New infinite families of Candelabra Systems with block size 6 and stem size 2

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Abstract

In this paper, we seek to generalize a construction of Hanani which produces a Steiner 3-design on a larger set of points starting from a smaller Steiner 3-design, with constant block size 6. We achieve this by constructing new infinite families of uniform Candelabra Systems with block size 6 and stem size 2.

Keywords: Steiner 3-designs, Candelabra systems, Lattice Candelabra systems, Lattice Transversal designs.

1 Introduction

Let v, k, t be positive integers. By a Steiner t-design (denoted $t-(v,k,1)$ or $S(t,k,v)$), we mean a pair $(X, \mathcal{B})$, where X is a finite set of size v and $\mathcal{B}$, a collection of k-subsets of X such that the following holds: For any subset T of X of size t, there is a unique element $B \in \mathcal{B}$ such that $T \subset B$. By elementary counting arguments, one can show that the following arithmetic conditions are necessary in order that a Steiner design $S(t,k,v)$ exists:

$$\binom{v-i}{t-i} \equiv 0 \pmod{\binom{k-i}{t-i}} \quad \text{for } i = 0, 1, \ldots, t-1.$$

It is known that these conditions, though necessary, are by no means sufficient. Indeed, there is no $S(2,6,36)$ since such a Steiner 2-design is an affine plane of order 6, and the nonexistence of an affine plane of order 6 is a well known consequence of the Bruck-Ryser-Chowla theorem.

Professor Dijen Ray-Chaudhuri conjectures (the 'v-large' conjecture) that the necessary conditions are however, 'asymptotically' sufficient, i.e., for fixed values of k, t, there exists $v_0(k,t) \in \mathbb{N}$

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such that for all $v \geq v_0$ and v satisfying the necessary conditions, a Steiner design $S(t, k, v)$ exists.

We are interested in the case $t = 3$ in this paper. The question of the existence of Steiner 3-designs has been completely resolved only for the case $k = 4$ (see [6]): There exists a Steiner design $S(3, 4, v)$ if and only if $v \equiv 2 \pmod 6$ or $v \equiv 4 \pmod 6, v \geq 4$.
For $t \geq 3, k \geq 5$, the problem is wide open. Even the task of constructing new Steiner designs has proven to be extraordinarily difficult.

One important technique for constructing new families of Steiner 3-designs starts with Steiner designs $3-(q^n+1, q+1, 1)$, where q is a prime power. These designs come about as a consequence of the 3-transitive action of the group $\mathrm{PGL}(2, q^n)$ (or the 3-transitive group, the 'twisted' $\mathrm{PGL}(2, q^2)$) on the finite projective line. Building upon these designs, one can obtain Steiner 3-designs on larger sets of points by what are essentially combinatorial constructions. A prototype result of this form is the following theorem:

Theorem 1 *[9] (Product theorem): If there exist Steiner* 3*-designs* $3-(a+1, q+1, 1), 3-(b+1, q+1, 1)$*, then there also exists a Steiner* 3*-design,* $3-(ab+1, q+1, 1)$*, where* q *is a prime power.*

The following 'mock-product' theorems are also known:

Theorem 2 *([3],[9])(Mock-product theorem) : Let* q *be a prime power and* a *be a positive integer. Suppose there exists a Steiner design* $S(3, q+1, a+1)$*. Then there exist integers* v_0 *and* $d_0 = d_0(q, v)$ *such that for any positive integer* $v \geq v_0$ *satisfying*

$$\begin{aligned} v-1 &\equiv 0 \pmod{q-1}, \\ v(v-1) &\equiv 0 \pmod{q(q-1)}, \\ v(v^2-1) &\equiv 0 \pmod{q(q^2-1)}, \end{aligned}$$

there is a Steiner design $S(3, q+1, va^d+1)$ *whenever* $d \geq d_0$*.*

Note that the mock-product theorem would follow from the product theorem as a consequence, if one proves the 'v-large' conjecture. Thus, these mock product theorems lend further support to the validity of the 'v-large' conjecture.

