover one has

$$\operatorname{reg}_A(H_0) \leq \max\{\alpha', \beta'\}, \tag{4}$$

where $\alpha' = \sup\{\operatorname{reg}_A(M_j) - j : j \geq 0\}$ and $\beta' = \sup\{\operatorname{reg}_A(H_j) - (j+1) : j \geq 1\}$.

Proof. Let Z_i, B_i, H_i denotes the i-th cycles, boundaries and homology modules respectively. We have short exact sequences

$$0 \to B_i \to Z_i \to H_i \to 0,$$

and

$$0 \to Z_{i+1} \to M_{i+1} \to B_i \to 0,$$

with $Z_0 = M_0$. Therefore, by [6, Lemma 2.2(a)], one has

$$\begin{aligned} t_i^A(H_0) &\leq \max\{t_i^A(M_0), t_{i-1}^A(B_0)\}, \\ t_{i-1}^A(B_0) &\leq \max\{t_{i-1}^A(M_1), t_{i-2}^A(Z_1)\}, \\ t_{i-2}^A(Z_1) &\leq \max\{t_{i-2}^A(B_1), t_{i-2}^A(H_1)\}, \\ t_{i-2}^A(B_1) &\leq \max\{t_{i-2}^A(M_2), t_{i-3}^A(Z_2)\}, \text{ and so on.} \end{aligned}$$

Summarizing the details, one has:

$$t_i^A(H_0) \leq \max\{t_i^A(M_0),\, t_{i-1}^A(M_1)\,,\, \ldots,\, t_{i-j}^A(M_j),\, t_{i-2}^A(H_1),\, \ldots,\, t_{i-j-1}^A(H_j)\}.$$

Take $\alpha = \sup\{t_{i-j}^A(M_j) : j = 0, \ldots, i\}$ and $\beta = \sup\{t_{i-j-1}^A(H_j) : j = 1, \ldots, i-1\}$, then we obtain the desired result (3) for $t_i^A(H_0)$. The second inequality (4) follows from (3). $\square$

Let $R = \oplus_{(i,j)\in\mathbb{Z}^2} R_{(i,j)}$ be a bigraded standard K-algebra. Here standard means that $R_{(0,0)} = K$ and R is generated as a K-algebra by the K-vector spaces $R_{(1,0)}$ and $R_{(0,1)}$ of finite dimension.

Definition 2.8. Diagonal subalgebra: Let R be a bigraded standard K-algebra. Let c and e be positive integers. Let $\triangle$ be the (c,e)-diagonal of $\mathbb{Z}^2$. The (c,e)-*diagonal subalgebra* $R_\triangle$ of R is defined as

$$R_\triangle = \bigoplus_{s\in\mathbb{Z}} R_{(cs,es)}.$$

We observe that $R_\triangle$ is the K-subalgebra of R generated by $R_{(c,e)}$ and hence it is a standard graded K-algebra. Similarly, one defines the (c,e)-diagonal submodule of any bigraded R-module $M = \oplus_{(i,j)\in\mathbb{Z}^2} M_{(i,j)}$ as $M_\triangle = \bigoplus_{s\in\mathbb{Z}} M_{(cs,es)}$. Notice that $M_\triangle$ is a module over $R_\triangle$. The map $M \longmapsto M_\triangle$, being a selection of homogeneous components, defines an exact functor from the category of bigraded R-modules and maps of degree 0 to the category of graded $R_\triangle$-modules with maps of degree 0.

Notation 2.9. We have $\triangle = \{(cs,es) : s \in \mathbb{Z}\}$, but the bounds on c and e change from time to time. Note that $c, s,\ \triangle$ will always be used in this way, with c and s changing as described. For a real number α, we use $\lceil \alpha \rceil$ for the smallest integer m such that $m \geq \alpha$.

For $(a,b) \in \mathbb{Z}^2$, let $R(-a,-b)$ be a shifted copy of R. By definition

$$R(-a,-b)_\triangle = \bigoplus_{s\in\mathbb{Z}} R_{(-a+cs,-b+es)}.$$

Since R is positively graded, we may consider only those $s \in \mathbb{Z}$ for which $-a+cs \geq 0$ and $-b+es \geq 0$. Assume $\max\left(\lceil \frac{a}{c} \rceil, \lceil \frac{b}{e} \rceil\right) = \lceil \frac{a}{c} \rceil$. Then

$$R(-a,-b)_\triangle = \bigoplus_{s\geq \lceil \frac{a}{c} \rceil} R_{(-a+cs,-b+es)}.$$

Therefore $R(-a,-b)_\triangle$ is a $R_\triangle$-submodule of R generated by $R_{(-a+c\lceil \frac{a}{c} \rceil, -b+e\lceil \frac{a}{c} \rceil)}$. The other case is similar, summarizing the details, one has:

$$R(-a,-b)_\triangle = \begin{cases} R(-a+c\lceil \frac{a}{c} \rceil, -b+e\lceil \frac{a}{c} \rceil)_\triangle(-\lceil \frac{a}{c} \rceil), & \text{if } \lceil \frac{a}{c} \rceil \geq \lceil \frac{b}{e} \rceil; \\ R(-a+c\lceil \frac{b}{e} \rceil, -b+e\lceil \frac{b}{e} \rceil)_\triangle(-\lceil \frac{b}{e} \rceil), & \text{if } \lceil \frac{a}{c} \rceil \leq \lceil \frac{b}{e} \rceil. \end{cases}$$

The homological properties of the shifted diagonal module $R(-a,-b)_\triangle$ play an important role in the transfer of homological information from R to $R_\triangle$. In the following proposition, we try to bound the homological invariants (regularity) of the shifted diagonal module $R(-a,-b)_\triangle$ as an $R_\triangle$-module. The following proposition is crucial in proving Theorem 3.1.

