Graphs with restricted valency and matching number

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Abstract

Consider the family of all finite graphs with maximum degree $\Delta(G) < d$ and matching number $\nu(G) < m$. In this paper we give a new proof to obtain the exact upper bound for the number of edges in such graphs and also characterize all the cases when the maximal graph is unique.

Key words: sunflower, Gallai's lemma, factor-critical graph.

1 Introduction

Consider the set $\mathcal{F}(d,m)$ of all finite maximal simple graphs that satisfy $\Delta(G) < d, \nu(G) < m$, where $\Delta(G)$ denotes the maximum degree among all the vertices of G and $\nu(G)$ denotes the size of a maximum matching in G. In particular, when $d = m = s$, this set consists of all those (finite) maximal graphs with both degree and matching size less than s.

A *sunflower* with s petals is a collection of sets $A_1, A_2 \ldots, A_s$ and a set X(possibly empty) such that $A_i \cap A_j = X$ whenever $i \neq j$. The set X is called the core of the sunflower. Hence the graphs in $\mathcal{F}(d,m)$ give set systems with uniform block size 2 containing no sunflower with s or more petals.

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It is a well known result (due to Erdős-Rado[3]) that a set system with uniform block size k, of size greater than $k!(s-1)^k$ admits a sunflower with s petals(for a proof see [1], for instance). Other bounds that ensure the existence of a sunflower with s petals are known in the case of $s = 3$ with block size k (see [2]), but the general case seems quite far away.

Our instance of this problem deals with $k = 2$. The Erdős - Rado bound for a set system $\mathcal{A}$ without sunflowers in this case is $|\mathcal{A}| \leq 2(s-1)^2$, and is trivial to prove. The precise upper bound in this case was first achieved in [5].

The exact bound for the general case, with arbitrary d, m, was first obtained in [4]. In this paper, we give different proof of the same result. Our proof is more 'structural' in approach as opposed to the methods in [4]. Moreover, our method enables us to give a simple characterization of all the cases where the extremal graph is unique.

2 Preliminaries

Throughout this paper, $V(G)$ shall denote the set of vertices of a graph G, and $E(G)$, the set of edges. A *Matching* of a graph G is a set of edges which are pairwise disjoint. A *Perfect Matching* of G is a matching $\mathcal{M}$ with every vertex of G incident with some edge of $\mathcal{M}$.
For $v \in V(G)$, we denote by $G \setminus v$, the induced subgraph of G on $V(G) \setminus \{v\}$. We call a vertex v *unsaturated* relative to a matching $\mathcal{M}$ if v is not covered by $\mathcal{M}$.

Definition 1 *: Let $d \geq 2$, $m \geq 2$ be integers. Let $\mathcal{F}(d, m)$ be the family of all finite graphs satisfying*

(1) $\Delta(G) < d$ and $\nu(G) < m$,
(2) $|E(G)|$ is maximal with property (1), i.e., if G' is a graph with $|E(G')| > |E(G)|$ then either $\Delta(G') \geq d$ or $\nu(G') \geq m$.

It is easy to see that for any $G \in \mathcal{F}(d, m), |E(G)| < \infty$. Indeed, if χ' denote the edge- chromatic index and ν denote the size of a maximum matching of G, the well-known theorem of Vizing (see [7]) states that $\chi' \leq \Delta(G) + 1$, so that $|E(G)| \leq \chi'\nu \leq (\Delta + 1)\nu \leq d(m-1)$.

Let $e(d, m) := |E(G)|$ for $G \in \mathcal{F}(d, m)$.

Our goal is to obtain a precise upper bound for $e(d, m)$. In particular, for the case where $d = m = s$, we are dealing with the Erdös-Rado problem of maximal set systems without sunflowers with s petals for the case $k = 2$.

Definition 2 *A graph G is called* Factor-Critical *if $G \setminus v$ has a perfect matching for all $v \in V(G)$.*

One of the most important results regarding factor critical graphs is Gallai's lemma; a proof of Gallai's lemma can be found in [6].