All the aforementioned product (and 'mock-product') theorems essentially construct certain Candelabra systems. Candelabra systems were first studied by A. Hartman ([7]) for block size 4 and then by Dijen Ray-Chaudhuri and Mohácsy for an arbitrary block size ([8]). Though their definitions are quite general, the Candelabra systems constructed concretely in these papers have a restriction: The stem size (We shall formally define this in the following section when we make the formal definitions) for all these Candelabra systems is one.

Our starting point is to construct new Candelabra systems with stem size 2 for the case $k = 6$. Having done so, we generalize the following theorem of Hanani:

Theorem 3 *('*$4v+2$*-theorem', [5]): Suppose there exists a Steiner design* $3-(v+1, 6, 1)$*. Then there exists a Steiner design* $3-(4v+2, 6, 1)$*.*

This enables us to obtain a new 'mock-product' theorem which gives us several infinite families of Steiner 3-designs with block size 6. Finally, we say a few words on attempts to attack the existence problem for Steiner 3-designs in general.

2 Preliminaries

We start with a few definitions. Though one can define t-Candelabra systems, we restrict our attention to the case $t = 3$.

Definition 4 *(**Candelabra system**, [8]): Let v, k be positive integers. A **Candelabra system (or CS)** of order v and block size k is a quadruple $(X, S, \Gamma, \mathcal{A})$ satisfying:*

- *X is a subset of v elements (also called **points**),*
- *S is a subset of X of size s (called the **stem** of the Candelabra),*
- *$\Gamma := \{G_1, G_2 \ldots\}$ is a partition of $X \setminus S$ into non-empty subsets (called **groups**),*
- *$\mathcal{A}$ is a family of k-subsets of X,*
- *Every subset $T \subset X$ of size 3 satisfying $|T \cap (S \cup G_i)| < 3$ for each i, is contained in precisely one block of $\mathcal{A}$ and any other 3-subset of X is not contained in any block.*

By the group type of a CS $(X, S, \Gamma, \mathcal{A})$, we mean the list $(|G|, G \in \Gamma : |S|)$ (the stem size is separated by a colon). If a CS has n_i groups of size g_i and has $|S| = s$, then we notate the group type as $(g_1^{n_1} g_2^{n_2} \ldots : s)$. If all the groups are of the same size, then the Candelabra system is called uniform.

A straightforward relation between Steiner 3-designs and Candelabra systems is as follows. Suppose there exists a Steiner 3-design on $v + 1$ points with block size k. Then there exists a Candelabra system on $v + 1$ points with group type $(1^v : 1)$ with block size k. This is clear: Suppose $(X, \mathcal{B})$ is a Steiner 3-design. Let $x_0 \in X$ be a fixed point. Consider the partition $\Gamma := \{\{x\} : x \in X, x \neq x_0\}$ of the set $X \setminus \{x_0\}$. Then $(X, \{x_0\}, \Gamma, \mathcal{B})$ is a Candelabra system with block size k and group type $(1^v : 1)$.

A less trivial connection between Steiner 3-designs and Candelabra systems comes through the following proposition.

Proposition 5 *: Let k be a positive integer, $k \geq 2$. Suppose there exists a Steiner design $S(3, k + 1, k^2 + 1)$. Then there exists a CS of group type $(k^k : 1)$ with block size $k + 1$. Conversely, if there exists a CS of group type $(k^k : 1)$ with block size $k+1$, there exists a Steiner design $S(3, k+1, k^2+1)$.*

Proof: Suppose $(X, \mathcal{B})$ is the given Steiner 3-design, where $|X| = k^2 + 1$. Fix a point $\infty \in X$. The derived design at the point ∞ is an affine plane of order k and hence admits a parallel class of

blocks, $B_1, B_2, \ldots, B_k$. Then clearly, $\Gamma := \{B_1, B_2, \ldots, B_k\}$ is a partition of the set $X \setminus \{\infty\}$. Let $\mathcal{A} := \mathcal{B} \setminus \{B_1 \cup \{\infty\}, B_2 \cup \{\infty\}, \ldots, B_k \cup \{\infty\}\}$. It is clear that $(X, \{\infty\}, \Gamma, \mathcal{A})$ is a CS with block size $k+1$ and group type $(k^k : 1)$.
The converse is equally simple and we omit the details. $\square$

Closely related objects to CS systems are **Lattice Candelabra systems**. These are basically uniform Candelabra systems with some additional structure on the point set and the block collection. However, these have stem size either zero or one.

