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Forbidden configurations and Steiner designs

Niranjan Balachandran

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Abstract Let $\mathcal{F}$ be a (0, 1) matrix. A (0, 1) matrix $\mathcal{M}$ is said to have $\mathcal{F}$ as a configuration if there is a submatrix of $\mathcal{M}$ which is a row and column permutation of $\mathcal{F}$. We say that a matrix $\mathcal{M}$ is *simple* if it has no repeated columns. For a given $v \in \mathbb{N}$, we shall denote by $\mathrm{forb}(v, \mathcal{F})$ the maximum number of columns in a simple (0, 1) matrix with v rows for which $\mathcal{F}$ *does not* occur as a configuration. We say that a matrix $\mathcal{M}$ is *maximal for* $\mathcal{F}$ if $\mathcal{M}$ has $\mathrm{forb}(v, \mathcal{F})$ columns. In this paper we show that for certain natural choices of $\mathcal{F}$, $\mathrm{forb}(v, \mathcal{F}) \leq \frac{\binom{v}{t}}{t+1}$. In particular this gives an extremal characterization for Steiner t-designs as maximal (0, 1) matrices in terms of certain forbidden configurations.

Keywords Forbidden configurations · Steiner designs · Nonuniform Ray–Chaudhuri–Wilson theorem

Mathematics Subject Classification 05B30 · 05D05

1 Introduction

Consider a matrix (0, 1) matrix $\mathcal{F}$. We say that a matrix $\mathcal{M}$ is *simple* if it has no repeated columns. A simple (0, 1) matrix $\mathcal{M}$ is said to have $\mathcal{F}$ as a configuration if there is a submatrix of $\mathcal{M}$ which is a row and column permutation of $\mathcal{F}$. For a given $v \in \mathbb{N}$, we shall denote by $\mathrm{forb}(v, \mathcal{F})$ the maximum number of columns in a simple (0, 1) matrix with v rows for which $\mathcal{F}$ *does not* occur as a configuration.

Several combinatorially 'nice' objects admit very natural descriptions as matrices with forbidden substructures. One of the earliest results of this kind is the following

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N. Balachandran (✉)
Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA
e-mail: nbalacha@caltech.edu

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Theorem 1 *Let v, k be positive integers with $k \leq v$. Call a set $A \subset \{0,1\}^v$ (v,k)-dense if for some set $S := \{i_1, i_2, \ldots, i_k\}$ of k coordinates, the projection of A onto S, namely, the set $A|_S := \{(w_{i_1}, w_{i_2}, \ldots, w_{i_k}) | w = (w_1, w_2, \ldots, w_v) \in A\}$ is the set $\{0,1\}^k$. Define*

$$H(v,k) := \sum_{i=0}^{k-1} \binom{v}{i}.$$

If $|A| > H(v,k)$ then A is (v,k)-dense. Moreover, the result is tight.

Thus, if a set $A \subset \{0,1\}^v$ is written in the form of a matrix with v rows whose columns correspond to the members of A (in a natural manner) then A is (v,k) dense if and only if the binary matrix $\mathcal{F}$ consisting of k rows and all possible 2^k columns is a configuration in A. This result, which was discovered by three sets of researchers independently, in the contexts of logic, extremal set theory, and probability theory respectively, and also at around the same time [12–14], determines $\text{forb}(v, \mathcal{F})$ for the aforementioned matrix $\mathcal{F}$.

Füredi [2] (unpublished) showed that for a fixed k and any (0, 1) matrix $\mathcal{F}$ with k rows, $\text{forb}(v, \mathcal{F}) = O(m^k)$. Anstee et al. [6] found the precise bound for $k = 2$ and Anstee-Sali[5] did the same for $k = 3$. What is rather remarkable in all the results proved here is that the asymptotics for any $\mathcal{F}$ satisfy $\text{forb}(v, \mathcal{F}) = \Theta(v^e)$ for some integer e. The Anstee-Sali conjecture (see [5]) states that such is the case for any $\mathcal{F}$ and attempts to explain this phenomenon by explicitly describing the integer e in terms of parameters associated with $\mathcal{F}$. This conjecture was settled in the affirmative in the case of $\mathcal{F}$ having 2 columns in [4]. We refer the interested reader to this paper for more results and details on this conjecture.

1.1 Forbidden configurations and designs

We start with some notation. In what follows, t, l, λ shall denote nonnegative integers. By $\mathbf{1_t 0_l}$ we shall mean the $(t+l) \times 1$ column consisting of t 1's atop l 0's. By $\mathcal{F}_{\lambda;t,l}$ we shall denote the $((t+l) \times (\lambda+1))$ matrix consisting of $\lambda + 1$ columns, each being $\mathbf{1_t 0_l}$. When $\lambda = 1$ we drop the λ in the notation and simply write $\mathcal{F}_{t,l}$.

Combinatorial designs also admit a natural description in terms of matrices with forbidden configurations. Let $v \geq k \geq t$ be positive integers. The incidence matrix of a $t - (v,k,\lambda)$ design (if there is one for these parameters) may be regarded as a $v \times b$ matrix $\mathcal{M}$ (where $b = \frac{\lambda\binom{v}{t}}{\binom{k}{t}}$ is the number of blocks in the design) which does not contain $\mathcal{F}_{\lambda;t,0}$ as a subconfiguration. Indeed, since such a subconfiguration is equivalent to a set of t points which are together contained in $\lambda + 1$ blocks, which by definition of a $t - (v,k,\lambda)$ design, is not allowed.

Conversely, if $\mathcal{M}$ is any $v \times n$ (0, 1)-matrix with $\mathcal{F}_{\lambda,t,0}$ as a forbidden configuration, and we know that every column of $\mathcal{M}$ has sum at least k then a simple counting argument shows that

$$n \leq \frac{\lambda\binom{v}{t}}{\binom{k}{t}}.$$

Thus, designs occur as extremal objects for certain matrices with forbidden configurations.

Anstee and Barekat [3] showed that block designs with block size 3 can be viewed as maximal matrices with forbidden configurations for certain other choices of $\mathcal{F}$ as well.