Theorem 3 *(Gallai's Lemma): If G is a simple connected graph such that for all $v \in V(G)$, $\nu(G) = \nu(G \setminus v)$ then G is factor-critical and hence $|V(G)| = 2\nu(G) + 1$.* □

Definition 4 *By a d-*star *of G for a graph G, we shall mean a connected component of G which is isomorphic to $K_{1,d-1}$.*

Let $s_d(G)$ denote the number of d-stars in a graph G. Note that if G satisfies $\Delta(G) < d$ and $\nu(G) < m$, then $s_d(G) \leq m - 1$. In particular,

$$\sup_{G \in \mathcal{F}(d,m)} s_d(G) \leq m - 1.$$

The following proposition is a vital observation vis-à-vis our estimate for $e(d, m)$. Note that since the family $\mathcal{F}(d, m)$ is finite, there is some $G \in \mathcal{F}(d, m)$ such that $s_d(G) = \sup_{G \in \mathcal{F}(d,m)} s_d(G)$.

Proposition 5 *: Let $G \in \mathcal{F}(d, m)$ be an extremal graph such that $s_d(G) = \sup_{G \in \mathcal{F}(d,m)} s_d(G)$. Then each connected component of G which is not a d-star is factor-critical.*

Proof: Suppose the contrary. Let $\mathcal{C}$ be a connected component of G which is not a d-star and not factor-critical. By Gallai's lemma there is a vertex $v \in V(\mathcal{C})$ such that $\nu(G \setminus v) < \nu(G)$.

Now let consider the graph G' consisting of the disjoint union of $G \setminus v$ and a d-star. Note that

$$\begin{align}
\nu(G') &\leq \nu(G \setminus v) + 1 \leq \nu(G) < m \tag{1}\\
\Delta(G') &\leq \max(d - 1, \Delta(G \setminus v)) = d - 1. \tag{2}
\end{align}$$

(2) follows from the assumption that $G \in \mathcal{F}(d, m)$. Further, $|E(G')| = |E(G \setminus v)| + d - 1 \geq |E(G)|$ since $|E(G \setminus v)| \geq |E(G)| - (d - 1)$. Consequently, $G' \in \mathcal{F}(d, m)$. However, $s_d(G') = s_d(G) + 1$. But this contradicts the assumption that $s_d(G) = \sup_{H \in \mathcal{F}(d,m)} s_d(H)$ and that completes the proof. □

3 An upper bound for $e(d, m)$

Let G be an extremal graph as defined in the proposition of the previous section,i.e., let G be such that $s_d(G)$ is maximum among all the graphs in $\mathcal{F}(d, m)$. By the last proposition of the preceding section it follows that G has two kinds of components:

(1) d-stars,
(2) Factor-critical compoents.

Let us denote by $\mathcal{FC}$, the set of factor-critical components in G.

Gallai's lemma assures us that for each component $\mathcal{C} \in \mathcal{FC}$, $\mathcal{C}$ has *exactly one unsaturated vertex*. Hence if $\mathcal{C} \in \mathcal{FC}$ has r matching edges in a maximum matching of G, then we have

$$|V(\mathcal{C})| = 2r + 1 \text{ and } |E(\mathcal{C})| \leq \min\left\{(2r + 1)r, \left\lfloor \frac{(2r + 1)(d - 1)}{2} \right\rfloor\right\},$$

where, as always, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Thus, if G has t components which are d-stars and k components $\mathcal{C}_i \in \mathcal{FC}$, with $|\nu(\mathcal{C}_i)| = r_i$, then we have $|E(G)| \leq e_0(G) = e_0(t, k, \{r_i\}_{i=1,2\ldots,k})$ where

$$e_0(G) := (d - 1)t + \sum_{i=1}^{k} \min\{(2r_i + 1)r_i, \lfloor \frac{(2r_i + 1)(d - 1)}{2} \rfloor\}.$$

Note that the parameters are subject to the constraint, $t + \sum_{i=1}^{k} r_i = m - 1$.

We now estimate the maximum value of $e(G)$ subject to the linear constraint, $t + \sum_{i=1}^{k} r_i = m - 1$. The rest of this section repeatedly uses the same simple idea and the calculations that follow are rather routine. Our calculations will also play an important role in characterizing some of the extremal graphs in the final section. We have included all the details for the sake of completeness.

We start with a few observations in order to rewrite the linear constraint and the expression for the number of edges.