Proposition 2.10. *Let $S = K[x_1,\ldots,x_m,t_1,\ldots,t_n]$ be a polynomial ring bigraded by $\deg x_i = (1,0)$ for $i = 1,\ldots,m$ and $\deg t_i = (0,1)$ for $i = 1,\ldots,n$. Let I be an ideal of S generated by a regular sequence with elements all of bidegree $(d,1)$ and $R = S/I$. Let $\frac{d}{2} \le c < \frac{2d}{3}$ and $e > 0$. Then:*

(a) $R_\triangle$ is Koszul.

(b) $\operatorname{reg}_{R_\triangle} R(-a,-b)_\triangle \le \max\{\lceil \frac{a}{c} \rceil, \lceil \frac{b}{e} \rceil\}$.

Proof. Let h be the codimension of I. The proof is by induction on h. If $h = 0$, then $R_\triangle$ is the Segre product of $K[x_1,\ldots,x_m]^{(c)}$ and $K[t_1,\ldots,t_n]^{(e)}$. Thus $R_\triangle$ is Koszul by [4]. For (b), see [9, proof of Theorem 6.2].

Assume $h > 0$. We may write $R = T/(f)$ where f is a T-regular element of bidegree $(d,1)$ and where T is defined as the quotient of S by an S-regular sequence of length $h-1$ of elements of bidegree $(d,1)$. We have a short exact sequence of T-modules:

$$0 \longrightarrow T(-d,-1) \longrightarrow T \longrightarrow R \longrightarrow 0 \tag{5}$$

and applying $-_\triangle$, we have an exact sequence of $T_\triangle$-modules:

$$0 \longrightarrow T(-d,-1)_\triangle \longrightarrow T_\triangle \longrightarrow R_\triangle \longrightarrow 0.$$

By induction we know $T_\triangle$ is Koszul and that $\operatorname{reg}_{T_\triangle} T(-d,-1)_\triangle \le \lceil \frac{d}{c} \rceil$. As $\frac{d}{c} \le 2$, one has

$$\operatorname{reg}_{T_\triangle} R_\triangle \le 1.$$

By [8, Lemma 2.1(3)], we may conclude that $R_\triangle$ is Koszul, as $T_\triangle$ is Koszul by induction. Now to prove (b) we consider the following two cases.

Case 1. Assume $\lceil \frac{a}{c} \rceil < \lceil \frac{b}{e} \rceil$. Shift (5) by $(-a,-b)$ and then apply $-_\triangle$, we get a short exact sequence of $T_\triangle$-modules:

$$0 \longrightarrow T(-a-d,-b-1)_\triangle \longrightarrow T(-a,-b)_\triangle \longrightarrow R(-a,-b)_\triangle \longrightarrow 0.$$

So we have:

$$\operatorname{reg}_{T_\triangle} R(-a,-b)_\triangle \le \max\{\ \operatorname{reg}_{T_\triangle} T(-a,-b)_\triangle,\ \operatorname{reg}_{T_\triangle} T(-a-d,-b-1)_\triangle - 1\ \}.$$

By induction, one has $\operatorname{reg}_{T_\triangle} T(-a,-b)_\triangle \le \lceil \frac{b}{e} \rceil$, and

$$\operatorname{reg}_{T_\triangle} T(-a-d,-b-1)_\triangle \le \max\{\lceil \frac{a+d}{c} \rceil, \lceil \frac{b+1}{e} \rceil\}.$$

Since

$$\lceil \frac{a+d}{c} \rceil \le \lceil \frac{a}{c} \rceil + \lceil \frac{d}{c} \rceil \le (\lceil \frac{b}{e} \rceil - 1) + 2 = \lceil \frac{b}{e} \rceil + 1 \text{ and } \lceil \frac{b+1}{e} \rceil \le \lceil \frac{b}{e} \rceil + 1,$$

we conclude that $\operatorname{reg}_{T_\triangle} T(-a-d,-b-1)_\triangle \le \lceil \frac{b}{e} \rceil + 1$. Thus we have

$$\operatorname{reg}_{T_\triangle} R(-a,-b)_\triangle \le \lceil \frac{b}{e} \rceil.$$

Since we have already shown that $\operatorname{reg}_{T_\triangle} R_\triangle \leq 1$, we may conclude by [8, Lemma 2.1(1)] that

$$\operatorname{reg}_{R_\triangle} R(-a,-b)_\triangle \leq \lceil \frac{b}{e} \rceil.$$

Case 2. Assume $\lceil \frac{a}{c} \rceil \geq \lceil \frac{b}{e} \rceil$. Set $P = (t_1, \ldots, t_n) \subset S$. We have

$$R(-a,-b)_\triangle = R(-a + c\lceil \frac{a}{c} \rceil, -b + e\lceil \frac{a}{c} \rceil)_\triangle(-\lceil \frac{a}{c} \rceil).$$