Theorem 2 ([3]) *Let λ, v be given integers.*

1. *There exists M so that for $v > M$, if $\mathcal{M}$ is a $v \times n$ (0,1) matrix with column sums at least 3 and at most $v - 1$ and $\mathcal{M}$ admits $\mathcal{F}_{\lambda;2,1}$ as a forbidden configuration, then $n \leq \frac{\lambda}{3}\binom{v}{2}$. Moreover, equality occurs if and only if the columns of $\mathcal{M}$ correspond to the set of blocks of a simple $2-(v, 3, \lambda)$-design.*
2. *There exists M so that for $v > M$, if $\mathcal{M}$ is a $v \times n$ (0,1) matrix with column sums at least 3 and at most $v - 3$ and $\mathcal{M}$ admits $\mathcal{F}_{\lambda;2,2}$ as a forbidden configuration, then $n \leq \frac{\lambda}{3}\binom{v}{2}$. Moreover, equality occurs if and only if there exist positive integers a, b such that $a + b = \lambda$, and $\mathcal{M}$ consists of*
   - *$\frac{a}{3}\binom{v}{2}$ columns with column sum 3 corresponding to the blocks of a $2-(v, 3, a)$-design,*
   - *$\frac{b}{3}\binom{v}{2}$ columns with column sum $v - 3$ corresponding to the complements of blocks of a $2-(v, 3, b)$-design.*

In this paper, our forbidden matrices of interest shall be $\mathcal{F}_{t,l}$ for $0 < l < t$, where $t \geq 3$. We first prove

Theorem 3 (Extremal Characterization for Steiner Quadruple Systems): *Let $\mathcal{M}$ be a simple $(0, 1)$-matrix with v rows such that each column in $\mathcal{M}$ has sum in $\{4, 5, \dots, v - 3\}$.*

1. *If $\mathcal{M}$ admits $\mathcal{F}_{3,1}$ as a forbidden configuration and $v > 4$, then $\mathcal{M}$ has at most $\frac{\binom{v}{3}}{4}$ columns. Furthermore equality occurs if and only if the columns of M describe the blocks of a $3-(v, 4, 1)$ design, i.e., a Steiner Quadruple System.*
2. *If $\mathcal{M}$ admits $\mathcal{F}_{3,2}$ as a forbidden configuration and $v > 80$, then $\mathcal{M}$ has at most $\frac{\binom{v}{3}}{4}$ columns. Furthermore equality occurs if and only if the columns of M describe the blocks of a $3-(v, 4, 1)$ design, i.e., a Steiner Quadruple System.*

We then follow the same lines of argument to prove

Theorem 4 *Let $0 < l < t, t \geq 3$ be integers. There exists $v_0 := v_0(t, l)$ such that for all $v \geq v_0$ the following holds. Suppose $\mathcal{M}$ is a simple $(0, 1)$ matrix with v rows and suppose that each column in $\mathcal{M}$ has sum in $\{t+1, t+2, \dots, v-t\}$. If $\mathcal{M}$ admits $\mathcal{F}_{t,l}$ as a forbidden configuration, then $\mathcal{M}$ has at most $\frac{\binom{v}{t}}{t+1}$ columns. Furthermore equality occurs if and only if the columns of $\mathcal{M}$ describe the blocks of a $t-(v, t+1, 1)$ design.*

One of the main tools in our proof is the Nonuniform Ray–Chaudhuri–Wilson theorem ([9]):

Theorem 5 (Nonuniform Ray–Chaudhuri–Wilson theorem): *Let $\mathcal{L}$ be a set of s non-negative integers, and let $\mathcal{A}$ be a family of subsets of $[v] := \{1, 2, \dots, v\}$ such that for any $A, B \in \mathcal{A}$ we have $|A \cap B| \in \mathcal{L}$. Then*

$$|\mathcal{A}| \leq 1 + \binom{v}{1} + \binom{v}{2} + \cdots + \binom{v}{s}.$$

The Nonuniform Ray–Chaudhuri–Wilson theorem has quite a few proofs, with one of the most interesting ones involving an elegant linear algebra argument. See for instance [7], chapter 5.

In the next section, we shall prove these two theorems. We first give the proof in detail for the case $t = 3$ and prove the general theorem in the subsequent subsection which closely follows the argument for $t = 3$. Since the argument is very close to the case for $t = 3$ we do not write all the details. There are however a few differences in the general case, which we shall address in more detail. The final section will include some remarks and suggest other questions for investigation along these lines.

2 Proofs of the theorems

Before we begin the proof of the theorem we shall rephrase some of the definitions above, in terms of set systems and make a few observations that will help simplify the presentation of the proof. Any simple matrix $\mathcal{M}$ which has $\mathcal{F}_{t,l}$ avoided as a configuration can be regarded (by abuse of notation) as a set system $\mathcal{M}$ of subsets of $[v] := \{1, 2, \ldots, v\}$ such that there are no two distinct $A, B \in \mathcal{M}$ with $|A \cap B| \geq t$ and $|\overline{A} \cap \overline{B}| \geq l$. The restriction on column sums is translated as requiring that $|A| \in \{t+1, t+2, \ldots, v-t\}$ for all $A \in \mathcal{M}$.

Proof of Theorem 3, part 1: Suppose $\mathcal{F} := \mathcal{F}_{3,1}$ is a forbidden configuration for $\mathcal{M}$. Let $\mathcal{T}$ be a 3-subset of $[v]$. Suppose there are distinct $A, B \in \mathcal{M}$ such that $\mathcal{T} \subset A \cap B$. We start with a few observations.

1. If $[v] \backslash (A \cup B)$ is non-empty, then A, B and any element of $[v] \setminus (A \cup B)$ induce $\mathcal{F}$ in $\mathcal{M}$ which is not possible by assumption. Hence we have $A \cup B = [v]$. Also, since $|A| \leq v - 3$ we have $|B - A| \geq 3$, and $|A - B| \geq 3$.
2. Suppose there exist $A, B \in \mathcal{M}$ such that $|A \cap B| \geq 3$, and suppose there exists another set $C \in \mathcal{M}$ satisfying $|C \cap A| \geq 3$. Then as noted above $A \cup C = [v]$, so that we have $B \cap C \supset \overline{A}$. Since $|A| \leq v - 3$ it follows that $|\overline{A}| \geq 3$, hence $|B \cap C| \geq 3$ and so by the previous observation we must have $B \cup C = [v]$ as well.