Definition 6 *(**Lattice Candelabra system or LCS**,[8]): Let k, n, m be positive integers. Let $(X, S, \Gamma, \mathcal{A})$ be a uniform candelabra system of group type $(m^n : |S|)$ over the set $X = I_m \times I_n \cup S$ with $S = \emptyset$ or $S = \{\infty\}$. $(X, S, \Gamma, \mathcal{A})$ is called a **Lattice Candelabra system (or LCS)** if for any block $A \in \mathcal{A}$, either $|A \cap (C_j \cup S)| < 3$ or $A \subseteq C_j \cup S$, where $C_j = \{j\} \times I_n$.*

Note that 'trivial' Lattice Candelabra systems can be constructed from Steiner 3-designs in a straightforward fashion. If $(X, \mathcal{B})$ is a Steiner 3-design, with $|X| = v+1$ and block size k, then let $x_0 \in X$ be the stem, and consider the partition $\Gamma := \{\{x\} : x \in X, x \neq x_0\}$. It is clear that $(X, \{x_0\}, \Gamma, \mathcal{B})$ is a Lattice Candelabra system with group type $(1^v : 1)$. Note that in this construction, we have $m = 1, n = v$. Though trivial, this observation is handy, as we shall see.
However, these are not the only instances of LCS.

Proposition 7 *(Existence of nontrivial LCS[1]): Suppose that for a positive integer k, there exists a Steiner design $S(3, k+1, k^2+1)$. Then there exists a $3-LCS$ with block size $k+1$ and group type $(k^k : 1)$.*

Proof: It suffices to construct a Candelabra system $(X, S, \Gamma, \mathcal{A})$ with group type $(k^k : 1)$ and block size $k+1$ such that S is a singleton set, and the Candelabra system admits a collection of k blocks $B_1, B_2, \ldots, B_k \in \mathcal{A}$ satisfying $B_i \cap B_j = S$ for any $i \neq j, 1 \leq i, j \leq k$.
To do that, note that there exists a $3-CS$ $(X, S, \Gamma, \mathcal{A})$ with group type $(k^k : 1)$, stem S of size one and block size $k+1$ as seen earlier. Observe that the Steiner 3-design $S(3, k+1, k^2+1)$ has as its derived design, an affine plane of order k. The partition Γ in the CS corresponds to a parallel class of blocks in the derived design. Now choose a second parallel class of blocks, Γ' of the derived design. The blocks in the CS corresponding to Γ' clearly satisfy the requirement above and that completes the proof. $\square$.

Since for each prime power q, there exists a Steiner design $S(3, q+1, q^2+1)$, there exist a LCS with group type $(q^q : 1)$ and block size $q+1$, whenever q is a prime power.

We now recall the definition of another very important class of combinatorial structures which are crucial in the construction of Steiner designs, namely, Group Divisible designs, and Transversal Designs.

[1]This also appears in [8].

Definition 8 ***(Group Divisible design or t-GDD)****: Let v, k be positive integers. A* ***t-Group Divisible Design(or t-GDD)*** *is a triple $(X, \Gamma, \mathcal{A})$ satisfying*

- *X is a set of size v(elements are called* ***points****),*
- *$\Gamma = \{G_1, G_2, \ldots\}$ is a partition of X into non-empty subsets (called* ***groups****),*
- *$\mathcal{A}$ is a family of subsets of X(called* ***blocks****), each of cardinality k such that, each block intersects any group in fewer than two points,*
- *Each t-set of points with each point from a different group is contained in exactly one block.*

A t-Group Divisible design with block size k and l groups, each of size k, is called a t-Transversal Design (denoted TD(t, k, l)). Analogously, we define Lattice Group Divisible designs (LGDD) and Lattice Transversal designs (LTD):