So we have to prove that $\operatorname{reg}_{R_\triangle} R(\alpha,\beta)_\triangle \leq 0$ where $\alpha = -a + c\lceil \frac{a}{c} \rceil$ and $\beta = -b + e\lceil \frac{a}{c} \rceil$. Consider the minimal free (bigraded) resolution of S/P^β as an S-module:

$$\mathbf{F}: \quad 0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0,$$

with $F_0 = S$ and $F_i = S^\sharp(0, -\beta - i + 1)$ for $i > 0$ where $\sharp$ denotes some integer depending on n, β and i that is irrelevant in our discussion. The homology of $\mathbf{F} \otimes R$ is $\operatorname{Tor}^S_\bullet(S/P^\beta, R)$. We may as well compute $\operatorname{Tor}^S_\bullet(S/P^\beta, R)$ as the homology of $S/P^\beta \otimes \mathbf{G}$ where $\mathbf{G}$ is a free resolution of R as an S-module. By assumption, we may take $\mathbf{G}$ to be a Koszul complex on a sequence of elements of bidegree $(d,1)$. It follows that:

$$H_i(\mathbf{F} \otimes R) = \begin{cases} \text{a subquotient of } (S/P^\beta)^\sharp(-di, -i), & \text{if } 0 \leq i \leq h; \\ 0, & \text{if } i > h. \end{cases}$$

Shifting with (α, β) and applying $-_\triangle$ we have a complex $(\mathbf{F} \otimes R(\alpha,\beta))_\triangle$. We claim this complex has no homology. Shifting and applying $-_\triangle$ are compatible operations with taking homology. Therefore to prove $(\mathbf{F} \otimes R(\alpha,\beta))_\triangle$ has no homology at all, we only need to check that

$$[(S/P^\beta)(-di + \alpha, -i + \beta)]_\triangle = 0 \quad \text{for all } i. \tag{6}$$

To prove (6), take the j-th degree component,

$$[[(S/P^\beta)(-di + \alpha, -i + \beta)]_\triangle]_j = (S/P^\beta)_{(cj - di + \alpha, ej - i + \beta)}.$$

We will show that

$$(S/P^\beta)_{(cj - di + \alpha, ej - i + \beta)} = 0. \tag{7}$$

The case $i = 0$ will be dealt separately. Assume $i > 0$. Clearly (7) holds if $ej - i + \beta \geq \beta$, that is, if $ej \geq i$. To complete the argument for (7), it is enough to show that $cj - di + \alpha < 0$ for $ej < i$. Let $a = qc + r$; $0 \leq r < c$, then

$$\lceil \frac{a}{c} \rceil = \begin{cases} q+1, & \text{if } r \neq 0; \\ q, & \text{if } r = 0. \end{cases}$$

Therefore

$$cj - di + \alpha = \begin{cases} cj - di + c - r, & \text{if } r \neq 0; \\ cj - di, & \text{if } r = 0. \end{cases} \tag{8}$$

Assume $r = 0$ and $ej < i$. It is easy to see that

$$cj - di < i(\frac{c}{e} - d) < i(\frac{2d}{3e} - d) < 0.$$

Assume $r \neq 0$ and $ej < i$. By assumption we have $j \leq \frac{i}{e} - \frac{1}{e}$ and $cj - di + c - r < c(j+1) - id$. One has

$$c(j+1) - di \leq c(\frac{i}{e} - \frac{1}{e} + 1) - di = i(\frac{c}{e} - d) - \frac{c}{e} + c. \tag{9}$$

As $\frac{d}{2} \leq c < \frac{2d}{3}$ and $e > 0$, we may write

$$\frac{c}{e} - d < \frac{2d}{3e} - d = \frac{d(2-3e)}{3e} \text{ and } -\frac{c}{e} \leq -\frac{d}{2e}. \tag{10}$$

Thus by (9) and (10), we have

$$c(j+1) - di < i(\frac{d(2-3e)}{3e}) - \frac{d}{2e} + \frac{2d}{3} = \frac{d}{6e}[i(4-6e) + (4e-3)].$$

It is easy to see that $i(4-6e) + (4e-3) < 0$ for all $i > 0$. Assume $i = 0$. Denote by $\mathbf{C}$, the complex $(\mathbf{F} \otimes R(\alpha,\beta))_\triangle$. If $H_0(\mathbf{C}) \neq 0$, then either $ej + \beta < \beta$ or $cj + \alpha \geq 0$ in (7). Thus $H_0(\mathbf{C}) \neq 0$ if $\frac{-\alpha}{c} \leq j < 0$, which is not possible, since j has to be an integer and by previous discussion in (8), one has $-1 < \frac{-\alpha}{c} \leq 0$. Since $H_i(\mathbf{C}) = 0$ for all $i \geq 0$, we have the following exact complex $\mathbf{C}$:

$$0 \longrightarrow R(\alpha, -i+1)_\triangle \longrightarrow \cdots \longrightarrow R(\alpha,-1)_\triangle \longrightarrow R(\alpha,0)_\triangle \longrightarrow R(\alpha,\beta)_\triangle \longrightarrow 0.$$

From the exact complex $\mathbf{C}$, we build another complex:

$$\mathbf{T}: \ 0 \longrightarrow R(\alpha, -i+1)_\triangle \longrightarrow \cdots \longrightarrow R(\alpha,-1)_\triangle \longrightarrow R(\alpha,0)_\triangle \longrightarrow 0.$$

Then the homology of the new complex $\mathbf{T}$ is given by

$$H_i(\mathbf{T}) = \begin{cases} R(\alpha,\beta)_\triangle, & \text{if } i = 0; \\ 0, & \text{if } i > 0. \end{cases}$$

