NOTE ON VANISHING POWER SUMS OF ROOTS OF UNITY

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Abstract. For fixed positive integers \( m \) and \( \ell \), we give a complete list of integers \( n \) for which their exist \( m \)th complex roots of unity \( x_1, \ldots, x_n \) such that \( x_1^\ell + \cdots + x_n^\ell = 0 \). This extends the earlier result of Lam and Leung on vanishing sums of roots of unity. Furthermore, we characterize all positive integers \( n \) with \( 2 \leq n \leq m \), for which there are distinct \( m \)th complex roots of unity \( x_1, \ldots, x_n \) such that \( x_1^\ell + \cdots + x_n^\ell = 0 \).

1. Introduction

Let \( m \) be a positive integer. By an \( m \)th root of unity, we mean a complex number \( \zeta \) such that \( \zeta^m = 1 \). That is, a root of the polynomial \( X^m - 1 \). One can easily see that the roots of \( X^m - 1 \) are distinct, in fact there are exactly \( m \), \( m \)th roots of unity. Using the relationship between the roots and the coefficients of a polynomial, we see that the sum of all \( m \)th roots of unity, which is the coefficient of \( X^{m-1} \) in \( X^m - 1 \), is zero. A natural question is: What are all the positive integers \( n \) for which there exist \( m \)th roots of unity \( x_1, \ldots, x_n \) (repetition is allowed) such that \( x_1 + \cdots + x_n = 0 \). A beautiful result of T. Y. Lam and K. H. Leung [1] gives a complete classification of all such integers. Suppose \( m \) has prime factorization \( p_1^{a_1} \cdots p_r^{a_r} \), where \( a_i > 0 \), then we have the following theorem due to Lam and Leung:

**Theorem 1.** Let \( n \) be a positive integer. Then there are \( m \)th roots of unity \( x_1, \ldots, x_n \) such that \( x_1 + \cdots + x_n = 0 \) if and only if \( n \) is of the form \( n_1 p_1 + \cdots + n_r p_r \) where each \( n_i \) is a non-negative integer for \( 1 \leq i \leq r \).

Theorem 1 motivate us to ask the following:

**Question 1.** Let \( m \) and \( \ell \) be positive integers. What are all the positive integers \( n \) for which there exist \( m \)th roots of unity \( x_1, \ldots, x_n \) such that \( x_1^\ell + \cdots + x_n^\ell = 0 \)?

Note that when \( \ell = 1 \), the complete answer to Question 1 is given by Theorem 1. However, for \( \ell \geq 2 \), we do not find any results in this direction in the literature. Our objective here is to study the case when \( \ell \geq 2 \). First, we fix some notations.

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Let $m$ be a positive integer, and let $\Omega_m$ denote the set of all $m$th roots of unity.

For a positive integer $\ell$, $W_\ell(m)$ denotes the set of all positive integers $n$ for which there exist $n$-elements $x_1, \ldots, x_n \in \Omega_m$ such that $x_1^\ell + \cdots + x_n^\ell = 0$. When $\ell = 1$, we simply denote $W_\ell(m)$ by $W(m)$. With this notation, Question 1 can be reformulated as follows: Let $m$ and $\ell$ be positive integers. What are all the positive integers in the set $W_\ell(m)$?

It is clear that if $m$ divides $\ell$ then $W_\ell(m)$ is an empty set. Suppose that there are $m$th complex roots of unity, say, $x_1, \ldots, x_n$ such that $x_1^\ell + \cdots + x_n^\ell = 0$. Since the $\ell$th power of an $m$th root of unity is still an $m$th root of unity, the equation $x_1^\ell + \cdots + x_n^\ell = 0$ with $x_i \in \Omega_m$ can be written in the form $y_1 + \cdots + y_n = 0$ with $y_i \in \Omega_m$. This shows that for any positive integer $m$ and $\ell$, $W_\ell(m)$ is a subset of $W(m)$. It follows from Theorem 1 that any positive integers in the set $W_\ell(m)$ must be of the form $n_1p_1 + \cdots + n_rp_r$ where each $n_i$ is a non-negative integer for $1 \leq i \leq r$. In Section 2, we give a complete list of integers in the set $W_\ell(m)$ (see Theorem 2). Moreover, in Section 3 we find all positive integers $n \in W_\ell(m)$ for which there are distinct $m$th complex roots of unity $x_1, \ldots, x_n$ such that $x_1^\ell + \cdots + x_n^\ell = 0$ (see Theorem 3).

There are algebraic aspects why Question 1 is important. For instance, for a positive integer $a$, denote by $p_a$ the power sum polynomial $X_1^a + \cdots + X_n^a$ of degree $a$. Let $\ell < k$ be two positive integers. In commutative algebra, one encounters the following situation: To show that the ideal $\langle p_\ell, p_k \rangle$ generated by the polynomials $p_\ell$ and $p_k$ is a prime ideal in $\mathbb{C}[X_1, \ldots, X_n]$, one needs to show that the power sum polynomial $X_1^\ell + \cdots + X_n^\ell$ does not vanish when one allows the $X_i$’s to take values among the $(\ell - k)$th roots of unity [2, see proof of Theorem 3.8].