Definition 9 ***(Lattice Group Divisible design or LGDD)****: Let t, k, n be positive integers with $1 \leq t \leq k$. Let $(X, \Gamma, \mathcal{A})$ be a uniform t-Group Divisible design of group type (m^n) constructed over the set $X := I_m \times I_n$, where $\Gamma = \{R_i : i = 1, 2 \ldots, n\}$, where $R_i := I_m \times \{i\}$. The triple $(X, \Gamma, \mathcal{A})$ is called a Lattice Group Divisible design (or $t - LGDD$) if for any block $A \in \mathcal{A}$ and any set $C_j := \{j\} \times I_n, j = 1, 2 \ldots, m$, either $|A \cap C_j| < t$ or $A \subseteq C_j$. A Lattice Group Divisible design with block size k and k groups of size m is called a Lattice Transversal design and is denoted LTD(t, k, m).*

Group divisible designs and transversal designs play a very important role in the construction of Candelabra systems. The following theorem, called the fundamental construction for Candelabra systems, appears in [8]; this is essentially a generalization of a construction of Hartman (see [7]) for Steiner Quadruple systems.

Theorem 10 ***(The fundamental construction for Candelabra systems****,[8]): Let $(X, S, \Gamma, \mathcal{A})$ be a Candelabra system with $S = \{s_0\}$ (called the 'ingredient CS) and let K be a set of positive integers. Let $\omega : X \to \mathbb{N} \cup \{0\}$ be a function(called a weight function). There exists a CS with block sizes in K and group type $(\sum_{x \in G} \omega(x) | G \in \Gamma : s)$ if the following conditions hold:*

- *For every block A of the CS not containing s_0, there exists a $3 - GDD$ with block sizes in K and group type $(\omega(x) | x \in A, \omega(x) \neq 0)$,*
- *For each block of the 'ingredient' candelabra system containing s_0, there is a CS with block sizes in K and group type $(\omega(x) | x \in A, x \neq s_0, \omega(x) \neq 0 : s)$.*

3 The Theorems

The main theorems of this paper are the following:

Theorem 11 *(CS theorem): There exists a Candelabra system with* 5 *groups with group size* $\frac{4^m-4}{3}$ *and stem size* 2*, and block size* 6*, for* $m \geq 2$*.*

Having obtained this, we prove a lemma,

Lemma 12 *(Steiner design lemma): Suppose there exist Steiner designs* $S(3,k,l+2), S(3,k,v+1)$*, a CS with group type* $(l^{k-1}:2)$ *and a TD*$(3,k,l)$*, then a Steiner design* $S(3,k,vl+2)$ *exists.*

which leads us to new families of Steiner designs with block size 6.

Theorem 13 *Let* $k \geq 2$ *be an integer and suppose* v *is a positive integer satisfying*

$$v \equiv 1 \pmod{20} \text{ or} \tag{1}$$
$$v \equiv 5 \pmod{20}. \tag{2}$$

Then, there exist integers m_0, v_0*, and* $d_0(v)$ *such that whenever* $m \geq m_0, v \geq v_0$ *and* $d \geq d_0(v)$*, there exist*

- *Steiner designs* $S(3,6,lv(4 \cdot 5^k+1)^d+2)$ *where* $l = \frac{4^m-4}{3}$*.*
- *Steiner designs* $S(3,6,lv \cdot 5^m+2)$

for $m \geq m_0(v)$ *for some fixed integer* $m_0(v)$*.*

4 The Proofs

We begin with the proof of the Steiner design lemma. By I_v we shall denote the set $I_v := \{1,2,\ldots,v\}$.

Proof of Steiner design lemma: We need to construct a Steiner design $S(3,k,vl+2)$ which we shall do so on the point set $X := I_l \times I_v \cup \{\infty_1, \infty_2\}$ of size $lv+2$. Since there exists a Steiner design $S(3,k,v+1)$, there also exists a CS $(Y, \{\infty\}, \mathcal{C})$ with group type $(1^v:1)$ on the point set $Y := I_v \cup \{\infty\}, |Y| = v+1$ by the observation following the definition of a Candelabra system in the preceding section.
Let $\mathcal{C}_\infty := \{B \in \mathcal{C} | \infty \in B\}$. Consider the weight function ω defined by $\omega(\infty) := 2, \omega(i) := l$ for $1 \leq i \leq v$. For each block $B \in \mathcal{C}_\infty$, let $Y_B := I_l \times (B \setminus \{\infty\}) \cup \{\infty_1, \infty_2\}$ and let $(Y_B, \mathcal{B}_B)$ be a CS with group type $(l^{k-1}:2)$ (this exists by the hypothesis). Similarly, for each block $B \in \mathcal{C} \setminus \mathcal{C}_\infty$ let $(Y_B, \mathcal{C}_B)$ be a TD$(3,k,l)$ (which again exists, by the assumption). Let