- Note that for $r_i \geq \lceil \frac{d-1}{2} \rceil$, we have

$$(2r_i + 1)r_i \geq (2r_i + 1)\lceil \frac{d - 1}{2} \rceil \geq \frac{(2r_i + 1)(d - 1)}{2} \geq \lfloor \frac{(2r_i + 1)(d - 1)}{2} \rfloor.$$

- If for any $i, r_i > \lceil \frac{d-1}{2} \rceil$, then we can redefine parameters t and r_i in the expression of $e(G)$ and the value of $e(G)$ is unaltered. More precisely, let

$t' = t+1, r_i = r_i - 1$. Since the linear constraint is satisfied, the change in e_0 is

$$(d-1)(t+1) + \lfloor \frac{(2r_i-1)(d-1)}{2} \rfloor - (d-1)t - \lfloor \frac{(2r_i+1)d-1}{2} \rfloor.$$

Thus

$$\begin{aligned}\text{Change in } e_0 &= d-1 + \lfloor \frac{(2r_i-1)(d-1)}{2} \rfloor - \lfloor \frac{(2r_i+1)d-1}{2} \rfloor \\ &= d-1 + \lfloor (r_i-1)(d-1) + \frac{d-1}{2} \rfloor - \lfloor r_i(d-1) + \frac{d-1}{2} \rfloor \\ &= d-1-(d-1) = 0.\end{aligned}$$

In view of the preceding observations we can assume without loss of generality that $r_i \le \lceil \frac{d-1}{2} \rceil$ for all $1 \le i \le k$.
Let $J := \{1 \le i \le k \mid r_i = \lceil \frac{d-1}{2} \rceil\}$ and let $I := \{1, 2 \ldots, k\} \setminus J$.

We consider the cases of d odd and d even separately.

CASE I: d is odd: Let $d = 2j+1$ for some non-negative integer j. Then

$$e_0 = t(2j) + \sum_{i \in I} (2r_i+1)r_i + (2j+1)j|J|$$

with I, J as described above, subject to the constraint,

$$t + \sum_{i \in I} r_i + j|J| = m-1$$

with $r_i \le j-1$ for all i.

Suppose $r_1 > 0$. Let $t' := t + r_1, r'_1 = 0$ and $r'_i = r_i$ for $i > 1$. Consider the parameters $(t', \{r'_i\}, J)$. If e' denotes the corresponding value of $e(G)$(as a function of these values), then $e' - e_0 = (t+r_1)(2j) - 2jt - (2r_1+1)r_1 = r_1(2j - 2r_1 - 1) \ge r_1(2j - 2j + 2 - 1) > 0$ and that contradicts the maximality of e_0.
Hence $r_i = 0 \ \forall \ i \in I$.

Thus we have the linear constraint $t + j|J| = m-1$ and we wish to maximize $(2j)t + (2j^2+j)|J|$.
It follows from elementary calculus that the maximum occurs at one of the extreme points, i.e., when $|J| = 0, t = m-1$ or when $|J| = \lfloor \frac{m-1}{j} \rfloor, t = m-1-j\lfloor \frac{m-1}{j} \rfloor$. A direct calculation shows that the maximum occurs precisely when $|J| = \lfloor \frac{m-1}{j} \rfloor, t = m-1-j\lfloor \frac{m-1}{j} \rfloor$ and maximum $e_0 = 2j(m-1) + j\lfloor \frac{m-1}{j} \rfloor$.

CASE II: d is even. Suppose $d = 2j$. In this case

$$e_0 = t(2j-1) + \sum_{i\in I}(2r_i+1)r_i + (2j^2-1)|J|$$

subject to the linear constraint

$$t + \sum_{i\in I} r_i + j|J| = m-1,$$

again with $r_i \le j-1$ for all i.
As before, suppose $r_1 > 0$. Then, defining t', r'_i and e' exactly as before, we note that $e' - e_0 = (t+r_1)(2j-1) - (2j-1)t - (2r_1+1)r_1 = r_1(2j-2r_1-2) \ge 0$. So here we can assume without loss of generality that $r_i = 0$ for all $i \in I$(note that equality can occur in the above chain of inequalities).