The preceding observations imply the following. Suppose there exist sets $A, B \in \mathcal{M}$ such that $|A \cap B| \geq 3$. Then for every set $C \in \mathcal{M}$ such that $|A \cap C| \geq 3$ we must have $A \cup B = A \cup C = B \cup C = [v]$. Furthermore, we must also have $C \supset A - B, B - A$. Consequently, the subfamily

$$\mathcal{M}_1 := \left\{C \in \mathcal{M} \middle| \, 3 \leq |C \cap A|\right\}$$

has a 1–1 correspondence with subsets of $A \cap B$. Moreover for $C_1 \neq C_2$ and $C_1, C_2 \in \mathcal{M}_1$ we have $C_i^* := C_i \setminus (A \Delta B)$, $i = 1, 2$ satisfying, $C_1^* \cup C_2^* = A \cap B$. It follows from a simple linear algebra argument that $|\mathcal{M}_1| \leq |A \cap B|$.

Now every member of $\mathcal{M}_0 := \mathcal{M} \setminus \mathcal{M}_1$ must by definition contain at most 2 elements of A. By the observation above, it also follows that every set in $\mathcal{M}_0$ must also contain at most 2 elements of B. Further, note that if there exists $C \in \mathcal{M}_0$ containing an element of $A \cap B$ then since $|C| \geq 4$ we must necessarily have $|C \cap A| \geq 3$ or $|C \cap B| \geq 3$ and that contradicts $C \notin \mathcal{M}_1$. Consequently, every $C \in \mathcal{M}_0$ contains precisely two elements in $A - B$ and two elements in $B - A$. Thus we may write $C := C_A \cup C_B$ with $C_A \subset A - B$, and $C_B \subset B - A$, and $|C_A| = |C_B| = 2$.

Suppose for distinct sets $C, C' \in \mathcal{M}_0$ we have $C_A = C'_A = \mathfrak{C}$, say. Then $C_B \cap C'_B$ must be empty. Otherwise, any $x \in C_B \cap C'_B$ together with $\mathfrak{C}$ is a subset of $C \cap C'$ and since neither C not C' contains any element of $A \cap B$ (which has at least 3 elements) the sets C, C' induce the forbidden configuration in $\mathcal{M}$.

Let us summarize our findings. We have already observed that $|\mathcal{M}_1| \leq |A \cap B|$. Furthermore, every $C \in \mathcal{M}_0$ has size exactly 4 and each such set contains precisely two elements of $A - B$ and two of $B - A$. Conversely, if $\alpha \subset A - B$ is a 2 element subset of $A - B$ then the collection

$$\mathfrak{B}_\alpha := \{\beta \subset B - A \,\big|\, \alpha \cup \beta \in \mathcal{M}_0\},$$

is a collection of pairwise disjoint subsets of $B-A$. Thus, if we write $|A-B|=a$, $|B-A|=b$, $|A\cap B|=c$, we have

$$|\mathcal{M}|=|\mathcal{M}_1|+|\mathcal{M}_0|\le c+\sum_{\substack{\alpha\subset A-B\\|\alpha|=2}}|\mathfrak{B}_\alpha|\le c+\binom{a}{2}\frac{b}{2}.$$

Note that by our notation, we have the constraint $a+b+c=v$.

Using the AM–GM[1] inequality, and writing $b=v-c-a$, we get
$|\mathcal{M}|\le c+\frac{a(a-1)b}{4}<\frac{(v-3)^3}{27}+v-3<\frac{\binom{v}{3}}{4}$ for $v\ge 5$.

In summary: If there are two sets $A,B\in\mathcal{M}$ such that $|A\cap B|\ge 3$ then the size of $\mathcal{M}$ is strictly less than the bound stated in the theorem.

So, suppose that for all $A,B\in\mathcal{M}$ we have $|A\cap B|\le 2$. For $B\in\mathcal{M}$ and $x\in B$, let B^x denote the set $B\setminus\{x\}$. Let $\mathcal{B}_B:=\{B^x|x\in B\}$, so that $|\mathcal{B}_B|=|B|$. Finally, let $b:=\sum_{B\in\mathcal{M}}|B|$.

Define $\mathfrak{M}$ to be the $b\times\binom{v}{3}$ incidence matrix with columns indexed by the 3 subsets of $[v]$, rows indexed by B^x with $x\in B$, $B\in\mathcal{M}$, and $\mathfrak{M}(B^x,T)=1$ if and only if $T\subset B^x$. Note that the matrix $\mathfrak{M}$ admits a block form with the rows grouped by $\mathcal{B}_B$, $B\in\mathcal{M}$. Let $\mathfrak{N}:=\mathfrak{M}\mathfrak{M}^T$.

Let $\mathfrak{N}(A^x,B^y)$ denote the $(A^x,B^y)^{th}$ element of $\mathfrak{N}$, where $A,B\in\mathcal{M}$ and $x\in A$, $y\in B$. If $A\ne B$ then $\mathfrak{N}(A^x,B^y)$ counts the number of three element subsets of $A^x\cap B^y$. By assumption, $|A\cap B|\le 2$, so $\mathfrak{N}(A^x,B^y)=0$ if $A\ne B$. Thus, $\mathfrak{N}$ is seen to be a block matrix will all the off-diagonal blocks being zero and the diagonal blocks $\mathcal{D}_A$ are matrices indexed by the members $A\in\mathcal{M}$. Clearly,

$$\mathcal{D}_A:=\left(\binom{|A|-1}{3}-\binom{|A|-2}{3}\right)\mathbf{I}_A+\binom{|B|-2}{3}\mathbf{J}_A,$$

where the matrices $\mathbf{I}_A,\mathbf{J}_A$ are matrices of order $|A|$. Since $|A|\ge 4$ these matrices are all invertible. Indeed, this is an easy consequence of the fact that the matrix $x\mathbf{I}_n+y\mathbf{J}_n$ has determinant $x^{n-1}(x+yn)$.