$$\mathcal{B} = (\bigcup_{B \in \mathcal{C}_\infty} \mathcal{B}_B) \cup (\bigcup_{B \in \mathcal{C} \setminus \mathcal{C}_\infty} \mathcal{C}_B).$$

By the fundamental construction for Candelabra systems, it follows that $(X, \mathcal{B})$ is a CS with group type $(l^v : 2)$. For each $1 \leq i \leq v$, let $(X_i, \mathcal{B}_i)$ be a Steiner 3-design $S(3, k, l+2)$ on the set $X_i := I_l \times \{i\} \cup (\{\infty_1, \infty_2\})$. Such a Steiner design again exists by the assumption of the existence of a Steiner design $S(3, k, l+2)$.

Finally let

$$\mathcal{A} := \mathcal{B} \cup \left(\bigcup_{i \in I_v} \mathcal{B}_i \right).$$

We claim that the pair $(X, \mathcal{A})$ gives us a Steiner design $S(3, k, lv+2)$.
Let x, y, z be distinct elements of X. Suppose, first that x, y, z are all distinct from ∞_1, ∞_2. If not all x, y and z lie in a subset of the form $I_l \times \{a\}$ for some $a \in I_v$, then there exists a unique block $B \in \mathcal{B}$ such that $x, y, z \in B$. If $x, y, z \in I_l \times \{a\}$ for some $a \in I_v$, then there exists a unique block $B \in \mathcal{B}_a$ such that $x, y, z \in B$.
If $\{x, y, z\} \cap \{\infty_1, \infty_2\} \neq \emptyset$, two cases arise:

1. $x = \infty_1, \infty_2 \neq y, z$: Suppose $y \in I_l \times \{a\}$ and $z \in I_l \times \{b\}$ with $a \neq b, a, b \in I_v$ then there exists a unique block (in the candelabra system) $B \in \mathcal{B}$ such that $\infty_1, y, z \in B$. If $a = b$, then there exists a unique block $B \in \mathcal{B}_a$ such that $\infty_1, y, z \in B$.

2. $x = \infty_1, y = \infty_2$: In this case if $z \in I_l \times \{a\}$ for some $a \in I_v$, there exists a unique block $B \in B_a$ such that $x, y, z \in B$.

Observe that the collections $\mathcal{B}, \mathcal{B}_i, i \in I_v$ are mutually disjoint, since any block in $\mathcal{B}_i$ is contained in $I_l \times \{i\}$, and any block of $\mathcal{B}$ intersects any set of the form $\{\infty_1, \infty_2\} \cup (I_l \times \{i\}), i \in I_v$ in fewer than 3 points. These observations establish the uniqueness of the block containing the set $\{x, y, z\}$ and so, the proof of the lemma is complete. □

In order to prove the CS theorem, we need a few other results.

Firstly, one can prove a 'Fundamental theorem for LCS' starting with appropriate 'ingredient' designs. Not surprisingly, the ingredient designs here, are LTDs and LCSs. We merely state the result here. One may look up [8] for a proof and further details. We mention in passing that the details are quite similar to that of the fundamental theorem of Candelabra systems.

Theorem 14 *(Fundamental construction for LCS): Let n, a and b be positive integers and let K be a set of positive integers. Let $(X, \{\infty\}, \Gamma, \mathcal{A})$ be a $3-LCS$ of group type $(a^n : 1)$. For any given positive integer b, there exists a $3-LCS$ of group type $((ab)^n : 1)$ with block sizes from K, if the following holds:*

1. *For each block A of the 3-LCS not containing ∞, there is a 3-LGDD of group type $b^{|A|}$ with block sizes from K.*

2. *For each block A of the 3-LCS containing ∞, there is a 3-LCS of group type $(b^{|A|-1} : 1)$ with block sizes from K.*

Thus, this theorem requires the existence of LCS and LGDDs as ingredients in order to be applicable. Note that a Lattice Transversal t-design is equivalent to a Transversal design admitting a parallel class of blocks. Thus if one can exhibit a parallel class of blocks in a Transversal design, one has a Lattice Transversal design.
The following theorem ([8]) provides a nontrivial family of Lattice Transversal designs.