Once again we are reduced to the case of maximizing $t(2j-1) + (2j^2-1)|J|$ subject to the constraint $t + j|J| = m-1$. Exactly as before we see that the maximum occurs at one of the extremities, i.e., at $t = m-1, |J| = 0$ or when $t = m-1-j\lfloor \frac{m-1}{j} \rfloor, |J| = \lfloor \frac{m-1}{j} \rfloor$. It is again trivial to see that the maximum occurs when $t = m-1-j\lfloor \frac{m-1}{j} \rfloor, |J| = \lfloor \frac{m-1}{j} \rfloor$ and in this case $e_0 = (2j-1)(m-1) + (j-1)\lfloor \frac{m-1}{j} \rfloor$.

So we finally have in either case, $e_0 = (d-1)(m-1) + \lfloor \frac{m-1}{\lceil \frac{d-1}{2} \rceil} \rfloor \lfloor \frac{d-1}{2} \rfloor$.

4 Extremal Graphs with $e(d,m)$ edges

To see that $e_0 = e(d,m)$[1] we construct graphs $G \in \mathcal{F}(d,m)$ such that $|E(G)| = e_0$. Firstly we construct a factor critical component $\mathcal{C}$ with $\nu(\mathcal{C}) = \lceil \frac{d-1}{2} \rceil$, $\Delta(\mathcal{C}) < d$ and $|E(\mathcal{C})|$ as large as possible.
For $d = 2j+1$, let $\mathcal{C} = K_d$ the complete graph on d vertices. It is easy to see that the $\mathcal{C}$ satisfies all the conditions above. Moreover, $\mathcal{C}$ is clearly the unique factor critical graph satisfying

$$\nu(\mathcal{C}) = \left\lceil \frac{d-1}{2} \right\rceil, \tag{3}$$

$$\Delta(\mathcal{C}) < d, \tag{4}$$

$$|E(\mathcal{C})| = \lfloor \frac{(2(\lceil \frac{d-1}{2} \rceil) + 1)(d-1)}{2} \rfloor. \tag{5}$$

[1] So far we only have $e(d,m) \le e_0$

If $d = 2j$ then let $\mathcal{E}$ be the graph obtained from a complete graph K_{2j} by removing a maximum matching $\mathcal{M}$. Connect any of the $2j-1$ vertices of $\mathcal{E}$ to a new vertex, $v \notin V(\mathcal{E})$. This new graph $\mathcal{C}$ clearly again satisfies all the requirements. To see that such a factor critical graph is unique (up to isomorphism) note that there is a unique vertex v of degree $2j-2$ among the $2j+1$ vertices of $\mathcal{C}$. Hence there is a (unique) vertex u in $V(\mathcal{C})$ which is not a neighbor of v. $\mathcal{C} \setminus u$ would be a regular graph of degree $2j-2$ on $2j$ vertices and hence its complement is a regular graph of degree one, namely, a matching of a complete graph on $2j$ vertices. This demonstrates the uniqueness of $\mathcal{C}$.

Now let G be the disjoint union of t d-stars with $t = m - 1 - \lceil\frac{d-1}{2}\rceil\lfloor\frac{m-1}{\lceil\frac{d-1}{2}\rceil}\rfloor$ and $\lfloor\frac{m-1}{\lceil\frac{d-1}{2}\rceil}\rfloor$ components isomorphic to $\mathcal{C}$ as defined above. The graph G has e_0 edges and satisfies $\Delta(G) \leq d-1, \nu(G) \leq m-1$.

Before we proceed to the final section, we make one important remark. While it is clear that any extremal graphs $G \in \mathcal{F}(d,m)$ with $d > 2$ necessarily satisfies $\nu(G) = m-1$, since we can always add isolated edges without violating either condition. However, extremal graphs $G \in \mathcal{F}(d,m)$ *need not satisfy* $\Delta(G) = d-1$. Indeed, if $d = 2m$, the graphs K_{2m-1} satisfy the required conditions and $e(2m,m) = |E(K_{2m-1}|$. However $\Delta(K_{2m-1}) = 2m-2 \neq d-1$ in this case.
We make another observation here that is relevant to us: For any graph G, we have

$$|E(G)| \leq (d-1)(m-1) + \lfloor\frac{m-1}{\lceil\frac{d-1}{2}\rceil}\rfloor\lfloor\frac{d-1}{2}\rfloor \leq (m-1)\left(d-1+\frac{\lfloor\frac{d-1}{2}\rfloor}{\lceil\frac{d-1}{2}\rceil}\right).$$

This follows as a simple consequence of our bound.