Thus $\mathfrak{N}$ is invertible which implies that $r(\mathfrak{M})=b$ and hence $b\le\binom{v}{3}$. Since $|A|\ge 4$ for all $A\in\mathcal{M}$, we have $4|\mathcal{M}|\le b\le\binom{v}{3}$ and that proves the upper bound.

In case of equality, i.e., when $|\mathcal{M}|=\frac{\binom{v}{3}}{4}$ then note that the matrix $\mathfrak{M}$ is a square matrix. Since all the rows and columns are indexed by the 3-elements subsets of $[v]$, it follows that $\mathfrak{M}$ is a permutation matrix of order $\binom{v}{3}$. Hence every 3-element subset of $[v]$ occurs in a unique member of $\mathcal{M}$ and that proves that $\mathcal{M}$ is indeed a Steiner Quadruple System. □

Remark One could alternately, use a counting argument to establish the upper bound instead of the matrix rank argument delineated above. But note that for $v\ge 27$, we can write

$$|\mathcal{M}|\le c+\frac{a(a-1)b}{4}<\frac{(v-3)^3}{27}+v-3<\frac{v^3}{27}<\frac{\binom{v}{3}}{4}.$$

In particular, we have shown that if $\mathcal{M}$ has $\mathcal{F}_{3,1}$ as a forbidden configuration and $|\mathcal{M}|\ge\frac{v^3}{27}$ then the incidence matrix of $\mathcal{M}$ has full row rank, which says something more than merely establishing the bound.

Proof of Theorem 3, part 2: We now consider the forbidden matrix $\mathcal{F}:=\mathcal{F}_{3,2}$ and look at maximal set systems with $\mathcal{F}$ as a forbidden configuration. We first prove the theorem when there is a set $A\in\mathcal{M}$ with size $v-3$. As before we consider sets with sizes in $\{4,5,\ldots,v-3\}$.

[1] Arithmetic mean–Geometric mean.

Lemma 6 *Suppose $\mathcal{M}$ is a set system on $[v]$ with $\mathcal{F}$ as a forbidden subconfiguration and suppose there exists $A \in \mathcal{M}$ such that $|A| = v - 3$. Then $|\mathcal{M}| \le 2 + 6(v-3) + \frac{(v-3)^2}{2}$. In particular, if $v \ge 19$, we have $|\mathcal{M}| < \frac{\binom{v}{3}}{4}$.*

Proof of Lemma 6: Let the elements not in A be x, y, z and write $X := \{x, y, z\}$. For any other set $B \in \mathcal{M}$ such that $|A \cap B| \ge 3$ we must have $|B \cap X| \ge 2$ otherwise $\mathcal{F}$ occurs as a sub-configuration.

We again start with a claim. For any $\mathfrak{a} \subset A$ satisfying $|\mathfrak{a}| \ge 2$ and $|A \setminus \mathfrak{a}| \ge 2$, there is at most one set of the form $\mathfrak{a} \cup \alpha \in \mathcal{M}$ with $\alpha \subset X$. To see why this must be so, suppose the contrary, i.e., suppose $\mathfrak{a}_1 := \mathfrak{a} \cup \alpha$, $\mathfrak{a}_2 := \mathfrak{a} \cup \beta$ are distinct sets in $\mathcal{M}$ for distinct $\alpha, \beta \subset X$. If one of α or β, say α has size less than 2 then it follows that $\mathfrak{a}$ has size at least 3 since $\mathfrak{a}_1$ has at least 4 elements. Consequently, $|A \cap \mathfrak{a}_1| \ge 3$ and $|\overline{A} \cap \overline{\mathfrak{a}_1}| \ge 2$, which induces $\mathcal{F}$ as a subconfiguration. Hence both α, β have at least 2 elements. In this case $|\alpha \cap \beta| \ge 1$, so $|\mathfrak{a}_1 \cap \mathfrak{a}_2| \ge 3$. Also since $A \setminus \mathfrak{a}$ contains at least two elements, we have $|\overline{\mathfrak{a}_1} \cap \overline{\mathfrak{a}_2}| \ge 2$ so that $\mathfrak{a}_1, \mathfrak{a}_2$ together induce the configuration $\mathcal{F}$.

A consequence of the claim is the following. For every set $C \in \mathcal{M}$, precisely one of the following holds:

1. $|C \cap A| = v - 4$. In this case $|C \cap X| \le 1$ since $|C| \le v - 3$.
2. $v - 4 > |C \cap A| \ge 2$. In this case, the claim proven above implies that the set $C \cap A$ determines C uniquely.
3. $|C \cap A| \le 1$.

The collection $\mathcal{M}_1 := \{C \in \mathcal{M} | |C \cap A| = v - 4\}$ has size at most $4(v-3)$. This follows since for every choice of $C \cap A$ of which there are $v - 3$ such sets, there are at most 4 subsets of X that can be appended to C.

Consider now, the collection $\mathcal{M}_2 := \{C \in \mathcal{M} | |C \cap A| \le 1\}$. Since the set sizes in $\mathcal{M}$ are at least 4, it follows that for any set $C \in \mathcal{M}_2$, we have $X \subset C$. Hence if there are two distinct $C_1, C_2 \in \mathcal{M}_2$ then $|C_1 \cap C_2| = 3$, $|\overline{C_1} \cap \overline{C_2}| \ge |A| - 2 \ge 2$ if $v \ge 7$ so that C_1, C_2 induce $\mathcal{F}_{3,2}$ as a configuration. Hence $|\mathcal{M}_2| \le 1$. It thus remains to bound the number of sets in $\mathcal{M}_0 := \{C \in \mathcal{M} | v - 4 > |C \cap A| \ge 2\}$.

For each subset $\alpha \subset X$, let $\mathcal{M}_0(\alpha) := \{C \in \mathcal{M}_0 | C \cap X = \alpha\}$, and let $m_0(\alpha) := |\mathcal{M}_0(\alpha)|$.