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2. VANISHING OF POWER SUMS OF ROOTS OF UNITY

Let $m$ and $\ell$ be positive integers. In this section, we completely characterize all the positive integers in the set $W_\ell(m)$. More precisely, we prove the following theorem:

**Theorem 2.** Let $m$ and $\ell$ be positive integers. Let $d = (m, \ell)$ be the greatest common divisor of $m$ and $\ell$. Then $W_\ell(m) = W(m/d)$.

In other words, Theorem 2 says that: For any positive integer $n$, $x_1^\ell + \cdots + x_n^\ell = 0$ with $x_i \in \Omega_m$ if and only if $y_1 + \cdots + y_n = 0$ with $y_i \in \Omega_{m/d}$. 
Proof. It is well known that $\Omega_m$, that is, the set of all $m$th roots of unity, form a group with respect to the multiplication of complex numbers. In fact, it is a cyclic group of order $m$, generated by the complex number $\zeta_m = \cos 2\pi/m + i\sin 2\pi/m$. There is a remarkable property about finite cyclic groups. Namely, if $G$ is a finite cyclic group and $\ell$ is a positive integer relatively prime to the order of $G$, then the map

$$x \mapsto x^\ell \quad (x \in G)$$

is an automorphism of $G$ (In fact, all the automorphisms of $G$ are of the form (1) for some integer $\ell$ which is relatively prime to the order of $G$). It follows that, if $\ell$ is a positive integer which is relatively prime to $m$ then every element of $\Omega_m$ is a $\ell$th power of some element of $\Omega_m$. Thus, for an integer $\ell$ which is relatively prime to $m$, the equation $x_1^\ell + \cdots + x_n^\ell = 0$ with $x_i \in \Omega_m$ can be replaced by $y_1 + \cdots + y_n = 0$ with $y_i \in \Omega_m$, and vice versa. This discussion proves Theorem 2 for the case when $\ell$ is relatively prime to $m$.

Now assume that $d > 1$. Consider the map

$$\psi_d : \Omega_m \longrightarrow \Omega_{m/d}$$

defined by $x \mapsto x^d$ for $x \in \Omega_m$. This map is clearly onto, and the kernel is exactly $\Omega_d$. Thus, $\Omega_m / \Omega_d \cong \Omega_{m/d}$. Now suppose that there are elements $x_1, \ldots, x_n \in \Omega_m$ such that $x_1^\ell + \cdots + x_n^\ell = 0$. Then this sum can be rewritten as $\left(x_1^{\ell/d}\right)^d + \cdots + \left(x_n^{\ell/d}\right)^d = 0$. Since $\ell/d$ and $m$ are relatively prime, by the above discussion, the latter equation can be rewritten in the form $y_1^d + \cdots + y_n^d = 0$ with $y_i \in \Omega_m$. Finally, using the map $\psi_d$, the latter sum can be realized as $z_1 + \cdots + z_n = 0$ where $z_i \in \Omega_{m/d}$ for $1 \leq i \leq n$. In fact, all these steps can be reversed. This completes the proof of Theorem 2.

Combining Theorems 1 and 2, we have the following corollary:

**Corollary 1.** Let $m$, $n$ and $\ell$ be positive integers. Let $d = (m, \ell)$ be the greatest common divisor of $m$ and $\ell$. Then there are $m$th roots of unity $x_1, \ldots, x_n$ such that $x_1^\ell + \cdots + x_n^\ell = 0$ if and only if $n$ is of the form $n_1q_1 + \cdots + n_sq_s$ where each $n_i$ is a non-negative integer for $1 \leq i \leq s$ and $q_1, \ldots, q_s$ are distinct prime divisors of $m/d$.

**Example.** Let $m = 60$, and let $\ell$ be an integer with $1 \leq \ell < 60$. By Theorem 2, $W_\ell(m) = W(m/d)$ where $d$ is the greatest common divisor of $m$ and $\ell$. When $d$ varies over the divisors of $m$, $m/d$ also varies over the divisors of $m$. Thus $W_\ell(m)$ coincides with $W(d)$ for some divisor $d$ of $m$. On the other hand, by Theorem 1, $W(d) = \sum_{i=1}^s q_i N$ where $d = q_1^{b_1} \cdots q_s^{b_s}$ is the prime factorization of $d$. Here $N$
denotes the set of non-negative integers. We thus have the following table which describe \( W(d) \) for all positive divisors \( d \) of \( m = 60 \).