Theorem 15 *(Existence of nontrivial LTDs): Suppose q is a prime power satisfying $(q-1, t-1) > 1$. Then one can obtain a transversal design $TD(t, q+1, q)$ admitting a parallel class of blocks. Hence if q is an odd prime power, then there exists an $LTD(3, q+1, q)$.*

Proof of the CS Theorem: To prove the theorem we first make the following claim:

Claim: There exists a Lattice Candelabra system, CS$(5^{l_0} : 1)$ with block size 6, where $l_0 = \frac{4^m-1}{3}, m \geq 2$.

We use the fundamental construction for LCSs. Firstly, note that there exists a Steiner design $S(3, 6, l_0+1)$ (note that $l_0+1 = \frac{4^m+2}{3}$). This follows by repeated application of the product theorem (theorem 1) and the existence of a Steiner design $3-(22, 6, 1)$. This 3-design occurs as a two point derivation of the well known Steiner 5-design on 24 points[2]. Hence there exists a 3-LCS with group type $(1^{l_0} : 1)$.

By the proposition on the existence of non-trivial LCS, there exists a 3-LCS with group type $(5^5 : 1)$. Again, by the existence of nontrivial LTDs, there exists a LTD(3, 6, 5)., Let $a = 1, b = 5, K = \{6\}, s = 1$, so that the conditions of the fundamental construction for LCS are satisfied. Thus we have a Lattice Candelabra system with group type $(l_0^5 : 1)$ and block size 6.

Now, let $X := I_5 \times I_{l_0}, \Gamma = \{I_5 \times \{i\}, i \in I_{l_0}\}, \mathcal{R} = \{\{i\} \times I_{l_0}, i \in I_5\}$, and let $(X, \{\infty\}, \Gamma, \mathcal{B})$ be a Lattice Candelabra system with group type $(l_0^5 : 1)$ and block size 6. Let us denote the elements of Γ by $G_i, i \in I_{l_0}$ for brevity. Let

$$\mathcal{B}_i = \{B \in \mathcal{B} | B \subset \{\infty\} \cup \{i\} \times I_{l_0}\}, i \in I_5.$$

Let

$$\mathcal{A} = \left(\mathcal{B} \setminus \bigcup_{i \in I_5} \mathcal{B}_i\right) \bigcup_{i \in I_{l_0}} (\{\infty\} \cup G_i).$$

Then $(X, \{\infty\}, \mathcal{R}, \mathcal{A})$ is a Lattice Candelabra system with group type $(5^{l_0} : 1)$ with block size 6.

To see that this is indeed the case, we only need to show that this is a Candelabra system; since $(\{\infty\} \cup G_i), i \in I_{l_0}$ is a block, it follows trivially that it is also a Lattice Candelabra system.

Consider any set $\mathcal{T} = \{x, y, z\}$ of size 3 in X. Suppose $\mathcal{T} \subset \{\infty\} \cup \{i\} \times I_{l_0}$ for some $i \in I_5$, and that if possible we have $\mathcal{T} \subset A$ for some $A \in \mathcal{A}$. Clearly, $A \in \mathcal{B} \setminus \bigcup_{i \in I_5} \mathcal{B}_i$, so $A \in \mathcal{B}$. But since $(X, \Gamma, \{\infty\}, \mathcal{B})$ is a LCS, we have $A \in \mathcal{B}_i$ and that is a contradiction.
We now consider two cases again:

1. $\infty \notin \mathcal{T}$: In this case, if $\mathcal{T} \subset G_i$ for some i, then $\mathcal{T} \subset \{\infty\} \cup G_i$ and this is the unique block of $\mathcal{A}$ containing $\mathcal{T}$. Suppose $\mathcal{T} \not\subset G_i$. Then there exists a unique block $A \in \mathcal{B} \setminus \bigcup_{i \in I_5} \mathcal{B}_i$ such that $\mathcal{T} \subset A$.