Firstly, note that the proof of the claim actually shows that if $|\alpha| = 0$ or 1 then any $C \in \mathcal{M}_0(\alpha)$ together with A will induce an $\mathcal{F}_{3,2}$. Hence we may assume $|\alpha| \ge 2$.

Let $\mathcal{M}_A(\alpha) := \{C \cap A | C \in \mathcal{M}_0(\alpha)\}$ and define $m_A(\alpha)$ analogously. By the observations made earlier, each set in $\mathcal{M}_A(\alpha)$ determines $\mathcal{M}_0(\alpha)$ uniquely.

If $|\alpha| = 2$, then for any distinct $C_1, C_2 \in \mathcal{M}_A(\alpha)$ either $|C_1 \cup C_2| = A$ or $C_1 \cap C_2 = \emptyset$, since $|C_i| < v - 4$ for all such C_i. In this case, we claim that $m_A(\alpha) \le \frac{v-3}{2}$. This is a simple consequence of the following general observation: Suppose $\mathcal{A} \subset \mathcal{P}(X)$ is a family of sets such that $X \notin \mathcal{A}$ and for any distinct non-empty $A, B \in \mathcal{A}$ we have $A \cap B = \emptyset$ or $A \cup B = X$. Then either $A \cap B = \emptyset$ for all distinct A, B or $A \cup B = X$ for all distinct A, B. Indeed, by taking complements if necessary, it follows that $m_A(\alpha)$ has no more members than the cardinality of a set of subsets of A of size 2 each, which are pairwise disjoint.

To prove the observation stated above, suppose $A_1 \cap B_1 = \emptyset$ and suppose there exists A_2 such that $A_2 \cap B_1 \ne \emptyset$. By the assumption, we must have $A_2 \cup B_1 = X$ and so it follows that $A_1 \subset A_2$. But then $A_1 \cup A_2 = A_2 = X$ by assumption which contradicts that $X \notin \mathcal{A}$.

Finally, if $\alpha = X$, then for $C_1, C_2 \in \mathcal{M}_A(\alpha)$ we have $|\overline{C_1} \cap \overline{C_2}| \le 1$. By the Nonuniform Ray–Chaudhuri–Wilson theorem, we have $m_A(\alpha) \le 1 + (v-3) + \binom{v-3}{2}$. Hence, summing over the $m_A(\alpha)$ we complete the proof of the lemma. □

Resuming the Proof of Theorem 3, part 2: The lemma above settles the proof of the theorem in the case where there is some set $A \in \mathcal{M}$ for which $|A| = v-3$. So, in what follows, we shall assume that for all $A \in \mathcal{M}$, we have $|A| < v-3$.

Suppose there are sets $A, B \in \mathcal{M}$ with $|A \cap B| \geq 3$. Since $\mathcal{F}_{3,2}$ is forbidden we have $|\overline{A \cup B}| = 0$ or 1. We consider each case separately.

1. Suppose there exist $A, B \in \mathcal{M}$ such that $|A \cap B| \geq 3$, $|A \cup B| = v$. Let
$$\mathcal{M}_1 := \{C \in \mathcal{M} \big| \, 3 \leq |C \cap A|\}.$$
Since we have $|A| \leq v-4$, $|B| \leq v-4$ we must have $|A \setminus B| \geq 4$, $|B \setminus A| \geq 4$.
Since $|A \cap C| \geq 3$ we must have $|\overline{A \cup C}| = 0$ or 1; in particular, C misses at most one element of $B \setminus A$, so $|B \cap C| \geq 3$ as well. Thus by symmetry, we must in particular have $|C \cap (A \setminus B)| \geq 3$. Consequently, for $C_1, C_2 \in \mathcal{M}_1$ we have $|(C_1 \cap C_2 \cap (A \setminus B)| \geq 2$ and $|(C_1 \cap C_2 \cap (B \setminus A)| \geq 2$, so that $|C_1 \cap C_2| > 3$. Hence for any two distinct sets C_1, C_2 in $\mathcal{M}_1$ we have $|C_1 \cup C_2| \geq v-1$. Taking complements and using the Nonuniform Ray–Chaudhuri–Wilson theorem again, it follows that $|\mathcal{M}_1| \leq 1 + v + \frac{v(v-1)}{2}$.
For any $C \in \mathcal{M}_0 := \mathcal{M} \setminus \mathcal{M}_1$ we must hence have $|C \cap A| \leq 2$ and also $|C \cap B| \leq 2$. Furthermore, the same reasoning as outlined in the proof of the first part shows that $A \cap B \cap C = \emptyset$. Thus, we conclude that for $C \in \mathcal{M}_0$ we must then have $|C \cap (A \setminus B)| = 2$, $|C \cap (B \setminus A)| = 2$, and $A \cap B \cap C = \emptyset$. Once again writing $a = |A \setminus B|$, $b = |B \setminus A|$ (so that we have the constraint $a + b \leq v-3$.), as in the proof of part 1, we could use the same upper bound for $|\mathcal{M}_0|$ as in part 1 of the theorem. But this time, observe that the argument outlined in the proof of the first part actually shows $|\mathcal{M}_0| \leq \min\left(\binom{a}{2}\frac{b}{2}, \binom{b}{2}\frac{a}{2}\right)$ by the symmetric nature of the argument, and this improvement will serve us better here. Let
$$f(t) := \max\left(\binom{a}{2}\binom{b}{2}\frac{ab}{4}\right) \text{ subject to } a+b = t-3, a \geq 0, b \geq 0.$$
We are interested in the maximum value of $f(t)$ subject to $0 \leq t \leq v$. However, note that the function $f(t)$ is clearly increasing in t and furthermore,
$$|\mathcal{M}_0| \leq \min\left(\binom{a}{2}\frac{b}{2}, \binom{b}{2}\frac{a}{2}\right) \leq \sqrt{f(v)}.$$
A simple calculation (involving nothing more than elementary calculus) shows that $\sqrt{f(v)} = \frac{(v-3)^2(v-5)}{32}$ for $v \geq 5$. Another simple calculation shows that for $v \geq 32$ we have,
$$|\mathcal{M}_1| + |\mathcal{M}_0| < \frac{\binom{v}{3}}{4}.$$
2. Suppose now that whenever $|A \cap B| \geq 3$ we have $|A \cup B| = v-1$. Let us denote the unique element of $[v] \backslash (A \cup B)$ by x_{AB}. Fix a pair of distinct sets A, B for which $|A \cap B| \geq 3$. Define
$$\mathcal{M}_0(A) := \{C \in \mathcal{M} \big| \, 3 \leq |C \cap A|\}, \mathcal{M}_0(B) := \{C \in \mathcal{M} \big| \, 3 \leq |C \cap B|\}.$$
We now estimate an upper bound for $|\mathcal{M}_0(A)|$. A similar estimate will also be applicable for $|\mathcal{M}_0(B)|$.
For each $x \notin A$, let $\mathcal{M}_A(x) := \{C \in \mathcal{M}_0(A) | x_{AC} = x\}$. As before, note that for $C, C' \in \mathcal{M}_A(x)$ we have $|C' \cap A| \geq 3$, $|C \cap A| \geq 3$, $A \cup C' = A \cup C = [v] \setminus \{x\}$. Since $|A| \leq v-4$ we have $\overline{A \cup \{x\}} \subseteq C' \cap C$, so $|C' \cap C| \geq 3$. By the hypothesis