<table>
<thead>
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<th>( d )</th>
<th>( W(d) )</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>( 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( 2N )</td>
</tr>
<tr>
<td>3</td>
<td>( 3N )</td>
</tr>
<tr>
<td>4</td>
<td>( 2N )</td>
</tr>
<tr>
<td>5</td>
<td>( 3N )</td>
</tr>
<tr>
<td>6</td>
<td>( 2N + 3N = N \setminus {1} )</td>
</tr>
<tr>
<td>10</td>
<td>( 2N + 5N = N \setminus {1,3} )</td>
</tr>
<tr>
<td>12</td>
<td>( 2N + 3N = N \setminus {1} )</td>
</tr>
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<td>15</td>
<td>( 3N + 5N = N \setminus {1,2,4,7} )</td>
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<tr>
<td>20</td>
<td>( 2N + 5N = N \setminus {1,3} )</td>
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<tr>
<td>30</td>
<td>( 2N + 3N + 5N = N \setminus {1} )</td>
</tr>
<tr>
<td>60</td>
<td>( 2N + 3N + 5N = N \setminus {1} )</td>
</tr>
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</table>

3. Vanishing of Power Sums of Distinct Roots of Unity

Let \( m \) and \( \ell \) be two positive integers. For an integer \( n \in W_\ell(m) \), the height \( H(n;\ell,m) \) of \( n \) is defined to be the smallest positive integer \( h \) for which there are \( m \)th roots of unity \( x_1, \ldots, x_n \) such that \( x_1^h + \cdots + x_n^h = 0 \) and the maximum among the repetition of \( x_i \)'s is \( h \), that is, \( h \) is the maximum among the \( h_i \), where \( h_i \) is the number of times \( x_i \) appears in the list \( x_1, \ldots, x_n \). When \( \ell = 1 \), we denote \( H(n;\ell,m) \) by \( H(n;m) \). Note that \( H(n;m) = 1 \) provided \( 2 \leq n \leq m \). Gary Sivak [3] refined the work of Lam and Leung by proving that for any integers \( m \geq 2 \) and \( 2 \leq n \leq m \), \( H(n;m) = 1 \) if and only if both \( n \) and \( m - n \) are expressible as a linear combination of the prime factors of \( m \) with non-negative integer coefficients. Here we extend Sivak's result to vanishing of power sums of distinct roots of unity:

**Theorem 3.** Let \( m \) and \( \ell \) be positive integers, and let \( n \) be an integer such that \( 2 \leq n \leq m \). Let \( d \) be the greatest common divisor of \( m \) and \( \ell \). Then \( H(n;\ell,m) = 1 \) if and only if \( H(n;m/d) \leq d \).

**Proof.** Let \( \Omega_{m/d} = \{z_1, \ldots, z_{m/d}\} \). Suppose that there are distinct \( m \)th roots of unity \( x_1, \ldots, x_n \) such that \( x_1^h + \cdots + x_n^h = 0 \). Since \( d \) is the greatest common divisor of \( \ell \) and \( m \), this equation can be rewritten in the form \( y_1^h + \cdots + y_n^h = 0 \) with \( y_1, \ldots, y_n \) are \( m \)th roots of unity. Using the map \( \psi_d \), the latter equation can be written as \( \sum_{i=1}^{m/d} a_i z_i = 0 \) where \( a_i \) is the cardinality of the set \( \{y_1, \ldots, y_n\} \cap \psi_d^{-1}(z_i) \) for \( 1 \leq i \leq m/d \). On the other hand, \( \psi_d^{-1}(z) \) has exactly \( d \) elements for each \( z \in \Omega_{m/d} \). It follows that \( H(n;m/d) \leq \max\{a_1, \ldots, a_{m/d}\} \leq d \). This proves that if \( H(n;\ell,m) = 1 \) then \( H(n;m/d) \leq d \).
Conversely, suppose that \( H(n; m/d) \leq d \). Then there is a partition \((a_1, \ldots, a_{m/d})\) of \( n \) into non-negative integers \( a_i \) with \( a_i \leq d \) for \( 1 \leq i \leq m/d \) and \( \sum_{i=1}^{m/d} a_i z_i = 0 \).

Let \( y_i \) be any element of \( \psi_d^{-1}(z_i) \) for \( 1 \leq i \leq m/d \). Then \( \psi_d^{-1}(z_i) = y_i \Omega_d = \{ y_i x \mid x \in \Omega_d \} \). Since \( a_i \leq d \), one can replace \( a_i z_i \) by \( y_i^d(x_1^d + \cdots + x_{a_i}^d) \) where \( x_1, \ldots, x_{a_i} \) are distinct elements of \( \Omega_d \). Hence \( H(n; \ell, m) = H(n; d, m) = 1 \) since \( \sum_{i=1}^{m/d} a_i = n \). This completes the proof of Theorem 3.

\[ \square \]

References


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