5 Unique graphs in $\mathcal{F}(d,m)$ with $e(d,m)$ edges

Now we characterize all the cases where the extremal graphs with no isolated vertices and with $e(d,m)$ edges are unique up to isomorphism.
For $d = 2$, the graph is easily seen to consist of $m-1$ copies of K_2.
If $m = 2$ and $d \neq 4$ then it is easy to see that a graph with $e(d,2) = d-1$ edges is a d-star.
Suppose $\lceil\frac{d-1}{2}\rceil$ divides $m-1$. In this case, following the calculation of e_0 we see that $t = 0$ (where t is as in section 4), hence there are no d-stars in any $G \in \mathcal{F}(d,m)$. Furthermore, $\nu(G) = \nu(G \setminus v)$ for all $v \in V(G)$ for $G \in \mathcal{F}(d,m)$. Gallai's lemma implies that all components of G are factor critical and hence if G has no isolated vertices, it follows that G is the graph with $\lfloor\frac{m-1}{\lceil\frac{d-1}{2}\rceil}\rfloor$ components, each isomorphic to $\mathcal{C}$, where $\mathcal{C}$ is the graph described

in the previous section. The uniqueness of G follows from the following two facts:

- For any $G \in \mathcal{F}(d,m)$, $\Delta(G) = d-1$ (contrast this with the last remark of the preceding section). This follows because if $\Delta(G) \leq d-2$, then $G \in \mathcal{F}(d-1,m)$ and so by the last inequality of the preceding section,

$$\frac{|E(G)|}{m-1} \leq d-2+\frac{\lfloor\frac{d-2}{2}\rfloor}{\lceil\frac{d-2}{2}\rceil} < d-1+\frac{\lfloor\frac{d-1}{2}\rfloor}{\lceil\frac{d-1}{2}\rceil} = \frac{|E(G)|}{m-1},$$

  and that is a contradiction. The last equality in the equation above follows because $\lceil\frac{d-1}{2}\rceil$ divides $m-1$.
- For any factor critical graph $\mathcal{C}$ with maximum degree Δ and maximum matching size ν,

$$r(G) := \frac{|E(G)|}{\nu(G)} \leq \Delta + \frac{\lfloor\Delta/2\rfloor}{\lceil\Delta)/2\rceil}.$$

  This follows as a consequence of the bound $e(d,m)$. Moreover, equality occurs if and only if $\nu = \lceil\frac{d-1}{2}\rceil$.
- $\mathcal{C}$ is the unique factor critical graph with $\lfloor\frac{(2\lceil\frac{d-1}{2}\rceil+1)(d-1)}{2}\rfloor$ edges, with matching size $\lceil\frac{d-1}{2}\rceil$, and maximum degree $d-1$.

However if $\lceil\frac{d-1}{2}\rceil$ does not divide $m-1$, $m>2$ and $d>2$ then many graphs achieve the bound $e(d,m)$. In this case $t \neq 0$ and G may include at least two components:

- two d-stars,
- a d-star and another component isomorphic to $\mathcal{C}$ as described in the previous section.

We can always coalesce the two components mentioned above into a single component without loss of number of edges. If there are two d-stars H_1, H_2, one could remove an edge of H_1 and then connect the vertex of degree $d-2$ to any of the degree one vertices of H_2. Note that the new graph thus formed has exactly the same number of edges and so is again a member of $\mathcal{F}(d,m)$. If there is a d-star and a component isomorphic to $\mathcal{C}$ then we can form a new component with $2(\lceil\frac{d-1}{2}\rceil+1)+1$ vertices and maximum degree less than d, having number of edges equal to the total number of edges in the original two components.
In the special case $d=m=s$, we see that the maximum is $e(s,s)=s(s-1)$ when s is odd and $\lfloor\frac{(2s-1)(s-1)}{2}\rfloor$ when s is even. Furthermore, when s is odd and $s=d=m$ then clearly, $\lceil\frac{d-1}{2}\rceil = \frac{s-1}{2}$ divides $m-1=s-1$ and so in this case the maximum is attained when the graph is isomorphic to two disjoint copies of K_s. In the case of s even, there are several graphs that attain the bound.

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