about the family $\mathcal{M}$ it follows that since $|C \cap C'| \geq 3$ we must have $|C \cup C'| = v - 1$ for distinct $C, C' \in \mathcal{M}_A(x)$. But since C, C' do not contain x we must necessarily have $C \cup C' = [v]\backslash\{x\}$. Equivalently, their complements (taken in the set $[v]\backslash\{x\}$) are pairwise disjoint. Again, by assumption, each member of $\mathcal{M}_A(x)$ has at most $v - 4$ elements, so the complements of these members (in $[v]\backslash\{x\}$) have at least 3 elements each. Hence, $|\mathcal{M}_A(x)| \leq \frac{(v-1)}{3}$. Therefore,

$$|\mathcal{M}_0(A)| = \left| \bigcup_{x \notin A} \mathcal{M}_A(x) \right| \leq \frac{(v-1)(v-4)}{3}.$$

A similar bound holds for $|\mathcal{M}_0(B)|$.

Finally, let $\mathcal{M}_1 := \mathcal{M} \setminus (\mathcal{M}_0(A) \cup \mathcal{M}_0(B))$. We give an upper bound for $|\mathcal{M}_1|$ by bounding the following two subfamilies of $\mathcal{M}_1$:

- $\mathcal{M}_1' := \{C \in \mathcal{M}_1 | x_{AB} \in C\}$. In this case, deleting x_{AB} from each of these sets gives us a family of subsets of $A \cup B$; each of the sets in $\mathcal{M}_1$ has at most two elements from each of A and B, hence any two miss at least 2 common elements if v is sufficiently large. Furthermore, since all these sets contain the element x_{AB} it follows that if $|C_1 \cap C_2| \geq 2$ then C_1, C_2 will induce $\mathcal{F}_{3,2}$. Hence we must have $|C_1 \cap C_2| < 2$ for $C_1, \mathcal{C}_2 \in \mathcal{M}_1'$. The Nonuniform Ray–Chaudhuri–Wilson theorem then gives $|\mathcal{M}_1'| \leq 1 + (v-1) + \frac{(v-1)^2}{2}$.
- $\mathcal{M}_1'' := \{C \in \mathcal{M}_1 | x_{AB} \notin C\}$. In this case, we argue exactly as in the previous case noting that any such set C must satisfy $|C \cap A| \leq 2, |C \cap B| \leq 2, A \cap B \cap C = \emptyset$. A very similar calculation as in the preceding part gives us $|\mathcal{M}_1''| \leq \frac{(v-4)^2(v-6)}{32}$.

Summing all these inequalities, we have

$$|\mathcal{M}| \leq \frac{(v-4)^2(v-6)}{32} + \frac{2(v-1)(v-4)}{3} + \left(1 + (v-1) + \frac{(v-1)^2}{2}\right) < \frac{\binom{v}{3}}{4},$$

for $v \geq 81$. Finally, if $\mathcal{M}$ satisfies that for any distinct $A, B \in \mathcal{M}$ we have $|A \cap B| \leq 2$ then we proceed exactly as in the proof of part 1. We omit the details.

2.1 The proof of the theorem for arbitrary t, l

We now proceed to show the proof in the general case ($0 < l < t$) with $\mathcal{F}_{t,l}$ being forbidden in $\mathcal{M}$. The proof is very similar to the case $t = 3$. We first prove a lemma analogous to Lemma 6 which deals with the case where one of the sets in $\mathcal{M}$ is 'large' in which case we shall show a stricter bound: $|\mathcal{M}| = O(v^{t-1})$. Then we shall complete the proof of the theorem in the general case.

To set up our notation, we have the following premise: $\mathcal{M}$ is a collection of subsets of $[v] := \{1, 2, \ldots, v\}$ such that there are no two distinct $A, B \in \mathcal{M}$ such that $|A \cap B| \geq t, |\overline{A} \cap \overline{B}| \geq l$. Also, $|A| \in \{t+1, t+2, \ldots, v-t\}$ for all $A \in \mathcal{M}$.

Lemma 7 *Let $0 < l < t$, and suppose $\mathcal{M}$ is a collections of subsets of $[v]$ with $\mathcal{F}_{t,l}$ being a forbidden configuration. Suppose $\max_{A \in \mathcal{M}} |A| > v + 1 - (t + l)$. Then $|\mathcal{M}| = O_{t,l}(v^{t-1}) = O(v^{t-1})$ as v goes to infinity.*

Remark The notation $O_{t,l}$ indicates that the associated constant in the statement of the lemma depends on t, l but is independent of v. We shall keep the t, l implicit in our O notation in what follows.