Thus Lemma 2.7 applied to the complex $\mathbf{T}$, one has

$$\operatorname{reg}_{R_\triangle} R(\alpha,\beta)_\triangle \leq \max\{\operatorname{reg}_{R_\triangle} R(\alpha,-i)_\triangle - i : i \geq 0\}.$$

Note that by Case 1, we have $\operatorname{reg}_{R_\triangle} R(\alpha,-i)_\triangle \leq \lceil \frac{i}{e} \rceil$, since $\lceil \frac{-\alpha}{c} \rceil \leq \lceil \frac{i}{e} \rceil$. Thus we conclude that $\operatorname{reg}_{R_\triangle} R(\alpha,\beta)_\triangle \leq 0$. Hence the claim (b) follows. $\square$

Remark 2.11. Note that the Proposition 2.10 is an extension of [9, Theorem 6.2] for certain bigraded complete intersection ideal. Note also that the statement of Proposition 2.10 is similar to (and more general then) [8, Proposition 2.2].

3. Improvement of bounds

Let $S = K[x_1,\ldots,x_n]$ be a polynomial ring. Let I be a homogeneous ideal in S generated by a regular sequence $f_1, f_2, \ldots, f_k$, of homogeneous forms of degrees d. Let c and e be positive integers. Consider the K-subalgebra of S generated by the homogeneous forms of degree $ed + c$ in the ideal I^e, that is, $K[(I^e)_{ed+c}]$. We have seen that $K[(I^e)_{ed+c}]$ is the (c,e)-diagonal subalgebra of $\operatorname{Rees}(I)$.

By [9, Corollary 6.10], $K[(I^e)_{ed+c}]$ is shown to be quadratic if $c \geq \frac{d}{2}$ and Koszul if $c \geq \frac{d(k-1)}{k}$. In [9, p.900] the authors mentioned that, they expect $K[(I^e)_{ed+c}]$ to be Koszul also for $\frac{d}{2} \leq c < \frac{d(k-1)}{k}$. For $k = 1,2$, it is obvious that $K[(I^e)_{ed+c}]$ is Koszul. The very first nontrivial instance of this problem occurs for $d = 2$ and $k = 3$, in which case, only possible value is $c = 1$. For $c = 1$, the answer is positive and solved by Caviglia and Conca [8].

In this Section, for $k = 3$ and for any d, we prove that $K[(I^e)_{ed+c}]$ is Koszul as soon as $c \geq \frac{d}{2}$ and $e > 0$. The main theorem of this section is as follows:

Theorem 3.1. *Let I be an ideal of the polynomial ring $K[x_1,\dots,x_n]$ generated by a regular sequence f_1, f_2, f_3, of homogeneous forms of degree d. Then $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq \frac{d}{2}$ and $e > 0$.*

We consider the Rees algebra, $\mathrm{Rees}(I) \subset S[t]$ of I with its standard bigraded structure induced by $\deg(x_i) = (1,0)$ and $\deg(f_j t) = (0,1)$. It can be realized as a quotient of the polynomial ring $S' = K[x_1,\dots,x_n,t_1,t_2,t_3]$ bigraded with $\deg(x_i) = (1,0)$ and $\deg(t_j) = (0,1)$, by the ideal J generated by the 2-minors of

$$M = \begin{pmatrix} f_1 & f_2 & f_3 \\ t_1 & t_2 & t_3 \end{pmatrix}.$$

Let h_1,h_2,h_3 be the 2-minors of M with the appropriate sign, say h_i equal to $(-1)^{i+1}$ times the minor of M obtained by deleting the i-th column. Hence

$$J = I_2(M) = (h_1,h_2,h_3).$$

The sign convention is chosen so that the rows of the matrix M are syzygies of h_1,h_2,h_3. We will use the following lemma to prove Theorem 3.1.

Lemma 3.2. [8, Lemma 3.1] (Technical Lemma)
(i) *h_1,h_2 form a regular S'-sequence.*
(ii) $(h_1,h_2) : h_3 = (f_3,t_3)$.
(iii) $(h_1,h_2) : t_3 = J$.
(iv) $(t_3,h_1,h_2) : f_3 = (t_1,t_2,t_3)$.

Remark 3.3. Note that Lemma 3.2 is proved for $d = 2$ in [8, Lemma 3.1]. We observe that the proof of [8, Lemma 3.1] is independent of the degree of polynomials d. Hence Lemma 3.2 also holds for all d.

We are now ready for the proof of Theorem 3.1:

Proof. Recall that $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq \frac{2d}{3}$ and $e > 0$ [9, Corollary 6.10]. We will show that $K[(I^e)_{ed+c}]$ is Koszul also for $\frac{d}{2} \leq c < \frac{2d}{3}$ and $e > 0$.

We set $B = S'/(h_1,h_2)$. By Lemma 3.2(i), we may apply to B the result of Proposition 2.10. Hence $B_\triangle$ is Koszul. One has also $\frac{B}{h_3 B} = \mathrm{Rees}(I)$. It is enough to show for the (c,e)-diagonal subalgebra of $\mathrm{Rees}(I)$, one has

$$\mathrm{reg}_{B_\triangle}(\mathrm{Rees}(I)_\triangle) \leq 1.$$

Since $h_3 t_3 = 0$ in B, we have a complex

$$\mathbf{F} : \cdots \xrightarrow{t_3} B(-2d,-3) \xrightarrow{h_3} B(-d,-2) \xrightarrow{t_3} B(-d,-1) \xrightarrow{h_3} B \longrightarrow 0, \quad (11)$$

where $F_0 = B$, $F_{2i} = B(-id,-2i)$, $F_{2i+1} = B(-(i+1)d,-2i-1)$. The homology of $\mathbf{F}$ can be described by using Lemma 3.2

$$H_k(\mathbf{F}) = \begin{cases} \mathrm{Rees}(I) & \text{if } k = 0, \\ 0 & \text{if } k = 2i \text{ and } i > 0, \\ [S'/(t_1,t_2,t_3)](-(i+1)d-d,-2i-1) & \text{if } k = 2i+1 \text{ and } i \geq 0. \end{cases}$$