[2] Assmus and Key in [1] also construct Steiner 3-designs with these parameters though their construction uses the points and lines of the underlying PG$(m-1, 4)$ in a direct manner.

2. $\mathcal{T} = \{\infty, x, y\}$: Suppose $x, y \in G_i$ for some i. Then again, $\mathcal{T} \subset \{\infty\} \cup G_i$ and this is the unique block of $\mathcal{A}$ containing $\mathcal{T}$. If $x \in G_i, y \in G_j, i \neq j$, then again there exists a unique block $A \in \mathcal{B} \setminus \bigcup_{i \in I_5} \mathcal{B}_i$ such that $\mathcal{T} \subset A$.

That completes the proof of the claim.

Before we complete the proof, we make one other observation. Let $(X, \mathcal{B})$ be the[3] Steiner 3-design $3-(22,6,1)$ which exists, as noted earlier. For any two distinct points $x, y \in X$, let $\mathcal{B}_{x,y} = \{B \in \mathcal{B} | x, y \in B\}$. Since $(X, \mathcal{B})$ is a Steiner design, $\mathcal{B}_{x,y}$ induces a partition Π on the set $X \setminus \{x, y\}$ into 5 groups of size 4 each. Then $(X, \{x, y\}, \Pi, \mathcal{B} \setminus \mathcal{B}_{x,y})$ is clearly a Candelabra system with group type $(4^5 : 2)$ and block size 6.

We finally complete the proof as follows: Let $(X, \{\infty\}, \Gamma, \mathcal{B})$ be a Candelabra system with group type $(l_0^5 : 1)$, with $l_0 = \frac{4^{m-1}-1}{3}$ and the element ∞ being the sole element of the stem. Consider the weight function $\omega : X \to \mathbb{N}$ defined by $\omega(\infty) := 2, \omega(x) := 4$ for $x \in X, x \neq \infty$. Since there exist a CS$(4^5 : 2)$ with block size 6 and a TD$(3,6,4)$ (this is due to Hanani; see [5]) the conditions for the fundamental theorem for Candelabra systems are satisfied and so there exists a CS$(l^5 : 2)$ with block size 6, where $l = \frac{4^m-4}{3}$. □

Before we turn to the proof of Theorem 13, we recall an important result due to Blanchard (see [2]) which states that transversal designs of strength 3 exist for 'sufficiently large' m, for a fixed block size k. More precisely, Blanchard proves in [2], the following result:

Theorem 16 *(Blanchard's theorem for transversal designs [2]): Let k be a given positive integer. There exists an integer $m_0 := m_0(k)$ such that for all integers $m \geq m_0$, there exists a Transversal design TD$(3, k, m)$.*

This is an extension of the well known Chowla-Erdös-Straus theorem for transversal designs of strength two.

Notationally, by writing $m \gg 0$, we mean that m is an element of the set $\{n \in \mathbb{N} : n \geq m_0\}$ for some absolute constant m_0. Thus, Blanchard's result ensures that for $m \gg 0$, there always exists a transversal design TD$(3,6,l)$ with $l = \frac{4^m-4}{3}$.

Proof of Theorem 13: For $l = \frac{4^m-4}{3}$, note that there exists a Steiner 3-design $S(3,6,l+2)$, since $l + 2 = \frac{4^m-4}{3} + 2 = \frac{4^m+2}{3}$ and a Steiner design $S(3, 6, \frac{4^m+2}{3})$ exists for all $m \geq 2$.

The CS theorem yields a Candelabra system with group type $(l^5 : 2)$ and block size 6. By Hanani's '$4v+2$-theorem', the Steiner designs $S(3,6,5^k+1), d \geq 2$, yield the Steiner design S$(3,6,4 \cdot 5^k+1)+1)$. Repeated application of the product theorem gives us Steiner designs $S(3,6,(4 \cdot 5^k + 1)^d + 1)$ for all $k, d \geq 2$. The mock-product theorem (theorem 1) implies that for each fixed $u \gg 0$, and u satisfying the conditions (1), (2) of the statement of theorem 13, there exists $d_0(u)$ such that for $d \geq d_0(u)$, a Steiner design $S(3,6,u \cdot (4 \cdot 5^k+1)^d + 1)$, for all $d \geq d_0(v), m \geq 2$ exists. Also note that by Blanchard's theorem, there exists a TD$(3,6,l)$ if $l \geq L_0$ for some fixed constant L_0. Let

[3] The uniqueness of this design was established first by Witt.