Proof of Lemma 7: Suppose A is a largest set in $\mathcal{M}$ with $|A| = a > (v+1) - (t+l)$, so, $|\overline{A}| < t + l - 1$. As in the Proof of Lemma 6, if $\mathfrak{a} \subset A$ such that $|\mathfrak{a}| \geq t-1$, and $|A \setminus \mathfrak{a}| \geq l$, then for any $\alpha, \beta \subset \overline{A}$ with $C_1 := \mathfrak{a} \cup \alpha, C_2 := \alpha \cup \beta \in \mathcal{M}$, we must have $\alpha \cap \beta = \emptyset$, else the sets C_1, C_2 induce $\mathcal{F}_{t,l}$ as a subconfiguration of $\mathcal{M}$. Hence, as before, it follows that every member of $\mathcal{M}$ is in one of the following subfamilies of $\mathcal{M}$:

1. $\mathcal{M}_1 := \{C \in \mathcal{M} \big| \ |A \cap \overline{C}| \leq l-1\}$. It is now easy to see that

$$|\mathcal{M}_1| \leq \sum_{i=0}^{l-1} \binom{|A|}{i} 2^{t+l-1} = O(v^{l-1}) 2^{t+l-1} = O(v^{l-1}).$$

2. $\mathcal{M}_2 := \{C \in \mathcal{M} \big| \ |C \cap A| < t-1\}$. Again, as in the above case, it is easy to see that $|\mathcal{M}_2| = O(v^{t-2})$.
3. $\mathcal{M}_A := \{C \in \mathcal{M} \big| \ |C \cap A| \geq t, |A \setminus C| \geq l\}$. Again, in this case, we analogously define $M_A(\alpha)$ for each $\alpha \subset \overline{A}$ as in the Proof of Lemma 6. Once again, we see that if $|\alpha| \leq t - l$ we have $|\mathcal{M}_A(\alpha)| = 0$. Furthermore, if $C_1, C_2 \in \mathcal{M}_A(\alpha)$ then we must have $|C_1 \cap C_2| \leq l - 1$ otherwise $\mathcal{F}_{t,l}$ occurs as a subconfiguration. Once again, the Nonuniform Ray–Chaudhuri–Wilson theorem implies that $|\mathcal{M}_A(\alpha)| = O(v^l)$.
   Thus

$$\mathcal{M}_A = \bigcup_{\alpha \subset \overline{A}} \mathcal{M}_A(\alpha),$$

   so in particular,

$$|\mathcal{M}_A| \leq \sum_{\alpha \subset \overline{A}} |\mathcal{M}_A(\alpha)|.$$

   We have just observed that $|\mathcal{M}_A(\alpha)| = O(v^l)$ for $\alpha \subset \overline{A}$. Since $|\overline{A}| < t + l - 1$, the number of choices for α is at most 2^{t+l-1}. Thus, $|\mathcal{M}_A| = O(v^{t-1})$.

The sizes of these three subfamilies + 1 (for the set A) is clearly an upper bound for $|\mathcal{M}|$. The preceding observations complete the proof. □

Proof of Theorem 4: By the previous lemma, we can assume that $|A| \leq (v+1) - (t+l)$ for all $A \in \mathcal{M}$. Suppose we have $A, B \in \mathcal{M}$ such that $|A \cap B| \geq t$. Fix such a pair A, B of distinct sets. Since $\mathcal{F}_{t,l}$ is forbidden we have $|\overline{A \cup B}| < l$. Once again define $\mathcal{M}_0(A), \mathcal{M}_0(B)$ as in the Proof of Theorem 4:

$$\mathcal{M}_0(A) := \{C \in \mathcal{M} \big| \ t \leq |C \cap A|\}, \mathcal{M}_0(B) := \{C \in \mathcal{M} \big| \ t \leq |C \cap B|\}.$$

For each $\mathfrak{a} \subset \overline{A}$ with $|\mathfrak{a}| = s < l$, let

$$\mathcal{M}_A(\mathfrak{a}) := \{C \in \mathcal{M}_0(A) | \overline{A \cup C} = \mathfrak{a}\}.$$

Then for $C_1, C_2 \in \mathcal{M}_A(\mathfrak{a})$ we have $|C_1 \cap C_2| \geq t$, which implies that $|C_1 \cup C_2| \geq v - (l-1)$. Thus, $|\overline{C_1 \cup C_2}| < l - s$ where the complementation is with respect to the set $[v] \setminus \mathfrak{a}$. So, the Nonuniform Ray–Chaudhuri–Wilson theorem gives us $|\mathcal{M}_A(\mathfrak{a})| = O(v^{l-s})$. Since there are at most $\binom{v-t-1}{s} = O(v^s)$ choices for sets $\mathfrak{a}$ with size s, we have

$$|\mathcal{M}_0(A)| \leq \sum_{\substack{\mathfrak{a} \subset \overline{A} \\ |\mathfrak{a}| \leq l-1}} |\mathcal{M}_A(\mathfrak{a})| = \sum_{0 \leq s \leq l-1} O(v^s) O(v^{l-s}) = O(v^l).$$

Since $l < t$ this bound is $O(v^{t-1})$. The same argument yields $|\mathcal{M}_0(B)| = O(v^{t-1})$, so we are left with bounding the subfamily $\mathcal{M}_1 := \mathcal{M} \setminus (\mathcal{M}_0(A) \cup \mathcal{M}_0(B))$.

This time, we obtain an upper bound for this family in a slightly differently manner from the proof techniques of Theorem 3. Firstly note that any $C \in \mathcal{M}_1$ satisfies $|C \cap A| \leq t-1, |C \cap B| \leq t-1$; hence we have $|C| \leq (t-1)+(t-1)+(l-1) = 2t+l-3$ for any $C \in \mathcal{M}_1$.