The assertion for $k=0$ holds by construction. For k even and positive holds because of Lemma 3.2. For k odd and positive by Lemma 3.2 (ii), we have

$$H_{2i+1}(\mathbf{F}) = \frac{(t_3,f_3)}{(t_3,h_1,h_2)}(-(i+1)d,-2i-1).$$

Hence $H_{2i+1}(\mathbf{F})$ is cyclic generated by the residue class of $f_3 \mod (t_3,h_1,h_2)$ that has degree $(-(i+1)d-d,-2i-1)$. Using Lemma 3.2(iv) and keeping track of the degree we get the desired result. Applying $-_\triangle$ functor to (11), we obtain a complex $\mathbf{F}_\triangle$:

$$\cdots \longrightarrow B(-2d,-3)_\triangle \longrightarrow B(-d,-2)_\triangle \longrightarrow B(-d,-1)_\triangle \longrightarrow B_\triangle \longrightarrow 0, \quad (12)$$

where

$$(F_k)_\triangle = \begin{cases} B_\triangle & \text{if } k=0, \\ B(-id,-2i)_\triangle & \text{if } k=2i, \\ B(-(i+1)d,-2i-1)_\triangle & \text{if } k=2i+1. \end{cases}$$

Note that $H_{2i}(\mathbf{F}_\triangle)=0$ and $H_0(\mathbf{F}_\triangle)=\operatorname{Rees}(I)_\triangle$. We observe that $H_{2i+1}(\mathbf{F}_\triangle)$ is not necessarily zero for all $e\geq 1$. Assume $e\geq 2$. Then we claim that $H_{2i+1}(\mathbf{F}_\triangle)=0$. Take the j-th degree component

$$(H_{2i+1}(\mathbf{F})_\triangle)_j = [S'/(t_1,t_2,t_3)]_{(-(i+1)d-d+jc,-2i-1+je)}.$$

We will show that

$$[S'/(t_1,t_2,t_3)]_{(-(i+1)d-d+jc,-2i-1+je)} = 0. \quad (13)$$

Clearly (13) holds if $-2i-1+ej\geq 1$, that is, if $ej\geq 2(i+1)$. So, it is enough to show that $-(i+1)d-d+jc<0$ for $ej<2(i+1)$, that is, $j<\frac{(i+2)d}{c}$ for $j<\frac{2(i+1)}{e}$. This is an easy consequence of the following inequalities:

$$\frac{2(i+1)}{e} < \frac{3(i+2)}{2} < \frac{(i+2)d}{c}.$$

Assume $e=1$. We know that $H_{2i+1}(\mathbf{F}_\triangle)=H_{2i+1}(\mathbf{F})_\triangle$. Take the j-th degree component of $H_{2i+1}(\mathbf{F})_\triangle$, then $(H_{2i+1}(\mathbf{F})_\triangle)_j=0$ if $-(2i+1)+j\geq 1$, that is, if $j\geq 2i+2$. So, the largest degree of a non zero component of $H_{2i+1}(\mathbf{F}_\triangle)$ is at most $2i+1$. Therefore by (1), one has $\operatorname{reg}_{B_\triangle} H_k(\mathbf{F}_\triangle)\leq \operatorname{reg}_{S'} H_k(\mathbf{F}_\triangle)\leq k$ for all $k\geq 1$. Applying Lemma 2.7 to (12), we obtain

$$\operatorname{reg}_{B_\triangle}(\operatorname{Rees}(I)_\triangle) \leq \sup\{\alpha',\beta'\}, \text{ where} \quad (14)$$

$\alpha' = \sup\{\operatorname{reg}_{B_\triangle}(F_k)_\triangle - k : k\geq 0\}$ and $\beta' = \sup\{\operatorname{reg}_{B_\triangle} H_k(\mathbf{F}_\triangle)-(k+1) : k\geq 1\}$. Since B is defined by a regular sequence of elements of bidegree $(d,1)$, we may apply Proposition 2.10 to (12):

$$\operatorname{reg}_{B_\triangle}(F_k)_\triangle \leq \begin{cases} \max\{\lceil\frac{id}{c}\rceil,\lceil\frac{2i}{e}\rceil\} & \text{if } k=2i, \\ \max\{\lceil\frac{(i+1)d}{c}\rceil,\lceil\frac{2i+1}{e}\rceil\} & \text{if } k=2i+1. \end{cases}$$

Since $\frac{3}{2}<\frac{d}{c}\leq 2$, we conclude that $\alpha'\leq 1$. Since $\operatorname{reg}_{B_\triangle} H_k(\mathbf{F}_\triangle)\leq k$ for all $k\geq 1$, we conclude that $\beta'\leq -1$. Therefore by (14), one has

$$\operatorname{reg}_{B_\triangle}(\operatorname{Rees}(I)_\triangle)\leq 1.$$

Thus we conclude that $\operatorname{Rees}(I)_\triangle$ is Koszul. $\square$

Remark 3.4. We observe that in the proof of [8, Theorem 3.2], $H_k(\mathbf{F})_\triangle = 0$ for all k, whereas in our case, this is true for all $e \geq 2$, and not for $e = 1$. This affects the proof of Theorem 3.1 very much from that of [8, Theorem 3.2]. To achieve our goal we first have to deduce Lemma 2.7. We use the fact that if the homology module is non zero and its regularity is bounded by the homology module at zero position, then by (1), Lemma 2.7 and Proposition 2.10, we conclude the proof.