$m_0 = \lfloor \log_4(3L_0 + 4) \rfloor$.
For $m > m_0, l = \frac{4^m - 4}{3}, k = 6, v = u \cdot (4 \cdot 5^k + 1)^d$, the hypotheses of the Steiner design lemma are satisfied, yielding a Steiner design $S(3, 6, lv(4 \cdot 5^k + 1)^d + 2)$ as desired.

For the second family of Steiner 3-designs, let $v = u \cdot 5^m + 1$ in the Steiner design lemma. □

Remark: The product theorem is basically used repeatedly in the last construction. However this construction presents us with certain shortcomings which will be addressed in the final section.

5 Some concrete constructions

The theorem proved in the previous section results in several new Steiner 3-designs with block size 6. We shall indicate a concrete instance of such an infinite family here. The bounds in the result of Blanchard ([2]) are too large and often, one can obtain smaller bounds. The following proposition gives us a concrete result in this context.

Proposition 17 *: Suppose k is a positive integer and q is a prime power with $k < q$. Then there exists a TD(t, k, q). Consequently, if for some m, all the prime factors q of m satisfy $q > k$, then there exits a TD(t, k, m).*

Proof: Consider a TD$(t, q + 1, q)$ which exists for prime power q. Now simply 'puncture' this transversal design at $q + 1 - k$ points to obtain the desired TD(t, k, q). □

Let $l = 340 = \frac{4^5 - 4}{3}$. All the theorems proved in the previous section hold for this particular value of l, except possibly the existence of a TD$(3, 6, 340)$. Now the observation made above implies that we can construct a transversal design TD$(3, 6, 17)$. One can also construct a TD$(3, 6, 5)$,TD$(3, 6, 4)$ (see [5]). Since $340 = 4 \cdot 5 \cdot 17$, one can also construct a TD$(3, 6, 340)$ as a consequence of MacNeish's product theorem for Transversal designs.

Thus there exist Steiner designs $S(3, 6, 340v(4 \cdot 5^k + 1)^d + 2)$ with $v \geq v_0$ and v satisfying (1), (2) of the statement of Theorem 13.

6 Concluding Remarks

Before we conclude, we make a remark on the general problem for constructing Steiner 3-designs. All known combinatorial constructions for Steiner 3-designs generally involve two kinds of constructions:

1. Constructions of Transversal designs/ Group divisible designs,

2. Recursive constructions.

Most constructions of the first type give Steiner designs with parameters of the type in [9]. Improvements in constructions involving the techniques of 'block spreading' (see [3],[9]) can lead to constructions of newer Steiner 3-designs (new parameters).

The harder problem seems to be in doing away with recursive constructions. While recursive constructions appear frequently in constructions of Steiner designs (for instance the fundamental construction for Candelabra systems is essentially a recursive construction), the new parameter values that result from these constructions increase in geometric progression while the necessary arithmetic conditions for the existence of Steiner 3-designs increase in arithmetic progression. What we seem to be missing fundamentally are what can be termed 'adjunction theorems' or 'additive theorems'.

In computational terms, it appears that brute force cannot take us very far in our quest for new Steiner 3-designs. For instance, since the classical Witt designs have block size $q+1$ where q is a prime power, all these combinatorial techniques again yield Steiner 3-designs with block size of the same type. It is indeed curious that there is no single known Steiner 3-design whose block size is not of the aforementioned type. The smallest such integer is $k = 7$ and *there is no known Steiner* 3*-design with block size* 7. The smallest possible such set of parameters for which there potentially exists a Steiner 3-design would be an $S(3,7,77)$. At the moment, it is not clear how one might construct Steiner designs with an arbitrary block size. Such designs are bound to bring in new techniques and ideas.

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