Suppose $C_1, C_2 \in \mathcal{M}_1$ and $|C_1 \cap C_2| \geq t$. Then $|C_1 \cap C_2| > v-l$. But we have $|C_1 \cup C_2| = |C_1| + |C_2| - |C_1 \cap C_2| \leq 2(2t+l-3)-t = 3t+2l-6$ and this is smaller than $v-l$ if $v \geq 3(t+l-2)$. In particular, every t-element subset of $[v]$ not contained in A or B is contained in at most one member of $\mathcal{M}_1$. Since every set in $\mathcal{M}$ has size at least $t+1$, we have,

$$|\mathcal{M}_1| \leq \frac{\binom{v}{t} - \left(\binom{a+c}{t} + \binom{b+c}{t} - \binom{c}{t}\right)}{t+1},$$

where as before, $a := |A \setminus B|, b := |B \setminus A|, c = |A \cap B|$. Since for $x, y \geq t > 1$ we have $\binom{x+y}{t} \geq \binom{x}{t} + \binom{y}{t}$ we have

$$\binom{a+c}{t} + \binom{b+c}{t} - \binom{c}{t} \geq \binom{a}{t} + \binom{b}{t} + \binom{c}{t}.$$

Furthermore, note that $f(x) = \binom{x}{t} := \frac{x(x-1)\cdots(x-t+1)}{t!}$ is convex for $x \geq t$; this gives us

$$|\mathcal{M}_1| \leq \frac{\binom{v}{t} - 3\binom{(v-l+1)/3}{t}}{t+1},$$

so we have

$$\begin{aligned} |\mathcal{M}| \leq |\mathcal{M}_0(A)| + |\mathcal{M}_0(B)| + |\mathcal{M}_1| &\leq \frac{\binom{v}{t}}{t+1} - \frac{3\binom{(v-l+1)/3}{t}}{t+1} + O(v^{t-1}) \\ &< \frac{\binom{v}{t}}{t+1} \text{ for large } v. \end{aligned}$$

Thus, if some for $A, B \in \mathcal{M}$ we have $|A \cap B| \geq t$ then the bound in the statement of the theorem is strict for large v.

Thus, we may assume that $|A \cap B| < t$ for all distinct $A, B \in \mathcal{M}$. Again, the same argument in the Proof of Theorem 3, part 1 completes the proof. The case of equality is also similar to the proof outlined in Theorem 3 part 1; equality is attained if and only if every t element subset of $[v]$ is contained in a unique member of $\mathcal{M}$ which is equivalent to saying that $([v], \mathcal{M})$ is a Steiner t-design. We omit the details. □

3 Concluding remarks

- The authors of [3] observe that equality in Theorem 2 is attained whenever we also have $\lambda v(v-1) \equiv 0 \pmod 6, \lambda(v-1) \equiv 0 \pmod 2$, and $v \geq \lambda + 2$ which invokes a result due to Dehon [8] that shows the existence of appropriate block designs. Hanani's result on existence of Steiner quadruple systems [10] implies that equality in Theorem 3 is attained if and only if $v \equiv 2$ or $4 \pmod 6$.

The general problem of existence of Steiner t-designs is unsettled for $t \geq 6$. Though it is widely believed that Steiner t-designs exist for all t, it is still conceivable that Steiner t-designs do not exist for some t, in which case equality in the aforementioned result cannot be attained. Nonetheless, our theorem is asymptotically tight as a consequence of the Erdős-Hanani conjecture settled by Rödl [11]: For any given integers $t > 0$ and any small real $\epsilon > 0$, there exists an integer $v_0 = v_0(t, \epsilon)$ such that for all $v \geq v_0$ there exists a collection $\mathcal{M}$ of $t+1$-subsets of $[v]$ such that no t element subset of $[v]$ is contained in two distinct members of $\mathcal{M}$ and $|M| \geq (1-\epsilon)\frac{\binom{v}{t}}{t+1}$.

- A closer inspection of the Proof of Theorem 4 shows that the estimates for $|\mathcal{M}_0(A)|$ and $|M_0(B)|$ remain valid even if we were to impose a restriction on column sums to be at least K for some fixed integer $K \geq t+1$. Thus, if $k \geq t$ is an integer such that $K+1 \geq \binom{k}{t}$ then the latter estimate for $|\mathcal{M}_1|$ on Theorem 4 shows that we can improve the lower bound to $|\mathcal{M}| \leq \frac{\binom{v}{t}}{\binom{k}{t}}$. If the condition on the column sums in $\mathcal{M}$ is made tighter, then one might end with bounds of a completely different nature. Indeed, as shown in [1] if for some $\epsilon > 0$ we have $\epsilon v \leq |A| \leq (1-\epsilon)v$ for all $A \in \mathcal{M}$ then if $|\mathcal{M}| \geq (\frac{4}{\epsilon})^2$ there must exist distinct $A, B \in \mathcal{M}$ such that $|A \cap B| \geq (\epsilon/4)^2 v$ and $|\overline{A} \cap \overline{B}| \geq (\epsilon/4)^2 v$, which follows from a probabilistic argument. In particular, if we were to make a much tighter constraint on the column sums of $\mathcal{M}$, then an uppoer bound for the number of columns of $\mathcal{M}$ with $\mathcal{F}_{t,l}$ being forbidden for any $0 \leq t, l \leq (\epsilon/4)^2 v$ is $O_\epsilon(1)$.
- Theorem 3 part 2 proves the theorem for $v \geq 81$. We do not believe that this value is best possible, i.e., it is quite possible that for smaller values of v, this bound could be bettered. Likewise, the implicit value of $v_0(t, l)$ as outlined by the Proof of Theorem 4 could perhaps be bettered by a different argument.
- We have seen that $\mathrm{forb}(v, \mathcal{F}_{t,l}) \leq \frac{\binom{v}{t}}{t+1}$ for all $l < t$. In the case $l = t$, if $\mathcal{M}$ is a family forbidding $\mathcal{F}_{t,t}$ then so is $\overline{\mathcal{M}}$ which consists of the complements of the sets in $\mathcal{M}$. Anstee [2] believes that the same bound extends to that case as well, and that equality occurs if and only if $\mathcal{M}$ describes the blocks or the complements of the blocks of a Steiner t-design.
- As in the paper of [3], it would be of interest to consider $\mathcal{F}_{\lambda;t,l}$ for fixed λ. We hope to address this problem in the near future.

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