4. MORE GENERAL BASE RINGS

In this Section, the two main results that we generalize are [9, Theorem 6.2] and [9, Corollary 6.10]. We show that the Koszulness property holds even if the assumption of polynomial ring is replaced by a Koszul ring.

Conca et al. in [9] posed two interesting questions at page 900 one of which was positively answered by Aramova, Crona and De Negri [1] who showed that for an arbitrary bigraded standard algebra R, the defining ideal of $R_\triangle$ has quadratic Grobner basis for $c,e \gg 0$, and another one by Blum [5], who showed that all the diagonal algebras of bigraded standard Koszul algebra R are Koszul. We will use these results in the proof of Theorem 4.1 and Theorem 4.5.

Let A and B are two standard graded Koszul algebras. Let $A = K[A_1]$, where $A_1 = \langle X_1, \dots X_m \rangle$ is a K vector space generated be linear forms with $\deg(X_i) = 1$. Similarly let $B = K[B_1]$, where $B_1 = \langle Y_1, \dots Y_n \rangle$ is a K vector space generated be linear forms with $\deg(Y_j) = 1$. We set $T = A \otimes_K B$. Then T is bigraded standard by setting $\deg(X_i) = (1,0)$ and $\deg(Y_j) = (0,1)$. Let R be a bigraded quotient of T, that is, $R = T/I$ for some bihomogeneous ideal I of T.

Let c and e be positive integers. We will study the Koszul property of (c,e)-diagonal subalgebra $R_\triangle$ of bigraded algebra R. Consider the bigraded free resolution of R over T:

$$\cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow T \longrightarrow R \longrightarrow 0, \tag{15}$$

where

$$F_i = \bigoplus_{(a,b)\in\mathbb{N}^2} T(-a,-b)^{\beta_{i,(a,b)}}.$$

Set

$$t_{i,1} = \max\{a : \exists\, b \text{ s.t. } \beta_{i,(a,b)} \neq 0\}, \text{ and } t_{i,2} = \max\{b : \exists\, a \text{ s.t. } \beta_{i,(a,b)} \neq 0\}. \tag{16}$$

With this notation, the following is a generalization of [9, Theorem 6.2]:

Theorem 4.1. *Let A and B are standard graded Koszul algebras. We set $T = A \otimes_K B$ and R a bi-graded quotient of T. Then:*

(i) *T is Koszul and $\operatorname{reg}_T R$ is finite.*
(ii) *If $c \geq \sup\{\frac{t_{i,1}}{i+1} : i \geq 1\} \in \mathbb{R}$ and $e \geq \sup\{\frac{t_{i,2}}{i+1} : i \geq 1\} \in \mathbb{R}$, then $R_\triangle$ is Koszul.*
(iii) *In particular, if $c \geq \frac{\operatorname{reg}_T R - 1}{2}$ and $e \geq \frac{\operatorname{reg}_T R - 1}{2}$, then $R_\triangle$ is Koszul.*

Proof. For (i) T is Koszul by [4] and $\operatorname{reg}_T R$ is finite by [3, Theorem 1]. For (ii) applying $-_\triangle$ functor to bigraded free resolution (15) of R over T, one obtains an exact complex

$$\cdots \longrightarrow (F_i)_\triangle \longrightarrow \cdots \longrightarrow (F_1)_\triangle \longrightarrow T_\triangle \longrightarrow R_\triangle \longrightarrow 0, \tag{17}$$

of $T_\triangle$-modules, where

$$(F_i)_\triangle = \bigoplus_{(a,b)\in\mathbb{N}^2} T(-a,-b)_\triangle^{\beta_{i,(a,b)}}.$$

Note that $T_\triangle = A^{(c)}\underline{\otimes}B^{(e)}$, where $A^{(c)}$ denotes the c-th Veronese subring of A, and $B^{(e)}$ denotes the e-th Veronese subring of B. Note that $T_\triangle$ is Koszul [5, Theorem 2.1]. To show that $R_\triangle$ is Koszul, it is enough to show that

$$\operatorname{reg}_{T_\triangle} R_\triangle \leq 1.$$

Applying [9, Lemma 6.3(ii)] to (17), we get

$$\operatorname{reg}_{T_\triangle} R_\triangle \leq \sup\{\operatorname{reg}_{T_\triangle}(F_i)_\triangle - i \,:\, i \geq 1\}.$$

Thus, it is enough to show that

$$\operatorname{reg}_{T_\triangle}(F_i)_\triangle - i \leq 1 \text{ for all } i \geq 1. \tag{18}$$

Since

$$(F_i)_\triangle = \bigoplus_{(a,b)\in\mathbb{N}^2} T(-a,-b)_\triangle^{\beta_{i,(a,b)}},$$

one has

$$\operatorname{reg}_{T_\triangle}(F_i)_\triangle = \max\{\operatorname{reg}_{T_\triangle} T(-a,-b)_\triangle \,:\, \beta_{i,(a,b)} \neq 0\}. \tag{19}$$

Now we need to evaluate $\operatorname{reg}_{T_\triangle} T(-a,-b)_\triangle$. We denote by $V_A(c,\alpha)$, the Veronese modules of A, that is, $V_A(c,\alpha) = \bigoplus_{s\in\mathbb{N}} A_{sc+\alpha}$ for $\alpha = 0,\dots,c-1$. Similarly denote $V_B(e,\beta)$, the Veronese modules of B. Hence for the shifted module $T(-a,-b)_\triangle$ we can write

$$T(-a,-b)_\triangle = \bigoplus_s [A_{sc-a} \otimes B_{se-b}] \;=\; V_A(c,\alpha)(-\lceil\frac{a}{c}\rceil) \,\underline{\otimes}\, V_B(e,\beta)(-\lceil\frac{b}{e}\rceil),$$

where $\alpha = -a \mod (c)$, $0 \leq \alpha \leq c-1$ and $\beta = -b \mod (e)$, $0 \leq \beta \leq e-1$.

The Veronese modules $V_A(c,\alpha)$ and $V_B(e,\beta)$ have a linear resolutions as a $A^{(c)}$-module and $B^{(c)}$-module respectively, see [6, Lemma 5.1]. Hence by (2), one has

$$\operatorname{reg}_{T_\triangle} T(-a,-b)_\triangle = \max\{\lceil\frac{a}{c}\rceil, \lceil\frac{b}{e}\rceil\}. \tag{20}$$

Thus by (16), (18), (19) and (20), we conclude that $R_\triangle$ is Koszul provided that

$$\max\{\lceil\frac{t_{i,1}}{c}\rceil,\, \lceil\frac{t_{i,2}}{e}\rceil\} \leq i+1 \text{ for all } i \geq 1. \tag{21}$$

From (21), we conclude that if $c \geq \sup\{\frac{t_{i,1}}{i+1} \,:\, i \geq 1\}$ and $e \geq \sup\{\frac{t_{i,2}}{i+1} \,:\, i \geq 1\}$, then $R_\triangle$ is Koszul. By definition, one has $t_{i,1} \leq t_i \leq \operatorname{reg}_T R - i$. Thus we have:

$$\frac{t_{i,1}}{i+1} \leq \frac{t_i}{i+1} \leq \frac{\operatorname{reg}_T R - i}{i+1}. \tag{22}$$

We know that $\operatorname{reg}_T R$ is finite. Notice that $\dfrac{\operatorname{reg}_T R - i}{i+1}$ is a decreasing function of i, as i vary over the natural numbers. Taking sup in (22), we get

$$\sup\{\frac{t_{i,1}}{i+1} \,:\, i \geq 1\} \leq \sup\{\frac{t_i}{i+1} \,:\, i \geq 1\} \leq \sup\{\frac{\operatorname{reg}_T R - i}{i+1} \,:\, i \geq 1\}.$$

Similarly the other case

$$\sup\{\frac{t_{i,2}}{i+1} : i \geq 1\} \leq \sup\{\frac{t_i}{i+1} : i \geq 1\} \leq \sup\{\frac{\operatorname{reg}_T R - i}{i+1} : i \geq 1\}.$$

Note that

$$\sup\{\frac{\operatorname{reg}_T R - i}{i+1} : i \geq 1\} \leq \frac{\operatorname{reg}_T R - 1}{2}.$$

Thus we observe that the numbers $\sup\{\frac{t_{i,1}}{i+1} : i \geq 1\}$ and $\sup\{\frac{t_{i,2}}{i+1} : i \geq 1\}$ are in fact some finite real numbers, bounded by $\dfrac{\operatorname{reg}_T R - 1}{2}$. Thus the claim (ii) and (iii) follows.

□

Remark 4.2. Note that in the Theorem 4.1, if we take $A = K[x_1, \cdots, x_m]$ and $B = K[y_1, \cdots, y_n]$, then we will re-obtain [9, Theorem 6.2].

The following is the generalization of [9, Corollary 6.9]:

Corollary 4.3. *Let I be a homogeneous ideal of a standard graded Koszul ring A. Let d denotes the highest degree of a minimal generator of I. Then there exists integers c_0, e_0 such that the K-algebra $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq c_0$ and $e \geq e_0$.*

Proof. Let $B = K[t_1, \ldots, t_k]$ in Theorem 4.1. Then $T = A[t_1, \ldots, t_k]$ is a polynomial extension of A. Note that T is a bigraded standard algebra by setting $\deg X_i = (1,0)$ and $\deg t_j = (0,1)$. By replacing I with the ideal generated by I_d, we may assume that I is generated by forms of degree d. Then $\operatorname{Rees}(I) \subset A[t]$ is a bigraded standard algebra by setting $\deg X_i = (1,0)$ and $\deg ft = (0,1)$ for all $f \in I_d$. Moreover $\operatorname{Rees}(I)$ can also be realized as the bigraded quotient of T.

By Theorem 4.1(i), we have T is Koszul and $\operatorname{reg}_T \operatorname{Rees}(I)$ is finite. Note that the numbers c and e exist from Theorem 4.1(ii) such that $\operatorname{Rees}(I)_\triangle$ is Koszul. In particular, if $c, e \geq \frac{\operatorname{reg}_T \operatorname{Rees}(I) - 1}{2}$, then $\operatorname{Rees}(I)_\triangle$ is Koszul by Theorem 4.1(iii). Thus the claim follows, since $\operatorname{Rees}(I)_\triangle = K[(I^e)_{ed+c}]$. □

Remark 4.4. In the Corollary 4.3, the integers c_0 and e_0 can be computed explicitly whenever one knows the shifts in the bigraded free resolution of $\operatorname{Rees}(I)$ over the Koszul ring T. For instance, when I is a complete intersection ideal generated by homogeneous forms of degree d, one has Theorem 4.5.

The following is the generalization of [9, Corollary 6.10]:

Theorem 4.5. *Let A be a standard graded Koszul ring. Let I be an ideal of A generated by a regular sequence $f_1, f_2, \ldots, f_k$, of homogeneous forms of degree d. Then $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq \frac{d(k-1)}{k}$ and $e > 0$.*

Proof. Let $A = K[A_1]$, where $A_1 = \langle X_1, \ldots X_m \rangle$ is a K vector space generated be linear forms with $\deg(X_i) = 1$. Let $A' = A[t_1, \ldots, t_k]$ be a polynomial extension of A. Then A' is a bigraded standard algebra by setting $\deg X_i = (1,0)$ and $\deg t_j = (0,1)$. Let I be an ideal of A generated by a regular sequence $f_1, f_2, \ldots, f_k$, of homogeneous forms of degree d. Then $\operatorname{Rees}(I) \subset A[t]$ is a bigraded standard algebra by setting $\deg X_i = (1,0)$ and $\deg f_j t = (0,1)$. Let

$$\phi : A[t_1, t_2, \ldots, t_k] \longmapsto A[It]$$

be the surjective map by sending t_j to f_jt. Since I is a complete intersection ideal, one has

$$\ker(\phi) = I_2\begin{pmatrix} f_1 & f_2 & \dots & f_k \\ t_1 & t_2 & \dots & t_k \end{pmatrix}.$$

The resolution of $A[It]$ over A' is given by the Eagon-Northcott complex (23). We know that $\ker(\phi)$ is a determinantal ideal and $\mathrm{grade}(\ker(\phi)) = k-1$, hence the Eagon-Northcott complex (23) is the minimal free resolution of $A[It]$ over A':

$$0 \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A[It] \longrightarrow 0, \tag{23}$$

where

$$F_i = \bigoplus_{j=1}^{i} A'(-jd, -i-1+j)^{\sharp_i}.$$

Here $\sharp_i$ denotes some integer which is irrelevant in our discussion. Applying $-_\triangle$ functor to (23), one obtains an exact complex:

$$0 \longrightarrow (F_{k-1})_\triangle \longrightarrow \cdots \longrightarrow (F_1)_\triangle \longrightarrow (F_0)_\triangle \longrightarrow A[It]_\triangle \longrightarrow 0 \tag{24}$$

of $A'_\triangle$-modules, where

$$(F_i)_\triangle = \bigoplus_{j=1}^{i} A'(-jd, -i-1+j)^{\sharp_i}_\triangle.$$

From the exact complex (24), we build another complex:

$$\mathbf{F}: \ 0 \longrightarrow (F_{k-1})_\triangle \longrightarrow \cdots \longrightarrow (F_1)_\triangle \longrightarrow (F_0)_\triangle \longrightarrow 0$$

of $A'_\triangle$-modules. Then the homology of the new complex $\mathbf{F}$ is given by

$$H_i(\mathbf{F}) = \begin{cases} A[It]_\triangle, & \text{if } i = 0; \\ 0, & \text{if } i > 0. \end{cases}$$

Thus Lemma 2.7 applied to the complex $\mathbf{F}$, one has

$$\mathrm{reg}_{A'_\triangle}(A[It]_\triangle) \leq \sup\{\mathrm{reg}_{A'_\triangle}(F_i)_\triangle - i \ : \ i = 1, 2, \dots, k-1\}.$$

Note that $A'_\triangle$ is Koszul [5, Theorem 2.1]. To show that $A[It]_\triangle$ is Koszul, it is enough to show that

$$\mathrm{reg}_{A'_\triangle}(F_i)_\triangle - i \leq 1 \text{ for all } i = 1, 2, \dots, k-1. \tag{25}$$

One has

$$\mathrm{reg}_{A'_\triangle}(F_i)_\triangle = \max\{\mathrm{reg}_{A'_\triangle} A'(-jd, -i-1+j)_\triangle \ : \ j = 1, 2, \dots, i\}.$$

By similar argument as used in Theorem 4.1 to obtain Equation (20), we get

$$\mathrm{reg}_{A'_\triangle}(F_i)_\triangle = \max\{\lceil \frac{jd}{c} \rceil, \lceil \frac{i+1-j}{e} \rceil \ : \ j = 1, 2, \dots, i\}.$$

Thus we have

$$\mathrm{reg}_{A'_\triangle}(F_i)_\triangle = \max\{\lceil \frac{id}{c} \rceil, \lceil \frac{i}{e} \rceil\}. \tag{26}$$

Therefore by (25) and (26), we conclude that $K[(I^e)_{ed+c}]$ is Koszul if $c \geq \frac{d(k-1)}{k}$ and $e > 0$. $\square$

We conclude the section with one final remark, and with an open question:

Remark 4.6. The claim by Conca, Herzog, Trung and Valla in [9, p. 900] together with Theorem 3.1 and Theorem 4.5 in this paper, suggest that the following question may have a positive answer.

Question 4.7. *Let I be an ideal of a Koszul ring A generated by a regular sequence $f_1, f_2, \dots, f_k$, of homogeneous forms of degree d. Is it true that $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq \frac{d}{2}$ and $e > 0$.*

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